



Numerical Algorithms for Direct Solution of Fourth Order Ordinary Differential Equations

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Abstract

This paper examines the derivation of hybrid numerical algorithms with step length(k) of five for solving fourth order initial value problems of ordinary differential equations directly. In developing the methods, interpolation and collocation techniques are considered. Approximated power series is used as interpolating polynomial and its fourth derivative as the collocating equation. These equations are solved using Gaussian-elimination approach in finding the unknown variables a_j , $j=0, \dots, 10$ which are substituted into basis function to give continuous implicit scheme. The discrete schemes and its derivatives that form the block are obtained by evaluating continuous implicit scheme at non-interpolating points. The developed methods are of order seven and the results generated when the methods were applied to fourth order initial value problems compared favourably with existing methods.

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1. Introduction

The general fourth order initial value problem of ordinary differential equations of the form

$$y^{(4)} = f(x, y(x), y'(x), y''(x), y'''(x)),$$
$$y(x_0) = y_1, y'(x_0) = y_2, y''(x_0) = y_3 \quad (1)$$

is considered in this article. In the past, solving fourth order ordinary differential equations (ODEs) requires reducing

the differentials to systems of first order ODEs and approximate numerical method for the first order would be used to solve the system. This approach is been attached with lots of setbacks which include: computational burden, lots of human effort, complexity in developing computer code which affects the accuracy of the method in terms of error. This was extensively discussed by researchers like Awoyemi [1], Fatunla [2] and Lambert [3]. Due to several disadvantages found in reduction method, the direct method of solving ODEs of higher order was developed by lots of scholars which include Akeremale et al. [4], Abolarin et al. [5], Kuboye et. al [6], Omar & Kuboye [7], Adeyefa [8], Abdullahi et al. [9], Adeniyi & Mohammed [10], Olabode [11], Adesanya et al.[12], Omar & Suleiman [13]

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and Familua & Omole [14]. Specifically, numerical methods for solving equation (1) were proposed by Omar and Kuboye[15], Areo and Omole[16] and Mohammed[17]. These current methods solved directly equation (1) but its accuracy in terms of error can still be improved. Therefore, this paper examines the derivation and implementation of the efficient numerical algorithm for solving fourth order ordinary differential equations directly and it focuses on improving the accuracy of the existing methods.

2. Methodology

This section considers derivation of block methods for direct solution of fourth order ODEs.

2.1. Derivation of First Block Method(FBM)

Power series approximate solution of the form

$$y(x) = \sum_{j=0}^{k+5} a_j x^j \tag{2}$$

is used as interpolating polynomial where k=5. The fourth derivative of equation(2) gives:

$$y^{iv}(x) = \sum_{j=4}^{k+5} j(j-1)(j-2)(j-3)a_j x^{j-4} \tag{3}$$

Equation (2) is interpolated at $x = x_{n+i}, i = 0(1)2$ and $\frac{5}{2}$ and equation (3) is collocated at $x = x_{n+i}, i = 0(1)5$ and $\frac{5}{2}$. Interpolation and collocation equations are combined together to give a non-linear system of equations of the form:

$$\begin{aligned} \sum_{j=0}^{k+5} a_j x_{n+i}^j &= y_{n+i} \\ \sum_{j=4}^{k+5} j(j-1)(j-2)(j-3)a_j x_{n+i}^{j-4} &= f_{n+i} \end{aligned} \tag{4}$$

The unknown variables a_j 's in (4) are gotten with the use of Gaussian elimination approach which are substituted into equation (2) and this yields a continuous implicit scheme of the form

$$\sum_{j=0}^{k-3} \alpha_j(t)y_{n+j} + \alpha_{\frac{5}{2}}y_{n+\frac{5}{2}} = h^4 \sum_{j=0}^k \beta_j(t)f_{n+j} + h^4 \lambda_{\frac{5}{2}} f_{n+\frac{5}{2}} \tag{5}$$

where $t = \frac{x-x_{n+k-1}}{h}$

$$\begin{bmatrix} \alpha_0(t) \\ \alpha_1(t) \\ \alpha_2(t) \\ \alpha_{\frac{5}{2}}(t) \end{bmatrix} = \begin{bmatrix} \frac{-9}{5} & \frac{-27}{10} & \frac{-13}{10} & \frac{-1}{5} \\ & 8 & \frac{34}{3} & 5 \\ & & \frac{45}{2} & \frac{17}{2} \\ \frac{64}{5} & \frac{208}{15} & \frac{24}{5} & \frac{8}{15} \end{bmatrix} \begin{bmatrix} t^0 \\ t^1 \\ t^2 \\ t^3 \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} \beta_0(t) \\ \beta_1(t) \\ \beta_2(t) \\ \beta_{\frac{5}{2}}(t) \\ \beta_3(t) \\ \beta_4(t) \\ \beta_5(t) \end{bmatrix} = T \begin{bmatrix} t^0 \\ t^1 \\ t^2 \\ t^3 \\ t^5 \\ t^6 \\ t^7 \\ t^8 \\ t^9 \\ t^{10} \end{bmatrix} \tag{7}$$

where T =

$\frac{297000}{290304000}$	$\frac{404694}{290304000}$	$\frac{75279}{290304000}$	$\frac{-102185}{290304000}$	$\frac{72576}{290304000}$	$\frac{36288}{290304000}$	$\frac{-2880}{290304000}$	$\frac{-7200}{290304000}$	$\frac{-2080}{290304000}$	$\frac{-192}{290304000}$
$\frac{3441528}{11612160}$	$\frac{5154498}{11612160}$	$\frac{2433933}{11612160}$	$\frac{323717}{11612160}$	$\frac{32256}{11612160}$	$\frac{15232}{11612160}$	$\frac{-1728}{11612160}$	$\frac{-3072}{11612160}$	$\frac{-800}{11612160}$	$\frac{-64}{11612160}$
$\frac{6797304}{5806080}$	$\frac{10732194}{5806080}$	$\frac{5745021}{5806080}$	$\frac{1207573}{5806080}$	$\frac{-145152}{5806080}$	$\frac{-6048}{5806080}$	$\frac{10944}{5806080}$	$\frac{12384}{5806080}$	$\frac{2720}{5806080}$	$\frac{192}{5806080}$
$\frac{560520}{2268000}$	$\frac{690174}{2268000}$	$\frac{419859}{2268000}$	$\frac{267195}{2268000}$	$\frac{-129024}{2268000}$	$\frac{-46592}{2268000}$	$\frac{11520}{2268000}$	$\frac{9600}{2268000}$	$\frac{1920}{2268000}$	$\frac{128}{2268000}$
$\frac{1716984}{5806080}$	$\frac{3760866}{5806080}$	$\frac{3494877}{5806080}$	$\frac{153384}{5806080}$	$\frac{-290304}{5806080}$	$\frac{-72576}{5806080}$	$\frac{27072}{5806080}$	$\frac{16704}{5806080}$	$\frac{3040}{5806080}$	$\frac{192}{5806080}$
$\frac{187272}{11612160}$	$\frac{129150}{11612160}$	$\frac{-353709}{11612160}$	$\frac{-650629}{11612160}$	$\frac{-169344}{11612160}$	$\frac{-3136}{11612160}$	$\frac{20160}{11612160}$	$\frac{7392}{11612160}$	$\frac{1120}{11612160}$	$\frac{64}{11612160}$
$\frac{428760}{290304000}$	$\frac{508266}{290304000}$	$\frac{-76719}{290304000}$	$\frac{-322779}{290304000}$	$\frac{290304}{290304000}$	$\frac{266112}{290304000}$	$\frac{118080}{290304000}$	$\frac{2880}{290304000}$	$\frac{3680}{290304000}$	$\frac{192}{290304000}$

The coefficient of first and higher derivatives of (5) give

$$\begin{bmatrix} \alpha'_0(t) \\ \alpha'_1(t) \\ \alpha'_2(t) \\ \alpha'_{\frac{5}{2}}(t) \end{bmatrix} = \begin{bmatrix} \frac{27}{10} & \frac{26}{10} & \frac{-6}{10} \\ \frac{34}{3} & \frac{30}{3} & \frac{6}{3} \\ \frac{45}{2} & \frac{34}{2} & \frac{-6}{2} \\ \frac{208}{15} & \frac{144}{15} & \frac{124}{15} \end{bmatrix} \frac{1}{h} \tag{8}$$

$$\begin{bmatrix} \beta'_0 \\ \beta'_1 \\ \beta'_2 \\ \beta'_{\frac{3}{2}} \\ \beta'_3(t) \\ \beta'_4(t) \\ \beta'_5(t) \end{bmatrix} = S \begin{bmatrix} t^0 \\ t^1 \\ t^2 \\ t^4 \\ t^5 \\ t^6 \\ t^7 \\ t^8 \\ t^9 \end{bmatrix} \tag{9}$$

where S =

$$\begin{bmatrix} \frac{-134898}{96768000} & \frac{-50186}{96768000} & \frac{102185}{96768000} & \frac{-120960}{96768000} & \frac{-72576}{96768000} & \frac{6720}{96768000} & \frac{19200}{96768000} & \frac{6240}{96768000} & \frac{640}{96768000} \\ \frac{5154498}{11612160} & \frac{4867866}{11612160} & \frac{971151}{11612160} & \frac{161280}{11612160} & \frac{91392}{11612160} & \frac{-12096}{11612160} & \frac{-24576}{11612160} & \frac{-7200}{11612160} & \frac{-640}{11612160} \\ \frac{3577398}{1935360} & \frac{3830014}{1935360} & \frac{1207573}{1935360} & \frac{-241920}{1935360} & \frac{-120960}{1935360} & \frac{25536}{1935360} & \frac{33024}{1935360} & \frac{8160}{1935360} & \frac{640}{1935360} \\ \frac{690174}{2268000} & \frac{839718}{2268000} & \frac{801585}{2268000} & \frac{-645120}{2268000} & \frac{-2795520}{2268000} & \frac{80640}{2268000} & \frac{76800}{2268000} & \frac{17280}{2268000} & \frac{1280}{2268000} \\ \frac{1253622}{1935360} & \frac{2329918}{1935360} & \frac{1533845}{1935360} & \frac{-483840}{1935360} & \frac{-145152}{1935360} & \frac{63168}{1935360} & \frac{44544}{1935360} & \frac{9120}{1935360} & \frac{640}{1935360} \\ \frac{-129150}{11612160} & \frac{707418}{11612160} & \frac{1951887}{11612160} & \frac{846720}{11612160} & \frac{18816}{11612160} & \frac{-141120}{11612160} & \frac{-59136}{11612160} & \frac{-10080}{11612160} & \frac{-640}{11612160} \\ \frac{169422}{96768000} & \frac{-51146}{96768000} & \frac{-322775}{96768000} & \frac{4838405}{96768000} & \frac{532224}{96768000} & \frac{275520}{96768000} & \frac{76800}{96768000} & \frac{11040}{96768000} & \frac{640}{96768000} \end{bmatrix}$$

$$\begin{bmatrix} \alpha''_0(t) \\ \alpha''_1(t) \\ \alpha''_2(t) \\ \alpha''_{\frac{3}{2}}(t) \end{bmatrix} = \begin{bmatrix} \frac{13}{5h^2} & \frac{-6}{5h^2} \\ \frac{10}{h^2} & \frac{4}{h^2} \\ \frac{17}{h^2} & \frac{-6}{h^2} \\ \frac{48}{5h^2} & \frac{16}{5h^2} \end{bmatrix} \begin{bmatrix} t^0 \\ t^1 \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} \beta_0''(t) \\ \beta_1''(t) \\ \beta_2''(t) \\ \beta_{\frac{5}{2}}''(t) \\ \beta_3''(t) \\ \beta_4''(t) \\ \beta_5''(t) \end{bmatrix} = U \begin{bmatrix} t^0 \\ t^1 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \\ t^7 \\ t^8 \end{bmatrix} \tag{11}$$

where $U =$

$$\begin{bmatrix} \frac{-25093}{48384000} & \frac{102185}{48384000} & \frac{-241920}{48384000} & \frac{-181440}{48384000} & \frac{20160}{48384000} & \frac{67200}{48384000} & \frac{24960}{48384000} & \frac{2880}{48384000} \\ \frac{811311}{1935360} & \frac{323717}{1935360} & \frac{107520}{1935360} & \frac{76160}{1935360} & \frac{-12096}{1935360} & \frac{-28672}{1935360} & \frac{-9600}{1935360} & \frac{-960}{1935360} \\ \frac{1915007}{967680} & \frac{1207573}{967680} & \frac{-483840}{967680} & \frac{-302400}{967680} & \frac{76608}{967680} & \frac{11558}{967680} & \frac{32640}{967680} & \frac{2880}{967680} \\ \frac{139953}{378000} & \frac{267195}{378000} & \frac{-430080}{378000} & \frac{-232960}{378000} & \frac{80640}{378000} & \frac{89600}{378000} & \frac{23040}{378000} & \frac{1920}{378000} \\ \frac{1164959}{967680} & \frac{1533845}{967680} & \frac{-967680}{967680} & \frac{-362880}{967680} & \frac{18950}{967680} & \frac{155904}{967680} & \frac{36480}{967680} & \frac{2880}{967680} \\ \frac{117903}{1935360} & \frac{650629}{1935360} & \frac{564480}{1935360} & \frac{15680}{1935360} & \frac{-141120}{1935360} & \frac{-68992}{1935360} & \frac{-13440}{1935360} & \frac{-960}{1935360} \\ \frac{-25573}{48384000} & \frac{-322775}{48384000} & \frac{967680}{48384000} & \frac{1330560}{48384000} & \frac{826560}{48384000} & \frac{268800}{48384000} & \frac{44160}{48384000} & \frac{2880}{48384000} \end{bmatrix}$$

$$\begin{bmatrix} \alpha_0'''(t) \\ \alpha_1'''(t) \\ \alpha_2'''(t) \\ \alpha_{\frac{5}{2}}'''(t) \end{bmatrix} = \begin{bmatrix} \frac{-6}{5} \\ 4 \\ -6 \\ \frac{16}{5} \end{bmatrix} t^0 \tag{12}$$

$$\begin{bmatrix} \beta_0'''(t) \\ \beta_1'''(t) \\ \beta_2'''(t) \\ \beta_{\frac{5}{2}}'''(t) \\ \beta_3'''(t) \\ \beta_4'''(t) \\ \beta_5'''(t) \end{bmatrix} = V \begin{bmatrix} t^0 \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \\ t^7 \end{bmatrix} \tag{13}$$

$$\text{where } V = \begin{bmatrix} \frac{20437}{9676800} & \frac{-14515}{9676800} & \frac{-145152}{9676800} & \frac{20160}{9676800} & \frac{80640}{9676800} & \frac{3494}{9676800} & \frac{4608}{9676800} \\ \frac{323717}{967680} & \frac{322560}{967680} & \frac{304640}{967680} & \frac{-60480}{967680} & \frac{-172032}{967680} & \frac{-67200}{967680} & \frac{-7680}{967680} \\ \frac{1207573}{967680} & \frac{-1451520}{967680} & \frac{-1209600}{967680} & \frac{383040}{967680} & \frac{69350}{967680} & \frac{228480}{967680} & \frac{23040}{967680} \\ \frac{-53439}{75600} & \frac{258048}{75600} & \frac{186368}{75600} & \frac{-80640}{75600} & \frac{-107520}{75600} & \frac{-32256}{75600} & \frac{-3072}{75600} \\ \frac{1533845}{967680} & \frac{-2903040}{967680} & \frac{-1451520}{967680} & \frac{947520}{967680} & \frac{935424}{967680} & \frac{255360}{967680} & \frac{23040}{967680} \\ \frac{650629}{1935360} & \frac{1935360}{1693440} & \frac{62720}{1935360} & \frac{-705600}{1935360} & \frac{-413952}{1935360} & \frac{-94080}{1935360} & \frac{-7680}{1935360} \\ \frac{-64555}{9676800} & \frac{580608}{9676800} & \frac{1064448}{9676800} & \frac{826560}{9676800} & \frac{322560}{9676800} & \frac{61824}{9676800} & \frac{4608}{9676800} \end{bmatrix}$$

Discrete schemes and its derivatives are derived by evaluating (5) as well as its derivatives at grid points and non-grid points which are used to form the block

$$\begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [y_n] + \begin{bmatrix} 1 \\ 2 \\ \frac{5}{2} \\ 3 \\ 4 \\ 5 \end{bmatrix} [hy'_n] + \begin{bmatrix} \frac{1}{2} \\ 2 \\ \frac{25}{8} \\ \frac{9}{2} \\ 8 \\ \frac{25}{2} \end{bmatrix} [h^2y''_n] + \begin{bmatrix} \frac{1}{6} \\ \frac{4}{3} \\ \frac{125}{48} \\ \frac{9}{2} \\ \frac{32}{3} \\ \frac{125}{6} \end{bmatrix} [h^3y'''_n] + h^4 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{57923}{2268000} \\ 0 & 0 & 0 & 0 & 0 & \frac{19676}{70875} \\ 0 & 0 & 0 & 0 & 0 & \frac{10754125}{18579456} \\ 0 & 0 & 0 & 0 & 0 & \frac{58509}{56000} \\ 0 & 0 & 0 & 0 & 0 & \frac{185344}{70875} \\ 0 & 0 & 0 & 0 & 0 & \frac{47875}{9072} \end{bmatrix} \begin{bmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-\frac{5}{2}} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}$$

$$+ h^4 \begin{bmatrix} \frac{413}{10368} & \frac{-8977}{90720} & \frac{9292}{70875} & \frac{-2279}{36288} & \frac{29}{3780} & \frac{-6593}{9072000} \\ \frac{428}{567} & \frac{-122}{81} & \frac{140288}{70875} & \frac{-2672}{2835} & \frac{4}{35} & \frac{764}{70875} \\ \frac{33640625}{18579456} & \frac{-30109375}{9289728} & \frac{89375}{20736} & \frac{-19109375}{9289728} & \frac{171875}{688128} & \frac{-437875}{18579456} \\ \frac{16137}{4480} & \frac{3267}{560} & \frac{6948}{875} & \frac{-243}{64} & \frac{207}{448} & \frac{-4887}{112000} \\ \frac{28928}{2835} & \frac{-7936}{567} & \frac{1441792}{70875} & \frac{-27136}{2835} & \frac{32}{27} & \frac{-7936}{70875} \\ \frac{1609375}{72576} & \frac{-484375}{18144} & \frac{23500}{567} & \frac{-671875}{36288} & \frac{15625}{6048} & \frac{-2375}{10368} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{bmatrix} \tag{14}$$

2.2. Derivation of Second Block Method(SBM)

Equation(2) is interpolated at $x = x_{n+i}, i = 0(1)2$ and $\frac{9}{4}$ and equation(3) is collocated at $x = x_{n+i}, i = 0(1)5$ and $\frac{9}{4}$. The same steps used in deriving the first block method are also employed and this produces the block method

$$\begin{aligned}
 \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+\frac{9}{4}} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \\ \frac{9}{4} \\ 3 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 2 \\ \frac{81}{32} \\ \frac{9}{2} \\ 8 \\ \frac{25}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{6} \\ \frac{4}{3} \\ \frac{243}{128} \\ \frac{9}{2} \\ \frac{32}{3} \\ \frac{125}{6} \end{bmatrix} \begin{bmatrix} h^3 y_n'''' \end{bmatrix} + h^4 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{69043}{2721600} \\ 0 & 0 & 0 & 0 & 0 & \frac{11696}{42525} \\ 0 & 0 & 0 & 0 & 0 & \frac{1191908439}{2936012800} \\ 0 & 0 & 0 & 0 & 0 & \frac{5793}{5600} \\ 0 & 0 & 0 & 0 & 0 & \frac{22016}{8505} \\ 0 & 0 & 0 & 0 & 0 & \frac{568625}{108864} \end{bmatrix} \begin{bmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-\frac{9}{4}} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} \\
 &+ h^4 \begin{bmatrix} \frac{76921}{1814400} & \frac{-1139}{6480} & \frac{594688}{3274425} & \frac{-6749}{181440} & \frac{7421}{1270080} & \frac{-11717}{19958400} \\ \frac{11248}{14175} & \frac{-7558}{2835} & \frac{8978432}{3274425} & \frac{-1576}{2835} & \frac{344}{3969} & \frac{-1352}{155925} \\ \frac{3705501897}{2936012800} & \frac{-169057287}{41943040} & \frac{229379121}{55193600} & \frac{-247763043}{293601280} & \frac{540947889}{4110417920} & \frac{-425211849}{32296140800} \\ \frac{84159}{22400} & \frac{-2349}{224} & \frac{148224}{13475} & \frac{-5031}{2240} & \frac{1377}{3920} & \frac{-8667}{246400} \\ \frac{150272}{14175} & \frac{-10496}{405} & \frac{8388608}{297675} & \frac{-15872}{2835} & \frac{17888}{19845} & \frac{-256}{2835} \\ \frac{1668125}{72576} & \frac{-115625}{2268} & \frac{7520000}{130977} & \frac{-378125}{36288} & \frac{509375}{254016} & \frac{-147625}{798336} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+\frac{9}{4}} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{bmatrix} \tag{15}
 \end{aligned}$$

3. Analysis of the method

Properties of the methods are examined in this section.

3.1. Order of block methods

In finding the order of the block methods, the method proposed by Lambert[3] is employed whereby Taylor series expansion are used in expanding the y and f-functions and by further comparing the coefficients of h, this gives the block methods to be of order [7, 7, 7, 7, 7, 7]^T.

3.2. Zero-stability

A linear multi-step method is said to be zero-stable if the roots $r_s, s=1,2,\dots, N$ (grid and non grid points) of the first characteristics polynomial defined by $p(r) = \det(rA' - B')$ satisfy $|r_s| < 1$ and the root $|r| = 1$ having multiplicity not exceeding one.(Lambert [3])

Where $A' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$$B' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore $r=0,0,0,0,0,1$. Hence the zero-stability of first block method is confirmed which is also applied to the second block method.

3.3. Convergence

According to Awoyemi[1], Equation(5) converges if it is zero-stable and consistent. This implies that the developed methods converged.

4. Test Problems

The following fourth order initial value problems[I.V.P] are solved in order to examine the accuracy of the methods

Problem 1:

$$y^{iv} = x, y(0) = 0, y'(0) = 1, y''(0) = y'''(0) = 0, h = 0.1$$

$$\text{Exact solution : } y(x) = \frac{x^5}{120} + x$$

Source: Mohammed[17]

Problem 2:

$$y^{iv} - y = 0, y(0) = 1, y'(0) = 0, y''(0) = -2, y'''(0) = 0, h = \frac{1}{320}$$

$$\text{Exact solution : } y(x) = \frac{-1}{4}e^x - \frac{1}{4}e^{-x} + \frac{3}{2}\cos(x)$$

Source: Areo and Omole[16]

Problem 3:

$$y^{iv} = (y')^2 - yy''' - 4x^2 + e^x(1 - 4x + x^2), y(0) = 1, y'(0) = 1, y''(0) = 3, y'''(0) = 1, h = \frac{0.1}{32}$$

$$\text{Exact solution : } y(x) = x^2 + e^x$$

Source: Familua and Omole [14]

The following acronyms are used in the Tables below

ES - Exact Solution

CS - Computed Solution

FBM – First Block Method

SBM – Second Block Method

EIM[17] - Error in Mohammed[17]

EIAO[16]- Error in Areo and Omole[16]

EIOK [15]- Error in Omar and Kuboye[15]

EIFBM - Error in First Block Method

EISBM - Error in Second Block Method

EIFO [14] - Error in Familua and Omole [14]

Table 1. ES and CS of FBM for Problem 1

x	ES	CS
0.1	0.10000083333333340	0.10000083333333340
0.2	0.20000266666666690	0.20000266666666690
0.3	0.30002025000000040	0.30002025000000040
0.4	0.40008533333333350	0.40008533333333350
0.5	0.500260416666666650	0.500260416666666650
0.6	0.60064799999999960	0.60064800000000070
0.7	0.70140058333333330	0.70140058333333330
0.8	0.802730666666666700	0.802730666666666700
0.9	0.904920750000000050	0.904920750000000100
1.0	1.00833333333333300	1.008333333333333500

Table 2. ES and CS of SBM for Problem 1

x	ES	CS
0.1	0.10000083333333340	0.10000083333333340
0.2	0.20000266666666690	0.20000266666666690
0.3	0.30002025000000040	0.300020250000000100
0.4	0.40008533333333350	0.400085333333333400
0.5	0.500260416666666650	0.500260416666666650
0.6	0.60064799999999960	0.60064800000000070
0.7	0.70140058333333330	0.701400583333333220
0.8	0.802730666666666700	0.802730666666666810
0.9	0.904920750000000050	0.904920750000000100
1.0	1.00833333333333300	1.008333333333333000

5. Discussion of Results

In Tables 1 and 2, exact and computed solutions of FBM and SBM for solving problem 1 are shown. Table 3 reveals the efficiency of these block methods (EIFBM and EISBM) as compared favourably with EIM[17] and EIOK[15]. Furthermore, exact and computed solutions of the newly developed block

Table 3. Comparison of EIFBM and EISBM with EIM[17] and EIOK[15] for solving problem 1

x	EIFBM	EISBM	EIM(2010)	EIOK(2016)
0.1	0.0000000e+00	0.0000000e+00	7.000024E-10	1.002087E-12
0.2	0.0000000e+00	0.0000000e+00	8.9999912E-10	0.000000E+00
0.3	0.0000000e+00	5.5511151E-17	2.599993E-09	0.000000E+00
0.4	5.5511151E-17	5.5511151E-17	5.100033E-09	0.000000E+00
0.5	1.1102230E-16	0.0000000e+00	7.799979E-09	1.002087E-12
0.6	1.1102230E-16	1.1102230E-16	1.180009E-08	2.755907E-12
0.7	2.2204460E-16	1.1102230E-16	1.180009E-08	3.507306E-12
0.8	0.0000000e+00	1.1102230E-16	1.410006E-08	3.507306E-12
0.9	1.1102230E-16	1.1102230E-16	1.880000E-08	4.175549E-12
1.0	2.2204460E-16	0.0000000e+00	1.008335E-08	4.759970E-12

Table 4. ES and CS of FBM for Problem 2

x	ES	CS
0.0031250	1.000009765628973500	1.000009765628973700
0.0062500	1.000039062563578400	1.000039062563578400
0.0093750	1.000087890946866900	1.000087890946867100
0.0125000	1.000156251017263000	1.000156251017263500
0.0156250	1.000244143108567400	1.000244143108567400
0.0187500	1.000351567649961900	1.000351567649962400
0.0218750	1.000478525166021100	1.000478525166021300
0.0250000	1.000625016276719800	1.000625016276719600
0.0281250	1.000791041697446400	1.000791041697446800
0.0312500	1.000976602239017000	1.000976602239017000

Table 5. ES and CS of SBM for Problem 2

x	ES	CS
0.0031250	1.000009765628973500	1.000009765628973900
0.0062500	1.000039062563578400	1.000039062563578400
0.0093750	1.000087890946866900	1.000087890946866900
0.0125000	1.000156251017263000	1.000156251017263500
0.0156250	1.000244143108567400	1.000244143108567100
0.0187500	1.000351567649961900	1.000351567649962100
0.0218750	1.000478525166021100	1.000478525166021100
0.0250000	1.000625016276719800	1.000625016276719600
0.0281250	1.000791041697446400	1.000791041697445900
0.0312500	1.000976602239017000	1.000976602239017200

Table 6. Comparison of EIFBM and EISBM with EIAO[16] for solving problem 2

x	EIFBM	EISBM	EIAO (2015)
0.0031250	2.2204460e-016	4.4408921e-016	4.440892e-16
0.0062500	0.0000000e+000	0.0000000e+000	2.176037e-14
0.0093750	2.2204460e-016	0.0000000e+000	.771916e-13
0.0125000	4.4408921e-016	4.4408921e-016	7.666090e-13
0.0156250	0.0000000e+000	2.2204460e-016	2.367773e-12
0.0187500	4.4408921e-016	2.2204460e-016	5.932477e-12
0.0218750	2.2204460e-016	0.0000000e+000	1.287681e-11
0.0250000	2.2204460e-016	2.2204460e-016	2.517841e-11
0.0281250	4.4408921e-016	4.4408921e-016	4.546752e-11
0.0312500	0.0000000e+000	2.2204460e-016	7.712331e-11

Table 7. ES and CS of FBM for Problem 3

x	ES	CS
0.103125	1.119264744787591900	1.119264744969084200
0.206250	1.271599493198048500	1.271599504741302500
0.306250	1.452110907065013100	1.452111029006491400
0.406250	1.666216862500122800	1.666217515460942200
0.506250	1.915347109920916500	1.915349507140536000
0.603125	2.191581593606204900	2.191588302867649500
0.703125	2.514440293333696500	2.514456732090109900
0.803125	2.877516387746607200	2.877551937602963200
0.903125	3.282936158805099100	3.283006004031709900
1.003125	3.733049511495175400	3.733176679391747100

Table 8. ES and CS of SBM for Problem 3

x	ES	CS
0.103125	1.119264744787591900	1.119264744966372600
0.206250	1.271599493198048500	1.271599504536039800
0.306250	1.452110907065013100	1.452111026692671300
0.406250	1.666216862500122800	1.666217502634746300
0.506250	1.915347109920916500	1.915349459022614800
0.603125	2.191581593606204900	2.191588166231517800
0.703125	2.514440293333696500	2.514456393591375100
0.803125	2.877516387746607200	2.880551715825508700
0.903125	3.282936158805099100	3.283004543615038400
1.003125	3.733049511495175400	3.733174005515217600

Table 9. Comparison of EIFBM and EISBM with EIFO [14] for solving problem 3

x	EIFBM	EISBM	EIFO[14]
0.103125	1.8149238e-010	1.7878077e-010	9.02145880e-10
0.206250	1.1543254e-008	1.1337991e-008	1.216821428e-09
0.306250	1.2194148e-007	1.1962766e-007	1.21681228e-09
0.406250	6.5296082e-007	6.4013462e-007	1.713796095e-09
0.506250	2.3972196e-006	2.3491017e-006	1.481970916e-08
0.603125	6.7092614e-006	6.5726253e-006	3.058338503e-08
0.703125	1.6438756e-005	1.6100258e-005	4.941858156e-08
0.803125	3.5549856e-005	3.5007632e-005	7.128679089e-08
0.903125	6.9845227e-005	6.8384810e-005	1.058773080e-07
1.003125	1.2716790e-004	1.2449402e-004	1.445520074e-07

methods for the solution of problems 2 and 3 are demonstrated in Tables 4, 5, 7 and 8. These methods outperform method proposed by Areo and Omole [16] in terms of accuracy. In addition, the performance of these methods in solving problem 3 is not encouraging as the accuracy is lower when the comparison is made with EIFO [14]. However, the capability of these methods in solving the nonlinear equation is established in Table 9. Finally, it is evident in Tables 3, 6 and 9 that SBM is better than FBM in solving fourth order ODEs.

6. Conclusion

In this paper, new numerical algorithms for solving fourth order initial value problems of ODEs via multistep collocation approach were developed. The use of approximated power series as a basis function and its fourth derivatives as collocating equation were considered. The derived methods are efficient in the solution of fourth order ODEs as depicted in Tables 3, 6 and 9. The accuracy of these numerical models is found better compared with some of the existing methods in terms of error. Hence, FBM and SBM are viable numerical methods for solving fourth order initial value problems.

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