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Approximate solution of space fractional order diffusion equations by Gegenbauer collocation and compact finite difference scheme

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Abstract

In this paper, approximation of space fractional order diffusion equation are considered using compact finite difference technique to discretize the time derivative, which was then approximated via shifted Gegenbauer polynomials using zeros of $(N-1)$ degree shifted Gegenbauer polynomial as collocation points. The important feature in this approach is that it reduces the problems to algebraic linear system of equations together with the boundary conditions gives $(N+1)$ linear equations. Some theorems are given to establish the convergence and the stability of the proposed method. To validate the efficiency and the accuracy of the method, obtained results are compared with the existing results in the literature. The graphical representation are also displayed for various values of $\beta-$ Gegenbauer polynomials. It can be observe in the tables of the results and figures that the proposed method performs better than the existing one in the literature.

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1. Introduction

In the last decade, significant interest has been given to the study of fractional calculus due to its application in applied Mathematics. Some of its application can be traced to modeling of some physical phenomena such as communication, robotics, transportation systems, finance, signal processing, genetic algorithms and the damping visco-elasticity as discussed in [1–4]. In recent time, the theory and applications of fractional calculus have expanded enormously, its first appearance was in a letter written by Gottfried Wilhelm Leibniz in 1695, as discussed in [5]. One of the popular areas of fractional calculus is the space fractional order diffusion equations (SFODEs). Several researchers have studied SFODEs and applied different methods to find the approximate solution, [6, 7] employed finite difference method (FDM) to discretize the time derivative, then used Gegenbauer polynomial as approximating polynomial, [8] developed tau method approach to find an approximant to SFODEs, [9] employed second kind shifted Chebyshev polynomials for solving SFODEs, [10] proposed generalized shifted Chebyshev polynomials for solving space fractional optimal

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control problems, new aspects of fractional Biswas-Milovic model with Mittag-Leffler law was reported in [11]. [12] employed Riesz derivative to solve high-order solvers. [13] solved fractional Rayleigh-Stokes model in their paper titled "Numerical solution of fractional Rayleigh-stokes model arising in a heated generalized second grade fluid". [14] investigate nonlinear variable order fractional reaction-diffusion equation with Mittag-Leffler kernel. [15] developed lines and splines methods for solving SFODEs. [16] employed spline method combined with Richardson extrapolation to solve SFODEs, [17] discussed shifted Legendre spectral method for fractional-order multi-point boundary value problems, [18] analysis the initial-boundary value problems for the linear time-fractional diffusion equations with a uniformly elliptic spatial differential operator of the second order and the Caputo type time-fractional derivative acting in the fractional Sobolev spaces, [19] proposed spatio-temporal homogenization function method for solving 1D fourth-order fractional diffusion-wave equations. [20] investigate Caputo-Fabrizio and Fractal fractional derivative operator in the of studied HIV/AIDS fractional-order model is studied, [21] studied the fractional-order COVID-19 epidemic model using Laplace homotopy analysis method and [22] investigated the numerical solution of SFODEs using Chebyshev collocation method of the fourth kind and compact finite difference scheme. The main aim of this paper is to extend the work reported in [22] by introducing shifted Gegenbauer polynomials as approximating polynomial $U_N(x,T)$ for solving SFODEs using shifted Gegenbauer collocation method to find the solution, since its solution generalizes the results of some other orthogonal polynomials such as Legendre, Jacobi polynomials $P_n^{\alpha,\beta}(x)$ of certain kinds with $\alpha = \beta$ and shifted Chebyshev polynomials of certain kinds. In this paper, we want to employ compact finite difference method (CFDM) to discretize the time derivative, then the discretized SFODEs will be collocated at $(N-1)$ points. These equations together with the boundary conditions generate $(N+1)$ system of linear equations.

This paper is organized as follows. In section 2, we define Caputo fractional order derivative, review some notable orthogonal polynomials, state the problem under consideration and some useful lemmas from the literature are recalled (see 2.1,2.2,2.3). Shifted Gegenbauer interms of differential operator is presented in section 3, we present the formulation of the scheme in section 3.1. We analyze the convergence and the stability of the proposed method in section 3.2, in section 4 numerical examples are presented with computational results and graphical representation to show the efficiency and the accuracy of the proposed method, concluding remarks are given in section 5.

2. Preliminaries

2.1. Caputo fractional order derivative operator

The Caputo fractional derivative operator D^ω of order ω is given as

$$D^\omega f(t) = \frac{1}{\Gamma(n-\omega)}\int_0^t \frac{f^{(i)}(t)}{(t-x)^{\omega+1-i}}dx,\ i-1<\omega<i, i\in\mathbb{N} \tag{1}$$

where $\omega > 0$ is the order of the derivative. The linearity property of the Caputo fractional derivative is:

$$D^\omega(\lambda f(t) + \mu g(t)) = \lambda D^\omega f(t) + \mu D^\omega g(t), \tag{2}$$

where, λ & μ are constants. The following results are obtained:

$$D^\omega t^j = \begin{cases} 0, \text{ for } \ j\in\mathbb{N}_0\ ,\ j<\lceil\omega\rceil \\ \frac{\Gamma(j+1)}{\Gamma(j+1-\omega)}t^{j-\omega}, \text{ for } j\in\mathbb{N}_0\ ,\ j\geq\lceil\omega\rceil \end{cases} \tag{3}$$

where $\lceil\omega\rceil$ is the smallest integer greater than or equal to ω.

2.2. Some of orthogonal polynomials

Some of the notable orthogonal polynomials $\psi(t)$ are defined here:
$\psi(t)$ is an orthogonal polynomial with respect to the weight function $\omega(t)$ in an interval $[a,b]$, if the inner product of $\psi(t)$ is zero.
That is,

$$\langle\psi_m(t),\psi_n(t)\rangle \ = \ \int_a^b \omega(t)\psi_m(t)\psi_n(t)dt = \begin{cases} 0, & m\neq n \\ \lambda_n, & m=n \end{cases} \tag{4}$$

Some of these orthogonal polynomials $\psi_i(t)$ are:

2.2.1. Legendre polynomials

An orthogonal polynomial $\psi_m(t)$ defined in an interval $[-1,1]$ is said to be a Legendre polynomial if the weight function $\omega(t)=1$. The recurrence relation of a legendre polynomial is given as:

$$P_{k+1}(u) = \frac{2k+1}{k+1}uP_k(u) - \frac{k}{k+1}P_{k-1}(u), k\geq 1, \tag{5}$$

with $P_0(u)=1, P_1(u)=u$.

2.2.2. Chebyshev polynomials

The prominent one listed with their respective weight functions $\omega(t)$ in the interval $[-1,\ 1]$ are found in [23] as:

$$\psi_m = \begin{cases} T_m(t) = \cos(mt), & \omega(t) = \dfrac{1}{\sqrt{1-t^2}}, \\ U_m = \dfrac{\sin{(m+1)\,t}}{\sin{(t)}}, & \omega(t) = \sqrt{1-t^2}, \\ V_m = \dfrac{\cos\left(m+\frac{1}{2}\right)t}{\cos\left(\frac{t}{2}\right)}, & \omega(t) = \sqrt{\dfrac{1+t}{1-t}}, \\ W_m = \dfrac{\sin\left(m+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}, & \omega(t) = \sqrt{\dfrac{1-t}{1+t}}. \end{cases} \tag{6}$$

The inner product for the third and fourth kinds Chebyshev polynomials in the interval $[-1,\ 1]$ are defined as:

$$\langle \psi_m, \psi_n \rangle = \begin{cases} \langle V_m, V_n \rangle = \int_{-1}^{1} \sqrt{\dfrac{1+t}{1-t}} V_m(t) V_n(t) dt = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}, \\ \langle W_m, W_n \rangle = \int_{-1}^{1} \sqrt{\dfrac{1-t}{1+t}} W_m(t) W_n(t) dt = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}. \end{cases} \tag{7}$$

while the fifth kind was reported in [14, 24].

2.2.3. Shifted Gegenbauer polynomials

Gegenbauer polynomials $C_i^{(\beta)}(t)$ defined in the interval $[-1,\ 1]$ with respect to weight function $\omega(t) = (1-t^2)^{(\beta-\frac{1}{2})}$ can be determined using

$$C_i^{(\beta)}(t) = \sum_{j=0}^{i} \frac{(-1)^j \Gamma(2\beta + 2i - j) \Gamma(\beta + \frac{1}{2})}{(i-j)!\, \Gamma(2\beta) \Gamma(j+1) \Gamma(i - j + \beta + \frac{1}{2})} t^{i-j}. \tag{8}$$

The recurrence relation is given as

$$C_i^{(\beta)}(t) = \frac{1}{i} \left[2(i + \beta - 1) t C_{i-1}^{(\beta)}(t) - (i + 2\beta - 2) C_{i-2}^{(\beta)}(t) \right], i \geq 2, \tag{9}$$

where $C_0^{\beta}(t) = 1$, $C_1^{\beta}(t) = 2\beta t$. To use this polynomial in the interval $t \in [a, b]$ a shifted Gegenbauer polynomial is defined by introducing the variable $\lambda = \frac{2t-(a+b)}{b-a}$. Hence, the shifted Gegenbauer polynomial in t is obtained as:

$$\begin{aligned} C_i^{(\beta)*}(t) = \frac{1}{i} \Bigg[& 2(i + \beta - 1) \left(\frac{2t - (a+b)}{b-a} \right) C_{i-1}^{(\beta)*}(t) \\ & - (i + 2\beta - 2) C_{i-2}^{(\beta)*}(t) \Bigg], i \geq 2 \end{aligned} \tag{10}$$

where $C_0^{(\beta)*}(t) = 1$, $C_1^{(\beta)*}(t) = 2\beta \left(\frac{2t-(a+b)}{b-a} \right)$.

The analytic form of the shifted Gegenbauer polynomial $C_i^{(\beta)*}(t)$ is given as:

$$C_i^{(\beta)*}(t) = \sum_{j=0}^{i} \frac{(-1)^j \Gamma(2\beta + 2i - j) \Gamma(\beta + \frac{1}{2})}{(i-j)!\, \Gamma(j+1) \Gamma(i - j + \beta + \frac{1}{2}) \Gamma(2\beta)} t^{i-j}. \tag{11}$$

The orthogonality condition is

$$\begin{aligned} \langle C_m^{(\beta)*}(t),\ C_n^{(\beta)*}(t) \rangle &= \int_0^1 \left(t - t^2 \right)^{(\alpha - \frac{1}{2})} C_m^{(\beta)*}(t) C_n^{(\beta)*}(t) dt \\ &= \begin{cases} 0, & \text{for } m \neq n \\ \frac{\pi 2^{1-4\alpha} \Gamma(n+2\alpha)}{n! [\Gamma(\alpha)]^2 (n+\alpha)}, & \text{for } m = n \end{cases} \end{aligned} \tag{12}$$

Let $f(x)$ be a square integrable function in $[0, 1]$, expressing it in terms of the shifted Gegenbauer polynomials as follows:

$$f(x) = \sum_{i=0}^{\infty} \alpha_i C_i^{(\beta)*}(x), \tag{13}$$

where α_i is defined as:

$$\alpha_i = \frac{i!\, [\Gamma(\beta)]^2 (i + \beta)}{\pi 2^{1-2\beta} \Gamma(i + 2\beta)} \int_{-1}^{1} (1 - x^2)^{\beta - \frac{1}{2}} f\left(\frac{x+1}{2} \right) C_i^{\beta}(x) dx, \tag{14}$$

or,

$$\alpha_i = \frac{n!\, [\Gamma(\beta)]^2 (i + \beta)}{\pi 2^{1-4\beta} \Gamma(i + 2\beta)} \int_{0}^{1} (x - x^2)^{\beta - \frac{1}{2}} f(x) C_i^{(\beta)*}(x) dx, \tag{15}$$

$(N+1)$ terms of $C_n^{(\beta)*}(x)$ are considered in the approximation. Then Eq. (13) becomes

$$f(x) = \sum_{i=0}^{N} \alpha_i C_i^{(\beta)*}(x). \tag{16}$$

2.3. Statement of the problem

The problem under consideration and its initial and boundary conditions are:

$$\frac{\partial u(x,t)}{\partial t} = a(x) \frac{\partial^{\omega} u(x,t)}{\partial x^{\omega}} + b(x,t) \quad 0 < x < \varsigma,\ 0 \leq t \leq \upsilon,\ 1 < \omega < 2 \tag{17}$$

$$u(x, 0) = g(x),\ \ 0 < x < \varsigma \tag{18}$$

$$\begin{aligned} u(0, t) &= \mu_0(t),\ \ 0 \leq t \leq \upsilon \\ u(\varsigma, t) &= \mu_1(t),\ \ 0 \leq t \leq \upsilon \end{aligned} \tag{19}$$

Eqs. (18) and (19) are initial and boundary conditions respectively.

When $\omega = 2$, Eq. (17) becomes a classical diffusion equation, that is:

$$\frac{\partial u(x,t)}{\partial t} = a(x) \frac{\partial^2 u(x,t)}{\partial x^2} + b(x,t). \tag{20}$$

Here, we try to reduce the fractional derivative to a system of ordinary differential equations by utilizing the shifted Gegenbauer polynomials and then solve the resulting system of equation using compact finite difference technique.

We recall some lemmas to justify our finding
Let η denote an open bounded domain in $\mathbb{R}^2$ space and $L_2(\eta)$ represents a Hilbert space with the inner product

$$\langle p(x), q(x) \rangle = \int_{\eta} p(x) q(x) dx, \tag{21}$$

together with the Euclidean norm $||u(x)|| = \langle u(x), \upsilon(x) \rangle^2$, and the Sobolev space as

$$H^s(\eta) = u \epsilon L_2(\eta), \frac{d^s u}{dx^s} \epsilon L_2(\eta)$$

Suppose that $b(x) > 0$ for $0 \leq x \leq 1$

Lemma 2.1. *For any* $p, q \in H^{\frac{\beta}{2}}(\eta)$, $\langle {}_aD_x^{\beta} p, q \rangle = \langle {}_aD_x^{\frac{\beta}{2}} p, {}_xD_b^{\frac{\beta}{2}} q \rangle$, $\langle {}_xD_b^{\beta} p, q \rangle = \langle {}_xD_b^{\frac{\beta}{2}} p, {}_aD_x^{\frac{\beta}{2}} q \rangle$, *for* $1 < \beta < 2$,
$\langle {}_aD_x^{\beta} p, {}_xD_b^{\beta} p \rangle = \cos(\beta\pi) ||{}_aD_x^{\beta} p||^2 = cos(\beta\pi) ||{}_xD_b^{\beta} p||^2$, *for* $\beta > 0$

Lemma 2.2. *For the functions $f(x)$ and $\langle {}_aD_x^{\beta} f(x)\rangle \epsilon H^{\beta}(\eta)$, $\exists\ \delta\tau > 0\ \ni$*

$$\|f(x) + \frac{\delta}{2} b(x_n)_a D_x^{\beta} f(x)\| \leq \|f(x)\|, \quad for\ \ 1 < \beta < 2$$

see [22] for details.

3. Shifted Gegenbauer CFDM

Here, we employed the formula for fractional derivative $f(x)$ derived in [7]:

Assume $\omega > 0$, now using Caputo linearity property, we have:

$$D^{\omega} f_N(x) = \sum_{i=0}^{N} \alpha_i D^{\omega}\left(C_i^{(\beta)*}(x)\right). \tag{22}$$

By linearity property and Eq. (3) we obtain:

$$D^{\omega}\left(C_i^{(\beta)*}(x)\right) = 0, i = 0, 1, \dots, \lceil\omega\rceil - 1, \omega > 0 \tag{23}$$

$$D^{\omega}\left(C_i^{(\beta)*}(x)\right) = \sum_{j=0}^{i} \frac{(-1)^j \Gamma(2\beta + 2i - j)\Gamma(\beta + \frac{1}{2})}{(i-j)!\,\Gamma(j+1)\Gamma(i - j + \beta + \frac{1}{2})\Gamma(2\beta)} \times D^{\omega} x^{i-j}, \quad i \geq \lceil\omega\rceil \tag{24}$$

Applying Eq. (3) to Eq. (24), we obtain:

$$D^{\omega}\left(C_i^{(\beta)*}(x)\right) = \sum_{j=0}^{i-\lceil\omega\rceil} \frac{(-1)^j \Gamma(2\beta + 2i - j)\Gamma(\beta + \frac{1}{2})}{\Gamma(j+1)\Gamma(i - k + \beta + \frac{1}{2})\Gamma(2\beta)\Gamma(i + 1 - j - \sigma)} \times x^{i-j-\omega} \tag{25}$$

Substituting (25) into (22) to obtain:

$$D^{\omega}(g_N(x)) = \sum_{i=\lceil\omega\rceil}^{N} \sum_{j=0}^{i-\lceil\omega\rceil} \alpha_i \frac{(-1)^j \Gamma(2\beta + 2i - j)\Gamma(\beta + \frac{1}{2})}{\Gamma(j+1)\Gamma(i - j + \beta + \frac{1}{2})\Gamma(2\beta)\Gamma(i + 1 - j - \sigma)} x^{i-j-\omega} \tag{26}$$

Let

$$N_{i,j} = \frac{(-1)^j \Gamma(2\beta + 2i - j)\Gamma(\beta + \frac{1}{2})}{\Gamma(j+1)\Gamma(i - j + \beta + \frac{1}{2})\Gamma(2\beta)\Gamma(i + 1 - j - \sigma)} \tag{27}$$

Simplifying Eq. (26) gives

$$D^{w}(f_N(x)) = \sum_{i=\lceil\omega\rceil}^{N} \sum_{j=0}^{i-\lceil\omega\rceil} \alpha_i N_{i,j} x^{i-j-\omega}. \tag{28}$$

See [7] for for more details

3.1. Formulation of the scheme

Solving Eq.(17) based on CFDM, using Gegenbauer collocation, given two integers $M,\ N > 0$ and two mesh points $\tau_{m-1} = (m-1)\delta\tau\ for\ m = 1, 2, ..., M+1, \delta\tau = \frac{T}{M}$.

Introducing Taylor's expansion for the time-discretizing that is:

$$\frac{\partial u(x_n, t_m)}{\partial t} = \delta_\tau u(x_n, t_m) + \frac{\delta\tau}{2} \frac{\partial^2 u(x_n, t_m)}{\partial t^2} + 0(\delta\tau^2) \tag{29}$$

where $\delta_\tau u(x_n, t_m) = \frac{u_n^m - u_n^{m-1}}{\delta\tau}$, substituting Eq.(29) into Eq.(3)

$$\delta_\tau u(x_n, t_m) + \frac{\delta\tau}{2} \frac{\partial^2 u(x_n, t_m)}{\partial t^2} + 0(\delta\tau^2) = a(x_n) \frac{\partial^w u(x_n, t_m)}{\partial x^w} + b(x_n, t_m) \tag{30}$$

Differentiating Eq.(17)

$$\frac{\partial^2 u(x, t)}{\partial t^2} = a(x_n)\delta_\tau \frac{\partial^w u(x_n, t_m)}{\partial x^w} + \delta_\tau b(x_n, t_m) \tag{31}$$

substituting Eq.(31) in Eq.(30) gives:

$$\delta_\tau u(x_n, t_m) = a(x_n) \frac{\partial^w u(x_n, t_m)}{\partial x^w} + b(x_n, t_m) - \frac{\delta\tau}{2}\left[a(x_n)\delta_\tau \frac{\partial^w u(x_n, t_m)}{\partial x^w} + \delta_\tau b(x_n, t_m)\right] + \cdots \tag{32}$$

Simplifying Eq.(32) gives

$$u_n^m - u_n^{m-1} = \frac{\delta\tau}{2} a(x_n) \frac{\partial^w u_n^m}{\partial x^w} + \frac{\delta\tau}{2} a(x_n) \frac{\partial^w u_n^{m-1}}{\partial x^w} + \frac{\delta\tau}{2}(b_n^m + b_n^{m-1}) + R^n(x)(\delta\tau)^3 \tag{33}$$

where,$R^n(x)$ is the truncation term, let $u(x_n, t_m) = U_n^m$

$$U_n^m - \frac{\delta\tau}{2} a(x_n) \frac{\partial^w U_n^m}{\partial x^w} = U_n^{m-1} + \frac{\delta\tau}{2} a(x_n) \frac{\partial^w U_n^{m-1}}{\partial x^w} + \frac{\delta\tau}{2}(b_n^m + b_n^{m-1}) + R^n(x)(\delta\tau)^3, \tag{34}$$

to obtain full discrete form, it is required to approximate the Caputo derivative in $\frac{\partial^w U_n^m}{\partial x^w}$. The approximant $u_N(x, t)$ is constructed using the Gegenbauer collocation technique.

$$u_N(x, t) = \sum_{k=0}^{N} u_k(t) C_k^{(\beta)*}(x) \tag{35}$$

Using Eqs. (34),(35) and the roots of shifted Gegenbauer polynomial of degree $N-1$, we obtain the simplified form as

$$\begin{aligned} &\sum_{k=0}^{N} u_k(t) C_k^{(\beta)*}(x_n) - \frac{\delta\tau}{2} a(x_n) \sum_{k=\lceil w\rceil}^{N} \sum_{i=0}^{k-\lceil w\rceil} u_k(t) N_{k,i} x^{k-i-w} \\ &= \sum_{k=0}^{N} u_k^{m-1} C_k^{(\beta)*}(x_n) + \frac{\delta\tau}{2} a(x_n) \sum_{k=\lceil w\rceil}^{N} \sum_{i=0}^{k-\lceil w\rceil} u_k(t) N_{k,i} x^{k-i-w} \\ &+ \frac{\delta\tau}{2}(b(x_n, t_m) + b(x_n, t_{m-1})) \end{aligned} \tag{36}$$

with the boundary conditions:

$$U(0,t) = \sum_{n=0}^{N} \frac{(-1)^n \Gamma(n+2\beta)}{n!\,\Gamma(2\beta)} u_n(t) = \mu_0, \\ U(1,t) = \sum_{n=0}^{N} \frac{\Gamma(n+2\beta)}{n!\,\Gamma(2\beta)} u_n(t) = \mu_1 \tag{37}$$

Eqs. (36) and (37) gives $(N+1)$ system of linear equations, then obtain u_N, $n = 0,1,\cdots,N$. To get the initial condition U_n^0 of Eq.(36), the initial condition of the problem $U(x,0)$ combining Eq. (15) will be used to obtain U_n^0.

3.2. Convergence and Stability Analysis

In this section, we consider convergence and stability of the proposed method. Let η denote an open bounded domain in $\mathbb{R}^2$ space and $L_2(\eta)$ represents a Hilbert space with the inner product

$$\langle p(x), q(x)\rangle = \int_\eta p(x)q(x)dx, \tag{38}$$

together with the Euclidean norm $\|u(x)\| = \langle u(x), \upsilon(x)\rangle^2$, and the Sobolev space as

$$H^s(\eta) = u \in L_2(\eta), \frac{d^s u}{dx^s} \in L_2(\eta)$$

Suppose that $b(x) > 0$ for $0 \le x \le 1$ the following lemmas are required to investigate the stability and convergence of the method.

Lemma 3.1. *Let $U^j \in H^1(\eta)$, $j = 1,2,...,J$ be the solution of equation (34)and U^0 be the initial condition, then the following inequality holds*

$$\|U^j\| \le \|U^0\| + \max_{0\le\tau\le N} \frac{\delta\tau}{2}(\|b_i^j\| + \|b_i^{j-1}), \tag{39}$$

where, $U^j = u(x,t_j)$

Proof: By Mathematical induction on j.
For $j = 1$ in Eq. (34), we obtain

$$U^1 - \frac{\delta\tau}{2} a(x_i)_a D_x^\omega U^1 = U^0 + \frac{\delta\tau}{2} a(x_i)\, _aD_x^\omega U^0 + \frac{\delta\tau}{2}(b^1 + b^0), \tag{40}$$

multiplying Eq. (40) by U^1 and integrating on η, we get:

$$\|U^1\|^2 - \frac{\delta\tau}{2} a(x_i)\langle _aD_x^\omega U^1, U^1\rangle = \langle U^0, U^1\rangle + \frac{\delta\tau}{2} a(x_i)\langle _aD_x^\omega U^0, U^1\rangle \\ + \frac{\delta\tau}{2}(\langle b^1, U^1\rangle + \langle b^0, U^1\rangle). \tag{41}$$

Since $\cos\left(\frac{\beta}{2}\pi\right) < 0$, using Lemma 2.1, we have

$$\langle _aD_x^\beta U^1, U^1\rangle = \langle _aD_x^{\frac{\beta}{2}} U^1, {}_x D_b^{\frac{\beta}{2}} U^1\rangle = \cos\left(\frac{\beta}{2}\pi\right) \|_aD_x^{\frac{\beta}{2}} U^1\|^2 < 0 \tag{42}$$

since $\cos(\frac{\beta}{2}\pi) < 0$, for $1 < \beta < 2$.
From the left side of Eq. (41), we obtain

$$\|U^1\|^2 \le \|U^1\|^2 - \frac{\delta\tau}{2} a(x_i)\langle _aD_x^\beta U^1, U^1\rangle \tag{43}$$

Using Cauchy-Schwartz inequality, the right side of Eq. (41) becomes

$$\|\langle U^0, U^1\rangle + \frac{\delta\tau}{2} a(x_i)\langle _aD_x^\omega U^0, U^1\rangle\| \\ \le \|U^0 + \frac{\delta\tau}{2} a(x_i)\, _aD_x^\omega U^0\|\|U^1| \le \|U^0\|\|U^1\| \tag{44}$$

Using Eqs. (42),(43), and (44), to obtain

$$\|U^1\| \le \|U^0\| + \max_{0\le n\le N} \frac{\delta\tau}{2}(\|b_i^j\| + \|b_i^{j-1}\|) \tag{45}$$

Assume Eq. (39) is true for $k = 1,2,..., j-1$, then

$$\|U^k\| \le \|U^0\| + \max_{0\le n\le N} \frac{\delta\tau}{2}(\|b_i^k\| + \|b_i^{k-1}\|) \tag{46}$$

Now, multiplying Eq. (34) by U^j and integrating on η

$$\|U^j\|^2 - \frac{\delta\tau}{2} a(x_i)\langle _aD_x^\omega U^j, U^j\rangle = \langle U^{j-1}, U^j\rangle + \frac{\delta\tau}{2} a(x_i)\langle _aD_x^\omega U^{j-1}, U^j\rangle \\ + \frac{\delta\tau}{2}(\langle b^j, U^j\rangle + \langle b^{j-1}, U^j\rangle) \tag{47}$$

From the previous procedure, Eq. (47) becomes

$$\|U^j\| \le \|U^{j-1}\| + \max_{0\le i\le N} \frac{\delta\tau}{2}(\|b_i^j\| + \|b_i^{j-1}\|)$$

Theorem 3.2. *The approximation method introduced in equation (34) is unconditionally stable.*

Proof Let U_i^j, $j = 1,2,...,J$ be the approximate solution obtained in Eq.(34) with the initial condition $U_i^0 = u(x_i,0)$, then the error $\varepsilon^j = u(x_i,t_j) - U_i^j$ satisfies

$$\varepsilon^j - \frac{\delta\tau}{2} p(x_i)\frac{\partial^\beta \varepsilon^j}{\partial x^\beta} - \frac{\delta\tau}{2} p(x_i)\frac{\partial^\beta \varepsilon^{j-1}}{\partial x^\beta} = \varepsilon^{j-1}. \tag{48}$$

According to lemma 3.1,

$$\|\varepsilon^j\| \le \|\varepsilon^0\| \text{ for, } j = 1,2,\ldots,M,$$

hence the proof.

4. Numerical Experiments

In this section, we validate the efficiency and the accuracy of the proposed method (PM) on some selected examples from the literature by comparing the maximum error ϵ_N with the existing results in the latest literature. Where

$$\epsilon_N = \max_{0\le i\le N} |U(x_i,T) - u_N(x_i,T)| \tag{49}$$

Example 4.1

Considering SFODE [7, 9, 22]

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} = \Gamma\left(\frac{6}{5}\right) x^{\frac{9}{5}} \frac{\partial^{\frac{9}{5}} u(x,t)}{\partial x^{\frac{9}{5}}} + 3x^2(2x-1)\exp(-t) \\ &u(x,0) = x^3(x^{-1}-1), \\ &u(0,t) = u(1,t) = 0, \quad t > 0. \end{aligned} \tag{50}$$

with the closed form solution $u(x,t) = x^3(x^{-1}-1)\exp(-t)$.

Comparing Eq. (50) with (17), we have

$$a(x) = \Gamma\left(\frac{6}{5}\right) x^{\frac{9}{5}}, \quad b(x,t) = 3x^2(2x-1)\exp(-t), \ \omega = \frac{9}{5}$$

For approximation of degree 3, that is $N = 3$, Eqs. (35) and (36) gives

$$u_3(x,t) = \sum_{k=0}^{3} u_k^m(t) C_k^{(\beta)*}(x) \tag{51}$$

$$\begin{aligned} &u_0^m C_0^{(\beta)*}(x_n) + u_1^m C_1^{(\beta)*}(x_n) + u_3^m C_3^{(\beta)*}(x_n) \\ &+ \frac{\delta\tau}{2} a(x_n)\left[u_2^m N_{2,0} x^{0.2} + u_3^m (N_{3,0} x^{1.2} + N_{3,1} x^{0.2})\right] \\ &= u_0^{m-1} C_0^{(\beta)*}(x_n) + u_1^{m-1} C_1^{(\beta)*}(x_n) + u_3^{m-1} C_3^{(\beta)*}(x_n) \\ &+ \frac{\delta\tau}{2} a(x_n)\left[u_2^{m-1} N_{2,0} x^{0.2} + u_3^{m-1} (N_{3,0} x^{1.2} + N_{3,1} x^{0.2})\right] \\ &+ \frac{\delta\tau}{2}(b(x_n,t_m) + b(x_n,t_{m-1})), \end{aligned} \tag{52}$$

respectively.

At $x = x_0$, Collocating at zeros of $C_2^{(\beta)*}(x)$, we obtain

$$\begin{aligned} &u_0^m C_0^{(\beta)*}(x_0) + u_1^m C_1^{(\beta)*}(x_0) + u_3^m C_3^{(\beta)*}(x_0) \\ &+ \frac{\delta\tau}{2} a(x_0)\left[u_2^m N_{2,0} x^{0.2} + u_3^m (N_{3,0} x^{1.2} + N_{3,1} x^{0.2})\right] \\ &= u_0^{m-1} C_0^{(\beta)*}(x_0) + u_1^{m-1} C_1^{(\beta)*}(x_0) + u_3^{m-1} C_3^{(\beta)*}(x_0) \\ &+ \frac{\delta\tau}{2} a(x_0)\left[u_2^{m-1} N_{2,0} x^{0.2} + u_3^{m-1} (N_{3,0} x^{1.2} + N_{3,1} x^{0.2})\right] \\ &+ \frac{\delta\tau}{2}(b(x_0,t_m) + b(x_0,t_{m-1})) \end{aligned} \tag{53}$$

we have $C_0^{(\beta)*}(x_0) = 1$. Let $G_1 = C_1^{(\beta)*}(x_0)$, $G_2 = C_3^{(\beta)*}(x_0)$, $H_1 = \frac{\delta\tau}{2} a(x_0) N_{2,0} x^{0.2}$, $H_2 = \frac{\delta\tau}{2} a(x_0)(N_{3,0} x^{1.2} + N_{3,1} x^{0.2})$

the equation becomes

$$\begin{aligned} &u_0^m + u_1^m G_1 + u_3^m G_2 + u_2^m H_1 + u_3^m H_2 \\ &\quad = u_0^{m-1} + u_1^{m-1} G_1 + u_3^{m-1} G_2 + u_2^{m-1} H_1 + u_3^{m-1} H_2 \end{aligned} \tag{54}$$

at $x = x_1$, we get

$$\begin{aligned} &u_0^m C_0^{(\beta)*}(x_1) + u_1^m C_1^{(\beta)*}(x_1) + u_3^m C_3^{(\beta)*}(x_1) \\ &+ \frac{\delta\tau}{2} a(x_1)\left[u_2^m N_{2,0} x^{0.2} + u_3^m (N_{3,0} x^{1.2} + N_{3,1} x^{0.2})\right] \\ &= u_0^{m-1} C_0^{(\beta)*}(x_1) + u_1^{m-1} C_1^{(\beta)*}(x_1) + u_3^{m-1} C_3^{(\beta)*}(x_1) \\ &+ \frac{\delta\tau}{2} a(x_1)\left[u_2^{m-1} N_{2,0} x^{0.2} + u_3^{m-1} (N_{3,0} x^{1.2} + N_{3,1} x^{0.2})\right] \\ &+ \frac{\delta\tau}{2}(b(x_1,t_m) - b(x_1,t_{m-1})). \end{aligned} \tag{55}$$

Let $G_{11} = C_1^{(\beta)*}(x_0)$, $G_{22} = C_3^{(\beta)*}(x_0)$, $H_{11} = \frac{\delta\tau}{2} a(x_0) N_{2,0} x^{0.2}$, $H_{22} = \frac{\delta\tau}{2} a(x_0)(N_{3,0} x^{1.2} + N_{3,1} x^{0.2})$

Eq. (55) becomes

$$\begin{aligned} &u_0^m + u_1^m G_{11} + u_3^m G_{22} + u_2^m H_{11} + u_3^m H_{22} \\ &\quad = u_0^{m-1} + u_1^{m-1} G_{11} + u_3^{m-1} G_{22} + u_2^{m-1} H_{11} + u_3^{m-1} H_{22}. \end{aligned} \tag{56}$$

From Eq. (37) we obtain

$$u_0^m(t) - \frac{\Gamma(1+2\beta)}{\Gamma(2\beta)} u_1^m(t) + \frac{\Gamma(2+2\beta)}{2\Gamma(2\beta)} u_2^m(t) - \frac{\Gamma(3+2\beta)}{6\Gamma(2\beta)} u_3^m(t) = \mu_0, \tag{57}$$

and

$$u_0^m(t) + \frac{\Gamma(1+2\beta)}{\Gamma(2\beta)} u_1^m(t) + \frac{\Gamma(2+2\beta)}{2\Gamma(2\beta)} u_2^m(t) + \frac{\Gamma(3+2\beta)}{6\Gamma(2\beta)} u_3^m(t) = \mu_1. \tag{58}$$

Eqs. (54),(56)-(58) can be written as

$$Au^m = Bu^{m-1} + \frac{\delta\tau}{2}\left(b_n^m - b_n^{m-1}\right), \tag{59}$$

or

$$u^m = A^{-1} Bu^{m-1} + \frac{\delta\tau}{2}\left(b_n^m - b_n^{m-1}\right), \tag{60}$$

where

$$A = \begin{pmatrix} 1 & G_1 & H_1 & G_2 + H_2 \\ 1 & G_{11} & H_{11} & G_{22} + H_{22} \\ 1 & -\frac{\Gamma(1+2\beta)}{\Gamma(2\beta)} & \frac{\Gamma(2+2\beta)}{2\Gamma(2\beta)} & -\frac{\Gamma(3+2\beta)}{6\Gamma(2\beta)} \\ 1 & \frac{\Gamma(1+2\beta)}{\Gamma(2\beta)} & \frac{\Gamma(2+2\beta)}{2\Gamma(2\beta)} & \frac{\Gamma(3+2\beta)}{6\Gamma(2\beta)} \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & G_1 & H_1 & G_2 + H_2 \\ 1 & G_{11} & H_{22} & G_{33} + H_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad u^m = \begin{pmatrix} u_0^m \\ u_1^m \\ u_2^m \\ u_3^m \end{pmatrix}$$

Table 1 shows the computational errors at different values of β while Figure 1a is the corresponding figure. Table 2 display the comparison of the absolute errors at $\beta = 0.5$ with the existing results in the literature at $\beta = 0.5$.

Example 4.2

Consider the space fractional order diffusion problem [7, 9]

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} = \frac{\Gamma(2.2)}{6} x^{2.8} \frac{\partial^{\frac{3}{2}} u(x,t)}{\partial x^{\frac{3}{2}}} - x^3(1+x)\exp(-t) \\ &u(x,0) = x^3 \\ &u(0,t) = 0, \quad u(1,t) = \exp(-t), \quad t > 0. \end{aligned}$$

The exact solution is $u(x,t) = x^3 e^{-t}$.

Table 3 display the comparison of the maximum errors at different values of β and at $\beta = 1$ in [7] is equivalent to the results obtained in [9]. Figure 2 is the graphical representation of the exact solution and its corresponding approximate solutions at different values of β while Figure 3 display the absolute errors at various values of β.

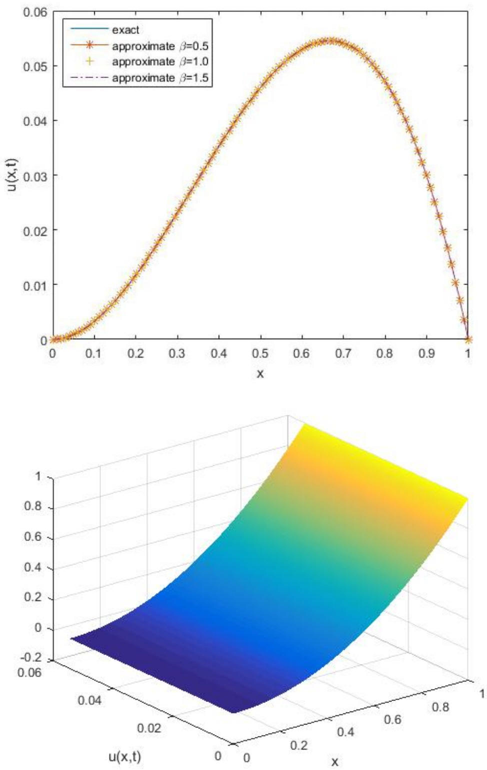

Figure 1: Example 4.1, relationship between exact and approximate solution at $N = 3,\ \&\ T = 1$

Example 4.3

Consider the space fractional diffusion problem [7]

$$\frac{\partial u(x,t)}{\partial t} = \Gamma\left(\frac{3}{2}\right) x^{\frac{1}{2}} \frac{\partial^{\frac{3}{2}} u(x,t)}{\partial x^{\frac{3}{2}}} - 2x\sin(t+1) + (x^2+1)\cos(t+1)$$
$$u(x,0) = (x^2+1)\sin(1)$$
$$u(0,t) = \sin(t+1), \quad u(1,t) = 2\sin(t+1), \quad t > 0.$$

The exact solution is $u(x,t) = (x^2+1)\sin(t+1)$

Comparison of the maximum errors are display in Table 4. Figure 4 is the graphical representation of the absolute errors at various values of β.

Table 1: Example 4.1, absolute errors for various values of β

x	$\beta = 0.5$	$\beta = 1$	$\beta = 1.5$
0	0	2.2959×10^{-41}	4.4409×10^{-16}
0.1	1.0738×10^{-9}	1.2324×10^{-9}	1.1618×10^{-9}
0.2	1.7375×10^{-9}	1.9252×10^{-9}	1.8596×10^{-9}
0.3	2.0553×10^{-9}	2.1777×10^{-9}	2.1705×10^{-9}
0.4	2.0916×10^{-9}	2.0890×10^{-9}	2.1718×10^{-9}
0.5	1.91075×10^{-9}	1.7583×10^{-9}	1.9406×10^{-9}
0.6	1.5770×10^{-9}	1.2849×10^{-9}	1.5542×10^{-9}
0.7	1.1547×10^{-9}	7.6791×10^{-10}	1.0897×10^{-9}
0.8	7.0821×10^{-10}	3.0655×10^{-10}	6.2437×10^{-10}
0.9	3.0187×10^{-10}	2.1667×10^{-17}	2.3540×10^{-10}
1	0	2.1667×10^{-17}	8.8818×10^{-16}

Table 2: Exapmle 4.1, comparison of absolute errors

x	[9]	[22]	[7]	PM
0	0	4.77×10^{-6}	0	0
0.1	5.33×10^{-6}	3.17×10^{-9}	5.46×10^{-6}	1.07×10^{-9}
0.2	8.06×10^{-6}	5.85×10^{-9}	8.51×10^{-6}	1.74×10^{-9}
0.3	8.72×10^{-6}	7.97×10^{-9}	9.60×10^{-6}	2.06×10^{-9}
0.4	7.84×10^{-6}	9.44×10^{-9}	9.18×10^{-6}	2.09×10^{-9}
0.5	5.96×10^{-6}	1.02×10^{-8}	7.69×10^{-6}	1.91×10^{-9}
0.6	3.59×10^{-6}	1.01×10^{-8}	5.60×10^{-6}	1.58×10^{-9}
0.7	1.29×10^{-6}	9.12×10^{-9}	3.33×10^{-6}	1.15×10^{-9}
0.8	4.32×10^{-7}	7.17×10^{-9}	1.34×10^{-6}	7.08×10^{-10}
0.9	4.04×10^{-6}	4.16×10^{-9}	8.39×10^{-8}	3.02×10^{-10}
1	0	7.55×10^{-17}	0	0

Table 3: Example 4.2, maximum error for various values of β

β	[7]	PM
$\beta = 0.5$	2.25×10^{-06}	6.30×10^{-09}
$\beta = 1$	8.83×10^{-06}	1.32×10^{-08}
$\beta = 1.5$	8.35×10^{-06}	3.69×10^{-08}

5. Discussion of results and conclusion

5.1. Discussion of Results

Table 1 shows the absolute errors of the proposed method for example 4.1 with respect to the values of β at $N = 3$ and $T = 1$. The absolute errors of the proposed method is compared with the results in [22],[7] and [9] as shown in Table 2. The approximate solutions at $\beta = 1$ and $\beta = 1.5$ correspond to the

Table 4: Maximum error for Example 4.3 relative to values of β

β	[7]	PM
$\beta = 0.5$	1.56×10^{-05}	8.23×10^{-09}
$\beta = 1$	1.36×10^{-05}	2.01×10^{-08}
$\beta = 1.5$	11.05×10^{-05}	1.06×10^{-09}

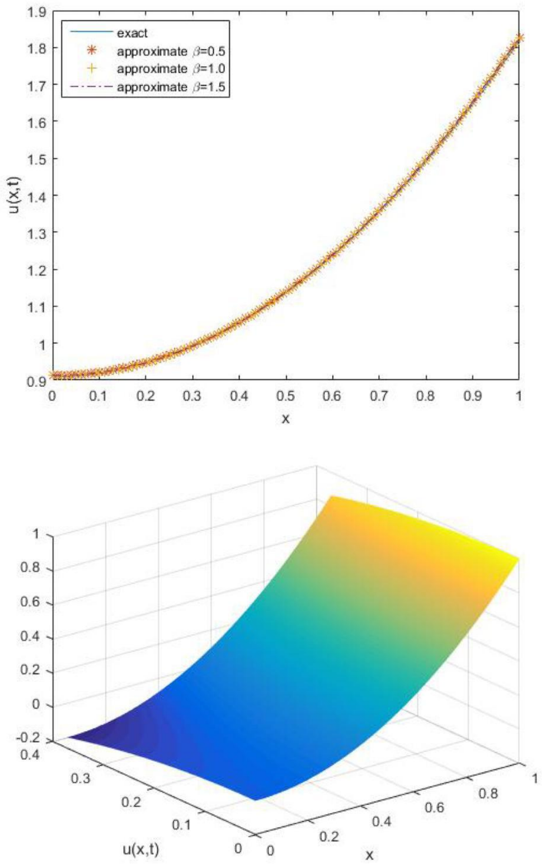

Figure 2: Example 4.2, relationship between exact and approximate solutions at $N = 3$, & $T = 1$

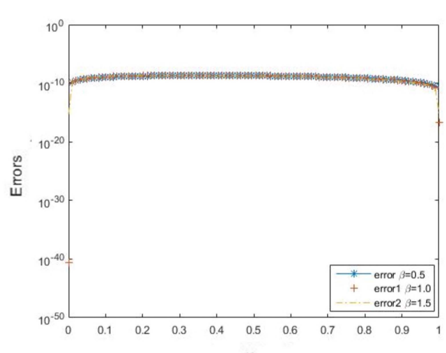

Figure 3: Absolute errors for Example 4.2 at $N = 3$, & $T = 1$

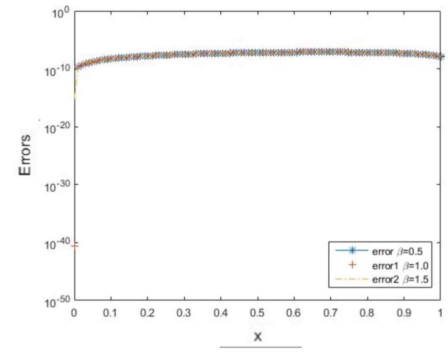

Figure 4: Absolute errors for Example 4.3 at $N = 3$, & $T = 1$

approximate solutions when using second kind shifted Chebyshev polynomial $U_j^*(x)$ and Jacobi $(j, 1, 1, x)$ (that's $P_j^{1,1}(x)$ as an approximation polynomials. Tables 2-4 show the comparison of the errors at various values of β relative to the existing results in the literature. Figures 1-2 are the comparison of the solution for various values of β, while figures 3 and 4 represent the absolute errors of the proposed method at various values of β.

5.2. Conclusion

This paper studied space fractional order diffusion equation by proposing a new technique for finding the approximate solution using compact finite difference method to discretize the time derivative, then use shifted Gegenbauer polynomials as approximating polynomial. The computational results are plotted and detailed in the tables to justify the accuracy of the propose method. It can be observe in the tables of the results and figures that the proposed method performs better than the existing one in the literature.

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