Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 5 (2023) 1221

Journal of the Nigerian Society of Physical Sciences

Solving fractional variable-order differential equations of the non-singular derivative using Jacobi operational matrix

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Abstract

This research derives the shifted Jacobi operational matrix (JOM) with respect to fractional derivatives, implemented with the spectral tau method for the numerical solution of the Atangana-Baleanu Caputo (ABC) derivative. The major aspect of this method is that it considerably simplifies problems by reducing them to ones that can be solved by solving a set of algebraic equations. The main advantage of this method is its high robustness and accuracy gained by a small number of Jacobi functions. The suggested approaches are applied in solving non-linear and linear ABC problems according to initial conditions, and the efficiency and applicability of the proposed method are proved by several test examples. A lot of focus is placed on contrasting the numerical outcomes discovered by the new algorithm together with those discovered by previously well-known methods.

Keywords: Fractional differential equations; Atangana-Baleanu Caputo; Variable order; Operational matrix.

Article History :
Received: 23 November 2022
Received in revised form: 31 January 2023
Accepted for publication: 14 February 2023
Published: 05 May 2023

Communicated by: O. Adeyeye

1. Introduction

Over the past three decades, the most focus was placed on fractional calculus study, together with its countless implementations in the fields of engineering and physics. Fractional differential equations (FDEs) can effectively represent the implementations of fractional calculus employed in a variety of fields, which includes signal processing, optics, statistics and probability, electrochemistry of corrosion, control theory of dynamical systems, electrical networks, as well as chemical physics. There have been a number of important early papers on fractional derivatives and FDEs, as may be seen in [1, 2]. These publications offer a systematic explanation of fractional calculus, including its uniqueness and existence, and are regarded as the introduction with respect to the FDEs and fractional derivative theory. Numerous other scholars have recently focused on the findings of the initial value problem (IVP) and boundary value problem (BVP) solutions for FDEs, which can be further read in [3-5]. It is crucial to determine approximate or exact solutions to FDEs. We have trouble locating their analytical solutions for any but a small subset of these equations. Many different types of differential equations in a variety of fields in science, engineering physical and natural applications can be solved, which are extremely effective [6-11]. Other than that, many authors have been inspired to adopt these approaches for various equations because of their high accuracy and simplicity of usage. The collocation, Galerkin, as well as tau methods

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are particular spectral methods that are more suitable and frequently employed.

When Saadatmandi and Dehghan [12] used spectral methods in solving multi-term linear as well as non-linear FDEs numerically, they instigated the shifted Legendre operational matrix with respect to fractional derivatives. To find approximate solutions for multi-term linear and non-linear FDEs, Doha et al. [13] developed a novel formula that explicitly expresses any fractional-order derivatives of shifted Chebyshev polynomials of any degree with respect to the shifted Chebyshev polynomials themselves. They combined this formula with tau and collocation spectral methods. Recently, Bhrawy et al. [14] treated multi-term linear FDEs having variable coefficients employing a quadrature-shifted Legendre tau technique. The shifted Chebyshev operational matrix was recently presented by Doha et al. [15] and used in conjunction with spectral methods to solve multi-term linear and non-linear FDEs according to IVPs and BVPs conditions.

Additionally, in [16, 17], the authors introduced the spectral tau method for the numerical solution of a few FDEs, while in [18], Pedas and Tamme created the spline collocation methods for solving FDEs. In recent years, Esmaeili and Shamsi [19] instigated a direct solution method for solving a particular family of fractional IVPs employing a pseudospectral method. Moreover, Esmaeili et al. [20] introduced a computational method with regard to the Müntz polynomials and collocation method for the FDEs solution. Furthermore, the algorithms utilized in the current work are associated with those applied by Saadatmandi and Dehghan [12], Doha *et al.* [13-15], as well as Bhrawy et al. [14] to create accurate algorithms for a variety of uses. The classical Jacobi polynomials, represented by $J_i^{u,v}(x)(i \geq 0, u, v > -1)$ [21] are crucial to the study and use of spectral methods and have been widely employed in mathematical analysis and real-world applications. The benefit of using general Jacobi polynomials is that they may be used to calculate solutions using the Jacobi parameters a and b (refer to [22]). Therefore, to generalize, it is beneficial to perform a systematic study with regards to Jacobi polynomials $(u, v > -1)$ having general indexes. This may then be immediately implemented in other contexts rather than generating approximation findings for each specific indices pair. This is the reason we introduce the Jacobi polynomials family having indexes $u, v > -1$ in this work.

To solve numerically linear fractional and variable order problems with initial conditions, this work introduces the shifted Jacobi operational matrix (JOM) of fractional derivative. This method relies on the Jacobi tau method. Additionally, we present an appropriate technique for approximating the non-linear fractional and variable order IVPs on the interval $[0, L]$ using the spectral shifted Jacobi collocation approach relying on JOM in order to determine the solution $y(t)$. At $(N - m + 1)$ points, the non-linear fractional and variable orders collocate. These equations produce $(N + 1)$ non-linear algebraic equations that may be resolved employing Newton's iterative method after being combined with m initial conditions. Finally, test problems are used to show how accurate the suggested algorithms are. We point out that Saadatmandi and Dehghan [12] and Doha et al. [15] introduced the two shifted Legendre and Chebyshev operational matrices, correspondingly, and that several more extremely interesting situations may be produced directly as special cases emerging from the shifted JOM [23]. As a result, we were inspired to pursue the shifted Jacobi polynomials because it is the most generalized of the orthogonal polynomials.

This paper is structured accordingly: First, we commence by going over several fundamental information about Jacobi polynomials and fractional calculus theory that is necessary for supporting our findings in Section 2. The JOM for Atangana-Baleanu Caputo (ABC) is obtained in Section 3. The spectral tau, JOM of ABC derivative, as well as collocation methods are all applied in Section 4 to solve general linear and non-linear ABC. The variable-order ABC-derivative JOM is found in section 5. The suggested methods are used in several cases in Section 6. In addition, Section 7 provides a conclusion.

2. Basic concepts and notations

This section defines the Caputo derivative, CF-derivative, as well as Atangana-Baleanu Caputo (ABC)-derivative in which fractional order and variable order are concisely highlighted.

2.1. Fractional derivatives

Caputo fractional-order differential equation is expressed by [1] as:

$$ {}^{C}D^{\alpha}y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y'(s)ds}{(t-s)^{\alpha}}, \tag{1} $$

where $0 < \alpha < 1$. The most popular fractional derivative is the Caputo derivative, which is frequently used in engineering and science domains.

Definition 2.1.1. To $0 < \alpha < 1$, $y(t) \in H^1(\imath, \jmath)$, $\jmath > \imath$, the fractional-order of the CF-derivative is defined by [29]:

$$ {}^{CF}D^{\alpha}y(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t y'(s) e^{\frac{-\alpha(t-s)}{1-\alpha}} ds, \tag{2} $$

in which $M(\alpha)$ denotes a normalization function.

Definition 2.1.2. For $0 < \alpha < 1$, $y(t) \in H^1(\imath, \jmath)$, $\jmath > \imath$, the fractional-order of the ABC-derivative is represented by [30]:

$$ {}^{ABC}D^{\alpha}y(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t y'(s) E_{\alpha}\left[\frac{-\alpha(t-s)^{\alpha}}{1-\alpha}\right] ds, \tag{3} $$

in which $0 < \alpha < 1$, $M(\alpha)$ resembles a normalization function, while E_{α} denotes Mittag-Leffler function.

To broaden the ABC-derivative for the case of $n < \alpha < n+1$ having $y^{(s)}(a) = 0 for \quad s = 1, 2, ..., n$, we have

$$ \begin{aligned} {}^{ABC}D^{\alpha}y(t) &= {}^{ABC}D^{\alpha}(D^{n}y(t)) \\ &= \frac{M(\alpha)}{1-\alpha} \int_0^{\chi} y^{(n+1)}(s) E_{\alpha}\left[\frac{-\alpha(t-s)^{\alpha}}{1-\alpha}\right] ds, \quad (4) \\ y^{(n+1)}(s) &= D^{(n+1)}y(s) = D^{\lceil \alpha \rceil}y(s), \end{aligned} $$

in which $\lceil \alpha \rceil$ represents ceil α.

Definition 2.1.3. The fractional integral of the ABC-derivative may be expressed by [30]:

$$ {}_0^{AB}I_t^{\alpha}\{y(t)\} = \frac{1-\alpha}{M(\alpha)}y(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t y(s)(t-s)^{\alpha-1}ds. \quad (5)$$

Definition 2.1.4. The ABC-derivative having variable-order $\alpha(t)$, $0<\alpha(t)<1$ of function $y(t)$ may be expressed by [31]:

$$ {}^{ABC}D^{\alpha(t)}y(t) = \frac{M(\alpha(t))}{1-\alpha(t)}\int_0^t y^{'}(s)E_{\alpha(t)}\left[\frac{-\alpha(t)}{1-\alpha(t)}(t-s)^{\alpha(t)}\right]ds, \quad (6)$$

where $E_{\alpha(t)}$ is the Mittag Leffer function.

Theorem 2.1. Suppose $\beta > 0$. Then, the variable-order ABC-derivative may be expressed as [31]:

$$ {}^{ABC}D^{\alpha(t)}t^{\beta} = \frac{M(\alpha(t))}{1-\alpha(t)}\Gamma(\beta+1)t^{\beta}E_{\alpha(t),\beta+1}\left[\frac{-\alpha(t)}{1-\alpha(t)}t^{\alpha(t)}\right]. \quad (7)$$

2.2. Some properties of SJPs

The following recurrence formula can be used to create the famous Jacobi polynomials, which are specified in the interval $[-1,1]$:

$$J_i^{(u,v)}(t) = \frac{(u+v+2i-1)[u^2-v^2+t(u+v+2i)(u+v+2i-2)]}{2i(u+v+i)(u+v+2i-2)} J_{i-1}^{(u,v)}(t) - \frac{(u+i-1)(v+i-1)(u+v+2i)}{i(u+v+2i)(u+v+2i-2)}J_{i-2}^{(u,v)},$$

for $i = 2,3,...,$
where

$$J_0^{(u,v)}(t) = 1 \quad and \quad J_1^{(u,v)}(t) = \frac{u+v+2}{2}t + \frac{u-v}{2}.$$

We now define the shifted Jacobi polynomials by instigating the change of variable $t = \frac{2t}{L} - 1$ to apply these polynomials with respect to the interval $t \in [0,L]$. Let the shifted Jacobi polynomials $J_i^{(u,v)}(\frac{2t}{L}-1)$ be expressed by $J_{L,i}^{(u,v)}(t)$. Then, $J_{L,i}^{(u,v)}(t)$ can be generated from:

$$J_{L,i}^{(u,v)}(t) = \frac{(u+v+2i-1)[u^2-v^2+(\frac{2t}{L}-1)(u+v+2i)(u+v+2i-2)}{2i(u+v+i)(u+v+2i-2)}J_{L,i-1}^{(u,v)}(t) - \frac{(u+i-1)(v+i-1)(u+v+2i)}{i(u+v+i)(u+v+2i-2)}J_{L,i-2}^{(u,v)}(t), \quad i=2,3,..., \quad (8)$$

where $J_{L,0}^{(u,v)}(t) = 1$ and $J_{L,1}^{(u,v)}(t) = \frac{u+v+2}{2}(\frac{2t}{L}-1) + \frac{u-v}{2}$.

The analytic form with regards to the shifted Jacobi polynomials $J_{L,i}^{(u,v)}(t)$ of degree i may be expressed as

$$J_{L,i}^{(u,v)}(t) = \sum_{k=0}^{i}(-1)^{i-k}\frac{\Gamma(i+v+1)\Gamma(i+k+u+v+1)}{\Gamma(k+v+1)\Gamma(i+u+v+1)(i-k)!k!L^k}t^k, \quad (9)$$

where

$$J_{L,i}^{(u,v)}(0) = (-1)^i\frac{\Gamma(i+v+1)}{\Gamma(v+1)i!}, \quad J_{L,i}^{(u,v)}(L) = \frac{\Gamma(i+u+1)}{\Gamma(u+1)i!}.$$

Of these polynomials, the most commonly utilized are the shifted Chebyshev polynomials with respect to the first kind $T_{L,i}(t)$, the shifted Legendre polynomials $P_{L,i}(t)$, as well as the shifted Chebyshev polynomials with respect to the second kind $U_{L,i}(t)$. Moreover, for the non-symmetric shifted Jacobi polynomials, two essential special cases with regards to shifted Chebyshev polynomials of third and fourth kinds $V_{L,i}(t)$ and $W_{L,i}(t)$ are also taken into account. The following relations connect these orthogonal polynomials with respect to the shifted Jacobi polynomials.

$$T_{L,i}(t) = \frac{i!\Gamma(0.5)}{\Gamma(i+0.5)}J_{L,i}^{(-0.5,-0.5)}(t), \quad P_{L,i}(t) = J_{L,i}^{(0,0)}(t),$$
$$U_{L,i}(t) = \frac{(i+1)!\Gamma(0.5)}{\Gamma(i+1.5)}J_{L,i}^{0.5,0.5}(t),$$

$$V_{L,i}(t) = \frac{(2i)!!}{(2i-1)!!}J_{L,i}^{(0.5,-0.5)}(t), \quad W_{L,i}(t) = \frac{(2i)!!}{(2i-1)!!}J_{L,i}^{(-0.5,0.5)}(t).$$

The orthogonality condition with respect to the shifted Jacobi polynomials is expressed as

$$\int_0^L J_{L,j}^{(u,v)}(t)J_{L,k}^{(u,v)}(t)W_L^{(u,v)}(t)dt = h_k, \quad (10)$$

where $W_L^{(u,v)}(t) = t^v(L-t)^u$ and

$$h_k = \begin{cases} \frac{L^{u+v+1}\Gamma(k+u+1)\Gamma(k+v+1)}{(2k+u+v+1)k!\Gamma(k+u+v+1)}, & i=j, \\ 0, & i \neq j. \end{cases}$$

Let $y(t)$ refers to a polynomial with degree n. Now, we may write these in terms of shifted Jacobi polynomials given by

$$y(t) = \sum_{j=0}^{N} c_j J_{L,j}^{(u,v)}(t) = c^T, \quad (11)$$

in which the coefficients c_j are provided as follows

$$c_j = \frac{1}{h_j}\int_0^L W_L^{(u,v)}(t)y(t)J_{L,j}^{(u,v)}(t)dt, \quad j=0,1,.... \quad (12)$$

Suppose the shifted Jacobi coefficient vector C, as well as the shifted Jacobi vector $\phi(t)$, is expressed as

$$C^T = [c_0, c_1, ..., c_N], \quad (13)$$

and

$$\phi(t) = [J_{L,0}^{(u,v)}(t), J_{L,1}^{(u,v)}(t), ..., J_{L,N}^{(u,v)}(t)]^T, \tag{14}$$

accordingly. Therefore, the first-order derivative with respect to the vector $\phi(t)$ may be written as

$$\frac{d\phi(t)}{dt} = D^{(1)}\phi(t), \tag{15}$$

in which $D^{(1)}$ denotes the $(N+1)\times(N+1)$ operational matrix of derivative written as

$$D^{(1)} = (d_{ij}) = \begin{cases} C_1(i,j), & i > j, \\ 0 & otherwise, \end{cases}$$

in which

$$C_1(i,j) = \frac{Lu + v(i+u+v+1)(i+u+v+2)_j(j+u+2)_{i-j-1}\Gamma(j+u+v+1)}{(i-j-1)!\Gamma(2j+u+v+1)} \times$$
$$ {}_3F_2\begin{pmatrix} -i+1+j, & i+j+u+v+2, & j+u+1 & \\ & & & ;1 \\ j+u+2, & 2j+u+v+2 & & \end{pmatrix}$$

(The proof can be found in [24], and the general definitions of a generalized hypergeometric series, as well as special ${}_3F_2$, may be found in [25], accordingly on pp. 41 and 103–104). For instance, for even N, we obtain

$$D^{(1)} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ C_1(1,0) & 0 & 0 & \dots & 0 & 0 \\ C_1(2,0) & C_1(2,1) & 0 & \dots & 0 & 0 \\ C_1(3,0) & C_1(3,1) & C_1(3,2) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ C_1(N,0) & C_1(N,1) & C_1(N,2) & \dots & C_1(N,N-1) & 0. \end{pmatrix} \tag{16}$$

3. Generalized SJPs operational matrix to fractional calculus

This section's major goal is to expand the Jacobi operational matrix (JOM) of derivatives for Atangana-Baleanu Caputo (ABC) [32, 33].

Theorem 3.1. Suppose $\Psi(t)$ vector be SJPs defined in Eq.(11) such that $\alpha > 0$. Then

$$^{ABC}D^{\alpha}\Psi(t) \simeq \ ^{ABC}D^{(\alpha)}\Psi(t), \tag{17}$$

in which D^{α} denotes the operational matrix $(m+1)\times(m+1)$ that may be expressed as:

Let $\phi(t)$ vector be SJPs defined in Eq.(14). Here, suppose $\alpha > 0$, then the ϑ in OM Eq.(18) is obtained using SJPs as follows

$$\vartheta_{i,j,k} = \frac{M(\alpha)}{(1-\alpha)} \sum_{\mathrm{ł}=0}^{j} \frac{(-1)^{i+j-k+\mathrm{ł}}\Gamma(i+v+1)\Gamma(i+k+u+v+1)\Gamma(j+v+1)\Gamma(j+\mathrm{ł}+u+v+1)}{h_j\Gamma(k+v+1)\Gamma(i+u+v+1)(i-k)!L^k\Gamma(j+v+1)\Gamma(j+u+v+1)(j-\mathrm{ł})!\mathrm{ł}!L^{\mathrm{ł}}} a_{j,\mathrm{ł}}. \tag{19}$$

Proof.

$$^{ABC}D^{\alpha}t^{v} = \frac{M(\alpha)}{1-\alpha}\int_0^t Y^{(n+1)}(s)E_{\alpha}\left[\frac{-\alpha(t-s)^{\alpha}}{1-\alpha}\right]ds, \quad v > 1, \quad v \geqslant \lceil\alpha\rceil$$
$$= \frac{M(\alpha)}{1-\alpha}\int_0^t \frac{\Gamma(v+1)}{\Gamma(v-n)}S^{v-n-1}E_{\alpha}\left[\frac{-\alpha(t-s)^{\alpha}}{1-\alpha}\right]ds$$
$$If \quad 0<\alpha<1, \quad \lfloor\alpha\rfloor = n = 0, \quad \psi(t) \quad is \quad solve \quad for \int_0^t S^{v-1}E_{\alpha}\left[\frac{-\alpha(t-s)^{\alpha}}{1-\alpha}\right]ds$$
$$^{ABC}D^{\alpha}J_{L,i}^{u,v}(t) = \sum_{k=\lceil\alpha\rceil}^{i}\frac{(-1)^{i-j}\Gamma(i+v+1)\Gamma(i+k+u+v+1)}{\Gamma(k+v+1)\Gamma(i+u+v+1)(i-k)!k!L^k}\ ^{ABC}D^{\alpha}t^{k}$$
$$= \sum_{k=\lceil\alpha\rceil}^{i}\frac{M(\alpha)}{1-\alpha}\psi(t)\frac{(-1)^{i-k}\Gamma(i+v+1)\Gamma(i+k+u+v+1)}{\Gamma(k+v+1)\Gamma(i+u+v+1)(i-k)!L^k\Gamma(k)}$$

$$where \quad \psi(t) = \sum_{j=0}^{\mu} a_{k,j} J_{L,j}^{u,v}(t)$$

$$a_{k,j} = \frac{1}{h_j} \sum_{\text{ł}=0}^{j} \frac{\Gamma(j+v+1)\Gamma(j+\text{ł}+u+v+1)}{\Gamma(j+v+1)\Gamma(j+u+v+1)(j-\text{ł})!\text{ł}!L^{\text{ł}}} \int_0^1 \psi(t) W_L^{(u,v)} t^{\text{ł}} dt$$

$$^{ABC}D^{\alpha} J_{L,i}^{u,v}(t) = \sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{\mu} \frac{(-1)^{i-k}\Gamma(i+v+1)\Gamma(i+k+u+v+1)}{\Gamma(k+v+1)\Gamma(i+u+v+1)(i-k)!k!L^k} a_{k,j} J_{L,j}^{u,v}(t)$$

$$= \sum_{j=0}^{m} \Big(\sum_{k=\lceil\alpha\rceil}^{i} \vartheta_{i,j,k} \Big) J_{L,j}^{u,v}(t).$$

□

Corollary 1. In the case of $u = v = 0$, it is apparent that the JOM of derivatives for integer calculus aligns with Legendre operational matrix of derivatives with respect to integer calculus as gained by Saadatmandi and Dehghan (refer [12] Eq. (11)).

Corollary 2. In the case of $u = v = -0.5$, it is evident that the JOM of derivatives for integer calculus aligns with Chebyshev's operational matrix of derivatives with respect to integer calculus as gained by Doha et al. (refer [15] Eq. (3.2)).

4. Applications of the operational matrix of fractional derivative

This section solves an FDE in order to demonstrate the great significance of an operational matrix based on SLPs of fractional derivatives. We follow the same steps when using SJPs.

4.1. Linear FDEs

Consider the linear FDEs

$$\begin{aligned} &^{ABC}D^{\alpha} y(t) = b_1 D^{\beta k} y(t) + ... + b_k D^{\beta 1} y(t) + b_{k+1} y(t) \\ &+ b_{k+2} q(t), \quad \text{for} \quad k = 1, 2, ... \end{aligned} \tag{20}$$

The initial conditions are

$$\mathrm{y}_0^{(v)} = d_v, \quad v = 0, ..., n, \tag{21}$$

in which b_k denotes real constant coefficients with $n < \alpha \leq n+1$, $0 < \beta_1 < \beta_2 < ... < \beta_k < \alpha$, in which $^{ABC}D^{\beta}$ refers to the ABC-derivative of order β.
To solve Eq.(20), we present an approximation of the function

$$q(t) \simeq \sum_{i=0}^{m} q_i J_{L,i}^{(u,v)}(t) = Q^T \Psi(t), \tag{22}$$

$$^{ABC}D^{\alpha} y(t) \simeq C^T \; ^{ABC}D^{(\alpha)} \Psi(t), \tag{23}$$

where $Q = [q_0, ..., q_m]^T$ is a known vector. Employing Eqs.(22),(23) and (13), the residual $R_m(t)$ for Eq.(20) can be written as

$$R_m(t) \simeq \left(C^T \; ^{ABC}D^{(\alpha)} - b_{k+1} C^T - b_{k+2} Q^T \right) \Psi(t). \tag{24}$$

We now establish $m - n$ linear equations as in a typical tau method by applying

$$\begin{aligned} \langle R_m(t), \mathfrak{p}_j(t) \rangle = \int_0^1 R_m(t) \mathfrak{p}_i(t) dt = 0, \\ i = 0, 1, ..., m - n - 1. \end{aligned} \tag{25}$$

Moreover, by substituting Eq.(13) and Eq.(14) with Eq.(21), we get

$$\begin{aligned} &y_0 = C^T \Psi(0) = d_0, \\ &y_0^{(1)} = C^T D^{(1)} \Psi(0) = d_1, \\ &\vdots \\ &y_0^{(n)} = C^T D^{(n)} \Psi(0) = d_n. \end{aligned} \tag{26}$$

Eqs. (25) and (26) generate $(m-n)$ linear equations, which may be solved using arbitrary coefficients with respect to the vector C.

4.2. Nonlinear FDEs

Consider the non-linear FDEs

$$^{ABC}D^{\alpha} y(t) = F(t, y(t), D^{\beta_1} y(t), ... D^{\beta_k} y(t)). \tag{27}$$

The initial conditions are

$$y_0^{(v)} = d_v, \quad v = 0, ..., n, \tag{28}$$

in which $n < \alpha \leq n+1$, $0 < \beta_1 < \beta_2 < ... < \beta_r < \alpha$, as well as $^{ABC}D^{\alpha}$ resembles the ABC-derivative of order α.
We put in mind that F can be non-linear in general.

$$\begin{aligned} &C^T \; ^{ABC}D^{(\alpha)} \Psi(t) \simeq \\ &F(t, C^T \Psi(t), C^T D^{(\beta_1)} \Psi(t), ..., C^T D^{(\beta_k)} \Psi(t)). \end{aligned} \tag{29}$$

Furthermore, upon substituting Eq.(13) and Eq.(14) with Eq.(28), we now have

$$\begin{aligned} &Y_0 = C^T \Psi(0) = d_0, \\ &Y_0^{(v)} = C^T D^{(v)} \Psi(0) = d_v, \quad v = 1, 2, ..., n. \end{aligned} \tag{30}$$

To discover the solution $y(t)$, we collocate Eq.(29) by employing first $(m-n)$ points shifted Legendre roots of $\bar{P}_{m+1}(t)$. These equations along with Eq.(30) establish $(m+1)$ non-linear equations, which can be resolved by employing Newton's iterative method.

5. JOM of variable-order ABC-derivative

This section is devoted to tackling the problem using the SJPs OM of variable order.

$$\lambda(t) = [1, t, t^2, ..., t^n]^T. \tag{31}$$

Thus, the vector $\Psi(t)$ can be expressed as:

$$\Psi(t) = A_{(u,v)}\lambda(t), \tag{32}$$

in which $A_{(u,v)}$ is $(n+1)\times(n+1)$ denotes a square matrix that specified by:

$$(a_{i,j})_{0\le i,j\le n} = \begin{cases} \frac{(-1)^{n-i}\Gamma(n+\beta+1)\Gamma(n+i+\alpha+\beta+1)}{\Gamma(i+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n-i+1)\Gamma(i+1)l^i}, & i \ge j, \\ 0, & otherwise. \end{cases} \tag{33}$$

Hence, by employing Eq.(32), we now have

$$\lambda(t) = A^{-1}\Psi(t). \tag{34}$$

Using the OM for variable-order fractional differential operator $D^{\alpha(t)}\Psi(t)$, and Eq.(32), we now have

$$D^{\alpha(t)}\Psi(t) = D^{\alpha(t)}(A\lambda(t)) = AD^{\alpha(t)}[1, t, t^2, ..., t^\iota]^T. \tag{35}$$

Here, the Atangana-Baleanu Caputo (ABC) derivative with respect to the variable order provided in Eq.(4) may be employed. Then, we may obtain Eq.(35) as given below:

$$D^{\alpha(t)}\Psi(t) = [0, \frac{\Gamma(2)}{\Gamma(1-\alpha(t))}t\sum_{k=0}^{\infty}\frac{(\frac{-\alpha(t)}{1-\alpha(t)}t^{\alpha(t)})^k}{\Gamma(k\alpha(t)+2)}, \frac{\Gamma(3)}{\Gamma(1-\alpha(t))}t^2\sum_{k=0}^{\infty}\frac{(\frac{-\alpha(t)}{1-\alpha(t)}t^{\alpha(t)})^k}{\Gamma(k\alpha(t)+3)},$$

$$..., \frac{\Gamma(\iota+1)}{\Gamma(1-\alpha(t))}t^\iota\sum_{k=0}^{\infty}\frac{(\frac{-\alpha(t)}{1-\alpha(t)}t^{\alpha(t)})^k}{\Gamma(k\alpha(t)+\iota+1)}]^T,$$

$$D^{\alpha(t)}\Psi(t) = AB(t)\lambda(t), \tag{36}$$

in which

$$B(t) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(1-\alpha(t))}\sum_{k=0}^{\infty}\frac{(\frac{-\alpha(t)}{1-\alpha(t)}t^{\alpha(t)})^k}{\Gamma(k\alpha(t)+2)} & 0 & \dots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(1-\alpha(t))}\sum_{k=0}^{\infty}\frac{(\frac{-\alpha(t)}{1-\alpha(t)}t^{\alpha(t)})^k}{\Gamma(k\alpha(t)+3)} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\Gamma(\iota+1)}{\Gamma(1-\alpha(t))}\sum_{k=0}^{\infty}\frac{(\frac{-\alpha(t)}{1-\alpha(t)}t^{\alpha(t)})^k}{\Gamma(k\alpha(t)+\iota+1)} \end{pmatrix} \tag{37}$$

Substituting Eq.(34) into Eq.(36), we get

$$D^{\alpha(t)}\Psi(t) = AB(t)A^{-1}\Psi(t), \tag{38}$$

in which $AB(t)A^{-1}$ denotes the OM of the variable-order ABC-derivative $D^{\alpha(t)}\Psi(t)$. Here, the approximate solution may be given as

$$D^{\alpha(t)}y(t) \simeq D^{\alpha(t)}(C^T\Psi(t)) = C^T D^{\alpha(t)}\Psi(t) = C^T AB(t)A^{-1}\Psi(t),$$

$$C^T AB(t)A^{-1}\Psi(t) = F[t, C^T\Psi(t), C^T AD^{(1)}A^{-1}\Psi(t),$$
$$..., C^T AD^{(n)}A^{-1}\Psi(t)], \tag{40}$$
$$0 \le t \le 1.$$

Here, we employ the collocation points, $t_u = \frac{2u+1}{2n+2}, u = 0, 1, ..., n$, in converting the system of equations given in Eq.(40) into an algebraic equations system as follows:

$$C^T AB(t_u)A^{-1}\Psi(t_u) = F[t_u, C^T\Psi(t_u), C^T AD^{(1)}A^{-1}\Psi(t_u),$$
$$..., C^T AD^{(n)}A^{-1}\Psi(t_u)], \tag{41}$$
$$C^T\Psi(0) = y_0$$

Ultimately, the arbitrary vector C in Eq.(9) may be gained by solving the algebraic equations system provided in Eq.(41).

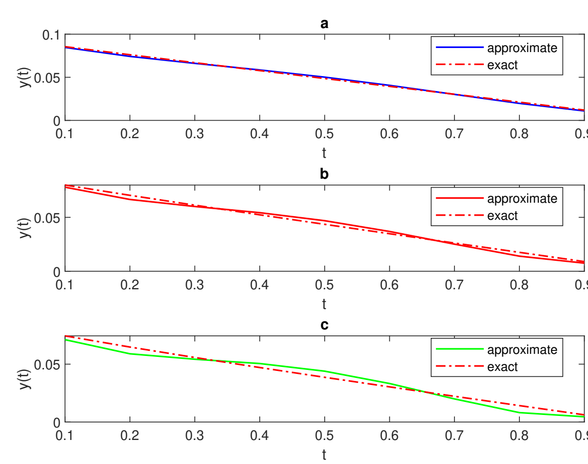

Figure 1: Comparison between exact and approximate solution for $(a)\alpha = 0.95$, $(b)\alpha = 0.9$ and $(c)\alpha = 0.85$ for Example 6.2.

Example 5.1. Suppose the following[36]

$$^{ABC}D^\alpha y(t) = y^2(t) - 2(t+1)^{-2},$$

α	Method	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
0.85	LOM	2.19158e-3	2.89152e-3	4.26507e-3	4.40040e-4	3.13683e-3
	PRCO	2.36581e-3	4.24947e-3	6.00249e-3	7.71664e-3	9.42102e-3
	MTSLP	1.08027e-2	5.75464e-3	3.35770e-3	1.33299e-3	5.46216e-4
	JOM(0,0.5)	3.00081e-3	5.85134e-4	2.97585e-3	2.07844e-3	1.57973e-4
	JOM(0.5,0)	3.28839e-3	1.50328e-3	5.29734e-3	2.26751e-3	1.67970e-3
0.9	LOM	1.26158e-3	2.08766e-3	2.57667e-3	5.91395e-4	1.94142e-3
	PRCO	1.18803e-3	2.39605e-3	3.56382e-3	4.72981e-3	5.90684e-3
	MTSLP	1.10428e-2	7.41747e-3	5.88865e-3	5.57577e-3	3.34146e-3
	JOM(0,0.5)	2.51233e-3	1.28073e-4	2.28507e-3	1.93179e-3	7.25019e-4
	JOM(0.5,0)	2.01129e-3	1.21534e-3	3.35330e-3	1.02220e-3	1.41226e-3
0.95	LOM	4.88352e-4	1.16544e-3	1.10247e-3	5.09878e-4	8.67245e-4
	PRCO	4.09639e-4	9.83473e-4	1.55920e-3	2.14596e-3	2.74713e-3
	MTSLP	9.83430e-3	8.58128e-3	7.92322e-3	7.35794e-3	6.82360e-3
	JOM(0,0.5)	1.77326e-3	6.45112e-4	1.54473e-3	1.53799e-3	1.01582e-3
	JOM(0.5,0)	8.75527e-4	7.70414e-4	1.55386e-3	1.52607e-4	9.24438e-4

Table 1: The absolute error obtained by employing various values of α for Example 6.2.

α	Method	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
0.85	LOM	8.14287e-4	1.92419e-3	1.00572e-2	4.67247e-2	1.15441e-1
	PRCO	1.02731e-3	7.40800e-3	1.19908e-2	5.84856e-2	1.50751e-1
	MTSLP	4e-3	2.74688e-3	1.97787e-3	1.91297e-2	7.45086e-2
	JOM(0,0.5)	8.01348e-4	1.93026e-3	1.00587e-2	4.67324e-2	1.15433e-1
	JOM(0.5,0)	8.16107e-4	1.91932e-3	1.00499e-2	4.67267e-2	1.15445e-1
0.9	LOM	6.70490e-4	1.42673e-3	8.20714e-3	3.66835e-2	8.89397e-2
	PRCO	8.64541e-4	1.70507e-3	9.89488e-3	4.55878e-2	1.14652e-1
	MTSLP	4e-3	2.06924e-3	2.35421e-3	1.18382e-2	3.19227e-1
	JOM(0,0.5)	6.62878e-4	1.42923e-3	8.20792e-3	3.66879e-2	8.89366e-2
	JOM(0.5,0)	6.71528e-4	1.42406e-3	8.20355e-3	3.66839e-2	8.89426e-2
0.95	LOM	4.13897e-4	7.86685e-4	5.01340e-3	2.15783e-2	5.13325e-2
	PRCO	5.90418e-4	8.64405e-4	6.50201e-3	2.77896e-2	2.90733e-1
	MTSLP	4e-3	6.57765e-3	3.02391e-3	1.94353e-3	1.90389e-2
	JOM(0,0.5)	4.11399e-4	7.86925e-4	5.01366e-3	2.15797e-2	5.13320e-2
	JOM(0.5,0)	2.94860e-4	3.13253e-4	3.61277e-3	1.25339e-2	2.53848e-2

Table 2: The absolute error obtained employing various values of α for Example 6.3.

For $y_0 = -2$ and the exact solution $y(t) = \frac{-2}{(t+1)}$ in case of $\alpha = t^0$, Figure 3 displays the approximate values of $\alpha = 0.85, 0.9, 0.95$ and $m = 6$. A good approximation that is comparable to the exact answer can be obtained via an operation matrix based on SJPs.

6. Numerical examples

The numerical examples of linear and non-linear fractional-order and variable-order scenarios will acquire some solutions in this section. Our computational findings will measure the difference between the exact and approximate solutions using absolute error. The MATLAB R2020b software is used to code and perform all of the numerical programs, whereas the CPU is for the next.

- **JOM** Jacobi Operational matrix method derived in this study.
- **CPSKOM** Chebyshev polynomials with respect to the second kind Operational matrix method derived in this study.
- **LOM** Legendre Operational matrix method [26].
- **PRCO** Predictor-Corrector method provided in [27].
- **MTSLP** Mixture two-step Lagrange polynomial as well

α	Method	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
0.85	LOM	-1.74986	-1.50024	-1.35680	-1.23342	-1.13618
	PRCO	-1.60319	-1.52292	-1.41361	-1.32635	-1.25359
	MTSLP	-1.67179	-1.45348	-1.31389	-1.22725	-1.17437
	JOM(0,0.5)	-1.76123	-1.50623	-1.35787	-1.23557	-1.13710
	JOM(0.5,0)	-1.74359	-1.49912	-1.35666	-1.23182	-1.13669
0.9	LOM	-1.78420	-1.52494	-1.36023	-1.22657	-1.11789
	PRCO	-1.65250	-1.54279	-1.42052	-1.32416	-1.24500
	MTSLP	-1.78773	-1.58943	-1.45396	-1.35011	-1.26552
	JOM(0,0.5)	-1.79249	-1.53108	-1.36213	-1.22826	-1.11925
	JOM(0.5,0)	-1.77931	-1.52306	-1.36002	-1.22533	-1.11773
0.95	LOM	-1.82307	-1.54990	-1.35819	-1.21279	-1.09254
	PRCO	-1.71332	-1.56366	-1.42509	-1.31793	-1.23183
	MTSLP	-1.89701	-1.64139	-1.47105	-1.34753	-1.25202
	JOM(0,0.5)	-1.82560	-1.55437	-1.36040	-1.21334	-1.09386
	JOM(0.5,0)	-1.82039	-1.54740	-1.35792	-1.21227	-1.09156
1	Exact	-1.81818	-1.53846	-1.33333	-1.17647	-1.05263

Table 3: The approximate solutions obtained employing various values of α for Example 6.4.

α	Method	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
0.85	LOM	0.13534	0.20345	0.24998	0.28275	0.31040
	PRCO	0.19782	0.22782	0.27041	0.30270	0.32864
	MTSLP	0.164103	0.22650	0.25520	0.28130	0.30738
	JOM(0,0.5)	0.12887	0.20271	0.24847	0.28210	0.30963
	JOM(0.5,0)	0.13737	0.20354	0.25049	0.28286	0.31040
0.9	LOM	0.11432	0.19514	0.24890	0.28804	0.31966
	PRCO	0.16976	0.21614	0.26616	0.30394	0.33403
	MTSLP	0.10613	0.18237	0.23502	0.27718	0.31138
	JOM(0,0.5)	0.10899	0.19386	0.24741	0.28717	0.31881
	JOM(0.5,0)	0.11608	0.19533	0.24944	0.28817	0.31989
0.95	LOM	0.08994	0.18737	0.24973	0.29612	0.33192
	PRCO	0.13780	0.20414	0.26292	0.30695	0.34158
	MTSLP	0.051491	0.15966	0.23010	0.28087	0.32011
	JOM(0,0.5)	0.08644	0.18564	0.24848	0.29510	0.33113
	JOM(0.5,0)	0.09129	0.18767	0.25026	0.29628	0.33233
1	Exact	0.08375	0.19264	0.26323	0.31422	0.35351

Table 4: The approximate solutions obtained using different values of α for Example 6.5.

as the fundamental theorem with respect to fractional calculus stated in [28].

Example 6.1. We now consider the Bagley–Torvik equation governing the motion of a rigid plate immersed in the Newtonian fluid given as follows

$$^{ABC}D^{1.5}y(t) + D^2 y(t) + y(t) = t + 1.$$

Here, $y_0 = t^0$, $y_0' = t^0$ and $y(t) = t + 1$ denotes the exact solution.

Using SLPs, the approximate solution for $m = 3$ is $y(t) = [1.5 \quad 0.5 \quad 0 \quad 0]\Psi(t) = t + 1$, which equals to the exact solution.

Using SJPs(0.5,0), the approximate solution for $m = 3$ is $y(t) = [1.4 \quad 0.4 \quad 0 \quad 0]\phi(t) = t + 1$, which equals to the exact solution.

For SJPs(0,0.5), the approximate solution for $m = 3$ is $y(t) = [1.6 \quad 0.4 \quad 0 \quad 0]\phi(t) = t + 1$, which equals to the exact solution.

t	Error LOM	Error PRCO	Error MTSLP	Error CPSKOM	Error JOM(0,0.5)	Error JOM(0.5,0)
0.1	7.71674e-12	1.09802e-4	1.1e-2	2.09480e-9	8.56178e-11	2.50289e-11
0.2	7.77495e-12	4.53577e-4	2.24064e-2	3.26627e-9	8.53458e-11	3.76301e-11
0.3	8.81307e-12	1.09934e-3	3.46062e-2	3.66969e-9	8.97760e-11	3.88097e-11
0.4	2.77689e-11	2.12757e-3	4.60297e-2	3.46035e-9	2.89525e-10	3.28610e-11
0.5	3.48143e-11	3.61364e-3	5.77915e-2	2.79354e-9	3.63677e-10	2.40773e-11
0.6	1.56708e-11	5.63079e-3	6.74063e-2	1.82454e-9	1.62011e-10	1.67516e-11
0.7	4.39398e-11	8.25300e-3	7.82267e-2	7.08644e-10	4.65697e-10	1.51773e-11
0.8	1.58296e-10	1.15567e-2	8.50985e-2	3.98864e-10	1.66967e-9	2.36475e-11
0.9	3.41676e-10	1.56209e-2	9.75044e-2	1.34269e-9	3.60013e-9	4.64555e-11

Table 5: The absolute error for Example 6.6 for $m = 4$

t	Error LOM	Error PRCO	Error MTSLP	Error CPSKOM	Error JOM(0,0.5)	Erorr JOM(0.5,0)
0.1	1.75338e-4	6.54123e-4	e-2	1.75338e-4	1.75338e-4	1.75338e-4
0.2	3.38732e-3	3.95957e-3	2.70108e-2	3.38732e-3	3.38732e-3	3.38732e-3
0.3	9.96953e-3	9.35838e-3	3.72260e-2	9.96953e-3	9.96953e-3	9.96953e-3
0.4	1.88528e-2	1.27167e-2	4.22493e-2	1.88528e-2	1.88528e-2	1.88528e-2
0.5	2.93188e-2	8.50431e-3	4.70844e-2	2.93188e-2	2.93188e-2	2.93188e-2
0.6	4.06491e-2	8.72007e-3	5.81601e-2	4.06490e-2	4.06490e-2	4.06490e-2
0.7	5.21251e-2	4.36191e-2	8.11371e-2	5.21250e-2	5.21250e-2	5.21250e-2
0.8	6.30285e-2	1.00115e-1	1.20052e-1	6.30284e-2	6.30284e-2	6.30284e-2
0.9	7.26408e-2	1.82050e-1	1.77509e-1	7.26406e-2	7.26406e-2	7.26406e-2

Table 6: The absolute error for Example 6.7 for $m = 4$

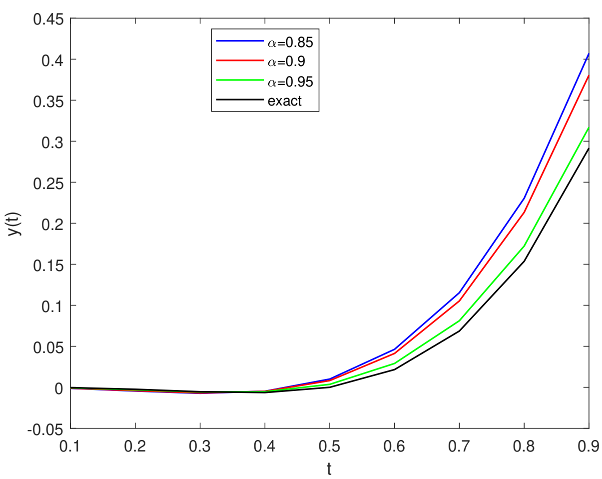

Figure 2: Comparison between approximate solutions for $\alpha = 0.85$, $\alpha = 0.9$ and $\alpha = 0.95$ with the exact solution for Example 6.3.

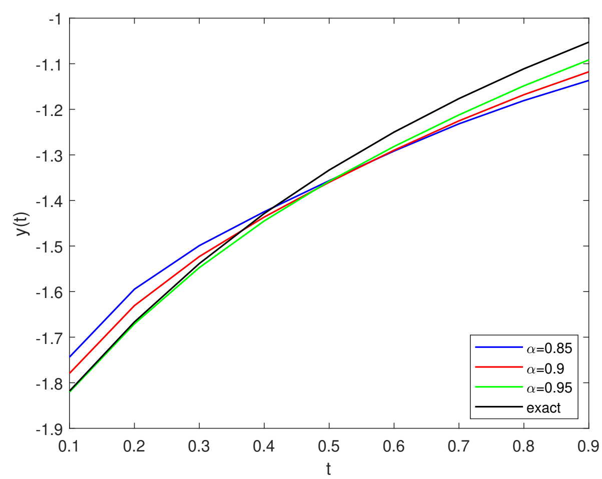

Figure 3: Comparison of approximate solutions for $\alpha = 0.85$, $\alpha = 0.9$ and $\alpha = 0.95$ with the exact solution in case $\alpha = t^0$ for Example 6.4.

Example 6.2. Suppose we have the following model[34]:

$$^{ABC}D^{\alpha}y(t) = -K(1 - y(t)), \quad 0 < \alpha < 1, \quad and \quad y_0 = 0.1$$

The exact solution is as follows:

$$y(t) = \frac{-K(1-\alpha)}{M(\alpha) - K(1-\alpha)} E_\alpha\left(\frac{K\alpha}{M(\alpha) - K(1-\alpha)} t^\alpha\right) + \left[1 - E_\alpha\left(\frac{K\alpha}{M(\alpha) - K(1-\alpha)} t^\alpha\right)\right] + \frac{M(\alpha)y(0)}{M(\alpha) - K(1-\alpha)} E_\alpha\left(\frac{K\alpha}{M(\alpha) - K(1-\alpha)} t^\alpha\right).$$

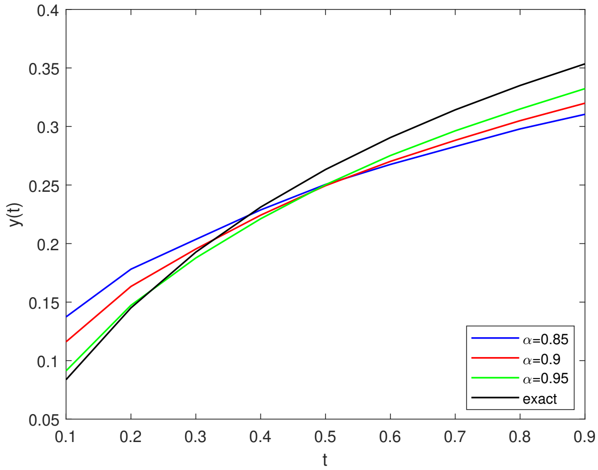

Figure 4: Comparison between approximate solutions for $\alpha = 0.85$, $\alpha = 0.9$ $\alpha = 0.95$ with exact solution in case of $\alpha = t^0$ for Example 6.5

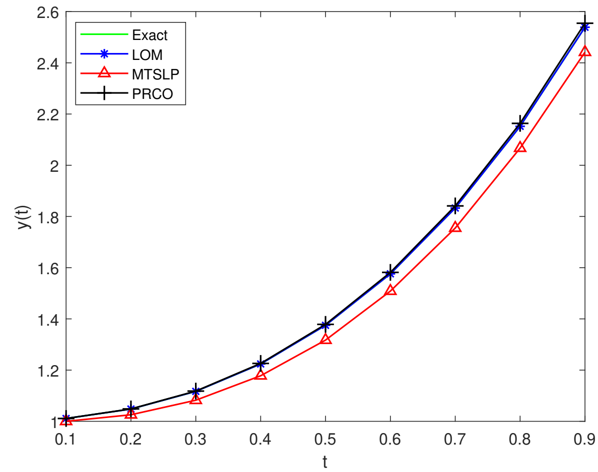

Figure 5: The approximate solution and exact solution for Example 6.6 for $m = 4$

Table 1 and Figure 1 compare the absolute error for the two methods for K is constant $K = 0.1, \alpha = 0.85, 0.9, 0.95$, and $m = 4$ respectively. An enhanced approximate solution comparable to the exact solution can be obtained using an operation matrix based on SJPs.

Example 6.3. Suppose we have the following[35]

$$^{ABC}D^{\alpha}y(t) = -y(t) + t^4 - 0.5t^3 - \frac{3}{\Gamma(4-\alpha)}t^{3-\alpha} + \frac{24}{\Gamma(5-\alpha)}t^{4-\alpha},$$

for $y_0 = 0$, while the exact solution is expressed by $y(t) = t^4 - 0.5t^3$.
Table 2 and Figure 2 compare the absolute error for the two approaches for $\alpha = 0.85, 0.9, 0.95$, and $m = 6$. A good approximate solution that is comparable to the exact answer can be obtained using an operation matrix based on SJPs.

Example 6.4. Suppose we have the following[37]

$$^{ABC}D^{\alpha}y(t) = (1 - y(t))^4,$$

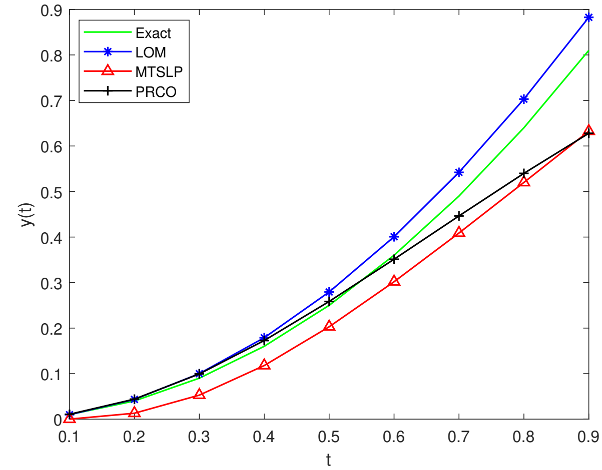

Figure 6: The approximate solution and exact solution for Example 6.7 for $m = 4$

The exact solution in case of $\alpha = t^0$ is expressed by $y(t) = \frac{1+3t-(1+6t+9t^2)^{1/3}}{1+3t}$ and $y_0 = 0$. The problem is solved with $m = 8$, and the numerical results are shown in Figure 4.

Example 6.5. Consider the following linear variable-order FDEs[38]:

$$^{ABC}D^{\alpha(t)}y(t) + e^t y(t) =$$
$$e^t(t^2 + t^3 + 1) + \frac{M(\alpha(t))}{1-\alpha(t)}2t^2 E_{\alpha(t),3}(-\frac{\alpha(t)}{1-\alpha(t)}t^{\alpha(t)})$$
$$+ \frac{M(\alpha(t))}{1-\alpha(t)}6t^3 E_{\alpha(t),4}(-\frac{\alpha(t)}{1-\alpha(t)}t^{\alpha(t)}),$$

where $\alpha(t) = 0.5t + 0.1$, $y_0 = t^0$ and the exact solution is provided by $y(t) = t^2 + t^3 + 1$.
The absolute error for $m = 4$ is shown in Table 3. An enhanced approximate solution that is comparable to the exact answer can be obtained using an operation matrix based on SJPs.

Example 6.6. We now consider the following non-linear variable-order FDEs given by[39]:

$$^{ABC}D^{\alpha(t)}y(t) + y^2(t) = t^2 + t^4 + \frac{2t^{2-sin(t)}}{\Gamma(3 - sin(t))}.$$

where $\alpha(t) = 0.5t + 0.6$, $y_0 = 0$ and the exact solution is expressed by $y(t) = t^2$.
The absolute error for $m = 4$ is shown in Figure 5. Operation matrix relying on SJPs may give an enhanced approximate solution that is comparable with the exact solution.

7. Conclusion

We came up with a general formulation for the fractional and variable order Jacobi operational matrix (JOM), which is utilized to approximate Atangana-Baleanu Caputo (ABC) derivatives in numerical solutions. The shifting Jacobi tau and collocation approaches served as the foundation for our strategy. Since the ABC fractional derivative enables the inclusion of traditional initial conditions in the formulation of the issue, the fractional derivatives are described in the ABC sense. Here, the findings presented in the preceding section show how accurate these algorithms are. Additionally, only a few shifted Jacobi polynomials are required to produce a good outcome. We can recommend some future studies for this paper: solve system of fractional and variable order differential equations by use Jacobi and more orthogonal polynomials.

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