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System of Non-Linear Volterra Integral Equations in a Direct-Sum of Hilbert Spaces

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Abstract

We use the contraction mapping theorem to present the existence and uniqueness of solutions in a short time to a system of non-linear Volterra integral equations in a certain type of direct-sum $H[a,b]$ of a Hilbert space $V[a,b]$. We extend the local existence and uniqueness of solutions to the global existence and uniqueness of solutions to the proposed problem. Because the kernel function is a transcendental function in $H[a,b]$ on the interval $[a,b]$, the results are novel and very important in numerical approximation.

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1. Introduction

Non-linear Volterra integral equations have many applications in several fields, such as physics, chemistry, biology, and engineering. For example, particle transport problems in astrophysics theory, electrostatics, potential theory, mathematical problems of radiative steady state, heat transfer problems, and many other mathematical modeling are described by Voltera integral equations [1-9]. In this paper, we introduce the following system of non-linear Volterra integral equations for $t \in [a,b]$:

$$f(t) = \alpha(t) + \int_a^t F(t,s,f(s),g(s))ds, \tag{1}$$

$$g(t) = \beta(t) + \int_a^t G(t,s,f(s),g(s))ds, \tag{2}$$

where $\alpha, \beta \in H[a,b]$ is a direct sum of reproducing kernel Hilbert space $V[a,b]$ consisting of those absolutely continuous functions whose derivative is square-integrable on $[a,b]$. F and G are given functions that satisfy fixed regularity conditions. f and g are unknown functions that need to be determined.

Recently, reproducing kernel Hilbert space methods have been widely studied by many researchers to solve linear and non-linear problems such as partial and ordinary differential equations, as well as integral, fractional, and integral differential equations [4,7,9]. Obviously, considering the existence and uniqueness of solutions to such kinds of problems is very important in pure and applied mathematics. In view of the fact that most real phenomena and non-linear problems in the world can not be solved analytically, researchers use numerical methods to obtain their approximate and numerical solutions in an

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appropriate space.

In this work, we examine the local and global existence and uniqueness of solutions to a system of non-linear Volterra integral equations in the reproducing kernel Hilbert space $H[a,b]$. This space is a very favorable space in numerical approximation since its reproducing kernel function is a transcendental function in $[a,b]$.

2. Preliminary Notation

This section is assigned to present basic notation, definitions, and theorems which will be used later.

Definition 1. *Let $S \neq \emptyset$. A Hilbert space $\mathcal{H}$ of continuous real-valued functions $f : S \to \mathbb{R}$ is called reproducing kernel Hilbert space if there exists a function $K : S \times S \to \mathbb{R}$ in $\mathcal{H}$ such that $\langle f(\cdot), K(\cdot, s)\rangle_{\mathcal{H}} = f(s)$, and $K(\cdot, s) \in \mathcal{H}$ for all $f \in \mathcal{H}$ and all $s \in S$. Such a function $K = K(\cdot,\cdot)$ is said to be a reproducing kernel function of $\mathcal{H}$ [4,6].*

Definition 2. *Let $V[a,b]$ be the space of all absolutely continuous functions $f : [a,b] \to \mathbb{R}$ such that $f' \in L^2[a,b]$ [4,6].*

Theorem 1. *The function space $V[a,b]$ equipped with the inner product [4]*

$$\langle f_1, f_2\rangle_{V[a,b]} = f_1(a)f_2(a) + \int_a^b f_1'(t)f_2'(t)dt,$$

and associated with the norm

$$\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle_{V[a,b]}},$$

is a reproducing kernel Hilbert space and the reproducing kernel function $K = K(\cdot,\cdot)$ is defined by:

$$K(t,\tau) = \frac{1}{2\sinh(b-a)}\big(\cosh(\tau + t - b + a) + \cosh(|\tau - t| - b - a)\big).$$

Definition 3. *The function space*

$$H[a,b] = V[a,b] \oplus V[a,b],$$

consists of those functions $\vec{h} : [a,b] \to \mathbb{R}^2$ where $\vec{h} = (h_1, h_2)$ such that h_1 and h_2 belong to $V[a,b]$.

Definition 4. *The inner product of the space $H[a,b]$ is defined by :*

$$\langle \vec{f}, \vec{g}\rangle_{H[a,b]} = \langle f_1, g_1\rangle_{V[a,b]} + \langle f_2, g_2\rangle_{V[a,b]},$$

where $f = (f_1, f_2)$ and $g = (g_1, g_2)$. Such a space is called direct sum of the reproducing kernel Hilbert space $V[a,b]$.

3. Existence and Uniqueness

In this section, we discuss the Banach fixed point theorem to show the local and global existence and uniqueness of solutions to (1)-(2). To do this first, we need to introduce some basic tools.

Let $\vec{h} \in H[a,b]$ and let $\mathcal{A} = \{(t,s) : a \leq s \leq t \leq b\}$. Define maps $T\vec{h} : [a,b] \to \mathbb{R}$ and $L\vec{h} : [a,b] \to \mathbb{R}$ by:

$$T\vec{h}(t) = \int_a^t F\Big(t, s, \vec{h}(s)\Big)ds,$$
$$L\vec{h}(t) = \int_a^t G\Big(t, s, \vec{h}(s)\Big)ds,$$

such that the following conditions are hold for ($k = 0, 1$):

C1) $\frac{\partial^k}{\partial t^k}F$ and $\frac{\partial^k}{\partial t^k}G$ are uniformly bounded functions on $\mathcal{A} \times \mathbb{R}^2$.

C2) For some positive constants M and N such that

i) $\left|\frac{\partial^k}{\partial t^k}F\Big(x, s, \vec{f_1}(s)\Big) - \frac{\partial^k}{\partial t^k}F\Big(y, s, \vec{f_2}(s)\Big)\right| \leqslant M\Big(|x-y| + \|\vec{f_1} - \vec{f_2}\|_2\Big)$;

ii) $\left|\frac{\partial^k}{\partial t^k}G\Big(x, s, \vec{g_1}(s)\Big) - \frac{\partial^k}{\partial t^k}G\Big(y, s, \vec{g_2}(s)\Big)\right| \leqslant N\Big(|x-y| + \|\vec{g_1} - \vec{g_2}\|_2\Big)$.

Theorem 2. *Let $\vec{h} \in H[a,b]$. Then $T\vec{h} \in H[a,b]$.*

We first assert that $T\vec{h}$ is absolutely continuous in $[a,b]$. By condition $(C1)$ for (k=0); F is uniformly bounded on $\mathcal{A} \times \mathbb{R}^2$ and condition $(C2)$ part (i) there are positive constants M and $M1$.

Let $I_j = \{[a_j, b_j]\}_{j=1}^n$ be a finite collection of non-over lapping intervals in $[a,b]$, and let $\varepsilon > 0$ such that:

$$\sum_{j=1}^n \left|b_j - a_j\right| < \frac{\varepsilon}{\Big(M1(b-a) + M\Big)}.$$

Since,

$$\begin{aligned}\sum_{j=1}^n \left|T\vec{h}(b_j) - T\vec{h}(a_j)\right| &= \sum_{j=1}^n \Big|\int_a^{b_j} F\Big(b_j, s, \vec{h}(s)\Big)ds \\ &\quad - \int_a^{a_j} F\Big(a_j, s, \vec{h}(s)\Big)ds\Big| \\ &= \sum_{j=1}^n \Big|\int_a^{a_j} F\Big(b_j, s, \vec{h}(s)\Big)ds \\ &\quad + \int_{a_j}^{b_j} F\Big(b_j, s, \vec{h}(s)\Big)ds \\ &\quad - \int_a^{a_j} F\Big(a_j, s, \vec{h}(s)\Big)ds\Big|\end{aligned}$$

$$\leqslant \sum_{j=1}^{n} \int_{a_j}^{a_j} \left|F\left(b_j, s, \vec{h}(s)\right) - F\left(a_j, s, \vec{h}(s)\right)\right| ds$$
$$+ \int_{a_j}^{b_j} \left|F\left(b_j, s, \vec{h}(s)\right) ds\right|$$
$$\leqslant \sum_{j=1}^{n} \int_{a}^{a_j} M1\left|b_j - a_j\right| ds + \int_{a_j}^{b_j} M ds$$
$$= \sum_{j=1}^{n} \left(M1(a_j - a)|b_j - a_j| + M|b_j - a_j|\right)$$
$$\leqslant \left(M1(b-a) + M\right) \sum_{j=1}^{n} \left|b_j - a_j\right|$$
$$< \varepsilon.$$

Hence, $T\vec{h}$ is absolutely continuous on in $[a, b]$. Next, we want to show $\frac{\partial}{\partial t} T\vec{h}(\cdot) \in L^2[a, b]$. Leibniz rule implies for almost every $t \in [a, b]$ that

$$\frac{\partial}{\partial t} T\vec{h}(t) = F\left(t, t, \vec{h}(t)\right) + \int_{a}^{t} \frac{\partial}{\partial t} F\left(t, s, \vec{h}(s)\right) ds.$$

Then,

$$\int_{a}^{b} \left|\frac{\partial}{\partial t} T\vec{h}(t)\right|^2 dt = \int_{a}^{b} \left|F\left(t, t, \vec{h}(t)\right) + \int_{a}^{t} \frac{\partial}{\partial t} F\left(t, s, \vec{h}(s)\right) ds\right|^2 dt$$
$$\leq 2 \int_{a}^{b} \left|F\left(t, t, \vec{h}(t)\right)\right|^2 dt + 2 \int_{a}^{b} \left|\int_{a}^{t} \frac{\partial}{\partial t} F\left(t, s, \vec{h}(s)\right) ds\right|^2 dt.$$

It follows from condition $(C1)$ for (k=0,1) there are positive constants N, D and the Cauchy-Schwartz inequality that

$$\int_{a}^{b} \left|\frac{\partial}{\partial t} T\vec{h}(t)\right|^2 dt \leq 2 \int_{a}^{b} N^2 dt + 2 \int_{a}^{b} \left(\int_{a}^{t} \left(\frac{\partial}{\partial t} F(t, s, \vec{h}(s))\right)^2 ds \int_{a}^{t} 1^2 ds\right) dt,$$

implies

$$\int_{a}^{b} \left|\frac{\partial}{\partial t} T\vec{h}(t)\right|^2 dt \leq 2 \int_{a}^{b} N^2 dt$$
$$+ 2 \int_{a}^{b} \left(\int_{a}^{t} \left(\frac{\partial}{\partial t} F(t, s, \vec{h}(s))\right)^2 ds \int_{a}^{t} 1^2 ds\right) dt$$
$$\leqslant 2N^2(b-a) + 2(b-a) \int_{a}^{b} \int_{a}^{t} \left(\frac{\partial}{\partial t} F(t, s, \vec{h}(t)\right)^2 dsdt$$
$$\leqslant 2N^2(b-a) + 2(b-a) \int_{a}^{b} \int_{a}^{b} D^2 dsdt$$
$$= 2N^2(b-a) + 2D^2(b-a)^3$$
$$< \infty.$$

Therefore, $T\vec{h}$ belongs to $H[a, b]$ by definitions (2) and (3). Similar arguments one can use to show that $L\vec{h}$ belongs to $H[a, b]$.

Theorem 3. *Let $\vec{f} \in H[a, b]$. Then $L\vec{f} \in H[a, b]$.*

The proof is analogous to the proof of Theorem 2. Set

$\alpha, \beta \in H[a, b]$. Define operators $\Gamma : H[a, b] \to H[a, b]$ and $\Lambda : H[a, b] \to H[a, b]$ such that:

$$\Gamma\vec{h}(t) = \alpha(t) + T\vec{h}(t);$$
$$\Lambda\vec{h}(t) = \beta(t) + L\vec{h}(t);$$

for all $\vec{h} \in H[a, b]$.

We divide the interval $[a, b]$into N equally sub-intervals $a \leq t_0 < t_1 < ... < t_n \leq b$; where $\triangle t = t_j - t_{j-1}$ $j = 1, 2, ..., N$ and $\triangle t = \frac{b-a}{N}$. The inner product in $H[t_j, t_j + \triangle t]$ is defined by:

$$\langle \vec{f}, \vec{g} \rangle_{H[t_j, t_j + \triangle t]} = \langle f_1, g_1 \rangle_{V[t_j, t_j + \triangle t]} + \langle f_2, g_2 \rangle_{V[t_j, t_j + \triangle t]},$$

for all $\vec{f}, \vec{g} \in H[t_j, t_j + \triangle t]$. As a result, we see that the operators $\Gamma : H[t_j, t_j + \triangle t] \to H[t_j, t_j + \triangle t]$ and $\Lambda : H[t_j, t_j + \triangle t] \to H[t_j, t_j + \triangle t]$ become

$$\Gamma\vec{h}(\mu) = \alpha(\mu) + \int_{t_j}^{\mu} F\left(\mu, s, \vec{h}(s)\right) ds;$$
$$\Lambda\vec{h}(\mu) = \beta(\mu) + \int_{t_j}^{\mu} G\left(\mu, s, \vec{h}(s)\right) ds;$$

for all $\mu \in H[t_j, t_j + \triangle t]$.

Lemma 1. *Let $\vec{h} \in H[t_j, t_j + \triangle t]$ and $\triangle t < 1$ [7]. Then*

$$\left\|\vec{h}\right\|_2 \leq \sqrt{2} \left\|\vec{h}\right\|_{H[t_j, t_j + \triangle t]}.$$

remark 1. *Assume that $\alpha(t_j) = \beta(t_j)$ for all $j = 0, 1, ..., N - 1$.*

Theorem 4. *Let $\vec{h}_1, \vec{h}_2 \in H[t_j, t_j + \triangle t]$. Then*

$$\left\|\Gamma\vec{h}_1 - \Gamma\vec{h}_2\right\|_{H[t_j, t_j + \triangle t]} \leqslant \delta(\triangle t) \left\|\vec{h}_1 - \vec{h}_2\right\|_{H[t_j, t_j + \triangle t]},$$

where $\delta(\triangle t) \leq C\sqrt{\triangle t}$, for some positive constant C.

Since

$$\begin{aligned}\left\|\Gamma\vec{h}_1-\Gamma\vec{h}_2\right\|^2_{H[t_j,t_j+\triangle t]} &= \left(\Gamma\vec{h}_1(t_j)-\Gamma\vec{h}_2(t_j)\right)^2\\ &+\int_{t_j}^{t_j+\triangle t}\Big(\frac{\partial}{\partial t}\Gamma\vec{h}_1(t)-\frac{\partial}{\partial t}\Gamma\vec{h}_2(t)\Big)^2dt\\ &=\left(\alpha(t_j)-\beta(t_j)\right)^2\\ &+\int_{t_j}^{t_j+\triangle t}\Big(F\big(t,t,\vec{h_1}(t)\big)-F\big(t,t,\vec{h_2}(t)\big)\\ &+\int_{t_i}^{t}\Big[\frac{\partial}{\partial t}F\big(t,s,\vec{h_1}(s)\big)-\frac{\partial}{\partial t}F\big(t,s,\vec{h_2}(s)\big)\Big]ds\Big)^2dt\end{aligned}$$

Implies,

$$\begin{aligned}\left\|\Gamma\vec{h}_1-\Gamma\vec{h}_2\right\|^2_{H[t_j,t_j+\triangle t]} &\leqslant 2\int_{t_j}^{t_j+\triangle t}\Big(F\big(t,t,\vec{h_1}(t)\big)\\ &\qquad -F\big(t,t,\vec{h_2}(t)\big)\Big)^2dt\\ &+2\int_{t_j}^{t_j+\triangle t}\Big(\int_{t_i}^{t}\Big[\frac{\partial}{\partial t}F\big(t,s,\vec{h_1}(s)\big)\\ &-\frac{\partial}{\partial t}F\big(t,s,\vec{h_2}(s)\big)\Big]ds\Big)^2dt\end{aligned}$$

By $(C2)$ we get constants M_1, M_2 such that

$$\begin{aligned}\left\|\Gamma\vec{h}_1-\Gamma\vec{h}_2\right\|^2_{H[t_j,t_j+\triangle t]} &\leq 2\int_{t_j}^{t_j+\triangle t}M_1^2\left\|\vec{h}_1(t)-\vec{h}_2(t)\right\|_2^2dt\\ &+2\int_{t_i}^{t_j+\triangle t}\Big(\int_{t_i}^{t}M_2\left\|\vec{h}_1(s)-\vec{h}_2(s)\right\|_2ds\Big)^2dt.\end{aligned}$$

Then,

$$\begin{aligned}\left\|\Gamma\vec{h}_1-\Gamma\vec{h}_2\right\|^2_{H[t_j,t_j+\triangle t]} &\leqslant 2\int_{t_i}^{t_j+\triangle t}M_1^2M_3^2\left\|\vec{h}_1-\vec{h}_2\right\|_2^2dt\\ &+2\int_{t_i}^{t_j+\triangle t}\Big(\int_{t_i}^{t}M_2M_3\left\|\vec{h}_1-\vec{h}_2\right\|_2ds\Big)^2dt\\ &\leqslant 2M_1^2M_3^2\left\|\vec{h}_1-\vec{h}_2\right\|_2^2\ (\triangle t)\\ &+\frac{2}{3}M_2^2M_3^2\left\|\vec{h}_1-\vec{h}_2\right\|_2^2\ (\triangle t)^3\\ &=\triangle t\left(2M_1^2M_3^2+\frac{2}{3}M_2^2M_3^2(\triangle t)^2\right)\left\|\vec{h}_1-\vec{h}_2\right\|_2^2.\end{aligned}$$

By using Lemma 1 that

$$\left\|\Gamma\vec{h}_1-\Gamma\vec{h}_2\right\|^2{}_{H[t_j,t_j+\triangle t]}\leqslant\delta^2(\triangle t)\left\|\vec{h}_1-\vec{h}_2\right\|^2_{H[t_j,t_j+\triangle t]},$$

Therefore,

$$\left\|\Gamma\vec{h}_1-\Gamma\vec{h}_2\right\|_{H[t_j,t_j+\triangle t]}\leqslant\delta(\triangle t)\left\|\vec{h}_1-\vec{h}_2\right\|_{H[t_j,t_j+\triangle t]},$$

Where $\delta(\triangle t) < C\ \sqrt{\triangle t}$, and

$$C=\sqrt{2M_3^2\big(M_1^2+\tfrac{1}{3}M_2^2\ \triangle^2\ t\big)}<\sqrt{2M_3^2\big(M_1^2+\tfrac{1}{3}M_2^2\big)}$$

if $\triangle t < 1$.

Theorem 5. *Let $f, g \in H[t_j, t_j + \triangle t]$. Then*

$$\left\|\Lambda f-\Lambda g\right\|_{H[t_j,t_j+\triangle t]}\leqslant\ \sigma(\triangle t)\ \left\|f-g\right\|_{H[t_j,t_j+\triangle t]},$$

where $\sigma(\triangle t) \leq C\sqrt{\triangle t}$, for some positive constant C.

The proof is similar to the proof of Theorem 4.

Theorem 6. *Let F and G satisfy conditions $(C1)$ and $(C2)$. Then there exists a unique solution $\vec{h} = (f, g) \in H[a, b]$ to (1) and (2).*

For all $\vec{h} = (f, g)$ in the space $H[a, b]$. It is clear that $\vec{h} \mapsto \Gamma\vec{h}$ and $\vec{h} \mapsto \Lambda\vec{h}$ are maps from $H[t_j, t_j+\triangle t]$ into $H[t_j, t_j+\triangle t]$. From Theorems 4 and 5; since $\triangle t$ is an arbitrary positive constant and if we pick $\triangle t$ small enough such that $\triangle t < \frac{1}{C^2}$ then we conclude that $\delta(\triangle t) < 1$ and $\sigma(\triangle t) < 1$. Therefore, by Theorems 4 and 5 the operators Γ and Λ are contraction mapping on $H[t_j, t_j + \triangle t]$, respectively. It is clear $\left(H[t_j, t_j+\triangle t], \|\cdot\|_{H[t_j,t_j+\triangle t]}\right)$ is a complete matrix space. Hence, the Banach contraction mapping theorem guarantees that the operators Γ and Λ have a unique fixed point $\vec{h} = (f, g)$ in $H[t_j, t_j + \triangle t]$.

Let $\varepsilon(\triangle t) = \min\{\delta(\triangle t), \sigma(\triangle t)\}$. The existence and uniqueness of solutions in the entire interval $[a, b]$ for (1) and (2) can be achieved by iterating the local existence result. This is accomplished by taking $[a, \varepsilon(\triangle t)], [\varepsilon(\triangle t), 2\varepsilon(\triangle t)], ...[n\varepsilon(\triangle t), b]$.

4. conclusion

We studied the local and global existence and uniqueness of solutions to a system of non-linear Volterra integral equations (1)-(2) in the reproducing kernel Hilbert spaces $V[a, b]$ and $H[a, b]$. The results are very significant in numerical methods since the reproducing kernel function of the space $V[a, b]$ is a smooth function on the compact interval $[a, b]$ and it can be used to solve a wide variety of linear and nonlinear problems.

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