Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 5 (2023) 1484

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Journal of the Nigerian Society of Physical Sciences

A class of single-step hybrid block methods with equally spaced points for general third-order ordinary differential equations

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Abstract

This study presents a class of single-step, self-starting hybrid block methods for directly solving general third-order ordinary differential equations (ODEs) without reduction to first order equations. The methods are developed through interpolation and collocation at systematically selected evenly spaced nodes with the aim of boosting the accuracy of the methods. The zero stability, consistency and convergence of the algorithms are established. Scalar and systems of linear and nonlinear ODEs are approximated to test the effectiveness of the schemes, and the results obtained are compared against other methods from the literature. Significantly, the study shows that an increase in the number of intra-step points improves the accuracy of the solutions obtained using the proposed methods.

DOI:10.46481/jnsps.2023.1484

Keywords: Third-order problems, Convergence, Hybrid block methods, Boundary value problems, Initial value problems

Article History :
Received: 01 April 2023
Received in revised form: 08 October 2023
Accepted for publication: 16 October 2023
Published: 28 November 2023

Communicated by: O. Adeyeye

1. Introduction

Many problems in the fields of mathematics and engineering are formulated using initial and boundary value problems (IVP and BVP) for third-order ODEs. Most physical problems, including the deflection of a curved beam with a constant or varying cross-section, a three-layer beam, the motion of a rocket, thin film flow, electromagnetic waves, gravity-driven flows, physical oceanography, and the context of a variational inequality involve the use of third-order ODEs of the form

$$y'''(x) = f(x, y, y', y''), \tag{1}$$

subject to initial conditions

$$y(x_0) = y_0, y'(x_0) = \delta_0, y''(x_0) = \omega_0,$$

boundary conditions

$$y(x_0) = y_0, y'(x_0) = \delta_0, y'(x_N) = \omega_0,$$

or with mixed boundary conditions

$$y(x_0) = y_0, y'(x_0) = \delta_0, y(x_N) - y'(x_N) = \omega_0,$$

where $x_0, x_N, y_0, \delta_0, \omega_0 \in \Re$, and f is a continuous function and satisfies a Lipschitz condition as given Henrici [1].

Researchers have published a considerable amount of literature on Linear Multistep Methods (LMM) for solving Equation

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(1). Through Taylor expansions, some 3 and 5-step schemes for the special third-order ODEs were presented by Rajabi *et al.* [2]. A one-step block method with four equidistant generalized hybrid points was presented by Adeyeye and Omar [3]. Via numerical integration, a two-point four-step direct implicit block method implemented at two points simultaneously in a block using four backward steps was proposed by Majid *et al.* [4]. Also, a fourth and fifth derivative, three-point implicit block method was presented by Allogmany and Ismail [5]. Using interpolation and collocation techniques, implicit continuous LMM for solving Equation (1) was presented by Jator [6]. Linear multistep methods have also been used to solve other types of differential equations in Refs. [7, 8].

Proposed to overcome the Dahlquist Barrier theorem, hybrid methods were introduced and have continued to generate interest among numerical analysts as given in Ref. [9, 10]. In this paper, we derive a class of hybrid block methods via interpolation and collocation technique, that directly solves general third-order ODEs. We reiterate that the most typical methods for attempting to solve (1) typically entail reducing the problem to a system of first-order differential equations and then solving the system using one of the available methods, which has been shown to have some significant computational drawbacks like requiring more time and labour from the user as as presented by Jator *et al.* [11]. Within the decade researchers have clearly shown an unequivocal dependence of the accuracy of block hybrid methods on the number and type of grid points incorporated in the derivation process as presented in Refs. [12-15].

In this paper, a self-starting class of single-step hybrid block methods are presented for numerically integrating general third-order ODEs. We formulate certain members of the single-step hybrid block methods, prove the convergence and perform numerical tests to check the precision of the suggested methods by contrasting them with other schemes from the literature.

2. Derivation and Analysis of Method

2.1. Derivation of the Methods

In this section, we describe the formulation of a class of one-step block hybrid methods via the interpolation and collocation approach as proposed by Onumanyi *et al.* [16], which will be utilized to generate a number of discrete solutions for solving (1).

We start by deriving a block hybrid method of the form

$$Y(x) = \alpha_0(x)y_n + \alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}} + \alpha_1(x)y_{n+1} + h^3\Big(\sum_{j=0}^{1}\beta_j(x)y_{n+j} + \sum_{v=1}^{m}\beta_{p_v}(x)f_{n+p_v}\Big), \quad (2)$$

where $p_v \in (0,k)$ are a countable number $m \in \mathbb{O}$ of equally spaced, non-integer off-grid points with $\frac{1}{2}$ as midpoint derived by

$$x_{n+p_v} = x_n + \frac{v}{m+1}h, \; v = 1,2,\cdots,m. \quad (3)$$

Equation (1) contains the first and second derivative, hence, the derivatives of Equation (2) are given as

$$Y'(x) = \frac{1}{h}\Big(\alpha_0'(x)y_n + \alpha_{\frac{1}{2}}'(x)y_{n+\frac{1}{2}} + \alpha_1'(x)y_{n+1} + h^3\Big(\sum_{j=0}^{1}\beta_j'(x)y_{n+j} + \sum_{v=1}^{m}\beta_{p_v}'(x)f_{n+p_v}\Big)\Big), \quad (4)$$

$$Y''(x) = \frac{1}{h^2}\Big(\alpha_0''(x)y_n + \alpha_{\frac{1}{2}}''(x)y_{n+\frac{1}{2}} + \alpha_1''(x)y_{n+1} + h^3\Big(\sum_{j=0}^{1}\beta_j''(x)y_{n+j} + \sum_{v=1}^{m}\beta_{p_v}''(x)f_{n+p_v}\Big)\Big). \quad (5)$$

The following conditions are imposed on Equations (4) and (5)

$$Y'(x) = \delta(x), \quad Y''(x) = \omega(x), \quad (6)$$

$$Y'(a) = \delta_0, \quad Y''(a) = \omega_0. \quad (7)$$

2.2. Specification of the methods

In this section, Equation (2) is utilized in obtaining a particular one-step block hybrid method with equally spaced off-grid points by specifying m. For illustrative purposes, the results for $m = 3$ is provided.

Evaluating Equation (2) at $x = \{x_{n+\frac{1}{4}},\ x_{n+\frac{1}{2}},\ x_{n+\frac{3}{4}}\}$, we generate the two main methods (8) and (9) given below as:

$$\left\{\begin{array}{l}\frac{h^3}{30720}\Big(f_n + f_{n+1} + 126f_{n+\frac{1}{2}} + 86f_{n+\frac{1}{4}} + 26f_{n+\frac{3}{4}}\Big)\\ +\frac{1}{8}\Big(3y_n - y_{n+1} + 6y_{n+\frac{1}{2}} - 8y_{n+\frac{1}{4}}\Big)\end{array}\right\} = 0, (8)$$

$$\left\{\begin{array}{l}\frac{h^3}{30720}\Big(-f_n - f_{n+1} - 126f_{n+\frac{1}{2}} - 26f_{n+\frac{1}{4}} - 86f_{n+\frac{3}{4}}\Big)\\ +\frac{1}{8}\Big(-y_n + 3y_{n+1} + 6y_{n+\frac{1}{2}} - 8y_{n+\frac{3}{4}}\Big)\end{array}\right\} = 0, (9)$$

The starters Equations (10) and (11) are obtained from Equation (7) and are given as:

$$\left\{\begin{array}{l}\frac{h^3}{10080}\Big(-41f_n + f_{n+1} - 288f_{n+\frac{1}{2}} - 424f_{n+\frac{1}{4}}\\ -88f_{n+\frac{3}{4}}\Big) + h\delta_n + 3y_n + y_{n+1} - 4y_{n+\frac{1}{2}}\end{array}\right\} = 0, \quad (10)$$

$$\left\{\begin{array}{l}\frac{1}{720}h^3\Big(59f_n - 3f_{n+1} + 48f_{n+\frac{1}{2}} + 216f_{n+\frac{1}{4}}\\ +40f_{n+\frac{3}{4}}\Big) + h^2\omega_n - 4\Big(y_n + y_{n+1} - 2y_{n+\frac{1}{2}}\Big)\end{array}\right\} = 0. \quad (11)$$

It is worth mentioning that the derivatives are generated by $\delta(x_{n+\tau_m}) = \delta_{n+\tau_m}$ and $\omega(x_{n+\tau_m}) = \omega_{n+\tau_m}$ as follows:

$$\left\{\begin{array}{l}\frac{h^3}{161280}\Big(79f_n - 5f_{n+1} + 54f_{n+\frac{1}{2}} + 1532f_{n+\frac{1}{4}}\\ +20f_{n+\frac{3}{4}}\Big) + h\delta_{n+\frac{1}{4}} + 2\Big(y_n - y_{n+\frac{1}{2}}\Big)\end{array}\right\} = 0, \quad (12)$$

$$\left\{\begin{array}{l}\frac{h^3}{10080}\Big(f_n + f_{n+1} + 258 f_{n+\frac{1}{2}} + 80 f_{n+\frac{1}{4}} \\ + 80 f_{n+\frac{3}{4}}\Big) + h\delta_{n+\frac{1}{2}} + y_n - y_{n+1}\end{array}\right\} = 0, \quad (13)$$

$$\left\{\begin{array}{l}\frac{h^3}{161280}\Big(-5 f_n + 79 f_{n+1} + 54 f_{n+\frac{1}{2}} + 20 f_{n+\frac{1}{4}} \\ + 1532 f_{n+\frac{3}{4}}\Big) + h\delta_{n+\frac{3}{4}} + 2\Big(y_{n+\frac{1}{2}} - y_{n+1}\Big)\end{array}\right\} = 0, \quad (14)$$

$$\left\{\begin{array}{l}\frac{h^3}{10080}\Big(f_n - 41 f_{n+1} - 288 f_{n+\frac{1}{2}} - 88 f_{n+\frac{1}{4}} \\ - 424 f_{n+\frac{3}{4}}\Big) + h\delta_{n+1} - y_n - 3y_{n+1} + 4y_{n+\frac{1}{2}}\end{array}\right\} = 0, \quad (15)$$

$$\left\{\begin{array}{l}\frac{h^3}{2880}\Big(-15 f_n + 7 f_{n+1} + 456 f_{n+\frac{1}{2}} + 218 f_{n+\frac{1}{4}} \\ + 54 f_{n+\frac{3}{4}}\Big) + h^2\omega_{n+\frac{1}{4}} - 4\Big(y_n + y_{n+1} - 2y_{n+\frac{1}{2}}\Big)\end{array}\right\} = 0, \quad (16)$$

$$\left\{\begin{array}{l}\frac{1}{720}h^3\Big(f_n - f_{n+1} - 32 f_{n+\frac{1}{4}} + 32 f_{n+\frac{3}{4}}\Big) \\ + h^2\omega_{n+\frac{1}{2}} - 4\Big(y_n + y_{n+1} - 2y_{n+\frac{1}{2}}\Big)\end{array}\right\} = 0, \quad (17)$$

$$\left\{\begin{array}{l}\frac{h^3}{2880}\Big(-7 f_n + 15 f_{n+1} - 456 f_{n+\frac{1}{2}} - 54 f_{n+\frac{1}{4}} \\ - 218 f_{n+\frac{3}{4}}\Big) + h^2\omega_{n+\frac{3}{4}} - 4\Big(y_n + y_{n+1} - 2y_{n+\frac{1}{2}}\Big)\end{array}\right\} = 0, \quad (18)$$

$$\left\{\begin{array}{l}\frac{1}{720}h^3\Big(3 f_n - 59 f_{n+1} - 48 f_{n+\frac{1}{2}} - 40 f_{n+\frac{1}{4}} \\ - 216 f_{n+\frac{3}{4}}\Big) + h^2\omega_{n+1} - 4\Big(y_n + y_{n+1} - 2y_{n+\frac{1}{2}}\Big)\end{array}\right\} = 0. \quad (19)$$

For the one-step block hybrid method with $m = 5$ equally spaced off-grid points, we evaluate Equation (2) at

$$x = \left\{x_{n+\frac{1}{6}},\ x_{n+\frac{1}{3}},\ x_{n+\frac{1}{2}},\ x_{n+\frac{2}{3}},\ x_{n+\frac{5}{6}},\ x_{n+1}\right\}.$$

We generate the four main methods (20), (21),(22) and (23) given below as:

$$\left\{\begin{array}{l}\frac{h^3}{19595520}\Big(412 f_n + 97 f_{n+1} + 44050 f_{n+\frac{1}{2}} \\ + 57525 f_{n+\frac{1}{3}} + 20670 f_{n+\frac{2}{3}} + 23673 f_{n+\frac{1}{6}} \\ + 4773 f_{n+\frac{5}{6}}\Big) + \frac{1}{9}\Big(5y_n - y_{n+1} + 5y_{n+\frac{1}{2}} - 9y_{n+\frac{1}{6}}\Big)\end{array}\right\} = 0, \quad (20)$$

$$\left\{\begin{array}{l}\frac{h^3}{19595520}(191 f_n + 65 f_{n+1} + 45824 f_{n+\frac{1}{2}} \\ + 40539 f_{n+\frac{1}{3}} + 19749 f_{n+\frac{2}{3}} + 9564 f_{n+\frac{1}{6}} \\ + 5028 f_{n+\frac{5}{6}}) + \frac{1}{9}(2y_n - y_{n+1} + 8y_{n+\frac{1}{2}} - 9y_{n+\frac{1}{3}})\end{array}\right\} = 0, \quad (21)$$

$$\left\{\begin{array}{l}\frac{h^3}{19595520}\Big(-65 f_n - 191 f_{n+1} - 45824 f_{n+\frac{1}{2}} \\ - 19749 f_{n+\frac{1}{3}} - 40539 f_{n+\frac{2}{3}} - 5028 f_{n+\frac{1}{6}} \\ - 9564 f_{n+\frac{5}{6}}\Big) + \frac{1}{9}\Big(-y_n + 2y_{n+1} + 8y_{n+\frac{1}{2}} \\ - 9y_{n+\frac{2}{3}}\Big)\end{array}\right\} = 0, \quad (22)$$

$$\left\{\begin{array}{l}\frac{h^3}{19595520}\Big(-97 f_n - 412 f_{n+1} - 44050 f_{n+\frac{1}{2}} \\ - 20670 f_{n+\frac{1}{3}} - 57525 f_{n+\frac{2}{3}} - 4773 f_{n+\frac{1}{6}} \\ - 23673 f_{n+\frac{5}{6}}\Big) + \frac{1}{9}\Big(-y_n + 5y_{n+1} + 5y_{n+\frac{1}{2}} \\ - 9y_{n+\frac{5}{6}}\Big)\end{array}\right\} = 0. \quad (23)$$

The starters Equations (24) and (25) are obtained from Equation (7) and are given as:

$$\left\{\begin{array}{l}\frac{h^3}{268800}\Big(-459 f_n + 11 f_{n+1} - 6400 f_{n+\frac{1}{2}} - 6615 f_{n+\frac{1}{3}} \\ - 2025 f_{n+\frac{2}{3}} - 6156 f_{n+\frac{1}{6}} - 756 f_{n+\frac{5}{6}}\Big) + h\delta_n + 3y_n \\ + y_{n+1} - 4y_{n+\frac{1}{2}}\end{array}\right\} = 0, \quad (24)$$

$$\left\{\begin{array}{l}\frac{h^3}{26880}\Big(1359 f_n - 47 f_{n+1} + 4352 f_{n+\frac{1}{2}} + 1035 f_{n+\frac{1}{3}} \\ - 171 f_{n+\frac{2}{3}} + 6300 f_{n+\frac{1}{6}} + 612 f_{n+\frac{5}{6}}\Big) + h^2\omega_n \\ - 4\Big(y_n + y_{n+1} - 2y_{n+\frac{1}{2}}\Big)\end{array}\right\} = 0. \quad (25)$$

It is worth mentioning that the derivatives are generated by $\delta(x_{n+\tau_m}) = \delta_{n+\tau_m}$ and $\omega(x_{n+\tau_m}) = \omega_{n+\tau_m}$

$$\left\{\begin{array}{l}\frac{h^3}{21772800}\Big(3541 f_n - 479 f_{n+1} - 143300 f_{n+\frac{1}{2}} \\ - 138825 f_{n+\frac{1}{3}} - 71565 f_{n+\frac{2}{3}} + 63114 f_{n+\frac{1}{6}} \\ - 14886 f_{n+\frac{5}{6}}\Big) + h\delta_{n+\frac{1}{6}} + \frac{1}{3}\Big(7y_n + y_{n+1} \\ - 8y_{n+\frac{1}{2}}\Big)\end{array}\right\} = 0, \quad (26)$$

$$\left\{\begin{array}{l}\frac{h^3}{21772800}\Big(1111 f_n + 361 f_{n+1} + 152320 f_{n+\frac{1}{2}} \\ + 281235 f_{n+\frac{1}{3}} + 69165 f_{n+\frac{2}{3}} + 84924 f_{n+\frac{1}{6}} \\ + 15684 f_{n+\frac{5}{6}}\Big) + h\delta_{n+\frac{1}{3}} + \frac{1}{3}\Big(5y_n - y_{n+1} - 4y_{n+\frac{1}{2}}\Big)\end{array}\right\} = 0, \quad (27)$$

$$\left\{\begin{array}{l}\frac{h^3}{268800}(11 f_n + 11 f_{n+1} + 4860 f_{n+\frac{1}{2}} + 2565 f_{n+\frac{1}{3}} \\ + 2565 f_{n+\frac{2}{3}} + 594 f_{n+\frac{1}{6}} + 594 f_{n+\frac{5}{6}}) + h\delta_{n+\frac{1}{2}} + y_n \\ - y_{n+1}\end{array}\right\} = 0, \quad (28)$$

$$\left\{\begin{array}{l}\frac{h^3}{21772800}\Big(361 f_n + 1111 f_{n+1} + 152320 f_{n+\frac{1}{2}} \\ + 69165 f_{n+\frac{1}{3}} + 281235 f_{n+\frac{2}{3}} + 15684 f_{n+\frac{1}{6}} \\ + 84924 f_{n+\frac{5}{6}}\Big) + h\delta_{n+\frac{2}{3}} + \frac{1}{3}\Big(y_n - 5y_{n+1} + 4y_{n+\frac{1}{2}}\Big)\end{array}\right\} = 0, \quad (29)$$

$$\left\{\begin{array}{l}\frac{h^3}{21772800}\Big(-479 f_n + 3541 f_{n+1} - 143300 f_{n+\frac{1}{2}} \\ - 71565 f_{n+\frac{1}{3}} - 138825 f_{n+\frac{2}{3}} - 14886 f_{n+\frac{1}{6}} \\ + 63114 f_{n+\frac{5}{6}}\Big) + h\delta_{n+\frac{5}{6}} + \frac{1}{3}\Big(-y_n - 7y_{n+1} \\ + 8y_{n+\frac{1}{2}}\Big)\end{array}\right\} = 0, \quad (30)$$

$$\begin{cases} \frac{h^3}{268800}(11f_n - 459f_{n+1} - 6400f_{n+\frac{1}{2}} - 2025f_{n+\frac{1}{3}} \\ - 6615f_{n+\frac{2}{3}} - 756f_{n+\frac{1}{6}} - 6156f_{n+\frac{5}{6}}) \\ + h\delta_{n+1} - y_n - 3y_{n+1} + 4y_{n+\frac{1}{2}} \end{cases} = 0, \quad (31)$$

$$\begin{cases} \frac{h^3}{725760}\Big(-1481f_n + 457f_{n+1} + 42496f_{n+\frac{1}{2}} \\ + 120867f_{n+\frac{1}{3}} + 35805f_{n+\frac{2}{3}} + 39876f_{n+\frac{1}{6}} \\ + 3900f_{n+\frac{5}{6}}\Big) + h^2\omega_{n+\frac{1}{6}} - 4\Big(y_n + y_{n+1} - 2y_{n+\frac{1}{2}}\Big) \end{cases} = 0, \quad (32)$$

$$\begin{cases} \frac{h^3}{725760}\Big(245f_n - 85f_{n+1} + 75008f_{n+\frac{1}{2}} \\ + 26889f_{n+\frac{1}{3}} + 21207f_{n+\frac{2}{3}} - 10380f_{n+\frac{1}{6}} \\ + 8076f_{n+\frac{5}{6}}\Big) + h^2\omega_{n+\frac{1}{3}} - 4\Big(y_n + y_{n+1} - 2y_{n+\frac{1}{2}}\Big) \end{cases} = 0, \quad (33)$$

$$\begin{cases} \frac{h^3}{26880}\Big(-11f_n + 11f_{n+1} - 1287f_{n+\frac{1}{3}} + 1287f_{n+\frac{2}{3}} \\ - 180f_{n+\frac{1}{6}} + 180f_{n+\frac{5}{6}}\Big) + h^2\omega_{n+\frac{1}{2}} - 4\Big(y_n + y_{n+1} \\ - 2y_{n+\frac{1}{2}}\Big) \end{cases} = 0, \quad (34)$$

$$\begin{cases} \frac{h^3}{725760}\Big(85f_n - 245f_{n+1} - 75008f_{n+\frac{1}{2}} \\ - 21207f_{n+\frac{1}{3}} - 26889f_{n+\frac{2}{3}} - 8076f_{n+\frac{1}{6}} \\ + 10380f_{n+\frac{5}{6}}\Big) + h^2\omega_{n+\frac{2}{3}} - 4\Big(y_n + y_{n+1} \\ - 2y_{n+\frac{1}{2}}\Big) \end{cases} = 0, \quad (35)$$

$$\begin{cases} \frac{h^3}{725760}\Big(-457f_n + 1481f_{n+1} - 42496f_{n+\frac{1}{2}} \\ - 35805f_{n+\frac{1}{3}} - 120867f_{n+\frac{2}{3}} - 3900f_{n+\frac{1}{6}} \\ - 39876f_{n+\frac{5}{6}}\Big) + h^2\omega_{n+\frac{5}{6}} - 4\Big(y_n + y_{n+1} - 2y_{n+\frac{1}{2}}\Big) \end{cases} = 0, \quad (36)$$

$$\begin{cases} \frac{h^3}{26880}\Big(47f_n - 1359f_{n+1} - 4352f_{n+\frac{1}{2}} + 171f_{n+\frac{1}{3}} \\ - 1035f_{n+\frac{2}{3}} - 612f_{n+\frac{1}{6}} - 6300f_{n+\frac{5}{6}}\Big) + h^2\omega_{n+1} \\ - 4\Big(y_n + y_{n+1} - 2y_{n+\frac{1}{2}}\Big) \end{cases} = 0. \quad (37)$$

For the one-step block hybrid method with $m = 7$ equally spaced off-grid points, we evaluate Equation (2) at

$$x = \Big\{x_{n+\frac{1}{8}},\ x_{n+\frac{1}{4}},\ x_{n+\frac{3}{8}},\ x_{n+\frac{1}{2}},\ x_{n+\frac{5}{8}},\ x_{n+\frac{3}{4}},\ x_{n+\frac{7}{8}},\ x_{n+1}\Big\}.$$

We generate the six main methods (38), (39), (40), (41), (42) and (43) given below as:

$$\begin{cases} \frac{h^3}{88473600}\Big(937f_n + 142f_{n+1} + 131885f_{n+\frac{1}{2}} \\ + 144631f_{n+\frac{1}{4}} + 33061f_{n+\frac{3}{4}} + 53474f_{n+\frac{1}{8}} \\ + 161518f_{n+\frac{3}{8}} + 71398f_{n+\frac{5}{8}} + 7754f_{n+\frac{7}{8}}\Big) \\ + \frac{1}{32}\Big(21y_n - 3y_{n+1} + 14y_{n+\frac{1}{2}} - 32y_{n+\frac{1}{8}}\Big) \end{cases} = 0, \quad (38)$$

$$\begin{cases} \frac{h^3}{9676800}\Big(59f_n + 19f_{n+1} + 18770f_{n+\frac{1}{2}} \\ + 14122f_{n+\frac{1}{4}} + 4742f_{n+\frac{3}{4}} + 3428f_{n+\frac{1}{8}} \\ + 22676f_{n+\frac{3}{8}} + 10636f_{n+\frac{5}{8}} + 1148f_{n+\frac{7}{8}}\Big) \\ + \frac{1}{8}\Big(3y_n - y_{n+1} + 6y_{n+\frac{1}{2}} - 8y_{n+\frac{1}{4}}\Big) \end{cases} = 0, \quad (39)$$

$$\begin{cases} \frac{h^3}{123863040}\Big(296f_n + 199f_{n+1} + 183445f_{n+\frac{1}{2}} \\ + 75337f_{n+\frac{1}{4}} + 46259f_{n+\frac{3}{4}} + 18466f_{n+\frac{1}{8}} \\ + 169886f_{n+\frac{3}{8}} + 100054f_{n+\frac{5}{8}} + 10858f_{n+\frac{7}{8}}\Big) \\ + \frac{1}{32}\Big(5y_n - 3y_{n+1} + 30y_{n+\frac{1}{2}} - 32y_{n+\frac{3}{8}}\Big) \end{cases} = 0, \quad (40)$$

$$\begin{cases} \frac{h^3}{123863040}\Big(-199f_n - 296f_{n+1} - 183445f_{n+\frac{1}{2}} \\ - 46259f_{n+\frac{1}{4}} - 75337f_{n+\frac{3}{4}} - 10858f_{n+\frac{1}{8}} \\ - 100054f_{n+\frac{3}{8}} - 169886f_{n+\frac{5}{8}} - 18466f_{n+\frac{7}{8}}\Big) \\ + \frac{1}{32}\Big(-3y_n + 5y_{n+1} + 30y_{n+\frac{1}{2}} - 32y_{n+\frac{5}{8}}\Big) \end{cases} = 0, \quad (41)$$

$$\begin{cases} \frac{h^3}{9676800}\Big(-19f_n - 59f_{n+1} - 18770f_{n+\frac{1}{2}} \\ - 4742f_{n+\frac{1}{4}} - 14122f_{n+\frac{3}{4}} - 1148f_{n+\frac{1}{8}} \\ - 10636f_{n+\frac{3}{8}} - 22676f_{n+\frac{5}{8}} - 3428f_{n+\frac{7}{8}}\Big) \\ + \frac{1}{8}(-y_n + 3y_{n+1} + 6y_{n+\frac{1}{2}} - 8y_{n+\frac{3}{4}}) \end{cases} = 0, \quad (42)$$

$$\begin{cases} \frac{h^3}{88473600}\Big(-142f_n - 937f_{n+1} - 131885f_{n+\frac{1}{2}} \\ - 33061f_{n+\frac{1}{4}} - 144631f_{n+\frac{3}{4}} - 7754f_{n+\frac{1}{8}} \\ - 71398f_{n+\frac{3}{8}} - 161518f_{n+\frac{5}{8}} - 53474f_{n+\frac{7}{8}}\Big) \\ + \frac{1}{32}\Big(-3y_n + 21y_{n+1} + 14y_{n+\frac{1}{2}} - 32y_{n+\frac{7}{8}}\Big) \end{cases} = 0. \quad (43)$$

The starters Equations (44) and (45) are obtained from Equation

(7) and are given as:

$$\left\{\begin{array}{l}\frac{h^3}{4989600}\Big(-4519f_n+79f_{n+1}-55520f_{n+\frac{1}{2}}\\-80024f_{n+\frac{1}{4}}-13672f_{n+\frac{3}{4}}-71648f_{n+\frac{1}{8}}\\-126304f_{n+\frac{3}{8}}-58016f_{n+\frac{5}{8}}-6176f_{n+\frac{7}{8}}\Big)\\+h\delta_n+3y_n+y_{n+1}-4y_{n+\frac{1}{2}}\end{array}\right\}=0, \quad (44)$$

$$\left\{\begin{array}{l}\frac{h^3}{226800}\Big(8121f_n-209f_{n+1}-18160f_{n+\frac{1}{2}}\\-2552f_{n+\frac{1}{4}}-4872f_{n+\frac{3}{4}}+44192f_{n+\frac{1}{8}}\\+55584f_{n+\frac{3}{8}}+28384f_{n+\frac{5}{8}}+2912f_{n+\frac{7}{8}}\Big)+h^2\omega_n\\-4\Big(y_n+y_{n+1}-2y_{n+\frac{1}{2}}\Big)\end{array}\right\}=0.(45)$$

It is worth mentioning that the derivatives are generated by $\delta(x_{n+\tau_m})=\delta_{n+\tau_m}$ and $\omega(x_{n+\tau_m})=\omega_{n+\tau_m}$

$$\left\{\begin{array}{l}\frac{h^3}{5109350400}\Big(371725f_n-65591f_{n+1}\\-43219630f_{n+\frac{1}{2}}-31649144f_{n+\frac{1}{4}}\\-10970288f_{n+\frac{3}{4}}+5710568f_{n+\frac{1}{8}}\\-44180864f_{n+\frac{3}{8}}-20167160f_{n+\frac{5}{8}}\\-2191216f_{n+\frac{7}{8}}\Big)+h\delta_{n+\frac{1}{8}}+\frac{1}{2}\Big(5y_n+y_{n+1}\\-6y_{n+\frac{1}{2}}\Big)\end{array}\right\}=0, \quad (46)$$

$$\left\{\begin{array}{l}\frac{h^3}{79833600}\Big(2221f_n+65f_{n+1}+10930f_{n+\frac{1}{2}}\\+515828f_{n+\frac{1}{4}}+2524f_{n+\frac{3}{4}}+156736f_{n+\frac{1}{8}}\\+149920f_{n+\frac{3}{8}}-6016f_{n+\frac{5}{8}}-608f_{n+\frac{7}{8}}\Big)\\+h\delta_{n+\frac{1}{4}}+2\Big(y_n-y_{n+\frac{1}{2}}\Big)\end{array}\right\}=0, \quad (47)$$

$$\left\{\begin{array}{l}\frac{h^3}{5109350400}\Big(128801f_n+39349f_{n+1}\\+40485970f_{n+\frac{1}{2}}+30702112f_{n+\frac{1}{4}}\\+9998264f_{n+\frac{3}{4}}+7176736f_{n+\frac{1}{8}}\\+59573336f_{n+\frac{3}{8}}+22437424f_{n+\frac{5}{8}}\\+2430808f_{n+\frac{7}{8}}\Big)+h\delta_{n+\frac{3}{8}}+\frac{1}{2}\Big(3y_n-y_{n+1}\\-2y_{n+\frac{1}{2}}\Big)\end{array}\right\}=0, \quad (48)$$

$$\left\{\begin{array}{l}\frac{h^3}{4989600}\Big(79f_n+79f_{n+1}+70430f_{n+\frac{1}{2}}\\+19504f_{n+\frac{1}{4}}+19504f_{n+\frac{3}{4}}+4736f_{n+\frac{1}{8}}\\+44416f_{n+\frac{3}{8}}\\+44416f_{n+\frac{5}{8}}+4736f_{n+\frac{7}{8}}\Big)+h\delta_{n+\frac{1}{2}}+y_n-y_{n+1}\end{array}\right\}=0,(49)$$

$$\left\{\begin{array}{l}\frac{h^3}{5109350400}\Big(39349f_n+128801f_{n+1}\\+40485970f_{n+\frac{1}{2}}+9998264f_{n+\frac{1}{4}}\\+30702112f_{n+\frac{3}{4}}+2430808f_{n+\frac{1}{8}}\\+22437424f_{n+\frac{3}{8}}+59573336f'_{n+\frac{5}{8}}\\+7176736f_{n+\frac{7}{8}})+h\delta_{n+\frac{5}{8}}+\frac{1}{2}(y_n-3y_{n+1}\\+2y_{n+\frac{1}{2}})\end{array}\right\}=0, \quad (50)$$

$$\left\{\begin{array}{l}\frac{h^3}{79833600}\Big(65f_n+2221f_{n+1}+10930f_{n+\frac{1}{2}}\\+2524f_{n+\frac{1}{4}}+515828f_{n+\frac{3}{4}}-608f_{n+\frac{1}{8}}\\-6016f_{n+\frac{3}{8}}+149920f_{n+\frac{5}{8}}+156736f_{n+\frac{7}{8}}\Big)\\+h\delta_{n+\frac{3}{4}}+2\Big(y_{n+\frac{1}{2}}-y_{n+1}\Big)\end{array}\right\}=0, \quad (51)$$

$$\left\{\begin{array}{l}\frac{h^3}{5109350400}\Big(-65591f_n+371725f_{n+1}\\-43219630f_{n+\frac{1}{2}}-10970288f_{n+\frac{1}{4}}\\-31649144f_{n+\frac{3}{4}}-2191216f_{n+\frac{1}{8}}\\-20167160f_{n+\frac{3}{8}}-44180864f_{n+\frac{5}{8}}\\+5710568f_{n+\frac{7}{8}}\Big)+h\delta_{n+\frac{7}{8}}+\frac{1}{2}\Big(-y_n-5y_{n+1}\\+6y_{n+\frac{1}{2}}\Big)\end{array}\right\}=0, \quad (52)$$

$$\left\{\begin{array}{l}\frac{h^3}{4989600}\Big(79f_n-4519f_{n+1}-55520f_{n+\frac{1}{2}}\\-13672f_{n+\frac{1}{4}}-80024f_{n+\frac{3}{4}}-6176f_{n+\frac{1}{8}}\\-58016f_{n+\frac{3}{8}}-126304f_{n+\frac{5}{8}}-71648f_{n+\frac{7}{8}}\Big)\\+h\delta_{n+1}-y_n-3y_{n+1}+4y_{n+\frac{1}{2}}\end{array}\right\}=0, \quad (53)$$

$$\left\{\begin{array}{l}\frac{h^3}{29030400}\Big(-30529f_n+7201f_{n+1}\\+2708640f_{n+\frac{1}{2}}+4277938f_{n+\frac{1}{4}}+667598f_{n+\frac{3}{4}}\\+1189482f_{n+\frac{1}{8}}+1519394f_{n+\frac{3}{8}}+486814f_{n+\frac{5}{8}}\\+59862f_{n+\frac{7}{8}}\Big)+h^2\omega_{n+\frac{1}{8}}-4\Big(y_n+y_{n+1}-2y_{n+\frac{1}{2}}\Big)\end{array}\right\}=0,(54)$$

$$\left\{\begin{array}{l}\frac{h^3}{907200}\Big(107f_n-3f_{n+1}+43480f_{n+\frac{1}{2}}\\+32286f_{n+\frac{1}{4}}+11666f_{n+\frac{3}{4}}-5816f_{n+\frac{1}{8}}\\+102248f_{n+\frac{3}{8}}+38808f_{n+\frac{5}{8}}\\+4024f_{n+\frac{7}{8}}\Big)+h^2\omega_{n+\frac{1}{4}}-4\Big(y_n+y_{n+1}-2y_{n+\frac{1}{2}}\Big)\end{array}\right\}=0,(55)$$

$$\left\{\begin{array}{l}\frac{h^3}{29030400}\Big(-3873f_n+3137f_{n+1}+2224480f_{n+\frac{1}{2}}\\-605134f_{n+\frac{1}{4}}+515406f_{n+\frac{3}{4}}-86486f_{n+\frac{1}{8}}\\+640098f_{n+\frac{3}{8}}+843998f_{n+\frac{5}{8}}+97174f_{n+\frac{7}{8}}\Big)\\+h^2\omega_{n+\frac{3}{8}}-4\Big(y_n+y_{n+1}-2y_{n+\frac{1}{2}}\Big)\end{array}\right\}=0,(56)$$

$$\left\{\begin{array}{l}\frac{h^3}{45360}\Big(-f_n+f_{n+1}-608f_{n+\frac{1}{4}}+608f_{n+\frac{3}{4}}\\-192f_{n+\frac{1}{8}}-1984f_{n+\frac{3}{8}}+1984f_{n+\frac{5}{8}}+192f_{n+\frac{7}{8}}\Big)\\+h^2\omega_{n+\frac{1}{2}}-4\Big(y_n+y_{n+1}-2y_{n+\frac{1}{2}}\Big)\end{array}\right\}=0,(57)$$

$$\left\{\begin{array}{l}\frac{h^3}{29030400}\Big(-3137f_n+3873f_{n+1}-2224480f_{n+\frac{1}{2}}\\-515406f_{n+\frac{1}{4}}+605134f_{n+\frac{3}{4}}-97174f_{n+\frac{1}{8}}\\-843998f_{n+\frac{3}{8}}-640098f_{n+\frac{5}{8}}+86486f_{n+\frac{7}{8}}\Big)\\+h^2\omega_{n+\frac{5}{8}}-4\Big(y_n+y_{n+1}-2y_{n+\frac{1}{2}}\Big)\end{array}\right\}=0,(58)$$

$$\left\{\begin{array}{l}\frac{h^3}{907200}\Big(3f_n-107f_{n+1}-43480f_{n+\frac{1}{2}}-11666f_{n+\frac{1}{4}}\\-32286f_{n+\frac{3}{4}}-4024f_{n+\frac{1}{8}}-38808f_{n+\frac{3}{8}}\\-102248f_{n+\frac{5}{8}}+5816f_{n+\frac{7}{8}}\Big)+h^2\omega_{n+\frac{3}{4}}\\-4\Big(y_n+y_{n+1}-2y_{n+\frac{1}{2}}\Big)\end{array}\right\}=0,(59)$$

$$\left\{\begin{array}{l}\frac{h^3}{29030400}\Big(-7201f_n+30529f_{n+1}\\-2708640f_{n+\frac{1}{2}}-667598f_{n+\frac{1}{4}}-4277938f_{n+\frac{3}{4}}\\-59862f_{n+\frac{1}{8}}-486814f_{n+\frac{3}{8}}-1519394f_{n+\frac{5}{8}}\\-1189482f_{n+\frac{7}{8}}\Big)\\+h^2\omega_{n+\frac{7}{8}}-4\Big(y_n+y_{n+1}-2y_{n+\frac{1}{2}}\Big)\end{array}\right\}=0,(60)$$

$$\left\{\begin{array}{l}\frac{h^3}{226800}\Big(209f_n-8121f_{n+1}+18160f_{n+\frac{1}{2}}\\+4872f_{n+\frac{1}{4}}+2552f_{n+\frac{3}{4}}-2912f_{n+\frac{1}{8}}\\-28384f_{n+\frac{3}{8}}-55584f_{n+\frac{5}{8}}-44192f_{n+\frac{7}{8}}\Big)\\+h^2\omega_{n+1}-4\Big(y_n+y_{n+1}-2y_{n+\frac{1}{2}}\Big)\end{array}\right\}=0. \quad (61)$$

For brevity, we have omitted the schemes for $m > 7$.

2.3. Analysis of the methods

We report the findings from our analysis of the characteristics of our proposed method with evenly spaced off-grid collocation nodes in this section. We focus in particular on the zero stability analysis and truncation error.

2.3.1. Local Truncation Errors and Order

We express our main methods and starters for third order IVPs in terms of a linear operator $\mathcal{L}$ defined as

$$\left.\begin{array}{l}\sum_{j=0}^{1}\alpha_j y(x_n+jh)+\alpha_{p_{\frac{1}{2}}}y(x_n+p_{\frac{1}{2}}h)\\-h^3(\sum_{i=0}^{1}\beta_i y'''(x_n+ih)+\sum_{\nu=1}^{m}\beta_{p_\nu}y'''(x_n+p_\nu h)),\\\sum_{j=0}^{1}\alpha_j y(x_n+jh)-hy'(x_n)\\-h^3(\sum_{i=0}^{1}\beta_i y'''(x_n+ih)+\sum_{\nu=1}^{m}\beta_{p_\nu}y'''(x_n+p_\nu h)),\\\sum_{j=0}^{1}\alpha_j y(x_n+jh)-hy''(x_n)\\-h^3(\sum_{i=0}^{1}\beta_i y'''(x_n+ih)+\sum_{\nu=1}^{m}\beta_{p_\nu}y'''(x_n+p_\nu h)),\end{array}\right\}=\mathcal{L}[Y(x_n);h],(62)$$

where $\zeta = 1, 2, \ldots, m$.

Assuming that $y(x_n)$ is sufficiently differentiable, we can expand the terms $y(x_n+jh), y(x_n+p_\zeta h), y'''(x_n+p_\nu h)$and $y'''(x_n+jh)$ as a Taylor series about the point x_n to obtain the expression

$$L[y(x);h]=\hat{C}_0 y(x_n)+\hat{C}_1 hy'(x_n)+\cdots+\hat{C}_p h^p y^p(x_n)+\cdots(63)$$

where $\hat{C}_0, \hat{C}_1, \ldots \hat{C}_p$ are constant vectors. As stated in Henrici [1], we say that the method has order p if

$$\hat{C}_0, \hat{C}_1, \cdots, \hat{C}_p, \hat{C}_{p+1}, \text{ and } \hat{C}_{p+2}=0 \quad C_{p+3}\neq 0.$$

The vector $\hat{C}_{p+3}$ is called the error constant and $\hat{C}_{p+3}h^{p+3}y^{(p+3)}(x_n)$ the principal part of the local truncation error at the end point x_n. It is standard from our computations that our methods have order $p > 1$ and with comparatively small error constants.

Consequently, for our method with $m = 3$, we have

$$\left.\begin{array}{l}\frac{h^8y^{(8)}(x)}{125829120}+O\left(h^9\right),\\\frac{h^8y^{(8)}(x)}{125829120}+O\left(h^9\right),\\\frac{h^8y^{(8)}(x)}{5160960}+O\left(h^9\right),\\-\frac{h^8y^{(8)}(x)}{33030144}+O\left(h^9\right)\end{array}\right\}=\mathcal{L}[Y(x_n);h]. \quad (64)$$

For our method with $m = 5$ and $m = 7$, we have

$$\left.\begin{array}{l} \dfrac{13h^{10}y^{(10)}(x)}{10970982973440} + O\left(h^{11}\right), \\ -\dfrac{h^{10}y^{(10)}(x)}{548549148672} + O\left(h^{11}\right), \\ -\dfrac{h^{10}y^{(10)}(x)}{548549148672} + O\left(h^{11}\right), \\ \dfrac{13h^{10}y^{(10)}(x)}{10970982973440} + O\left(h^{11}\right), \\ \dfrac{h^{10}y^{(10)}(x)}{4180377600} + O\left(h^{11}\right), \\ -\dfrac{h^{10}y^{(10)}(x)}{179159040} + O\left(h^{11}\right) \end{array}\right\} = \mathcal{L}[Y(x_n); h], \qquad (65)$$

$$\left.\begin{array}{l} \dfrac{7h^{12}y^{(12)}(x)}{94997804639846400} + O\left(h^{13}\right), \\ -\dfrac{h^{12}y^{(12)}(x)}{2597596220620800} + O\left(h^{13}\right), \\ \dfrac{h^{12}y^{(12)}(x)}{2955487255461888} + O\left(h^{13}\right), \\ \dfrac{h^{12}y^{(12)}(x)}{2955487255461888} + O\left(h^{13}\right), \\ -\dfrac{h^{12}y^{(12)}(x)}{2597596220620800} + O\left(h^{13}\right), \\ \dfrac{h^{12}y^{(12)}(x)}{4783519825920} + O\left(h^{13}\right), \\ -\dfrac{h^{12}y^{(12)}(x)}{2597596220620800} + O\left(h^{13}\right), \\ -\dfrac{317h^{12}y^{(12)}(x)}{50226958172160} + O\left(h^{13}\right) \end{array}\right\} = \mathcal{L}[Y(x_n); h]. \qquad (66)$$

2.3.2. *Zero Stability*

The starters and primary methods are represented in matrix-vector form as

$$A_m^{(1)} Y_{\lambda+1} = A_m^{(0)} Y_\lambda + h^3 [B_m^{(1)} F_{\lambda+1} + B_m^{(0)} F_\lambda] + h^2 C_m^{(0)} \omega_\lambda + h D_m^{(0)} \delta_\lambda, \quad (67)$$

where

$$Y_{\lambda+1} = (\cdots, y_{n+\frac{1}{2}}, \cdots, y_{n+1})^T,$$

$$Y_\lambda = (\cdots, y_{n-\frac{1}{2}}, \cdots, y_n)^T,$$

$$F_{\lambda+1} = (\cdots, f_{n+y_{n+\frac{1}{2}}}, \cdots, f_{n+2})^T,$$

$$F_\lambda = (\cdots, f_{n+y_{n-\frac{1}{2}}}, \cdots, f_n)^T,$$

$$\delta_\lambda = (\cdots, \delta_{n-\frac{1}{2}}, \cdots, \delta_n)^T,$$

$$\omega_\lambda = (\cdots, \omega_{n-\frac{1}{2}}, \cdots, \omega_n)^T,$$

$A_m^{(1)}, A_m^{(0)}, B_m^{(1)}, B_m^{(0)}, C_m^{(0)}$ and $D_m^{(0)}$ are square matrices.

For $m = 3$ we obtain we obtain the square matrices

$$A_3^{(1)} = \begin{pmatrix} 1 & -\frac{3}{4} & 0 & \frac{1}{8} \\ 0 & -\frac{3}{4} & 1 & -\frac{3}{8} \\ 0 & 4 & 0 & -1 \\ 0 & -8 & 0 & 4 \end{pmatrix}, A_0^{(0)} = \begin{pmatrix} 0 & 0 & 0 & -\frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 4 \end{pmatrix},$$

$$B_3^{(1)} = \begin{pmatrix} \frac{43}{15360} & \frac{21}{5120} & \frac{13}{15360} & \frac{1}{30720} \\ -\frac{13}{15360} & -\frac{21}{5120} & -\frac{43}{15360} & -\frac{1}{30720} \\ -\frac{53}{1260} & -\frac{1}{35} & -\frac{11}{1260} & \frac{1}{10080} \\ \frac{3}{10} & \frac{1}{15} & \frac{1}{18} & -\frac{1}{240} \end{pmatrix},$$

$$B_3^{(0)} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{30720} \\ 0 & 0 & 0 & -\frac{1}{30720} \\ 0 & 0 & 0 & -\frac{41}{10080} \\ 0 & 0 & 0 & \frac{59}{720} \end{pmatrix}, C_3^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$D_3^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $m = 5$ we obtain

$$A_5^{(1)} = \begin{pmatrix} 1 & 0 & -\frac{5}{9} & 0 & 0 & \frac{1}{9} \\ 0 & 1 & -\frac{8}{9} & 0 & 0 & \frac{1}{9} \\ 0 & 0 & -\frac{8}{9} & 1 & 0 & -\frac{2}{9} \\ 0 & 0 & -\frac{5}{9} & 0 & 1 & -\frac{5}{9} \\ 0 & 0 & 4 & 0 & 0 & -1 \\ 0 & 0 & -8 & 0 & 0 & 4 \end{pmatrix},$$

$$B_5^{(1)} =$$

$$\begin{pmatrix} \frac{7891}{6531840} & \frac{3835}{1306368} & \frac{4405}{1959552} & \frac{689}{653184} & \frac{1591}{6531840} & \frac{97}{19595520} \\ \frac{797}{1632960} & \frac{13513}{6531840} & \frac{179}{76545} & \frac{6583}{6531840} & \frac{419}{1632960} & \frac{13}{3919104} \\ -\frac{419}{1632960} & -\frac{6583}{6531840} & -\frac{179}{76545} & -\frac{13513}{6531840} & -\frac{797}{1632960} & -\frac{191}{19595520} \\ -\frac{1591}{6531840} & -\frac{689}{653184} & -\frac{4405}{1959552} & -\frac{3835}{1306368} & -\frac{7891}{6531840} & -\frac{103}{4898880} \\ -\frac{513}{22400} & -\frac{63}{2560} & -\frac{1}{42} & -\frac{27}{3584} & -\frac{9}{3200} & \frac{11}{268800} \\ \frac{15}{64} & \frac{69}{1792} & \frac{17}{105} & -\frac{57}{8960} & \frac{51}{2240} & -\frac{47}{26880} \end{pmatrix},$$

$$A_5^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{5}{9} \\ 0 & 0 & 0 & 0 & 0 & -\frac{2}{9} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{9} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{9} \\ 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix},$$

$$B_5^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{103}{4898880} \\ 0 & 0 & 0 & 0 & 0 & \frac{191}{19595520} \\ 0 & 0 & 0 & 0 & 0 & -\frac{13}{3919104} \\ 0 & 0 & 0 & 0 & 0 & -\frac{97}{19595520} \\ 0 & 0 & 0 & 0 & 0 & -\frac{153}{89600} \\ 0 & 0 & 0 & 0 & 0 & \frac{453}{8960} \end{pmatrix}.$$

$$C_5^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D_5^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For brevity, we omit the square matrices for the method $m = 7$. The matrix system (67) is reduced to

$$A_m^{(1)} Y_{\lambda+1} = A_m^{(0)} Y_\lambda. \tag{68}$$

The zero stability is established from the nature of the zeros of the first characteristic polynomial defined as

$$\rho(R) = det[A^{(1)} - A^{(0)}]. \tag{69}$$

A method is said to be zero stable if the roots of $\rho(R)$ satisfy $| R_j | \leq 1$ and all the roots with $| R_j | = 1$ have multiplicity that does not exceed 2. For our method with $m = 3$, we obtain the first characteristic polynomial given as

$$\rho(R) = -8R^3(1 + R), \tag{70}$$

which has principal root $| R_0 | = 1$ and spurious roots $R_j = 0, j = 1(1)3$. For our method with $m = 5$, we obtain the first characteristic polynomial given as

$$\rho(R) = 8R^5(1 + R), \tag{71}$$

which has principal root $| R_0 | = 1$ and spurious roots $R_j = 0, j = 1(1)5$. For our method with $m = 7$, we obtain the first characteristic polynomial given as

$$\rho(R) = -8R^7(1 + R), \tag{72}$$

which has principal root $| R_0 | = 1$ and spurious roots $R_j = 0, j = 1(1)7$. In general, the first characteristic polynomial of our class of hybrid block methods alternates between

$$\rho(R) = -R^m(R + 1), \tag{73}$$

and

$$\rho(R) = R^m(R + 1). \tag{74}$$

Following Jator [17], our class of hybrid block methods are zero stable, consistent and convergent.

3. Numerical Problems & Discussions

In this section, we have tested the performance of our methods on both scalar and system ODEs of the linear and nonlinear types. All results in this study were compared with the theoretical solutions, hence, the Absolute Errors (AE) and Maximum Absolute Errors (MAE) obtained. We compare our methods with the Generalized Linear Block Method (GLBM) of order 5 in Ref. [3], the Direct Two-Point Four-Step Variable Step (D2P4VS) methods of order 6 and 7 in Ref. [4], the Implicit Three-Point Block Method of order 9 (ITPBMO9) in Ref. [5], and the Boundary Value Methods (BVM5) of order 6 in Ref. [11] . All computations were implemented using Mathematica 13.

Table 1. Comparison of the MAE of $m = 3$ and BVM5 (Problem 1)

N	$m = 3$	BVM5
10	2.91×10^{-9}	5.07×10^{-6}
20	4.65×10^{-11}	1.16×10^{-7}
40	7.39×10^{-13}	2.07×10^{-9}
80	1.84×10^{-14}	3.35×10^{-11}
160	1.65×10^{-13}	5.92×10^{-13}
320	2.52×10^{-12}	8.28×10^{-14}

3.1. Problem 1

Consider the non-linear third order IVP

$$y'''(x) + 2e^{-3y(x)} = 4(1 + x)^{-3}, \tag{75}$$

which was solved in Ref. [11], with the initial conditions, interval of integration and exact solution given as

$$y(0) = 0,\ y'(0) = 1, y''(0) = -1,$$

$$x \in [0, 1],$$

$$Exact : y(x) = \ln(1 + x).$$

In this example, a comparison is made between our method $m = 3$ and the BVM5 in Ref. [11], both orders 5 and 6 respectively. In Table 1, we compare the MAE of our method $m = 3$ with the BVM5 at different values of N. It is evident from the results that our method $m = 3$ of order 5 performs favourably well against the BVM5 of order 6. Our method $m = 3$ outperforms the BVM5 for all the values of N provided except at 320. Table 2 display the MAE for our method $m = 3, 5$ and 7. The results were computed using different values for N.

3.2. Problem 2

Consider the singularly perturbed third order BVP

$$-\epsilon y'''(x) + 4y'(x) = 1,\ x \in [0, 1], \tag{76}$$

which was solved in Refs. [6, 11], with the boundary conditions, domain of integration and exact solution given as

$$y(0) = y'(0) = y'(1) = 1,$$

$$Exact : y(x) = \frac{x}{4} - \frac{c_1\sqrt{\epsilon}}{2}\exp\left(\frac{-2x}{\sqrt{\epsilon}}\right) + \frac{c_2\sqrt{\epsilon}}{2}\exp\left(\frac{2x}{\sqrt{\epsilon}}\right) + c_3,$$

where

$$c_1 = \frac{3}{4} - \frac{3}{4\left(1 + \exp\left(\frac{2}{\sqrt{\epsilon}}\right)\right)},$$

Table 2. MAE methods $m = 3, 5$ and 7 with various N (Problem 1)

N	$m = 3$	$m = 5$	$m = 7$
6	5.93×10^{-8}	1.52×10^{-10}	1.69×10^{-6}
10	2.91×10^{-9}	2.76×10^{-12}	2.28×10^{-8}
12	9.84×10^{-10}	6.52×10^{-13}	4.45×10^{-9}
20	4.65×10^{-11}	7.97×10^{-15}	3.73×10^{-11}
24	1.56×10^{-11}	5.53×10^{-15}	6.49×10^{-12}
40	7.39×10^{-13}	5.76×10^{-15}	1.08×10^{-14}
48	2.32×10^{-13}	1.34×10^{-14}	1.71×10^{-14}
80	1.84×10^{-14}	2.41×10^{-14}	8.40×10^{-14}
96	1.76×10^{-14}	7.64×10^{-14}	1.84×10^{-13}
160	1.65×10^{-13}	1.36×10^{-13}	5.79×10^{-14}
192	5.48×10^{-14}	1.58×10^{-13}	1.60×10^{-13}
320	2.52×10^{-12}	3.19×10^{-12}	1.01×10^{-12}

$$c_2 = \frac{3}{4\left(1 + \exp\left(\frac{2}{\sqrt{\epsilon}}\right)\right)},$$

$$c_3 = \frac{8 + 8\exp\left(\frac{2}{\sqrt{\epsilon}}\right) - 3\sqrt{\epsilon} + 3\sqrt{\epsilon}\exp\left(\frac{2}{\sqrt{\epsilon}}\right)}{8\left(1 + \exp\left(\frac{2}{\sqrt{\epsilon}}\right)\right)}.$$

The singularly perturbed third-order BVP has been solved with a variety of values ϵ, and it is shown that the results are still acceptable as $\epsilon \to 0$. In the Tables 3 and 4, the MAE of our method $m = 3$ of order 5 is compared with the BVM5 of order 6. From the results presented, our method $m = 3$ outperforms the BVM5 for $\epsilon = 0.01$. Although, our methods perform better with a decreased number of steps, for example, our methods $m = 3,\ 5$ and 7 outperform those presented in Ref. [11] for smaller values of N, the BVM5 outperforms our methods at $N = 320$ and with $\epsilon = 0.1$.

For our methods, $m = 3, 5$ and 7, the numerical solutions were contrasted with the theoretical and the MAE presented in Table 5 and 6.

3.3. Problem 3

Consider the third order IVP

$$y''' + 2y'' - 9y' - 18y = -18x^2 - 18x + 22, \quad (77)$$

with the initial conditions, domain of integration and exact solution given as

Table 3. Comparison of the MAE $m = 3$ and BVM5 (Problem 2, $\epsilon = 0.01$)

N	$m = 3$	BVM5
10	3.7×10^{-6}	2.6×10^{-3}
20	8.4×10^{-8}	1.1×10^{-4}
40	1.5×10^{-9}	6.9×10^{-6}
80	2.3×10^{-11}	4.3×10^{-7}
160	3.7×10^{-13}	2.7×10^{-12}
320	1.3×10^{-14}	1.7×10^{-9}

Table 4. Comparison of the MAE $m = 3$ and BVM5 (Problem 2, $\epsilon = 0.1$)

N	$m = 3$	BVM5
10	1.8×10^{-8}	2.4×10^{-5}
20	2.9×10^{-10}	4.8×10^{-7}
40	4.6×10^{-12}	8.0×10^{-9}
80	7.7×10^{-14}	1.3×10^{-10}
160	1.4×10^{-14}	2.0×10^{-12}
320	1.8×10^{-13}	1.8×10^{-14}

$$y(0) = -2,\ y'(0) = -8, y''(0) = 12,\ x \in [0, b],$$

$$Exact : y(x) = -2e^{-3x} + e^{-2x} + x^2 - 1.$$

Tables 7 and 8 display the AE at the endpoints $x = 1$ and $x = 4$ of the integration domain. The results were computed using different values for N.

We compare our methods $m = 3$ of order 5, with the two-point four-step implicit block method (D2P4VS) of orders 6 and 7 presented in Ref. [4].

In the Tables 9 and 10, we provide the MAE within the integration domain $x \in [0, b]$. From the results displayed in Table 9, our methods outperform the D2P4VS in a lesser number of steps with $b = 1$. In the Table 10, our method $m = 3$ outperforms the D2P4VS.

3.4. Problem 4

Consider the BVP with mixed boundary conditions

$$y'''(x) - \frac{1}{\sqrt{1 + x}} + 2y(x) = f(x), \quad (78)$$

Table 5. MAE method $m = 3$, 5 and 7 with various N (Problem 2, $\epsilon = 0.01$)

N	$m = 3$	$m = 5$	$m = 7$
6	3.9×10^{-5}	8.01×10^{-7}	1.10×10^{-8}
10	3.7×10^{-6}	2.63×10^{-8}	1.27×10^{-10}
12	1.4×10^{-6}	7.01×10^{-9}	2.33×10^{-11}
20	8.4×10^{-8}	1.46×10^{-10}	1.75×10^{-13}
24	2.9×10^{-8}	3.54×10^{-11}	3.22×10^{-14}
40	1.5×10^{-9}	6.31×10^{-13}	8.88×10^{-16}
48	4.9×10^{-10}	1.51×10^{-13}	2.58×10^{-14}
80	2.3×10^{-11}	3.33×10^{-15}	4.00×10^{-15}
96	7.9×10^{-12}	7.55×10^{-15}	1.55×10^{-15}
160	3.7×10^{-13}	7.59×10^{-14}	6.88×10^{-15}
192	1.2×10^{-13}	9.99×10^{-15}	7.37×10^{-13}
320	1.3×10^{-14}	8.00×10^{-15}	5.45×10^{-12}

which was solved in Ref. [11], with the mixed boundary conditions, domain of integration and exact solution given as

$$y(0) = y^{'}(0) = 1, y(1) - y^{'}(1) = 0,\ \ x \in [0, 1],$$

$$Exact:\ \ y(x) = \frac{1}{2}x^3.$$

In this problem, $f(x)$ is deduced from the exact solution. All methods solve this problem accurately and are consequently exact to machine precision. Table 11 shows the computational comparison of the different methods for this problem.
In Table 12, we display the MAE within the integration domain of the problem for $m = 3$, 5 and 7. The results were computed using different values of N.

3.5. Problem 5

Consider the third order IVP

$$y^{'''} + e^x = 0, \tag{79}$$

which was solved in Ref. [3], with the initial conditions, domain of integration and exact solution given as

$$y(0) = 1,\ \ y^{'}(0) = -1, y^{''}(0) = 3,\ \ x \in [0, 1]\,,$$

$$Exact: y(x) = 2x^2 - e^x + 2.$$

Tables 13 and 14 show the AE at selected points in the integration domain for the problem solved with step size $h = \frac{1}{10}$ and compared with the theoretical solution. In the Table 13, we compare the AE of our methods $m = 3$ against the GLBM in Ref. [3]. It is instructive to note that both methods are of order 5. It is easy to see that our method $m = 3$ performs relatively well against the GLBM. From Table 13, both methods perform slightly better than the other at 5 selected points each within the domain of integration. The AE of our methods $m = 5$ and $m = 7$ is presented in Table 14.

Table 6. MAE methods $m = 3$, 5 and 7 with various N (problem 2, $\epsilon = 0.1$)

N	$m = 3$	$m = 5$	$m = 7$
6	3.5×10^{-7}	6.74×10^{-10}	8.93×10^{-13}
10	1.8×10^{-8}	1.25×10^{-11}	6.22×10^{-15}
12	6.1×10^{-9}	2.95×10^{-12}	3.11×10^{-15}
20	2.9×10^{-10}	4.51×10^{-14}	4.22×10^{-15}
24	9.9×10^{-11}	1.31×10^{-14}	4.44×10^{-15}
40	4.6×10^{-12}	1.58×10^{-14}	7.33×10^{-15}
48	1.5×10^{-12}	3.33×10^{-15}	7.33×10^{-15}
80	7.7×10^{-14}	2.89×10^{-14}	2.02×10^{-14}
96	4.8×10^{-14}	4.06×10^{-14}	1.26×10^{-12}
160	1.4×10^{-14}	2.91×10^{-14}	5.13×10^{-14}
192	1.1×10^{-13}	7.86×10^{-13}	4.57×10^{-14}
320	1.8×10^{-13}	2.25×10^{-12}	5.20×10^{-14}

Table 7. AE at the end of interval (Problem 3, $x = 1$)

N	$m = 3$	$m = 5$	$m = 7$
8	2.456667×10^{-7}	6.023624×10^{-11}	1.010961×10^{-14}
10	6.421349×10^{-8}	1.007798×10^{-11}	1.082454×10^{-15}
22	5.640075×10^{-10}	1.829226×10^{-14}	4.059099×10^{-19}

Table 8. AE at the end of interval (Problem 3, $x = 4$)

N	$m = 3$	$m = 5$	$m = 7$
59	1.901826×10^{-4}	1.406541×10^{-8}	7.038723×10^{-13}
99	8.512268×10^{-6}	2.236027×10^{-10}	3.974098×10^{-15}
120	2.683452×10^{-6}	4.797736×10^{-11}	5.803680×10^{-16}

Table 9. Comparison of MAE $m = 3$ and D2P4VS (problem 3, $x = 1$)

N	$m = 3$	N	D2P4VS
8	2.46×10^{-7}	41	9.33×10^{-7}
10	6.42×10^{-8}	54	7.82×10^{-8}
22	5.64×10^{-10}	64	8.16×10^{-10}

Table 10. Comparison of MAE $m = 3$ and D2P4VS (problem 3, $x = 4$)

N	$m = 3$	D2P4VS
41	1.41×10^{-8}	2.26×10^{-6}
54	6.42×10^{-8}	7.82×10^{-8}
64	5.64×10^{-10}	8.16×10^{-10}

Table 11. Comparison of the MAE methods $m = 3$ and BVM5 (Problem 4)

N	$m = 3$	BVM5
10	1.55×10^{-15}	1.38×10^{-15}
20	4.44×10^{-15}	4.44×10^{-16}
40	2.44×10^{-15}	2.87×10^{-14}
80	1.65×10^{-14}	6.86×10^{-14}
160	7.07×10^{-14}	2.45×10^{-14}
320	1.94×10^{-14}	5.23×10^{-14}

3.6. *Problem 6*

Consider the nonlinear third order IVP

$$y^{'''} - xy^{''} + x^2y^2 = x\sin x - \cos x + x^2\sin^2 x, \qquad (80)$$

which was solved in Ref. [3], with the initial conditions, domain of integration and exact solution given as

$$y(0) = 0,\ y^{'}(0) = 1, y^{''}(0) = 0,\ x \in [0, 1],$$

$$Exact : y(x) = \sin x.$$

In this example, a comparison is made between our method $m = 3$ and the GLBM in Ref. [3]. In Table 15, we compare the AE of our method $m = 3$ with the GLBM at selected points within the integration domain of the problem solved with step size $h = \frac{1}{10}$. It is evident from the results that our method $m = 3$ performs relatively well against the GLBM. Our scheme $m = 3$ slightly outperforms the GLBM at the first five selected points in the Table 15, while the GLBM marginally outperforms our method $m = 3$ at the points $x = 0.6, 0.7, 0.8, 0.9$ and 1.0. The AE of our methods $m = 5$ and $m = 7$ is presented in Table 16.

Table 12. MAE methods $m = 3$, 5 and 7 with various N (Problem 4)

N	$m = 3$	$m = 5$	$m = 7$
6	1.10×10^{-16}	1.33×10^{-15}	8.22×10^{-18}
10	1.55×10^{-15}	1.44×10^{-15}	1.57×10^{-16}
12	1.22×10^{-15}	1.77×10^{-15}	1.17×10^{-15}
20	4.44×10^{-15}	1.44×10^{-15}	4.00×10^{-15}
24	8.10×10^{-16}	8.33×10^{-16}	5.55×10^{-16}
40	2.44×10^{-15}	1.11×10^{-15}	3.33×10^{-15}
48	1.37×10^{-16}	4.11×10^{-15}	8.22×10^{-15}
80	1.65×10^{-14}	1.99×10^{-14}	9.21×10^{-15}
96	3.15×10^{-16}	1.37×10^{-14}	9.38×10^{-15}
160	7.07×10^{-14}	3.21×10^{-14}	6.36×10^{-14}
192	4.15×10^{-16}	9.03×10^{-14}	2.89×10^{-14}
320	1.94×10^{-14}	1.22×10^{-13}	2.10×10^{-13}

Table 13. Comparison of the AE methods $m = 3$ and GLBM (Problem 5)

x	$m = 3$	GLBM
0.1	1.617997×10^{-14}	1.620926×10^{-14}
0.2	6.611369×10^{-14}	6.605827×10^{-14}
0.3	1.528079×10^{-13}	1.527667×10^{-13}
0.4	2.795853×10^{-13}	2.795542×10^{-13}
0.5	4.501185×10^{-13}	4.501954×10^{-13}
0.6	6.684659×10^{-13}	6.683543×10^{-13}
0.7	9.391131×10^{-13}	9.392487×10^{-13}
0.8	1.267017×10^{-12}	1.266987×10^{-12}
0.9	1.657656×10^{-12}	1.657785×10^{-12}
1.0	2.117086×10^{-12}	2.117417×10^{-12}

3.7. *Problem 7*

Consider the linear system

Table 14. AE at selected points for method $m = 5$ and $m = 7$ (Problem 5)

x	$m = 5$	$m = 7$
0.1	2.667406×10^{-19}	3.083489×10^{-24}
0.2	1.091358×10^{-18}	1.262510×10^{-23}
0.3	2.525817×10^{-18}	2.924209×10^{-23}
0.4	4.627550×10^{-18}	5.361661×10^{-23}
0.5	7.460028×10^{-18}	8.650259×10^{-23}
0.6	1.109340×10^{-17}	1.287332×10^{-22}
0.7	1.560518×10^{-17}	1.812293×10^{-22}
0.8	2.108106×10^{-17}	2.450086×10^{-22}
0.9	2.761572×10^{-17}	3.211957×10^{-22}
1.0	3.531380×10^{-17}	4.110336×10^{-22}

Table 15. Comparison of the AE methods $m = 3$ and GLBM (Problem 6)

x	$m = 3$	GLBM
0.1	6.6369×10^{-16}	6.661338×10^{-16}
0.2	3.9126×10^{-15}	3.913536×10^{-15}
0.3	1.23524×10^{-14}	1.243450×10^{-14}
0.4	2.86685×10^{-14}	2.886580×10^{-14}
0.5	5.59501×10^{-14}	5.601075×10^{-14}
0.6	9.70193×10^{-14}	9.692247×10^{-14}
0.7	1.55024×10^{-13}	1.546541×10^{-13}
0.8	2.33561×10^{-13}	2.325917×10^{-13}
0.9	3.35928×10^{-13}	3.346212×10^{-13}
1.0	4.65962×10^{-13}	4.644063×10^{-12}

$$\begin{cases} y_1'''(x) = \frac{1}{68}(817y_1 + 1393y_2 + 448y_3) \\ y_2'''(x) = -\frac{1}{68}(1141y_1 + 2837y_2 + 896y_3) \\ y_3'''(x) = \frac{1}{136}(3059y_1 + 4319y_2 + 1592y_3) \end{cases}, \quad (81)$$

which was solved in Ref. [11], with the initial conditions

Table 16. AE at selected points for method $m = 5$ and $m = 7$ (Problem 6)

x	$m = 5$	$m = 7$
0.1	2.44337×10^{-18}	2.44337×10^{-18}
0.2	9.39821×10^{-19}	9.39821×10^{-19}
0.3	2.65894×10^{-17}	8.44329×10^{-17}
0.4	3.0799×10^{-17}	1.35734×10^{-16}
0.5	1.05918×10^{-16}	3.27963×10^{-16}
0.6	9.68481×10^{-17}	4.29915×10^{-16}
0.7	1.47862×10^{-16}	4.80929×10^{-16}
0.8	2.97549×10^{-17}	4.14334×10^{-16}
0.9	2.47817×10^{-16}	2.57721×10^{-17}
1.0	6.64357×10^{-17}	6.64357×10^{-16}

$$\begin{cases} y_1(0) = 2,\ y_1'(0) = -12,\ y_1''(0) = 20 \\ y_2(0) = -2,\ y_1'(0) = 28,\ y_1''(0) = -52 \\ y_3(0) = -12,\ y_3'(0) = -33,\ y_1''(0) = 5 \end{cases},$$

and exact solutions

$$\begin{cases} y_1(x) = e^x - 2e^{2x} + 3e^{-3x} \\ y_2(x) = 3e^x + 2e^{2x} - 7e^{-3x} \\ y_3(x) = -11e^x - 5e^{2x} + 4e^{-3x} \end{cases}.$$

We compare our method $m = 3$ of order 5 with the BVM5 in Ref. [11] and ITPBO9 in Ref. [5] of orders 6 and 9 respectively. From the results presented in Table 17, our method $m = 3$ performs favourably well against the BVM5 and ITPBO9.
Table 18 shows the maximum error of our methods $m = 3, 5, 7$. This problem is solved with various values N.

3.8. Problem 8

Consider the non-linear system

$$\begin{cases} y_1'''(x) = \frac{1}{2}exp(4x)\ y_3(x)\ y_2'(x) \\ y_2'''(x) = \frac{8}{3}exp(2x)\ y_1(x)\ y_3'(x) \\ y_3'''(x) = 27\ y_2(x)\ y_1'(x) \end{cases}, \quad (82)$$

which was solved in Ref. [5], with the initial conditions

Table 17. MAE Comparison methods $m = 3$, BVM5 and ITPBO9 (Problem 7)

N	$m = 3$	BVM5	ITPBO9
10	1.51×10^{-7}	5.45×10^{-3}	1.42×10^{-12}
20	2.36×10^{-9}	9.59×10^{-5}	9.52×10^{-13}
40	3.68×10^{-11}	1.80×10^{-6}	5.40×10^{-13}
80	5.76×10^{-13}	2.98×10^{-8}	3.13×10^{-13}
160	8.99×10^{-15}	5.29×10^{-10}	1.22×10^{-13}
320	1.41×10^{-16}	1.07×10^{-11}	*not given*

Table 18. MAE methods $m = 5$ and $m = 7$ with various N (Problem 7)

N	$m = 3$	$m = 5$	$m = 7$
6	3.21×10^{-6}	1.16×10^{-9}	3.13×10^{-13}
10	1.51×10^{-7}	1.96×10^{-11}	1.90×10^{-15}
12	5.05×10^{-8}	4.57×10^{-12}	3.08×10^{-16}
20	2.36×10^{-9}	7.69×10^{-14}	1.86×10^{-18}
24	7.89×10^{-10}	1.79×10^{-14}	3.00×10^{-19}
40	3.68×10^{-11}	3.00×10^{-16}	1.82×10^{-21}
48	1.23×10^{-11}	6.99×10^{-17}	2.94×10^{-22}
80	5.76×10^{-13}	1.17×10^{-18}	1.78×10^{-24}
96	1.93×10^{-13}	2.73×10^{-19}	2.87×10^{-25}
160	8.99×10^{-15}	4.58×10^{-21}	1.74×10^{-27}
192	3.01×10^{-15}	1.07×10^{-21}	2.80×10^{-28}
320	1.41×10^{-16}	1.79×10^{-23}	1.70×10^{-30}

$$\begin{cases} y_1(0) = 1,\ y_1'(0) = -1,\ y_1''(0) = 1 \\ y_2(0) = 1,\ y_1'(0) = -1,\ y_1''(0) = 1 \\ y_3(0) = 1,\ y_3'(0) = -3,\ y_1''(0) = 9 \end{cases},$$

and exact solutions

$$\begin{cases} y_1(x) = e^{x} \\ y_2(x) = e^{-2x} \\ y_3(x) = e^{-3x} \end{cases}.$$

Table 19. AE at grid points for method $m = 3$ (Problem 8)

x	y_1	y_2	y_3
0.1	1.51546×10^{-14}	3.83364×10^{-12}	8.95258×10^{-11}
0.2	3.10129×10^{-14}	1.73926×10^{-11}	3.39825×10^{-10}
0.3	1.7698×10^{-13}	4.74889×10^{-11}	7.1774×10^{-10}
0.4	1.31187×10^{-12}	1.06305×10^{-10}	1.19818×10^{-9}
0.5	4.92628×10^{-12}	2.11419×10^{-10}	1.76148×10^{-9}
0.6	1.3897×10^{-11}	3.85917×10^{-10}	2.39116×10^{-9}
0.7	3.30345×10^{-11}	6.58552×10^{-10}	3.0719×10^{-9}
0.8	6.98675×10^{-11}	1.06386×10^{-9}	3.78767×10^{-9}
0.9	1.35634×10^{-10}	1.64216×10^{-9}	4.51995×10^{-9}
1.0	2.46510×10^{-10}	2.43917×10^{-9}	5.2460×10^{-9}

Table 20. AE at grid points for method $m = 5$ (Problem 8)

x	y_1	y_2	y_3
0.1	5.55827×10^{-17}	4.81773×10^{-16}	1.29796×10^{-14}
0.2	2.95383×10^{-16}	1.52751×10^{-15}	4.95592×10^{-14}
0.3	8.98209×10^{-16}	4.7062×10^{-15}	1.04197×10^{-13}
0.4	1.91418×10^{-15}	1.1297×10^{-14}	1.72933×10^{-13}
0.5	2.99826×10^{-15}	2.41904×10^{-14}	2.53058×10^{-13}
0.6	5.17479×10^{-15}	4.68669×10^{-14}	3.41539×10^{-13}
0.7	8.61645×10^{-15}	8.32541×10^{-14}	4.3678×10^{-13}
0.8	1.46822×10^{-14}	1.38331×10^{-13}	5.36118×10^{-13}
0.9	2.44228×10^{-14}	2.1786×10^{-13}	6.38609×10^{-13}
1.0	4.11462×10^{-14}	3.27866×10^{-13}	7.38401×10^{-13}

Tables 19, 20 and 21 shows the AE at selected points in the integration domain for the problem solved with step size $h = \frac{1}{10}$ and compared with the theoretical solution. The numerical results for this problem are presented in Tables 19, 20 and 21.

Table 21. AE at grid points for method $m = 7$ (Problem 8)

x	y_1	y_2	y_3
0.1	5.55827×10^{-17}	1.48706×10^{-16}	1.21053×10^{-16}
0.2	2.95383×10^{-16}	3.06269×10^{-16}	1.54285×10^{-16}
0.3	7.87186×10^{-16}	8.20419×10^{-16}	3.35219×10^{-16}
0.4	1.35907×10^{-15}	8.6094×10^{-16}	6.26151×10^{-16}
0.5	2.44315×10^{-15}	1.26433×10^{-15}	6.49266×10^{-16}
0.6	3.39843×10^{-15}	1.79189×10^{-15}	3.95042×10^{-16}
0.7	4.67516×10^{-15}	2.37434×10^{-15}	3.1434×10^{-16}
0.8	5.63386×10^{-15}	3.02269×10^{-15}	1.57661×10^{-15}
0.9	6.93674×10^{-15}	3.58693×10^{-15}	3.53019×10^{-15}
1.0	7.89501×10^{-15}	3.95862×10^{-15}	6.27424×10^{-15}

4. Conclusion

We proposed a class of single-step hybrid block methods for solving third-order ODEs without first converting to an analogous first-order system. The convergence of the new methods was established, and the schemes are implemented without the use of starting values or predictors, avoiding the necessity for complex subroutines. To illustrate the effectiveness of the new methods, eight test problems of varied degrees of difficulty are considered. From the results, it is shown that the improvement in accuracy increases as the number of intra-step points m is increased. Tables 1-21 discuss the numerical outcomes in further depth. Our future research will focus on developing optimized single-step hybrid block schemes for solving third-order ODEs directly.

Acknowledgment

We thank the referees for the positive enlightening comments and suggestions, which have greatly helped us in making improvements to this paper. This research is supported by the Department of Science and Innovation - National Research Foundation Centre of Excellence in Mathematical and Statistical Sciences (DSI-NRF CoE-MaSS), South Africa.

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