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Numerical Solution of Second Order Fuzzy Ordinary Differential Equations using Two-Step Block Method with Third and Fourth Derivatives

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Abstract

Fuzzy differential equation models are suitable where uncertainty exists for real-world phenomena. Numerical techniques are used to provide an approximate solution to these models in the absence of an exact solution. However, existing studies that have developed numerical techniques for solving second-order fuzzy ordinary differential equations (FODEs) possess an absolute error accuracy that could be improved. Therefore, this article developed a more accurate higher derivative self-starting block scheme for the numerical solution of second-order FODEs with fuzzy initial and boundary conditions imposed. Linear block approach using Taylor series expansion is adopted for the derivation of the proposed method and the basic properties are established using the definitions of stability and consistency for block methods. According to the numerical results, when compared to the exact solution in terms of absolute error, the new method proposed in this article outperformed existing numerical methods. It is thus concluded that the proposed method is effective for solving second-order FODEs directly.

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1. Introduction

Second-order differential equations have many applications, especially in the field of engineering, biology, chemistry, electronics, physics, etc. Unfortunately, unpredictable scenarios may be encountered which introduced the concept of uncertainty [1] and the application of fuzzy derivatives in fuzzy differential equations (FDEs) to handle these situations [2]. There are three differentiations used to describe the differential or derivative of a fuzzy function. The first is the Hukuhara derivative (H-derivative), which was introduced in [3], the second is the Seikkala derivative introduced in [4], and the third is the generalized derivative (g-derivative) introduced in [5]. This study focuses on the H-derivative in order to define the differential equations considered in this article, which follows the definition by the authors whose results were considered for comparison in the numerical examples with the newly developed block method.

The second-order FODE of the form given in the equation below is considered in this article,

$$\widetilde{y}''(x) = f(x, \widetilde{y}(x), \widetilde{y}'(x)), \forall x \in [a, b] \tag{1}$$

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From Equation 1, $\hat{y}''(x) = \frac{d^2\hat{y}}{dt^2} = f(x, \widehat{y}(x), \widehat{y}'(x))$ is a H-derivative and $\hat{y}$ is a fuzzy function of crisp variable x. Since the function is fuzzy, there exist solutions known as lower and upper solutions because the parametric form of the α-level is given as

$$\underline{\overline{\widehat{y}}}''(x,\alpha) = \underline{\overline{f}}(x, \widehat{y}(x,\alpha), \widehat{y}'(x,\alpha)), \forall\alpha \in [0,1],$$

where

$$\underline{f} = \min\left\{f(x, \underline{\widehat{y}}(x,\alpha), \underline{\widehat{y}}'(x,\alpha))\right\}$$

and

$$\overline{f} = \max\left\{f(x, \overline{\widehat{y}}(x,\alpha), \overline{\widehat{y}}'(x,\alpha))\right\}.$$

The above types of problems in the parametric form of fuzzy function may be difficult to solve directly, and sometimes it is not possible to obtain exact solutions. As a result, researchers were interested in employing various numerical approaches to obtain an approximate solution for second-order FODEs. Several types of numerical methods developed by numerous researchers for second-order FODEs with initial and boundary conditions include the homotopy analysis method in [6, 7], decomposition method [8], Laplace and differential transformation method in [9, 10], least-square method [11], and Runge-Kutta method in [12-14]. The biggest drawback of these approaches is the reduction of the second-order FODEs to the system of first-order FODEs, which leads to computational burden and also impacts solution accuracy. To bypass the rigor of reduction, block methods were introduced for the direct solution of second-order FODEs in [15-17]. However, due to the order of the block methods developed by these studies, it is observed that there is still room to improve the accuracy of their obtained results in terms of absolute error. Hence, the motivation of this study is to develop a new block method with the presence of two higher derivative terms with the aim of obtaining better accuracy. In comparison to existing methods, the newly developed method has the advantages of better accuracy, being self-starting, and incurring a low computational burden in the development and implementation of the block method.

The following is how this article is structured: The essential definitions for fuzzy set theory are presented in Section 2, and the construction of the two-step block method with third and fourth derivatives is presented in Section 3 with the use of the linear block approach. Section 4 highlights the block method's properties, Section 5 considers linear and nonlinear numerical examples, and Section 6 concludes the article.

2. Preliminaries

This section recalls some definitions which will be adopted in this article. The section discusses basic definitions of triangular fuzzy numbers, trapezoidal fuzzy numbers, fuzzy set support, α-level set, and Hukuhara differential. These concepts are required to establish the different parameters of the crisp theory's uncertain behavior. These concepts play an important role when fuzzy differential equations model real-life situations.

Definition 1: Triangular Fuzzy Number [18]

Consider three numbers $(\mu, v, w) \in \mathbb{R}^3, \mu \le v \le w$, then $M(x)$ denotes the triangular fuzzy number given as:

$$M(x,\mu,v,w) = \begin{cases} 0, & x < \mu \\ \frac{x-\mu}{v-\mu}, & \mu \le x \le v \\ \frac{w-x}{w-v}, & v < x \le w \\ 0, & x > w \end{cases} \tag{2}$$

The corresponding α-level set is defined as

$$M_\alpha = \left[\mu + \alpha\,(v-\mu)\,, w - \alpha(w-v)\right], \alpha \in [0,1]. \tag{3}$$

Definition 2: Trapezoidal Fuzzy Numbers [18]

Consider four numbers $(\mu, v, w, \delta) \in \mathbb{R}^4, \mu \le v \le w \le \delta$, then the trapezoidal fuzzy number $M(x)$ is given as:

$$M(x,\mu,v,w,\delta) = \begin{cases} 0, & x < \mu \\ \frac{x-\mu}{v-\mu}, & \mu \le x < v \\ 1, & v \le x \le w \\ \frac{w-x}{w-v}, & w < x \le \delta \\ 0, & x > \delta \end{cases} \tag{4}$$

The corresponding α-level set is defined as

$$M_\alpha = \left[\mu + \alpha\,(v-\mu)\,, \delta - \alpha(\delta-w)\right], \alpha \in [0,1]. \tag{5}$$

Definition 3: Fuzzy Set Support [18]

A set $\widehat{A}$ has fuzzy set support with X universal set defined as,

$$Supp(\widehat{A}) = \left\{x \in X | M_{\widehat{A}(x)} > 0\right\} \tag{6}$$

It contains all elements in X which have membership degree of fuzzy element greater than zero.

Definition 4: α-Level Set [18]

Consider that, $M \in \mathbb{R}_f$, the α-level set is defined as,

$$M_\alpha = \begin{cases} \{x \in \mathbb{R} | M(x) > 0\}, & \alpha \in [0,1] \\ cl(suppM), & \alpha = 0 \end{cases}, \tag{7}$$

with its closed, bounded interval $[\underline{M}(x), \overline{M}(x)]$. $\underline{M}(x)$ and $\overline{M}(x)$ are lower and upper bound of M_α respectively.

Definition 5: Hukuhara Differential [3]

A function $f : (u,v) \to \mathbb{R}_f$ is called H-differentiable, if for $h > 0$ sufficiently small, then H-difference

$$f(x) - f(x-h), f(x+h) - f(x)$$

exists and $\exists$ an element $f'(x) \in \mathbb{R}_f$ such that,

$$\lim_{h\to 0} \frac{f(x) - f(x-h)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x). \tag{8}$$

Then $f'(x)$ is called the H-derivative of f at x.

3. Methodology

Given that the second-order FODE defined in Equation 1 be a mapping $f : \mathbb{R}_f \rightarrow \mathbb{R}_f$ and $\widehat{y}_0 \in \mathbb{R}_f$ with α-level set $\widehat{y}_0 \in \left(\underline{y}(0,\alpha), \overline{\overline{y}}(0,\alpha)\right)^{\overline{\alpha}}_{\underline{\alpha}}, \alpha \in [0,1]$. The partition of the has the set of grid points $0 = x_0 < x_1 < x_2 <, ..., < x_n = X$ with exact solution as $\left(\widehat{Y}(x_n,\alpha)\right)^{\overline{\alpha}}_{\underline{\alpha}} = \left(\underline{Y}(x_n,\alpha), \overline{\overline{Y}}(x_n,\alpha)\right)^{\overline{\alpha}}_{\underline{\alpha}}$ and approximation solution also denoted as $(\widehat{y}(x_n,\alpha))^{\overline{\alpha}}_{\underline{\alpha}} = \left(\underline{y}(x_n,\alpha), \overline{\overline{y}}(x_n,\alpha)\right)^{\overline{\alpha}}_{\underline{\alpha}}$ at which points, $h = \frac{X - x_0}{n}$, $x_n = x_0 + nh, 0 \leq n \leq N$.

The two-step linear block method with the presence of third and fourth derivatives in second-order form is stated below as,

$$\left(\widehat{y}_{n+\eta}\right)^{\overline{\alpha}}_{\underline{\alpha}} = \left(\sum_{v=0}^{1} \frac{(\eta h)^v}{v!}\widehat{y}_n^{(v)} + \sum_{d=0}^{2}\left[\sum_{v=0}^{2} \psi_{dv\eta} f_{n+v}^{(d)}\right]\right)^{\overline{\alpha}}_{\underline{\alpha}}, \eta = 1,2 \quad (9)$$

with the first derivative expression for the block method form given as

$$\left(\widehat{y}'_{n+\eta}\right)^{\overline{\alpha}}_{\underline{\alpha}} = \left(\widehat{y}'_n + \sum_{d=0}^{2}\left[\sum_{v=0}^{2} \omega_{dv\eta} f_{n+v}^{(d)}\right]\right)^{\overline{\alpha}}_{\underline{\alpha}}, \eta = 1,2 \quad (10)$$

Expanding Equations 9 and 10 produces the expressions in Equations 11, 12, 13, and , 14.

$$(\widehat{y}_{n+1})^{\overline{\alpha}}_{\underline{\alpha}} = \begin{pmatrix} \widehat{y}_n + h\widehat{y}'_n + [\psi_{001}f_n + \psi_{011}f_{n+1} + \psi_{021}f_{n+2} \\ +\psi_{101}f'_n + \psi_{111}f'_{n+1} + \psi_{121}f'_{n+2} + \psi_{201}f''_n \\ +\psi_{211}f''_{n+1} + \psi_{221}f''_{n+2}] \end{pmatrix}^{\overline{\alpha}}_{\underline{\alpha}} \quad (11)$$

$$(\widehat{y}_{n+2})^{\overline{\alpha}}_{\underline{\alpha}} = \begin{pmatrix} \widehat{y}_n + 2h\widehat{y}'_n + [\psi_{002}f_n + \psi_{012}f_{n+1} + \psi_{022}f_{n+2} \\ +\psi_{102}f'_{n+1} + \psi_{112}f'_{n+1} + \psi_{122}f'_{n+2} + \psi_{202}f''_n \\ +\psi_{212}f''_{n+1} + \psi_{222}f''_{n+2}] \end{pmatrix}^{\overline{\alpha}}_{\underline{\alpha}} \quad (12)$$

$$\left(\widehat{y}'_{n+1}\right)^{\overline{\alpha}}_{\underline{\alpha}} = \begin{pmatrix} \widehat{y}'_n + [\omega_{001}f_n + \omega_{011}f_{n+1} + \omega_{021}f_{n+2} + \omega_{101}f'_n \\ +\omega_{111}f'_{n+1} + \omega_{121}f'_{n+2} + \omega_{201}f''_n + \omega_{211}f''_{n+1} \\ +\omega_{221}f''_{n+2}] \end{pmatrix}^{\overline{\alpha}}_{\underline{\alpha}} \quad (13)$$

$$\left(\widehat{y}'_{n+2}\right)^{\overline{\alpha}}_{\underline{\alpha}} = \begin{pmatrix} \widehat{y}'_n + [\omega_{002}f_n + \omega_{012}f_{n+1} + \omega_{022}f_{n+2} + \omega_{102}f'_n \\ +\omega_{112}f'_{n+1} + \omega_{122}f'_{n+2} + \omega_{202}f''_n + \omega_{212}f''_{n+1} \\ +\omega_{222}f''_{n+2}] \end{pmatrix}^{\overline{\alpha}}_{\underline{\alpha}} \quad (14)$$

By applying Taylor series expansions

$$(\widehat{y}(x+h;\alpha))^{\overline{\alpha}}_{\underline{\alpha}} = \left(\sum_{i=0}^{n} \frac{h^i}{i!} f^i(x;\alpha)\right)^{\overline{\alpha}}_{\underline{\alpha}} \quad (15)$$

which is given in [19] to expand each term in Equations 11-14 yields

$$\left(\widehat{y}_{n+j}\right)^{\overline{\alpha}}_{\underline{\alpha}} = (\widehat{y}(x_n + jh;\alpha))^{\overline{\alpha}}_{\underline{\alpha}} = \left(\sum_{i=0}^{n} \frac{(jh)^i}{i!} f^i(x_n;\alpha)\right)^{\overline{\alpha}}_{\underline{\alpha}}, j = 0,1,2, \quad (16)$$

$$\left(\widehat{y}_{n+j}\right)^{\overline{\alpha}}_{\underline{\alpha}} = \begin{pmatrix} \widehat{y}(x_n;\alpha) + jh\widehat{y}'(x_n;\alpha) + \frac{(jh)^2}{2!}\widehat{y}''(x_n;\alpha) \\ + \frac{(jh)^3}{3!}\widehat{y}'''(x_n;\alpha) + ... + \frac{(jh)^n}{n!}\widehat{y}^n(x_n;\alpha) \end{pmatrix}^{\overline{\alpha}}_{\underline{\alpha}}. \quad (17)$$

After that, the unknown coefficients ψ_{dvn} and ω_{dvn} are obtained from $\psi_{dvn} = A^{-1}B$ and $\omega_{dvn} = A^{-1}D$, where

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h & 2h & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & \frac{h^2}{2!} & \frac{2^2h^2}{2!} & 0 & h & 2h & 1 & 1 & 1 \\ 0 & \frac{h^3}{3!} & \frac{2^3h^3}{3!} & 0 & \frac{h^2}{2!} & \frac{2^2h^2}{2!} & 0 & h & 2h \\ 0 & \frac{h^4}{4!} & \frac{2^4h^4}{4!} & 0 & \frac{h^3}{3!} & \frac{2^3h^3}{31} & 0 & \frac{h^2}{2!} & \frac{2^2h^2}{2!} \\ 0 & \frac{h^5}{5!} & \frac{2^5h^5}{5!} & 0 & \frac{h^4}{4!} & \frac{2^4h^4}{4!} & 0 & \frac{h^3}{3!} & \frac{2^3h^3}{3!} \\ 0 & \frac{h^6}{6!} & \frac{2^6h^6}{6!} & 0 & \frac{h^5}{5!} & \frac{2^5h^5}{5!} & 0 & \frac{h^4}{4!} & \frac{2^4h^4}{4!} \\ 0 & \frac{h^7}{7!} & \frac{2^7h^7}{7!} & 0 & \frac{h^6}{6!} & \frac{2^6h^6}{6!} & 0 & \frac{h^5}{5!} & \frac{2^5h^5}{5!} \\ 0 & \frac{h^8}{8!} & \frac{2^8h^8}{8!} & 0 & \frac{h^7}{7!} & \frac{2^7h^7}{7!} & 0 & \frac{h^6}{61} & \frac{2^6h^6}{6!} \end{pmatrix}^{\overline{\alpha}}_{\underline{\alpha}},$$

$$B = \overset{\overline{\alpha}}{\underset{\underline{\alpha}}{\begin{pmatrix} \frac{(\eta h)^2}{2!} \\ \frac{(\eta h)^3}{3!} \\ \frac{(\eta h)^4}{4!} \\ \frac{(\eta h)^5}{5!} \\ \frac{(\eta h)^6}{6!} \\ \frac{(\eta h)^7}{7!} \\ \frac{(\eta h)^8}{8!} \\ \frac{(\eta h)^9}{9!} \\ \frac{(\eta h)^{10}}{10!} \end{pmatrix}}}, D = \begin{pmatrix} \eta h \\ \frac{(\eta h)^2}{2!} \\ \frac{(\eta h)^3}{3!} \\ \frac{(\eta h)^4}{4!} \\ \frac{(\eta h)^5}{5!} \\ \frac{(\eta h)^6}{6!} \\ \frac{(\eta h)^7}{7!} \\ \frac{(\eta h)^8}{8!} \\ \frac{(\eta h)^9}{9!} \end{pmatrix}^{\overline{\alpha}}_{\underline{\alpha}}.$$

Therefore,

$$\begin{pmatrix} \psi_{001} \\ \psi_{011} \\ \psi_{021} \\ \psi_{101} \\ \psi_{111} \\ \psi_{121} \\ \psi_{201} \\ \psi_{211} \\ \psi_{221} \end{pmatrix}^{\overline{\alpha}}_{\underline{\alpha}} = \begin{pmatrix} \frac{19h^2}{60} \\ \frac{h^2}{5} \\ \frac{-h^2}{60} \\ \frac{911h^3}{20160} \\ \frac{-16h^3}{315} \\ \frac{113h^3}{20160} \\ \frac{53h^4}{20160} \\ \frac{h^4}{80} \\ \frac{-11h^4}{20160} \end{pmatrix}, \begin{pmatrix} \psi_{002} \\ \psi_{012} \\ \psi_{022} \\ \psi_{102} \\ \psi_{112} \\ \psi_{122} \\ \psi_{202} \\ \psi_{212} \\ \psi_{222} \end{pmatrix}^{\overline{\alpha}}_{\underline{\alpha}} = \begin{pmatrix} \frac{76h^2}{105} \\ \frac{128h^2}{105} \\ \frac{2h^2}{35} \\ \frac{34h^3}{315} \\ \frac{-32h^3}{315} \\ \frac{-2h^3}{315} \\ \frac{2h^4}{315} \\ \frac{16h^4}{315} \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \omega_{001} \\ \omega_{011} \\ \omega_{021} \\ \omega_{101} \\ \omega_{111} \\ \omega_{121} \\ \omega_{201} \\ \omega_{211} \\ \omega_{221} \end{pmatrix}^{\overline{\alpha}}_{\underline{\alpha}} = \begin{pmatrix} \frac{5669h}{13440} \\ \frac{64h}{105} \\ \frac{-42h}{13440} \\ \frac{303h^2}{4480} \\ \frac{-1h^2}{8} \\ \frac{47h^2}{4480} \\ \frac{169h^3}{40320} \\ \frac{8h^3}{315} \\ \frac{-41h^3}{40320} \end{pmatrix}, \begin{pmatrix} \omega_{002} \\ \omega_{012} \\ \omega_{022} \\ \omega_{102} \\ \omega_{112} \\ \omega_{122} \\ \omega_{202} \\ \omega_{212} \\ \omega_{222} \end{pmatrix}^{\overline{\alpha}}_{\underline{\alpha}} = \begin{pmatrix} \frac{41h}{105} \\ \frac{128h}{105} \\ \frac{41h}{105} \\ \frac{2h^2}{35} \\ 0 \\ \frac{-2h^2}{35} \\ \frac{1h^3}{315} \\ \frac{16h^3}{315} \\ \frac{1h^3}{315} \end{pmatrix}.$$

The obtained values of the coefficients are substituted in Equations 11-14 which is the required two-step block method with the presence of third and fourth derivatives as given below.

$$\begin{aligned}(\widehat{y}_{n+1})_{\underline{\alpha}}^{\overline{\alpha}} = \widehat{y}_n + h\widehat{y}'_n + h^2\left[\tfrac{19}{60}f_n + \tfrac{1}{5}f_{n+1} - \tfrac{1}{60}f_{n+2}\right] \\ +h^3\left[\tfrac{911}{20160}g_n - \tfrac{16}{315}g_{n+1} + \tfrac{113}{20160}g_{n+2}\right] \\ +h^4\left[\tfrac{53}{20160}m_n + \tfrac{1}{80}m_{n+1} - \tfrac{11}{20160}m_{n+2}\right],\end{aligned}$$

$$\begin{aligned}(\widehat{y}_{n+2})_{\underline{\alpha}}^{\overline{\alpha}} = \widehat{y}_n + 2h\widehat{y}'_n + h^2\left[\tfrac{76}{105}f_n + \tfrac{128}{105}f_{n+1} + \tfrac{2}{35}f_{n+2}\right] \\ +h^3\left[\tfrac{34}{315}g_n - \tfrac{32}{315}g_{n+1} - \tfrac{2}{315}g_{n+2}\right] + h^4\left[\tfrac{2}{315}m_n + \tfrac{16}{315}m_{n+1}\right],\end{aligned} \tag{18}$$

$$\begin{aligned}(\widehat{y}'_{n+1})_{\underline{\alpha}}^{\overline{\alpha}} = \widehat{y}'_n + h\left[\tfrac{5669}{13440}f_n + \tfrac{64}{105}f_{n+1} - \tfrac{421}{13440}f_{n+2}\right] \\ +h^2\left[\tfrac{303}{4480}g_n - \tfrac{1}{8}g_{n+1} + \tfrac{47}{4480}g_{n+2}\right] \\ +h^3\left[\tfrac{169}{40320}m_n + \tfrac{8}{315}m_{n+1} - \tfrac{41}{40320}m_{n+2}\right],\end{aligned} \tag{19}$$

$$\begin{aligned}(\widehat{y}'_{n+2})_{\underline{\alpha}}^{\overline{\alpha}} = \widehat{y}'_n + h\left[\tfrac{41}{105}f_n + \tfrac{128}{105}f_{n+1} + \tfrac{41}{105}f_{n+2}\right] \\ +h^2\left[\tfrac{2}{35}g_n - \tfrac{2}{35}g_{n+2}\right] + h^3\left[\tfrac{1}{315}m_n + \tfrac{16}{315}m_{n+1} + \tfrac{1}{315}m_{n+2}\right]\end{aligned}$$

where $g = \frac{df(x,\alpha)}{dx}$, $m = \frac{d^2f(x,\alpha)}{dx}$.

The block method in Equation 18 has corrector form,

$$\begin{aligned}\left(A^0\widehat{Y}_{n+k}\right)_{\underline{\alpha}}^{\overline{\alpha}} = \left(A^1\widehat{Y}_{n-k}\right)_{\underline{\alpha}}^{\overline{\alpha}} + h\left(B^1\widehat{Y}'_{n-k}\right)_{\underline{\alpha}}^{\overline{\alpha}} + h^2\left(C^0F_{n+k} + C^1F_{n-k}\right)_{\underline{\alpha}}^{\overline{\alpha}} \\ +h^3\left(D^0G_{n+k} + D^1G_{n-k}\right)_{\underline{\alpha}}^{\overline{\alpha}} + h^4\left(E^0M_{n+k} + E^1M_{n-k}\right)_{\underline{\alpha}}^{\overline{\alpha}}\end{aligned}$$

where,

$$A^0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}, A^1 = \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}, B^1 = \begin{pmatrix}0 & 1\\ 0 & 2\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}, C^0 = \begin{pmatrix}\frac{1}{5} & \frac{-1}{60}\\ \frac{128}{105} & \frac{2}{35}\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}},$$

$$C^1 = \begin{pmatrix}0 & \frac{19}{60}\\ 0 & \frac{76}{105}\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}, D^0 = \begin{pmatrix}\frac{-16}{315} & \frac{113}{20160}\\ \frac{-32}{315} & \frac{-2}{315}\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}, D^1 = \begin{pmatrix}0 & \frac{911}{20160}\\ 0 & \frac{34}{315}\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}},$$

$$E^0 = \begin{pmatrix}\frac{1}{80} & \frac{-11}{20160}\\ \frac{16}{315} & \frac{-2}{315}\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}, E^1 = \begin{pmatrix}0 & \frac{53}{20160}\\ 0 & \frac{2}{315}\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}, \widehat{Y}_{n+k} = \begin{pmatrix}\widehat{y}_{n+1}\\ \widehat{y}_{n+2}\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}},$$

$$\widehat{Y}_{n-k} = \begin{pmatrix}\widehat{y}_{n-1}\\ \widehat{y}_n\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}, \widehat{Y}'_{n-k} = \begin{pmatrix}\widehat{y}'_{n-1}\\ \widehat{y}'_n\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}, F_{n+k} = \begin{pmatrix}f_{n+1}\\ f_{n+2}\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}},$$

$$F_{n-k} = \begin{pmatrix}f_{n-1}\\ f_n\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}, G_{n+k} = \begin{pmatrix}g_{n+1}\\ g_{n+2}\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}} G_{n-k} = \begin{pmatrix}g_{n-1}\\ g_n\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}},$$

$$M_{n+k} = \begin{pmatrix}m_{n+1}\\ m_{n+2}\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}, M_{n-k} = \begin{pmatrix}m_{n-1}\\ m_n\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}.$$

4. Properties of the Proposed Method

This section will first mention the required definitions and theorems to investigate the properties of the developed two-step third-fourth derivative scheme, and thereafter apply these theorems and definitions to the method.

4.1. Convergence and Stability Properties

Theorem 1:

A block method is convergent iff it is consistent and zero-stable. [22]

Proof

The aim of the proof is to show that zero stability and consistency are necessary conditions for convergence. Suppose that the block method defined in Equation 9 is convergent, the first condition for zero-stability follows by considering Equation 1 with a trivial solution $\widehat{y}(x) = 0$. Applying Equation 9 to this problem yields the difference equation

$$\left(\widehat{y}_{n+\eta} - \sum_{v=0}^{1}\frac{(\eta h)^v}{v!}\widehat{y}_n^{(v)} - \sum_{d=0}^{2}\left[\sum_{v=0}^{2}\psi_{dv\eta}f_{n+v}^{(d)}\right]\right)_{\underline{\alpha}}^{\overline{\alpha}}, \eta = 1,2 \tag{20}$$

Since the method is assumed to be convergent, for any $x > 0$, then

$$\lim_{\substack{h\to 0\\ nh\to 0}}\widehat{y}_{n+\eta} = 0 \tag{21}$$

for all solutions of Equation 20 satisfying $\widehat{y}_s = \varsigma_s(h)$, $s = 0, 1, ..., k-1$ where

$$\lim_{h\to 0}\widehat{y}_s = 0 \tag{22}$$

Let $\psi = re^{i\phi}$ be a root of the first characteristic polynomial $P(\psi) = 0$, $r \geq 0$, $0 \leq \phi \leq 2\pi$. It can be verified then that the numbers

$$\widehat{y}_{n+\eta} = hr^n\cos(n\phi) \tag{23}$$

define a solution to Equation 20 satisfying Equation 22. If $\phi = 0$, $\phi \neq \pi$, then

$$\frac{\widehat{y}_{n+\eta} - \widehat{y}_n - \widehat{y}'_n}{\sin^2\phi} = h^2r^{2n} \tag{24}$$

Since the left-hand side of this identity converges to 0 as $h \to 0$, $n \to \infty$, $nh = x$ the same must be true of the right-hand side; therefore,

$$\lim_{n\to\infty}\left(\frac{x}{n}\right)^{\infty}r^{2n} = 0 \tag{25}$$

This implies that $r \leq 1$. In other words, it is proven that any root of the first characteristic polynomial of (9) lies in the closed unit disc. Note that any root of the first characteristic polynomial of Equation 9 that lies on the unit circle must be simple.

For the other condition, which is consistency, let us first show that $C_0 = 0$. Consider Equation 1 with trivial solution, $\widehat{y}(x) = 1$. Applying Equation 9 to this problem yields the difference equation Equation 20. Choose $\widehat{y}_s = 1$, $s = 0, 1, ..., k-1$. Given that by hypothesis the method is convergent, it is deduced that

$$\lim_{h\to 0}\widehat{y}_s = 1 \tag{26}$$

Since in the present case $\widehat{y}_n$ is independent of the choice of h, Equation 26 is equivalent to saying that

$$\lim_{h\to\infty}\widehat{y}_n = 1, \tag{27}$$

and passing to the limit $n \to \infty$ in Equation 20, it is deduced that

$$\alpha_k + \alpha_{k-1}+, ..., +\alpha_0 = 0. \tag{28}$$

Recalling the definition of C_0, Equation 28 is equivalent to $C_0 = 0$ (i.e. $P(1) = 0$).

To show that $C_1 = 0$, consider Equation 1 with trivial solution, $\widehat{y}(x) = x$. Applying Equation 9 to this problem yields the difference equation in Equation 20. For a convergent method every solution of Equation 20 satisfying

$$\lim_{h\to 0} \varsigma_s(h) = 0, \;\; s = 0, 1, ..., k-1 \tag{29}$$

where $\widehat{y}_s = \varsigma_s(h), \;\; s = 0, 1, ..., k-1$, must also satisfy

$$\lim_{h\to 0} \widehat{y}_{n+\eta} = x. \tag{30}$$

Since according to the previous theorem zero-stability is necessary for convergence, we may take it for granted that the first characteristic polynomial $P(\psi)$ of the method does not have multiple roots on the unit circle $|\psi| = 1$, therefore

$$P^{'}(1) = k\alpha_k+, ..., +2\alpha_2 + \alpha_1 \neq 0. \tag{31}$$

Let the sequence $(x_n)_n^N = 0$ be defined by $\widehat{y}_n = Knh$, where

$$K = \frac{\psi_{dk\eta} + \psi_{d(k-1)\eta}+, ..., +\psi_{d2\eta} + \psi_{d1\eta} + \psi_{d0\eta}}{k\alpha_k+, ..., +2\alpha_2 + \alpha_1}. \tag{32}$$

This sequence clearly satisfies Equation 30 and is the solution of Equation 20. Furthermore, Equation 31 implies that

$$x = \widehat{y}(x) = \lim_{\substack{h\to 0\\ nh=x}} \widehat{y}_{n+\eta} = \lim_{\substack{h\to 0\\ nh=x}} Knh = Kx \tag{33}$$

$$\begin{aligned} C_1 &= (k\alpha_k+, ..., +2\alpha_2 + \alpha_1) \\ &-(\psi_{dk\eta} + \psi_{d(k-1)\eta}+, ..., +\psi_{d2\eta} + \psi_{d1\eta} + \psi_{d0\eta}) = 0. \end{aligned} \tag{34}$$

Equivalently, $P^{'}(1) = \sigma(1)$. Thus, since the necessary conditions in terms of zero-stability and consistency is satisfied, so the block method is convergent.

Definition 6: Consistency [20]

A block method is consistent if it has order $\rho \geq 1$.

Definition 7: Zero-Stability [20]

A block method with matrix difference equation in the following form

$$\begin{aligned} A^0\widehat{Y}_{n+k} &= A^1\widehat{Y}_{n-k} + B^1\widehat{Y}^{''}_{n-k} + B^2\widehat{Y}^{'}_{n-k} + \cdots + B^1\widehat{Y}^{(m-1)}_{n-k} \\ &+h^m\left(C^0\widehat{Y}^m_{n+k} + C^1\widehat{Y}^m_{n-k}\right)^{\overline{\alpha}}_{\underline{\alpha}} + h^{(m+1)}\left(D^0\widehat{Y}^{(m+1)}_{n+k} + D^1\widehat{Y}^{(m+1)}_{n-k}\right)^{\overline{\alpha}}_{\underline{\alpha}} \\ &+h^{(m+2)}\left(E^0\widehat{Y}^{(m+2)}_{n+k} + E^1\widehat{Y}^{(m+2)}_{n-k}\right)^{\overline{\alpha}}_{\underline{\alpha}}, \end{aligned} \tag{35}$$

with $\widehat{Y}^a_{n+k} = \left(\widehat{y}^a_{n+1}, \widehat{y}^a_{n+2}, ..., \widehat{y}^a_{n+k}\right)^T$ and $\widehat{Y}^a_{n-k} = \left(\widehat{y}^a_{n1(k-1)}, \widehat{y}^a_{n-(k-2)}, ..., \widehat{y}^a_{n}\right)^T$, is zero-stable if the first characteristic polynomial takes form

$$P(\psi) = \det(\psi_v A^0 - A^1), \tag{36}$$

and the root of $P(\psi) = 0$ satisfy $|\psi_v| \leq 1, v = 1, ..., , k$.

Definition 8: Region of Absolute Stability [26]

To obtain the polynomial for the absolute stability region of the block method. The expressions for the corrector take the form:

$$\left(\det\left[\begin{aligned} -(w)^k + A^1 + q\left[\sum_{j=0}^{k} B^j w^{k-j}\right] + q^2\left[\sum_{j=0}^{k} theC^j w^{k-j}\right] \\ +q^3\left[\sum_{j=0}^{k} D^j w^{k-j}\right] + q^4\left[\sum_{j=0}^{k} E^j w^{k-j}\right]\end{aligned}\right]\right)^{\overline{\alpha}}_{\underline{\alpha}},$$

$$q = \lambda h.$$

The absolute stability region is then obtained by plotting the polynomial roots using the boundary locus technique. If the obtained roots of the polynomial lie in the unit circle, then the block method is absolutely stable and its region is called the region of absolute stability. Note that large absolute stability regions mean that large time-step size can be used during the implementation of the method to solve the differential equation [27-29].

Definition 9: A-stable

According to [20], a numerical method is said to be A-stable if its region of absolute stability contains the whole of the left-hand half-plane.

Definition 10: L-stable

According to [20] a general linear multistep method is L-stable if it is A-stable and, in addition, when applied to the scalar test equation $\widehat{y}^{'} = \lambda y$, λ is a complex constant with $Re\lambda < 0$, it yields $\widehat{y}_{n+1} = R(h\lambda)\widehat{y}_n$, where, $|R(h\lambda)| \to 0$ as $Re(h\lambda) \to \infty$. However, a clause is encountered as given in the following definition

Definition 11

According to [21] an A-stable linear multistep method cannot have an order greater than two.

Therefore, based on Definition 8, the properties of A-stability and L-stability cannot be explored for the block methods developed in this article. This is because the block method developed have order greater than two. Hence, the stability property with respect to choosing a stepsize value is limited to just absolute stability alone. Although, much attention was not placed on choosing h-values from the stability region because the h-values were chosen the same as the authors for comparison.

These definitions for block methods in crisp form is adopted to the proposed method for FODEs to prove the convergence properties for the proposed method in the next subsection.

4.2. *Convergence and Stability Analysis of Proposed Method*

Order and Error constant

The linear operator associated with Equation 9 is defined as:

$$L(\widehat{y}(x), h) = \left(\widehat{y}_{n+\eta} - \sum_{v=0}^{1} \frac{(\eta h)^v}{v!}\widehat{y}^{(v)}_n + \sum_{d=0}^{2}\left[\sum_{v=0}^{2} \psi_{dv\eta} f^{(d)}_{n+v}\right]\right)^{\overline{\alpha}}_{\underline{\alpha}}, \tag{37}$$

$$\eta = 1, 2,$$

with

$$L(\widehat{y}(x),h)=\begin{pmatrix} C_0\widehat{y}(x_n)+C_1h\widehat{y}'(x_n)+C_2h^2\widehat{y}''(x_n)+...\\ +C_{z+1}h^{z+1}\widehat{y}^{z+1}(x_n)+C_{z+2}h^{z+2}\widehat{y}^{z+2}(x_n)\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}.$$

The method is said to be of order z if $C_0=C_1=\cdots=C_z=C_{z+1}=0, C_{z+2}\neq 0$, and C_{z+2} is the error constant.

Following the approach by [28], The order of the two-step third-fourth derivatives block method with corrector Equation 18 is nine with an error constant
$C_{11}=(3.8174e-08,\ 7.617e-08)^T$, and the order of the derivative part ten with an error constant
$C_{12}=(6.5076e-08,\ -7.6349e-09)^T$. The derivative formulae will be used to obtain the first derivative term in Equation 1. Expressing the corrector scheme 18 as blocks using previous definitions for the block methods. A simple iteration has been implemented to approximate the value of $\widehat{y}_{n+1}$ and $\widehat{y}_{n+2}$. In the code, we iterate the corrector to convergent and the convergence test employed, and the order of the correctors in nine [23]

Zero-Stability

Applying above definition in fuzzy form for the proposed method gives

$$P(\psi)=\det(\psi_v A^0-A^1)_{\underline{\alpha}}^{\overline{\alpha}}, \tag{38}$$

$$P(\psi)=\left|\psi_v\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}0&1\\0&1\end{pmatrix}\right|_{\underline{\alpha}}^{\overline{\alpha}}.$$

The root of $P(\psi)=0$ satisfies the condition $|\psi_v|\leq 1, v=1,2.$

Convergence

The proposed method is convergent because it is zero stable and consistent.

Absolute Stability Region

The polynomial of the proposed block method to plot its region of absolute stability is obtained as:

$$\left(\left|\begin{matrix}\begin{pmatrix}w&0\\0&w^2\end{pmatrix}+\begin{pmatrix}0&1\\0&1\end{pmatrix}+q\begin{pmatrix}0&1\\0&2\end{pmatrix}\\ +q^2\left[\begin{pmatrix}\frac{1w}{5}&\frac{1w^2}{60}\\ \frac{128w}{105}&\frac{2w^2}{35}\end{pmatrix}+\begin{pmatrix}0&\frac{19}{60}\\0&\frac{76}{105}\end{pmatrix}\right]\\ +q^3\left[\begin{pmatrix}\frac{-16w}{315}&\frac{113w^2}{20160}\\ \frac{-32w}{315}&\frac{-2w^2}{315}\end{pmatrix}+\begin{pmatrix}0&\frac{911}{20160}\\0&\frac{34}{315}\end{pmatrix}\right]\\ +q^4\left[\begin{pmatrix}\frac{1w}{80}&\frac{-11w^2}{20160}\\ \frac{16w}{315}&0\end{pmatrix}+\begin{pmatrix}0&\frac{53}{20160}\\0&\frac{2}{315}\end{pmatrix}\right]\end{matrix}\right|\right)_{\underline{\alpha}}^{\overline{\alpha}}, \tag{39}$$

$$R(w)=\left(\begin{pmatrix}\frac{11q^8}{396900}-\frac{37q^7}{88200}+\frac{19q^6}{52920}+\frac{44q^5}{4725}+\frac{13q^4}{3600}-\frac{2q^3}{35}\\ -\frac{9q^2}{35}+1]w^3+[\frac{43q^8}{793800}+\frac{53q^7}{52920}+\frac{508327q^6}{33868800}\\ -\frac{31547q^5}{793800}-\frac{35639q^4}{100800}-\frac{44q^3}{45}-\frac{20597q^2}{20160}-2q-1\end{pmatrix}w\right)_{\underline{\alpha}}^{\overline{\alpha}}$$

The absolute stability region is thus plotted as shown in Figure 1, which implies that large time-stepsizes can be utilised with the method. From Figure 1, it is seen that for the absolute stability region, all the roots of polynomial lie on the unit circle.

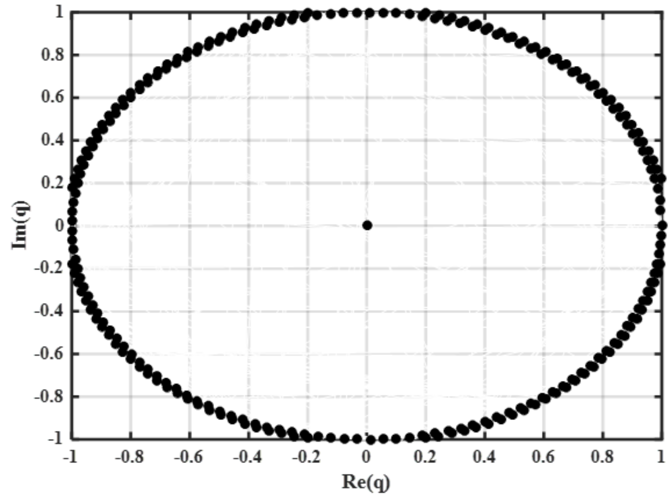

Figure 1. Absolute stability region of proposed method

5. Results

This section details the application of the developed block method for the solution of second-order (linear and nonlinear) FODEs (FIVPs and FBVPs) and the obtained results are compared with the exact solution and existing methods. Comparisons between exact and approximate solutions are shown in tables and graphs.
$x-$axis shows the value of the approximation solution,
$y-$axis show the value of α-level values,
$\underline{\widehat{Y}}$, $\overline{\widehat{Y}}$ are the lower and upper bounds of the exact solution respectively,
$\underline{\widehat{y}}$, $\overline{\widehat{y}}$ are the lower and upper bounds of the approximate solution respectively,
$\underline{E}=\left|\underline{\widehat{Y}}-\underline{\widehat{y}}\right|$ computes the absolute error of the lower bound approximation,
$\overline{E}=\left|\overline{\widehat{Y}}-\overline{\widehat{y}}\right|$ computes the absolute error of the upper bound approximation,
h is the step size,
TSBM: Two-step Block Method with Third and Fourth Derivatives,
EBHDEF: Extended Block Hybrid Backward Differentiation Formula [16],
BDF: Block Differentiation Formula [15],
BBDF: Block Backward Differentiation Formula [15],
OOMB: Optimization of One-Step Block Method [17],
RK5: Runge Kutta Method Order Five [14],
OHAM: Optimal Homotopy Asymptotic Method [7],
FDM: Finite Difference Method [30].

Example 1. Given the second-order linear FIVP

$$\widehat{y}''(x)=-\widehat{y}(x), \widehat{y}(0,\alpha)=0, \widehat{y}'(0,\alpha)=(0.9+0.1\alpha,1.1-0.1\alpha),$$

with exact solution

$$\underline{\widehat{Y}}(x,\alpha)=(0.9+0.1\alpha)\sin(x),\ \overline{\widehat{Y}}(x,\alpha)=(1.1-0.1\alpha)\sin(x),$$

Table 1. Lower and Upper solution of Example 1

α	TSBM $\underline{E}$ $h=0.1$	EBHDEF $\underline{E}$ $h=0.01$	BBDF $\underline{E}$ $h=0.01$	BDF $\underline{E}$ $h=0.01$
0	0.0000e+00	2.8094e-11	5.4048e-08	3.0991e-08
0.2	0.0000e+00	2.8719e-11	5.5249e-08	3.1647e-08
0.4	0.0000e+00	2.9343e-11	5.6450e-08	3.2335e-08
0.6	0.0000e+00	2.9966e-11	5.7651e-08	3.3023e-08
0.8	0.0000e+00	3.0592e-11	5.8853e-08	3.3711e-08
1	0.0000e+00	3.1216e-11	6.0054e-08	3.4399e-08

α	TSBM $\overline{E}$ $h=0.1$	EBHDEF $\overline{E}$ $h=0.01$	BBDF $\overline{E}$ $h=0.01$	BDF $\overline{E}$ $h=0.01$
0	1.1102e-16	3.4337e-11	5.4048e-08	3.7838e-08
0.2	1.1102e-16	3.3713e-11	5.5249e-08	3.7151e-08
0.4	1.1102e-16	3.3089e-11	5.6450e-08	3.6463e-08
0.6	0.0000e+00	3.2464e-11	5.7651e-08	3.5775e-08
0.8	0.0000e+00	3.1840e-11	6.1255e-08	3.5087e-08
1	0.0000e+00	3.1216e-11	6.0054e-08	3.4399e-08

Table 2. Lower and Upper solution of Example 2

α	TSBM $\underline{E}$ $h=0.1$	EBHDEF $\underline{E}$ $h=0.01$	BBDF $\underline{E}$ $h=0.01$	BDF $\underline{E}$ $h=0.01$
0	6.661338e-16	9.8449e-14	2.4250e-10	1.5988e-10
0.2	3.663736e-15	3.4927e-13	5.7971e-10	3.9122e-10
0.4	1.532108e-14	9.7144e-13	1.2016e-09	3.7933e-09
0.6	5.129230e-14	2.2859e-12	2.2597e-09	2.6125e-09
0.8	1.498801e-13	6.4525e-12	3.9207e-09	6.6967e-08
1	3.850253e-13	4.7628e-12	6.3971e-09	1.1110e-08

α	TSBM $\overline{E}$ $h=0.1$	EBHDEF $\overline{E}$ $h=0.01$	BBDF $\overline{E}$ $h=0.01$	BDF $\overline{E}$ $h=0.01$
0	1.345191e-13	4.2267e-11	4.07084e-08	1.26440e-07
0.2	7.431833e-12	2.6623e-11	2.98235e-08	9.1889e-08
0.4	3.907541e-12	1.5982e-11	2.13238e-08	7.34552e-08
0.6	2.774669e-12	9.0485e-11	1.48046e-08	3.52946e-08
0.8	9.001688e-13	8.1274e-11	9.93257e-09	1.51728e-08
1	3.854694e-13	4.7628e-12	6.39707e-09	1.11097e-08

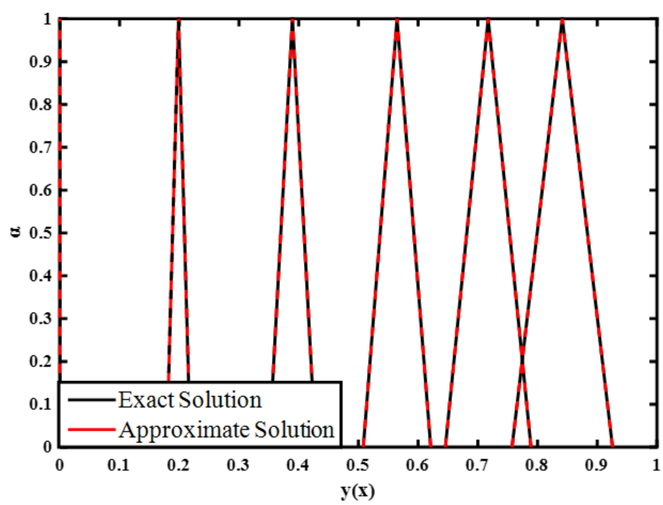

Figure 2. Numerical solution of Example 1 with Lower/Upper solution

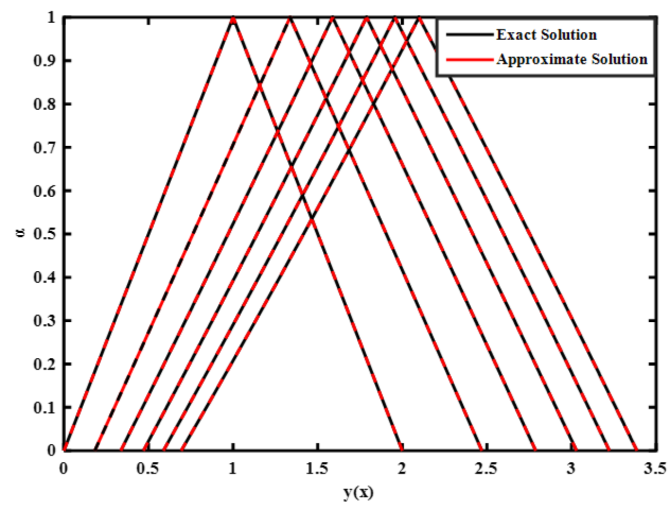

Figure 3. Numerical solution of Example 2 with Lower/Upper solution

and at $x = 1$, $\widehat{Y}(1,\alpha) = \left[\underline{Y}(1,\alpha), \overline{Y}(1,\alpha)\right], 0 \leq \alpha \leq 1$.

The results obtained for Example 1 are shown in Table 1 and Figure 2 displays the complete iterations graph with stepsize $h = 0.1$ and $h = 0.01$ partition of the time interval $x \in [0, 1]$.

It is observed from Table 1 that the approximate solution obtained by new proposed method in comparison to the exact solution in terms of absolute error is very impressive, as it give same results as the exact solution at certain points. The results are graphically shown in Figure 2. In the figure the behaviour of the linear FIVP solution is seen to monotonically increase as shown in the graph. This follows from the property that a function's output will not appear more than once during the course of a monotonically rising interval. It is worth noting that $y(x)$ rises in lockstep with x. The exact and approximate solutions are also compared using the graph and it shows the approximate solution completely overlapping the exact solution which indicates high accuracy of the proposed method.

Example 2. Given the second-order non-linear FIVP

$$\widetilde{y}''(x) = -(\widetilde{y}'(x))^2, \widetilde{y}(0,\alpha) = (\alpha, 2-\alpha), \widetilde{y}'(0,\alpha) = (1+\alpha, 3-\alpha),$$

with exact solution

$$\underline{\widehat{Y}}(x,\alpha) = \ln((x\alpha + x + 1)e^{\alpha}),\ \overline{\widehat{Y}}(x,\alpha) = \ln((3x - x\alpha + 1)e^{\alpha-2}),$$

and at $x = 1$, $\widehat{Y}(1,\alpha) = \left[\underline{Y}(1,\alpha), \overline{Y}(1,\alpha)\right], 0 \leq \alpha \leq 1$.

The results obtained for Example 2 are shown in Table 2 and Figure 3 displays the complete iterations graph with stepsize $h = 0.1$ and $h = 0.01$ partition of the time interval $x \in [0, 1]$.

It is observed from Table 2 that the approximate solution obtained by the new proposed method in comparison to the exact solution in terms of absolute error is very impressive. Just as the previous example, the results graphically shown in Figure 3 are monotonically increasing showing the behaviour of the nonlinear FIVP. Likewise, the approximate solution completely

Table 3. Lower and Upper solution of Example 3

α	Exact Solution	TSBM $\underline{E}$ $h = 0.1$	EBHDEF $\underline{E}$ $h = 0.1$
0	-0.100004086851013030	1.94289e-16	4.131e-07
0.2	-0.080004095094799887	8.32667e-17	4.137e-07
0.4	-0.060004103338586523	1.59594e-16	4.141e-07
0.6	-0.040004111582373492	6.93889e-17	4.149e-07
0.8	-0.020004119826159905	2.56739e-16	4.149e-07
1	-0.000004128069946763	1.35559e-16	4.161e-07

α	Exact Solution	TSBM $\overline{E}$ $h = 0.1$	EBHDEF $\overline{E}$ $h = 0.1$
0	0.100003908832573600	2.77555e-17	9.094e-03
0.2	0.080003917076360453	1.52655e-16	9.094e-03
0.4	0.060003925320147089	4.85722e-17	1.267e-01
0.6	0.040003933563933947	1.66533e-16	8.459e-02
0.8	0.020003941807720582	6.24500e-17	4.291e-02
1	-0.000004128069946763	1.35559e-16	4.161e-07

overlaps the exact solution which indicates high accuracy of the proposed method.

Example 3. Given the second-order linear FBVP

$$\widehat{y}''(x)+\widehat{y}(x)+x=0, \widehat{y}(0,\alpha)=\widehat{y}(1,\alpha)=(0.1\alpha-0.1,0.1-0.1\alpha),$$

with exact solution

$$\widehat{\underline{Y}}(x,\alpha) = -x + (0.1\alpha - 0.1)\cos(x) + (1.13376 + 0.054630\alpha)\sin(x)$$
$$\widehat{\overline{Y}}(x,\alpha) = -x + (0.1 - 0.1\alpha)\cos(x) + (1.24303 - 0.054630\alpha)\sin(x)$$

and at $x = 1$, $\widehat{Y}(1,\alpha) = \left[\underline{Y}(1,\alpha), \overline{Y}(1,\alpha)\right], 0 \le \alpha \le 1$.

The results obtained for Example 3 are shown in Table 3 and Figure 4 displays the complete iterations graph with stepsize $h = 0.1$ partition of the time interval $x \in [0,1]$.

From Table 3 and the graph in Figure 4, impressive monotonocally dereasing results are still observed. The absolute error accuracy is high compared with the existing EBHDEF method and the overlapping behaviour of the approximate solution with the exact solution is evident.

Example 4. Given the second-order non-linear FBVP

$$\widehat{y}''(x) = -\frac{[\widehat{y}'(x)]^2}{\widehat{y}(x)}, x \in [0,1],$$

$$\widehat{y}(0,\alpha)=(0.9+0.1\alpha, 1.1-0.1\alpha), \widehat{y}(1,\alpha)=(0.9+0.1\alpha, 2.1-0.1\alpha),$$

with exact solution

$$\widehat{\underline{Y}}(x,\alpha) = \sqrt{1.4+0.1\alpha}\sqrt{\frac{0.1(9+\alpha)^2}{14+\alpha}+2x},$$

$$\widehat{\overline{Y}}(x,\alpha) = \sqrt{1.6-0.1\alpha}\sqrt{\frac{-0.1(-11+\alpha)^2}{-16+\alpha}+2x},$$

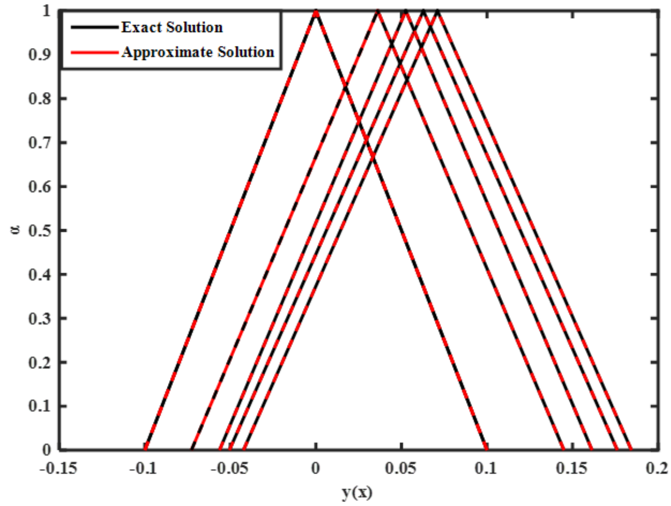

Figure 4. Numerical solution of Example 3 with Lower/Upper solution

Table 4. Lower and Upper solution of Example 4

α	TSBM $\underline{E}$ $h = 0.1$	EBHDEF $\underline{E}$ $h = 0.008$	FDM $\underline{E}$ $h = 0.008$
0	0.000000e+00	0	0
0.2	4.4408920e-16	2.57e-06	9.27e-07
0.4	2.4424906e-15	2e-06	8.55e-07
0.6	1.5321077e-14	1.26e-06	5.92e-07
0.8	9.6207486e-12	5.88e-07	2.94e-07
1	0.000000e+00	0	0

α	TSBM $\overline{E}$ $h = 0.1$	EBHDEF $\overline{E}$ $h = 0.1$	FDM $\overline{E}$ $h = 0.008$
0	0.000000e+00	0	0
0.2	4.4408920e-16	2.05e-06	8.15e-07
0.4	8.8817841e-15	1.63e-06	7.65e-07
0.6	1.7763568e-15	1.03e-06	5.35e-07
0.8	1.3322676e-15	4.87e-07	2.67e-07
1	0.000000e+00	0	0

and at $x = 1$, $\widehat{Y}(1,\alpha) = \left[\underline{Y}(1,\alpha), \overline{Y}(1,\alpha)\right], 0 \le \alpha \le 1$.

The results obtained for Example 4 are shown in Table 4 and Figure 5 displays the complete iterations graph with stepsize $h = 0.1$ and $h = 0.008$ partition of the time interval $x \in [0,1]$.

It is observed from Table 4 that the approximate solution obtained by the new proposed method in comparison to the exact solution in terms of absolute error is very impressive as it give same results as the interval boundaries. The results are graphically shown in Figure 5 and the behaviour of the nonlinear FBVP solution is seen to monotonically increase. The comparison of the exact and approximate solutions on the graph also shows high accuracy as the plots overlap. This indicates the high accuracy of the proposed method.

In addition, the time in seconds required to compute the approximate solution of the numerical examples is given in the table below. The program code was written with MATLAB 2015a on a laptop with 8GB RAM and Intel Core i5-3427U CPU.

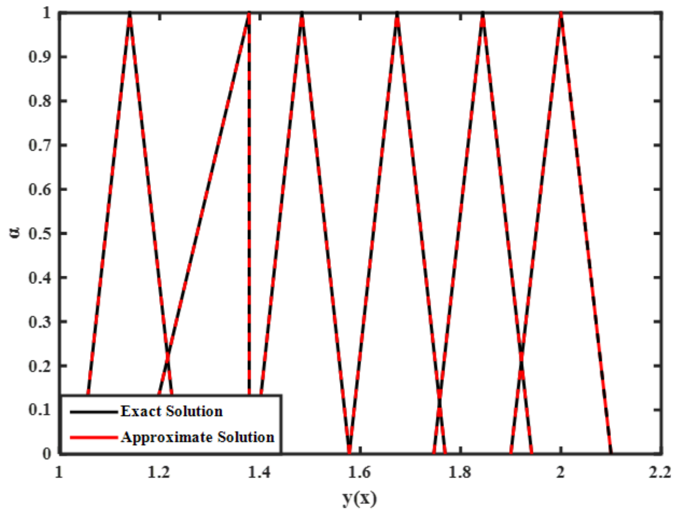

Figure 5. Numerical solution of Example 4 with Lower/Upper solution

Table 5. Time Taken to Compute Approximate Solutions

α	Example 1 $time/sec$	Example 2 $time/sec$	Example 3 $time/sec$	Example 4 $time/sec$
0	0.4767	2.0844	0.9204	0.4457
0.2	1.3682	1.9451	1.5024	0.4097
0.4	1.3742	2.0163	1.4550	0.4072
0.6	1.0563	1.9995	1.3675	0.3741
0.8	1.8364	2.1498	1.4603	0.3982
1	0.8937	2.0415	1.4031	0.3813
α	Example 1 $time/sec$	Example 2 $time/sec$	Example 3 $time/sec$	Example 4 $time/sec$
0	0.9037	2.2057	1.3941	0.3928
0.2	0.8487	2.0648	1.3404	0.3877
0.4	0.8703	2.7846	1.4259	0.4361
0.6	0.8650	2.1738	1.4331	0.2778
0.8	0.8650	1.9303	1.4310	0.3267
1	0.8937	2.0415	1.4031	0.3813

6. Conclusion

The major objective of this research to enhance the accuracy of the solution (in terms of absolute error) by developing a numerical technique for solving second order FODEs (FIVPs and FBVPs) directly. As a result, this article developed a two-step block method for second-order FODEs with the presence of third and fourth derivatives. The proposed method outperforms other methods discovered in the literature as shown in the tables and graphs of the numerical results obtained. In addition, the method eliminates the requirement for complicated subroutines in conventional methods that require starting values or predictors. The proposed block method has proven to be a viable strategy with increased accuracy for solving both linear and nonlinear FIVPs and FBVPs. The method developed using linear block approach with low computational complexity also satisfied all convergence conditions for the block methods. Hence, the proposed method in this article is more suitable for obtaining the approximate solutions of second order FIVPs and FBVPs.

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