



New Invariant Quantity to Measure the Entanglement in the Braids

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Abstract

In this work, we demonstrate that the integral formula for a generalised Sato-Levine invariant is consistent in certain situations with Evans and Berger's formula for the fourth-order winding number. Also, we found that, in principle, one can derive analogous high-order winding numbers from which one can calculate the entanglement of braids. The winding number for the Brunnian 4-braid is calculated algebraically using the cup product on the cohomology of a finite regular CW-space, which is the complement $\mathbb{R}^3 \setminus \mathcal{B}_4$.

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1. Introduction

A braid (or n -braid) is a collection of n strands stretching between 2 parallel planes that twist and cross. One example is the pure braid in which the strands start and end with the same permutation, see for instance, a 3-braid in Fig. 1 (left). Identifying the two parallel planes of this 3-braid yield a link, Fig. 1 (right).

Braids and links are used to model many physical systems that include entanglement in their force field lines (integral curves), see for example the coronal loops above the Sun's photosphere Fig. 1 (left). In this situation, the magnetic reconnection occurs

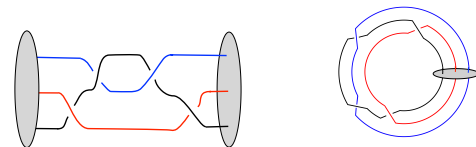


Figure 1. A 3-braid (left) and a link obtained from the braid (right)

due to the entanglement of the magnetic field lines and then the magnetic topology changes. This process leads to a conversion of the magnetic energy into kinetic energy and a heating of the plasma.

Understanding of magnetic reconnection is based on the Sweet-Parker model [1]. In a magnetic field, the connectivity of the field lines can occur only at null points, where the magnetic

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field vanishes. This model gives a rough scaling law predicting the rate of reconnection at the null point. On the other hand, there is no requirement that reconnection be limited to such locations; indeed, numerical experiments show the development of turbulent reconnection in magnetic fields with no null points.

In astrophysical environments, the magnetic fields are usually inherently disordered, being characterised by field lines that are tangled with one another (right), [2, 3], typical picture for braided magnetic field is shown in Fig.

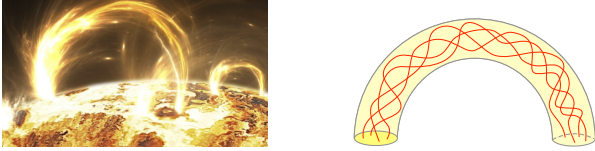


Figure 2. Coronal loops above the Sun's photosphere (left), some typical field lines are braided and contained within a tubular domain (right).

Winding numbers are used easily to do many topological calculations for example braid invariants which have applications in fluid mechanics, molecular biology and astrophysics [4].

The idea of introducing integrals of magnetic fields in the form of the Massey products first appeared in [5], and has been developed in [6, 7, 8]. Different types of higher-order invariants have been introduced, for instance, a fourth-order linking was introduced by [9] and another integral formula for the generalised Sato-Levine invariant was introduced by [10]. The permutation problem is a serious obstacle for calculating third-order linking numbers. A requirement for higher-order winding numbers is to avoid such problems [11]. In this paper, we are showing that the Evans-Berger and Akhmetev's formulas coincide in certain cases. Also, we will introduce another formula for the fourth-order linking number in analogy to the formula of lower-order. This formula is introduced in section (7), it allows to measure the linking of the field lines of a magnetic field (e.g. magnetic braid) when the lower order linking numbers are trivial. This formula is tested on a pure braid and also the result consist with that obtained algebraically in section 8.

2. Linking of two tubes

Suppose we have two closed curves α_1 and α_2 . These curves are parametrised by arclengths t and s , respectively. The positions along α_1 and α_2 are given by $\mathbf{x}_1(t)$ and $\mathbf{x}_2(s)$. Let $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ be the relative position vector. The linking number between α_1 and α_2 , introduced by Gauss (1867) has the following representation,

$$\mathcal{L}_{12} = \frac{1}{4\pi} \oint_{\alpha_1} \oint_{\alpha_2} \frac{d\mathbf{x}_1}{dt} \cdot \frac{d\mathbf{x}_2}{ds} \times \nabla_{\mathbf{x}_1} \frac{1}{|\mathbf{r}|} ds dt. \quad (1)$$

Gauss emphasised that \mathcal{L}_{12} has a value equal to the signed number of crossing in a projection of the link. This Gauss integral

is invariant under deformations of the two curves in which intersecting not allowed.

Now, in order to define the linking of two flux tubes we take two thin flux tubes U_1 and U_2 with unit fluxes. The curves α_1 and α_2 are enclosed in U_1 and U_2 , respectively. The vector potential \mathbf{A}_2 due to U_2 is defined according to the Ampér's theorem in the following form

$$\mathbf{A}_2 = \frac{1}{4\pi} \oint_{\alpha_2} \frac{d\mathbf{x}_2}{ds} \times \nabla_{\mathbf{x}_1} \frac{1}{|\mathbf{r}|} ds, \quad (2)$$

where the vector functions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(s)$ define points lying on the curve α_1 and α_2 , which are the axes of the thin flux tubes U_1 and U_2 , respectively. Now assume we have unit fluxes, Gauss linking integral can be written in the following way

$$\mathcal{L}_{12} = \oint_{\alpha_1} \mathbf{A}_2 \cdot d\mathbf{l} = \oint_{\alpha_2} \mathbf{A}_1 \cdot d\mathbf{l}. \quad (3)$$

Definition 1.

Let \mathbf{B}_1 and \mathbf{B}_2 be two divergence-free magnetic field and V be a simply connected volume such that the boundary ∂V is a so-called flux surface, i.e. $\mathbf{B}_1 \cdot \mathbf{n}|_{\partial V} = \mathbf{B}_2 \cdot \mathbf{n}|_{\partial V} = 0$. The *cross-helicity* of these magnetic fields is defined in the following form

$$\begin{aligned} \mathcal{H}(\mathbf{B}_1, \mathbf{B}_2) &= \int_V \mathbf{A}_1 \cdot \mathbf{B}_2 d^3x \\ &= \int_V \mathbf{A}_2 \cdot \mathbf{B}_1 d^3x. \end{aligned} \quad (4)$$

The cross-helicity measures the cross-linkage of flux between \mathbf{B}_1 and \mathbf{B}_2 . It can be written in a form to include the linking number. If we apply it to a system of two isolated thin flux tubes U_1 and U_2 which correspond to the fields \mathbf{B}_1 and \mathbf{B}_2 , we obtain

$$\mathcal{H}(\mathbf{B}_1, \mathbf{B}_2) = \phi_1 \phi_2 \mathcal{L}_{12}(U_1, U_2). \quad (5)$$

For this reason it is known as a second-order topological invariant (i.e. it is quadratic in magnetic flux ϕ_i). Eq. (3) can be reformulated as a volume integral over the cross-helicity as follows

$$\begin{aligned} \mathcal{L}_{12} &= \int_{U_1} \mathbf{A}_2 \cdot \mathbf{B}_1 dV \\ &= \int_{U_2} \mathbf{A}_1 \cdot \mathbf{B}_2 dV. \end{aligned} \quad (6)$$

Now, define a new field $\mathbf{M}_{12} = \mathbf{A}_1 \times \mathbf{A}_2$, which is divergence-free everywhere in the space $\tilde{U} = \mathbb{R}^3 \setminus \cup_{i=1}^2 U_i$, and apply the divergence theorem with the help of the vector identity [28],

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}).$$

Then Eq. (3) becomes

$$\begin{aligned} \mathcal{L}_{12} &= \int_{\partial U_1} \mathbf{M}_{12} \cdot \hat{\mathbf{n}} dS \\ &= \int_{\partial U_2} \mathbf{M}_{12} \cdot \hat{\mathbf{n}} dS. \end{aligned} \quad (7)$$

3. Third-Order Linking of Three Tubes

The Gauss integral cannot distinguish between the Whitehead link or the Borromean rings on one hand and the trivial link on the other hand. So, this integral is the lowest order used to measure the entanglement of curves and interlocking of flux tubes. Based on the Massey triple Evans and Berger (1992) constructed a third-order link integral which can be used to show that the Borromean rings are not unlinked. This integral is one of a whole hierarchy of linking integrals.

Now, suppose that the closed curves α_i , $i = 1, 2, 3$ of the Borromean rings are enclosed in tori U_i with vector potentials \mathbf{A}_i of magnetic fields \mathbf{B}_i such that these fields vanish outside the toroidal volume U_i , and let U' be the complement of the subspace $\cup_{i=1}^3 U_i \subset \mathbb{R}^3$.

From algebraic topology, use the fact 'the homology $H_1(U_i; \mathbb{R})$ and the cohomology $H^1(U_i; \mathbb{R})$ are isomorphic', by using the Alexander Duality Theorem¹, these are isomorphic to the homology and cohomology of the complement of U_i , and consequently, any basis of $H_1(U_i; \mathbb{R})$ and any basis of $H^1(\mathbb{R}^3 \setminus U_i; \mathbb{R})$ could be isomorphic. For this reason each tube U_i corresponds to a unique cohomology class in $H^1(\mathbb{R}^3 \setminus U_i; \mathbb{R})$.

Also, we can identify the vector potential \mathbf{A}_i with a differential one-form A_i such that $\mathbf{dA}_i = \mathbf{B}_i$, where \mathbf{B}_i is a differential two-form corresponding to the magnetic field \mathbf{B}_i in the tube U_i , and \mathbf{d} is the exterior derivative. Define the two-form M_{ij} which are the dual of the divergence-free vector fields $\mathbf{M}_{ij} = \mathbf{A}_i \times \mathbf{A}_j$ (defined recently). In this sense, the Gauss linking integral will take the following form

$$\mathcal{L}_{ij} = \int_{\partial U_i} M_{ij} = - \int_{\partial U_j} M_{ij}. \quad (8)$$

Using Stokes' Theorem the last integral becomes

$$\begin{aligned} \mathcal{L}_{ij} &= \int_{U_i} \mathbf{dM}_{ij} = \int_{U_i} \mathbf{B}_i \wedge A_j \\ &= - \int_{U_j} A_i \wedge \mathbf{B}_j. \end{aligned} \quad (9)$$

The linking numbers \mathcal{L}_{ij} ($i \neq j$) of the curves of the Borromean rings vanish.

The one-forms A_q represent a cohomology classes in $H^1(U'; \mathbb{R})$ and the two-form M_{pq} represent a cohomology classes in $H^2(U', \mathbb{R})$. We can add the two-form $\phi_i|_{U_j} B_j$ to the form M_{ij} inside the tube U_j and take away the form $\phi_j|_{U_i} B_i$ from M_{ij} inside the tube U_i , where $\phi_i|_{U_j}$ is a scalar potential satisfying $A_j = \mathbf{d}\phi_i|_{U_j}$. This is an approach by Berger to make the M_{ij} closed everywhere, and this allows us to find a potential N_{ij} such that $\mathbf{dN}_{ij} = M_{ij}$. Now the modified form is

$$M_{ij} = \begin{cases} A_i \wedge A_j - \phi_j|_{U_i} B_i & \text{on } U_i, \\ A_i \wedge A_j + \phi_i|_{U_j} B_j & \text{on } U_j, \\ A_i \wedge A_j & \text{o.w.} \end{cases} \quad (10)$$

¹Alexander Duality Theorem applies to the homology theory properties of the complement of a subspace X^p in e.g. Euclidean space \mathbb{R}^n , then $H_q(\mathbb{R}^n \setminus X)$ and $H^{n-q-1}(X)$ are isomorphic.

The Poincaré lemma is helpful to show that closed forms represent cohomology classes, i.e. the closed forms are exact at least locally. Here we remind the reader of the lemma :

Theorem 1 (Poincaré lemma).

Suppose X is a smooth manifold, $\Omega^k(X)$ is the set of all smooth differential k -form on X , and suppose ω is a closed form in $\Omega^k(X)$ for some $k > 0$. Then for every $x \in X$, there is a neighbourhood $U \subset X$ and a $(k-1)$ -form $\eta \in \Omega^{k-1}(U)$, such that $\mathbf{d}\eta = \omega$, where ι is the inclusion $\iota : U \rightarrow X$, (this is a proper Poincaré lemma) [12].

Moreover, if X is contractible, this η exists globally; there exists a $(k-1)$ -form $\eta \in \Omega^{k-1}(X)$ such that $\mathbf{d}\eta = \omega$.

With the help of the Poincaré lemma, there exists a one-form N_{ij} such that $\mathbf{dN}_{ij} = M_{ij}$. In the space $\cup_{i=1}^3 U_i$, the Massey product associated with the cohomology classes is defined as follows [13]. Let

$$\begin{aligned} M_{ijk}^1 &= A_i \wedge N_{jk}, \\ M_{ijk}^2 &= A_k \wedge N_{ij}, \end{aligned}$$

and define a two-form M_{ijk} by

$$M_{ijk} = M_{ijk}^1 - M_{ijk}^2. \quad (11)$$

This form represents an element in the cohomology class in $H^2(U', \mathbb{R})$ [13, 14]. The Massey product is introduced firstly by the following volume integral, which is known as the third-order linking integral

$$\mathcal{L}_{ijk} = \int_{U_i} \mathbf{dM}_{ijk} = - \int_{U_k} \mathbf{dM}_{ijk}, \quad (12)$$

expanding the integrand term in the last equation yields

$$\begin{aligned} \mathcal{L}_{ijk} &= \int_{U_i} \mathbf{d}(A_i \wedge N_{jk} - A_k \wedge N_{ij}) \\ &= \int_{U_i} \mathbf{B}_i \wedge N_{jk} - \phi_j|_{U_i} B_i \wedge A_k \\ &= \int_{U_i} \mathbf{B}_i \wedge (N_{jk} - \phi_j|_{U_i} A_k). \end{aligned} \quad (13)$$

By considering that the flux tube is thin with unit flux, the last integral can be reformulated as a line integral (see e.g. [15], p. 14-15) to take the following form

$$\begin{aligned} \mathcal{L}_{ijk} &= \oint_{\alpha_i} N_{jk} - \phi_j|_{U_i} A_k \\ &= \oint_{\alpha_k} N_{ij} + \phi_j|_{U_k} A_i. \end{aligned} \quad (14)$$

We will show how the line integral of the Massey product Eq. (14) and the third-order winding number

$$\psi_{abc} = \lambda_{ab}\omega_{bc} + \lambda_{bc}\omega_{ca} + \lambda_{ca}\omega_{ab}. \quad (15)$$

are related to each other in the particular case in which the braid is closed (periodic) and then can be considered as a link.

The third-order winding number is an invariant quantity up to smooth deformation and since every pure braid can be combed,

it is enough to consider a combed braid only. Here, we take a pure 3-braid as an example, comb it such that two strands, α_j and α_k say, are fixed and the other, α_i , winds around them.

For this braid the third-order integral, Eq. (15), becomes

$$\begin{aligned}\Psi_{ijk} &= \Re \left(\int_{\alpha_i} \lambda_{ki} \mathbf{d}\lambda_{ij} \right) \\ &= -\frac{1}{4\pi^2} \int_{\alpha_i} \left(\ln r_{ki} \frac{d}{dt} \ln r_{ij} - \theta_{ki} \frac{d}{dt} \theta_{ij} \right) dt. \quad (16)\end{aligned}$$

This is achieved by considering that α_i and α_j are complex functions. The terms r_{ij} and θ_{ij} are the modulus and argument of $\alpha_{ij} = \alpha_i - \alpha_j$.

Now, we want to show how Eq. (16) can match (14). For this purpose, the integral of the one-form A_k along the path α_i equal the flux induced by \mathbf{B}_k weighted by the winding angle θ_{ik}

$$\int_{\alpha_i(0)}^{\alpha_i(\tau)} A_k dt = \frac{\Phi_k}{2\pi} \theta_{ik}(\alpha_i(\tau)). \quad (17)$$

Now differentiate with respect to τ to obtain

$$A_k = \frac{\Phi_k}{2\pi} \mathbf{d}\theta_{ik}. \quad (18)$$

Since A_j on $U_i = \mathbf{d}\phi_j|_{U_i}$, then $\phi_j|_{U_i}$ can be written as follows

$$\phi_j|_{U_i} = \frac{\Phi_j}{2\pi} \theta_{ij}. \quad (19)$$

Also, the two-form M_{jk} on U_i is just

$$\begin{aligned}M_{jk} &= A_j \wedge A_k \\ &= \frac{\Phi_j \Phi_k}{4\pi^2} \mathbf{d}\theta_{ij} \wedge \mathbf{d}\theta_{ki} \\ &:= \mathbf{d}N_{jk}.\end{aligned} \quad (20)$$

Since, $z_{ij} = r_{ij} \exp(-i\theta_{ij})$ and $z_{ki} = r_{ki} \exp(-i\theta_{ki})$ and since, $\mathbf{d} \ln z_{ij} \wedge \mathbf{d} \ln z_{ki} = 0$, then we can get the following relation

$$\mathbf{d}\theta_{ij} \wedge \mathbf{d}\theta_{ki} = \frac{\mathbf{d}r_{ij}}{r_{ij}} \wedge \frac{\mathbf{d}r_{ki}}{r_{ki}}. \quad (21)$$

Now, by using the above identity in the Eq. (20), the following form can be used as a potential for the two-form M_{jk}

$$N_{jk} = -\frac{\Phi_j \Phi_k}{4\pi^2} \ln r_{ij} \mathbf{d} \ln r_{ki}. \quad (22)$$

Assuming that the tubes U_i are filled with unit longitudinal fluxes, Eq. (14) takes a new form equivalent to Eq. (16) after defining its terms as in Eqs. (17), (19) and (22).

4. Evans-Berger's Formula for the Fourth-Order Linking Number

Evans and Berger (1992) [9] have used the same idea of construction of the third-order winding to build a formula for the fourth-order linking number. They formulated the Massey quadruple in vector notation so as to construct the fourth-order link invariant for magnetic flux tubes in the form of a volume integral. They used the example of a link including four curves,

say α_i , $i = 1, 4$, which are enclosed in thin toroidal volumes U_i and each tube filled with divergence-free magnetic fields \mathbf{B}_i . These fields point in the same direction as the axial curves α_i and do not cross the boundary (i.e. $\mathbf{B}_i \cdot \hat{\mathbf{n}} = 0$). For each field a flux in the directions of the central line Φ_i . The magnetic fields have vector potentials \mathbf{A}_i (i.e. $\nabla \times \mathbf{A}_i = \mathbf{B}_i$). For this link the second and third order linkings vanish (i.e. $\mathcal{L}_{ij} = \mathcal{L}_{ijk} = 0$), and since we are working in a simply connected space, there exists a one-form N_{ijk} everywhere such that

$$M_{ijk} = \mathbf{d}N_{ijk},$$

where M_{ijk} defined as follows

$$M_{ijk} = \begin{cases} A_i \wedge N_{jk} + N_{ij} \wedge A_k - \omega_{jk}|_{U_i} B_i & \text{on } U_i, \\ A_i \wedge N_{jk} + N_{ij} \wedge A_k - \phi_i|_{U_j} \phi_k|_{U_j} B_j & \text{on } U_j, \\ A_i \wedge N_{jk} + N_{ij} \wedge A_k + \omega_{ij}|_{U_k} B_k & \text{on } U_k, \\ A_i \wedge N_{jk} + N_{ij} \wedge A_k & \text{o.s.} \end{cases} \quad (23)$$

One can check that the two-form M_{ijk} is closed, (i.e. $\mathbf{d}M_{ijk} = 0$). The scalar potentials $\omega_{jk}|_{U_i}$ and $\omega_{ij}|_{U_k}$ are defined in the way that the following condition hold

$$\mathbf{d}\omega_{jk}|_{U_i} = N_{jk} - \phi_j|_{U_i} A_k, \quad (24)$$

$$\mathbf{d}\omega_{ij}|_{U_k} = N_{ij} + \phi_j|_{U_k} A_i. \quad (25)$$

The integrand of the fourth-order linking integral, known as ‘‘Massey quadruple product’’, is defined as

$$M_{ijkl} = A_i \wedge N_{jkl} + N_{ij} \wedge N_{kl} + N_{ijk} \wedge A_l, \quad (26)$$

then the fourth-order linking can be written as a surface integral in the following way

$$\mathcal{L}_{ijkl} = \int_{\partial U_i} M_{ijkl} = - \int_{\partial U_i} M_{ijkl}. \quad (27)$$

Now, by applying Stokes' theorem, Eq. (27) can be reformulated as a volume integral as follows

$$\mathcal{L}_{ijkl} = \int_{U_i} \mathbf{d}M_{ijkl} = - \int_{U_i} \mathbf{d}M_{ijkl}. \quad (28)$$

5. Integral formula for a generalised Sato-Levine invariant

A fourth-order integral W has been proposed by [10], for a pair of divergence-free magnetic fields \mathbf{B}_i and \mathbf{B}_j respectively localised in two oriented tubes U_i and U_j in \mathbb{R}^3 . The integral W is invariant up to deformations of the configuration space, its value is preserved in the motion of tubes in an ideal medium. This integral is a generalisation of the Sato-Levine invariant which is defined for two tubes with zero linking number [16]. The Sato-Levine invariant β has been discovered independently by Jerome Levine, also, [17] and [18] have also studied this invariant.

The invariant is defined for a semiboundary link. Such a link has the property that every a pair of its components has a zero linking number, and the invariant is an integer which can be calculated by studying the intersections of Seifert surfaces¹ of the components of the link. It's proper definition is just for a pair of two linked curves [16]. Later, [19] generalised it for a higher order.

Akhmetiev start with a particular case of two tubes with zero linking number to construct W [20], and later they improved the formula to be applicable in more general cases, and they described the dependence between the integral W and the generalised Sato-Levine invariant.

In the next section, we will review the construction of the integral formula (W) of Akhmetiev and Ruzmaikin. We plan to show how this formula is related to the fourth-order invariant of Evans and Berger, so we will restrict ourselves to the case in which the linking numbers is zero.

5.1. Formula of the invariant W

The following construction is taken from [10] to derive the fourth-order integral W , but here it is restricted to case for which the linking numbers vanish.

We denote by U_i and U_j the magnetic flux tubes with fixed orientation of their axial lines α_i and α_j respectively, equipped by magnetic fields \mathbf{B}_i and \mathbf{B}_j respectively. The vector potentials of these fields are \mathbf{A}_i and \mathbf{A}_j respectively.

Now, consider the restriction $\mathbf{A}_j|_{U_i}$ and define a multivalued function $\tau_i : U_i \rightarrow \mathbb{R}$ as follows

$$\tau_i(x) = \int_{x_0}^x \mathbf{A}_j \cdot d\mathbf{s}, \quad x \in U_i, \quad \tau_i(x_0) = 0,$$

where x_0 is a fixed point in U_i , and \mathbf{s} is a path connecting the point x_0 and x in the torus U_i . The function τ_i can be defined by $\mathbf{A}_j|_{U_i}$ say λ_j .

In this case, the following relations hold for the tubes U_i and U_j respectively,

$$\mathbf{A}_j|_{U_i} = \nabla \lambda_j, \quad (29)$$

$$\mathbf{A}_i|_{U_j} = \nabla \lambda_i. \quad (30)$$

Now consider the vector field associated with a certain closed 2-differential form if the axial lines of the tubes U_i and U_j have zero linking number

$$\mathbf{F} = \mathbf{A}_i \times \mathbf{A}_j + \lambda_i \mathbf{B}_j - \lambda_j \mathbf{B}_i - \nabla \varphi.$$

The choosing for the potential φ suitably to makes \mathbf{F} divergence-free accordingly, since

$$\nabla \cdot (\mathbf{A}_i \times \mathbf{A}_j) = \mathbf{B}_i \cdot \mathbf{A}_j|_{U_i} - \mathbf{A}_i|_{U_j} \cdot \mathbf{B}_j$$

$$\nabla \cdot (\lambda_i \mathbf{B}_j - \lambda_j \mathbf{B}_i) = \nabla \lambda_i \cdot \mathbf{B}_j - \nabla \lambda_j \cdot \mathbf{B}_i$$

$$= \mathbf{B}_j \cdot \mathbf{A}_i|_{U_j} - \mathbf{B}_i \cdot \mathbf{A}_j|_{U_i}.$$

There exists a vector potential \mathbf{G} for the field \mathbf{F} , (i.e. $\nabla \times \mathbf{G} = \mathbf{F}$). If we suppose the requirement $\nabla \times \mathbf{G} = 0$ with a suitable boundary condition, then the vector field \mathbf{G} is uniquely determined by \mathbf{F} . Akhmetiev and Kunakovskaya have introduced the integral W by the formula

$$W(\mathbf{B}_i, \mathbf{B}_j) = \int_{\mathbb{R}^3} (\mathbf{G} \cdot \mathbf{F} - \lambda_i^2 \mathbf{B}_j \cdot \mathbf{A}_j - \lambda_j^2 \mathbf{B}_i \cdot \mathbf{A}_i) dx. \quad (31)$$

6. The Consistency of Evans-Berger and Akhmetiev's Formulas

In this part, we will show how the formula of the integral formula of Akhmetiev is consistent with the fourth order-linking of Evans and Berger after grouping the strands in pairs. Now we will start with expanding the Formula (28) by calculating the term dM_{ijkl}

$$\begin{aligned} dM_{ijkl} &= d\mathbf{A}_i \wedge N_{jkl} - A_i \wedge dN_{jkl} \\ &\quad + dN_{ij} \wedge N_{kl} + N_{ij} \wedge dN_{kl} \\ &\quad + dN_{ijk} \wedge A_l - N_{ijk} \wedge dA_l \\ &= B_i \wedge N_{jkl} - A_i \wedge M_{jkl} + \\ &\quad M_{ij} \wedge N_{kl} + N_{ij} \wedge M_{kl} + \\ &\quad M_{ijk} \wedge A_l - N_{ijk} \wedge B_l. \end{aligned}$$

Now we will identify strand i and k and also strand j and l . We can do this by renaming $k = i$ and $l = j$ and then do the calculation

$$\begin{aligned} dM_{ijij} &= B_i \wedge N_{jij} - A_i \wedge M_{jij} + \\ &\quad M_{ij} \wedge N_{ij} + N_{ij} \wedge M_{ij} + \\ &\quad M_{iji} \wedge A_j - N_{iji} \wedge B_j. \end{aligned} \quad (32)$$

We can ignore the sixth term in Eq. (32) because we will integrate over the volume U_i where B_j vanishes, and by summing the third and fourth terms, we will get

$$\begin{aligned} dM_{ijij}|_{U_i} &= 2M_{ij} \wedge N_{ij} + B_i \wedge \\ &\quad N_{jij} - A_i \wedge M_{jij} + M_{iji} \wedge A_j. \end{aligned} \quad (33)$$

Let us denote by K the last two terms in Eq. (33) and using definition of M_{ijk} , as well as Eqs. (23) and (14), K becomes

$$\begin{aligned} K|_{U_i} &= -A_i \wedge (A_j \wedge N_{ij} + N_{ji} \wedge \\ &\quad A_j - \phi_j^2|_{U_i} B_i) + (A_i \wedge N_{ji} \\ &\quad + N_{ij} \wedge A_i - \omega_{ji}|_{U_i} B_i \\ &\quad + \omega_{ij}|_{U_i} B_i) \wedge A_j = \phi_j^2|_{U_i} A_i \wedge B_i, \end{aligned} \quad (34)$$

(since, the terms $\omega_{ji}|_{U_i}$ and $\omega_{ij}|_{U_i}$ represent the third-order integrals of only two tubes, they are equal to zero).

Now, by using Eqs. (10) and (23), one can verify that the second term in Eq. (33) equal to $-\phi_j^2|_{U_i} B_i \wedge A_i$, then we will get

$$dM_{ijij}|_{U_i} = 2 M_{ij} \wedge N_{ij}. \quad (35)$$

¹A Seifert surface is a surface bounded by one component of the link.

Evans and Berger’s formula for the fourth-order linking number becomes

$$\mathcal{L}_{ijij} = \int_{U_i} 2 M_{ij} \wedge N_{ij}. \tag{36}$$

Now, by restricting the integration in Akhmetev’s formula, Eq. (31), to one flux tube, e.g. U_i , and translate Eq. (36) into vectorial language, the resultant forms are consistent.

7. A Higher Analog of the Winding Number

Accounting of the higher-order winding integral is eased by using differential forms. In the Brunnian 4-braid,

$$\mathcal{B}_4 = \sigma_3^{-1} \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_3 \sigma_3 \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \sigma_3^{-1}$$

all the second and third-order winding numbers vanish, see Figure 3 (left).

In order to describe the intricate tangling in such braid, we need to define a higher-order invariant.

Consider four complex functions $a = a(s), b = b(s), c = c(s)$, and $d = d(s)$ which define curves in the phase space $\mathbb{C} \times \mathbb{I}$, such that intersections among these curves are not allowed, and they define periodic orbits. We obtain a pure 4-braid, say κ . Assume $a(0), b(0)$ and $c(0)$ are all lie on the real line of the complex plane, and $|a(0) - b(0)| = |b(0) - c(0)| = |c(0) - a(0)| = r$ and let midpoints are $u = |b(0) - a(0)|/2, v = |b(0) - c(0)|/2$ and $w = |c(0) - a(0)|/2$.

The string $a(s)$ can be written in the following parametrisation form:

$$a(s) = \begin{cases} C_0 & 0 \leq s < \pi \\ C_1 & \pi \leq s < 2\pi \\ C_2 & 2\pi \leq s < 3\pi \\ \vdots & \vdots \\ C_{13} & 13\pi \leq s \leq 14\pi \end{cases}$$

where

$$\begin{aligned} C_0 &= u + (-\cos(s), \sin(s), s) \\ C_1 &= v + (-\cos(s - \pi), \sin(s - \pi), s) \\ C_2 &= w + (-\cos(s - 2\pi), -\sin(s - 2\pi), s) \\ &\vdots \\ C_{13} &= u + (\cos(s - 13\pi), -\sin(s - 13\pi), s) \end{aligned}$$



Figure 3. Brunnian 4-braid $\mathcal{B}_4 = \sigma_3^{-1} \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_3 \sigma_3 \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \sigma_3^{-1}$.

This also can be obtained from κ where $a(s) = \kappa(a(0))$. The other strings can be written in a similar way. Observe that all the lower winding numbers (λ_{ij} and Ψ_{ijk}) vanish.

The following one-form

$$\begin{aligned} \sigma_{abcd} &= (\lambda_{ab} - \lambda_{ad})\psi_{bcd} \\ &+ (\lambda_{bc} - \lambda_{ba})\psi_{cda} \\ &+ (\lambda_{cd} - \lambda_{cb})\psi_{dab} \\ &+ (\lambda_{da} - \lambda_{dc})\psi_{abc}, \end{aligned} \tag{37}$$

is the analogy of ψ_{abc} (third-order winding number), and it is a closed form as shown below

$$\begin{aligned} \mathbf{d}\sigma_{abcd} &= (\omega_{ab} - \omega_{ad})\psi_{bcd} \\ &+ (\omega_{bc} - \omega_{ba})\psi_{cda} \\ &+ (\omega_{cd} - \omega_{cb})\psi_{dab} \\ &+ (\omega_{da} - \omega_{dc})\psi_{abc}. \end{aligned} \tag{38}$$

Since the integrand ψ_{ijk} is a closed form, then all the terms which include $\mathbf{d}\psi_{ijk}$ vanish. Expanding the last equation and using $\omega_{ij} = \omega_{ji}$ and Arnold’s identity,

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ki} + \omega_{ki} \wedge \omega_{ij} = 0, \tag{39}$$

then we will get

$$\mathbf{d}\sigma_{abcd} = 0. \tag{40}$$

The real part of the integral

$$\Sigma_{abcd} = \int_{\kappa} \sigma_{abcd}, \tag{41}$$

provides a fourth-order winding number, which is an invariant quantity up to deformation measuring the linkage of the four strands of the braid κ .

In the following part, we will test this formula for a braid corresponding to a well known link “Brunnian ring”.

Now, we will apply the above formula for the braid \mathcal{B}_4 in Figure 3. First, we label the strings of the colours (red, blue, green and black) by a, b, c and d, respectively.

Integrating the above form along the braid $\tilde{\mathcal{B}}(s), 0 \leq s \leq 1$, we get

$$\Sigma_{abcd} = \int_0^1 -\lambda_{ab}\lambda_{da}\omega_{ac} + \lambda_{da}\lambda_{ac}\omega_{ab} = 1. \tag{42}$$

Figure 4 shows the fourth-order winding number of the Brunnian braid \mathcal{B}_4 . The value of of this integral grows starting from zero when there is no entanglement and then eventually becomes one which is the desired result.

The fourth-order winding number given by Eq. (41) can be used to distinguish four unlinked strands from the entanglement of the Brunnian braid.

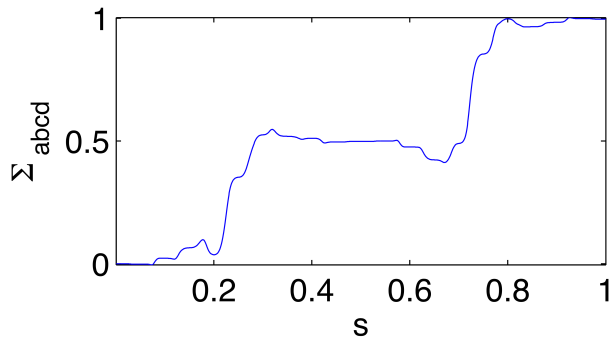


Figure 4. The fourth order invariant winding number for the braid \mathcal{B}_4 .

8. Cup product is a generalization of the linking number

Practical algorithm for finding a finite presentation of the fundamental group $\pi_1(X, x_0)$ of an arbitrary finite regular CW-space X which was illustrated in [21] and described in detail in [22]. From such a presentation, one can calculate the cup product

$$\cup: H^1(X, \mathbb{Z}) \times H^1(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z}) \quad (43)$$

without need for any further significant computations since this product is essentially an invariant of $\pi_1(X, x_0)$ details in [23].

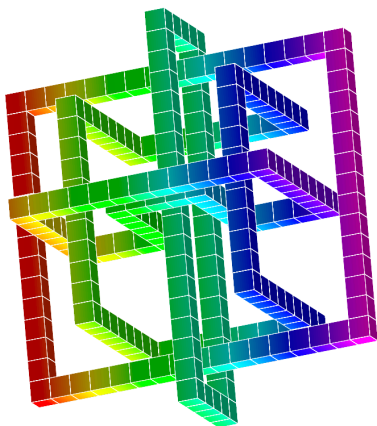


Figure 5. Pure cubical complex L representing the Brunnian link.

Figure 5 shows a pure permutahedral complex L representing a link with four components that represents the Brunnian link. To investigate the link we embed it into the interior of a contractible pure permutahedral complex R and form the complement $M = R \setminus \overset{\circ}{L}$ of the interior of L .

Groups, Algorithms, Programming GAP [24] is a system for computational Group Theory.

The following GAP session loads such a complex M from the file `purecubicalcomplex.txt` available at [25]. The pure permutahedral complex M . The session first constructs a smaller homotopy equivalent pure permutahedral complex $XM \simeq M$.

The space XM is constructed using a *zig-zag deformation retract* technique based on simple homotopy collapses which is described explicitly in [21]. The session then uses XM to compute the cohomology group $H^n(M, \mathbb{Z})$ and a presentation for $G = \pi_1 M$ using the algorithm of [22].

```
gap> Read("purecubicalcomplex.txt");;
gap> XM:=RegularCWComplex(
ZigZagContractedComplex(M));
Regular CW-complex of dimension 3
gap> G:=FundamentalGroup(XM);
<fp group of size infinity on the generators
[f1,f2,f3,f4,f5,f6,f7]>
```

The following continuation of the GAP session uses the method described in [23] to compute the cup product $\cup_i \alpha_i$ where α_i are free generators of $H^1(M, \mathbb{Z})$.

```
gap> cup:=CupProduct(G);
[ -1, 1 ]
```

It is well known that the cup product $\cup_i \alpha_i$ can be interpreted in terms of the linking number $Lk(K_1, K_2, K_3, K_4)$ where K_i are the four components in the Brunnian link (see for instance [26]).

These GAP functions are provided by HAP [27] which is one of GAP packages.

9. Conclusions

It is important to look for high-order linking numbers to measure the entanglement of the field lines of the magnetic braid, and by which one can avoid the problem of the permutation. In this regard, we first found that the integral formula for a generalised Sato-Levine invariant is consistent in certain situations with Evans and Berger's formula for the fourth-order winding number. Also, we found that, in principle, one can go forward to derive analogous high-order winding numbers. For example we applied the fourth-order linking formula to Brunnian rings and verified that the linking number is 1. It can also be calculated by computing the cup product on the cohomology of a finite regular CW-space which is the complement $\mathbb{R}^3 \setminus \mathcal{B}_4$.

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References

- [1] H. Baty, "On the Sweet-Parker model for incompressible visco-resistive magnetic reconnection in two dimensions associated to ideal magnetohydrodynamic instabilities", <https://doi.org/https://doi.org/10.48550/arXiv.2201.01135>.
- [2] M. Berger, "Introduction to magnetic helicity", IOP Publishing: Plasma Physics and Controlled Fusion **41**(1999) 167. <https://doi.org/10.1088/0741-3335/41/12b/312>.

- [3] H. K. Moffatt, "The degree of knottedness of tangled vortex lines", Cambridge University Press: Journal of Fluid Mechanics, **35** (1969) 117. <https://doi.org/https://doi.org/10.1017/S0022112069000991>.
- [4] M. A. Berger, "Third-order braid invariants", IOP Publishing, Journal of Physics A: Mathematical and General **24** (1991) 4027. <https://doi.org/https://iopscience.iop.org/article/10.1088/0305-4470/24/17/019>
- [5] M. I. Monastyrsky & V. Retakh, "Topology of linked defects in condensed matter", Commun. Math. Phys. **3** (1986) 445. <https://doi.org/https://doi.org/10.1007/BF01211760>.
- [6] Ch. Mayer, *Topological Link Invariants of Magnetic Fields*, PhD Thesis, Bochum University, 2003. <https://d-nb.info/969161964/34>.
- [7] M. I. Monastyrsky & P. V. Sasorov, "Topological invariants in magnetohydrodynamics", Soviet Physics JETP **4** (1987) 66. <http://www.jetp.ras.ru/cgi-bin/dn/e066040683.pdf>.
- [8] P. Akhmetiev, "On a new integral formula for an invariant of 3-component oriented links", J. Geom. Phys. **53** (2005) 180. <https://doi.org/https://doi.org/10.1016/j.geomphys.2004.06.002>.
- [9] N. Evans & M. A. Berger, *Hierarchy of linking integrals*, Kluwer Acad. Publ. 1992. https://doi.org/https://doi.org/10.1007/978-94-017-3550-6_12.
- [10] P. Akhmetiev & O. Kunakovskaya, "Integral formula for a generalized Sato-Levine invariant in magnetic hydrodynamics", Mathematical Notes **85** (2009) 503. <https://doi.org/https://doi.org/10.1134/S0001434609030225>.
- [11] A. B. Familus & E. O. Omoleb & L. A. Ukpebor, "A Higher-order Block Method for Numerical Approximation of Third-order Boundary Value Problems in ODEs", J. Nig. Soc. Phys. Sci. **4**, <https://doi.org/10.46481/jnsps.2022.706>, (2022) 706.
- [12] R. Bott & W. Loring, *Differential Forms in Algebraic Topology*, Springer-Verlag New York Incp, <https://link.springer.com/book/10.1007/978-1-4757-3951-0>, (1982).
- [13] W. S. Massey, *Some higher order cohomology operations*, Symposium internacional de topologia algebraica, UNESCO, Mexico. <https://mathscinet.ams.org/mathscinet-getitem?mr=0098366>, (1958).
- [14] R. A. Fenn, *Techniques of Geometric Topology*, Cambridge University Press: London Maths. Soc. Lecture Note Series, **57**, (1983).
- [15] H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids*, Cambridge University Press, ISBN 0521216400, (1983).
- [16] N. Sato, "Cobordisms of semiboundary links", Topology Appl, **18** (1984) 225.
- [17] D. Ruberman, "Concordance of links in S^4 ", Contemp. Math. **35** (1984) 481.
- [18] T. Cochran, "On an invariant of link cobordism in dimension four", Topology and its Applications, **18** (1984) 97. [https://doi.org/10.1016/0166-8641\(84\)90002-6](https://doi.org/10.1016/0166-8641(84)90002-6), 18.
- [19] T. Cochran, "Geometric invariants of link cobordism", Comment. Math. Helv. pp. 291-311, **60**, <https://doi.org/10.1007/BF02567416>
- [20] P. Akhmetiev & A. Ruzmaikin, "A fourth-order topological invariant of magnetic or vortex lines", J Geom Phys **15** (1995) 5.
- [21] G. Ellis & F. Hegarty, "Computational homotopy of finite regular CW-spaces", Journal of Homotopy and Related Structures **9** (2014) 25. <https://doi.org/10.1007/s40062-013-0029-4>.
- [22] P. Brendel, P. Dlotko, G. Ellis, M. Juda & M. Mrozek, "Computing fundamental groups from point clouds", Applicable Algebra in Engineering, Communication and Computing **26** (2015) 27. <https://doi.org/10.1007/s00200-014-0244-1>.
- [23] Nisreen Alokbi & G. Ellis, "Distributed computation of low-dimensional cup products", International Press of Boston: Homology, Homotopy and Applications **20** (2018) 41, <https://doi.org/10.4310/HHA.2018.v20.n2.a3>.
- [24] GAP Version 4.5.6, *Manual for GAP - Groups, Algorithms, and Programming*, The GAP Group 2013. <http://www.gap-system.org>.
- [25] Nisreen Alokbi, *FpGd - Finitely Presented Groupoid (GAP package)*, 2019. <https://github.com/nalokbi/FpGd>.
- [26] M. Morishita, *Knots and primes*, Springer, London, 2012 <https://link.springer.com/book/10.1007/978-1-4471-2158-9>.
- [27] G. Ellis, *HAP - Homological Algebra Programming*, Version 1.10.13. <http://www.gap-system.org/Packages/hap.html> (2013).
- [28] Jerrold Franklin, *Understanding Vector Calculus: Practical Development and Solved Problems*, Dover Publications, ISBN. 9780486835907, 2020.