Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 5 (2023) 1514

Journal of the Nigerian Society of Physical Sciences

The Efficiency of the K-L Estimator for the Seemingly Unrelated Regression Model: Simulation and Application

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Abstract

This paper considers the Ridge Feasible Generalized Least Squares Estimator (RFGLSE), Ridge Seemingly Unrelated Regression R_{SUR} and proposes the Kibria-Lukman KL_{SUR} estimator for the parameters of the Seemingly Unrelated Regression (SUR) model when the regressors of the models are collinear. A simulation study was conducted to compare the performance of the three different types of estimators for the SUR model. Different correlation levels $(0.0, 0.1, 0.2, \cdots, 0.9)$ among the independent variables, sample sizes replicated 10000 times and contemporaneous error correlation $(0.0, 0.1, 0.2, \cdots, 0.9)$ among the equations were assumed for the simulation study. The efficiency of the three (RFGLSE, R_{SUR}, and KL_{SUR} estimators for SUR, when the predictors are correlated, was investigated using the Trace Mean Square Error (TMSE). The results showed that the KL_{SUR} estimator outperformed the other estimators except for a few cases when the sample size is small.

DOI:10.46481/jnsps.2023.1514

Keywords: $K - L_{SUR}$; Multicollinearity; Ridge feasible generalized least squares estimator; Seemingly Unrelated Regression; Trace Mean Square Error

Article History :
Received: 22 April 2023
Received in revised form: 21 May 2023
Accepted for publication: 24 May 2023
Published: 14 June 2023

Communicated by: B. J. Falaye

1. Introduction

One of the most ingenious and groundbreaking research in econometrics is combining different single equations into a system of equations to improve the efficiency of the parameter estimation [1], [2]. The Seemingly unrelated regression (SUR) model with M equations and T observations is given as,

$$Y = X\beta + \varepsilon \tag{1}$$

$$\underset{mn\times 1}{\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}} = \underset{mn\times \sum k_i}{\begin{bmatrix} X_1 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & X_m \end{bmatrix}} \underset{\sum k_i \times 1}{\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}} \underset{mn\times 1}{\begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_m \end{bmatrix}} \tag{2}$$

where y_i is an $nm \times 1$ vector of observations on the i^{th} response variable, X_i is a fixed $mn \times \sum k_i$ matrix of explanatory variables, β_i is a $\sum k_i \times 1$ vector of unknown regression parameters, ε_i is an $nm \times 1$ vector of disturbances such that $cov(\varepsilon) = E[\varepsilon' \varepsilon] \otimes I_n$, $E(\varepsilon) = 0$.

The Ordinary Least Squares (OLS) estimator is widely used to estimate the unknown regression parameters one equation

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at a time since each equation is a classical regression. The SUR estimator simultaneously captures different regression equations. However, the efficiency gain in SUR is premised on the high level of contemporaneous correlation between or among each classical regression equation. Notable works on the efficiency gained in SUR which takes cognizance of the contemporaneous correlation of error terms in the joint equations include [3], [4], [5], [6], [7], [8], [9], [10], [11] among others. The Generalized Least Squares Estimator (GLSE) is used to estimate the variance-covariance matrix of the disturbances in SUR model.

It is a statistical fallacy to assume that the relationship between or among explanatory variables plays an insignificant effect on the error structure of the model. The severity of the correlation levels among the predictors can affect the efficiency and sensitivity of the estimators [12], [13]. Hence, the variance of the estimator is inflated, unreliable inference and the confidence interval due to multicollinearity is wider which may increase the probability of a type-II error in hypothesis testing of unknown parameters [14]. Numerous research on single equation models when the problem of multicollinearity is inherent are available in literature. Notable works include [15], [16], [17], [18], [19], [20], [21], [22] among others. Studies on multicollinearity on systems of equations of regression models are still lacking or scarce in literature. However, a notable exception is [23], [24].

Recently, shrinkage estimators such as ridge regression estimators have gained attraction among researchers such that quite a number of exciting estimators emerged [17], [25], [26]. [19] proposed the K-L estimator to tackle the correlated regressors problem for the classical linear regression model, which outperformed both the Generalized Least Squares Estimator (GLSE) and Ridge Feasible Generalized Least Squares Estimator

(RFGLSE). The objective of this paper is to develop an estimator which is suitable for the joint modelling of the K-L estimator, Kibria-Lukman Seemingly Unrelated Regression $K-L_{SUR}$ estimator when the predictors are correlated as well as compare the newly developed estimator with the existing Ridge Seemingly Unrelated Regression estimator (R_{SUR}) and Ridge Feasible Generalized Least Squares Estimator (RFGLSE).

The organization of the paper is as follows: The estimators and their Trace Mean Square Error (TMSE) expressions are given in Section 2. A simulation study is presented in Section 3. To illustrate the findings of the paper, real-life data are analysed in Section 4. The paper ends with some concluding remarks in Section 5.

2. Statistical Methodology

The GLSE is given as

$$\hat{\beta}_{GLSE} = \left(X'\left(\Sigma^{-1}\otimes I\right)X\right)^{-1}X'\left(\Sigma^{-1}\otimes I\right)y \quad (3)$$

The ridge parameter estimator is an important tool when explanatory variables are correlated in classical linear regression analysis. The ridge estimation technique was pioneered by [15] and extended to the SUR model by [25], [26], [27]. The ridge estimator for GLSE is given as:

$$\hat{\beta}_{RGLSE} = \left[X'\left(\Sigma^{-1}\otimes I\right)X+G\right]^{-1}X'\left(\Sigma^{-1}\otimes I\right)y$$
$$=\left[X'\psi^{-1}X+G\right]^{-1}X'\psi^{-1}y \text{ where } \psi=\Sigma^{-1}I_T, G=kI_{\Sigma k_i}$$

$$\hat{\beta}_{RGLSE} = \left[I_{\Sigma K_i}+\left(X'\psi^{-1}X\right)^{-1}G\right]^{-1}\hat{\beta}_{GLSE} \quad (4)$$

where G is a $k\times k$ matrix of non-negative elements characterizing the estimator. To circumvent the problem that σ is unknown, a Ridge Feasible Generalized Least Squares estimator (RFGLSE) is given as:

$$\hat{\beta}_{RFGLSE} = \left[X'\left(S^{-1}\otimes I\right)X+G\right]^{-1}X'\left(S^{-1}\otimes I\right)y$$

$$\hat{\beta}_{RFGLSE} = \left[I_{\Sigma k_i}+\left(X'\tilde{\psi}^{-1}X\right)^{-1}G\right]^{-1}\hat{\beta}_{FGLSE} \quad (5)$$

where $\hat{\beta}_{FGLSE} = \left(X'\left(S^{-1}\otimes I\right)X\right)^{-1}X'\left(S^{-1}\otimes I\right)y$ and $\tilde{\psi}^{-1} = S^{-1}\otimes I$

From (1), given Λ as the diagonal matrix of the eigenvalues and ψ a matrix whose columns are eigenvectors of $X^{*'}X^{*}$ of the systems of equations, SUR. The canonical version of (1) is defined as

$$Y^{*} = Z^{*}\alpha^{*}+e^{*} \quad (6)$$

where $Z^{*}=X^{*}\psi, \alpha^{*}=\psi'\beta$ and $Z^{*'}Z^{*}=\left(\psi'X^{*'}X^{*}\psi\right)=\Lambda$

The GLSE of SUR for α^{*} is;

$$\hat{\alpha}^{*}_{GLS} = \left(Z^{*'}Z^{*}\right)^{-1}Z^{*'}Y^{*} \quad (7)$$

$$bias\left(\hat{\alpha}^{*}_{GLS}\right) = E\left(\hat{\alpha}^{*}_{GLS}-\alpha^{*}\right) \quad (8)$$

$$Var\left(\hat{\alpha}^{*}_{GLS}\right) = E\left(\hat{\alpha}^{*}_{GLS}-\alpha^{*}\right)E\left(\hat{\alpha}^{*}_{GLS}-\alpha^{*}\right)'$$
$$=\sigma^{2}\left(\psi'X^{*'}X^{*}\psi\right)^{-1} \quad (9)$$
$$=\sigma^{2}\Lambda^{-1}$$

The ridge regression estimator is;

$$\hat{\alpha}_{R_{SUR}} = \left(Z^{*'}Z^{*}+\psi^{*}R\psi^{*'}\right)^{-1}Z^{*'}Y^{*} \quad (10)$$

$$bias\left(\hat{\alpha}^{*}_{R_{SUR}}\right) = E\left(\hat{\alpha}^{*}_{R_{SUR}}-\alpha^{*}\right) \quad (11)$$

using(4)and(7)

$$Var\left(\hat{\alpha}^{*}_{R_{SUR}}\right) = \left[X'\left(\Sigma^{-1}\otimes I\right)X+kI_{p}\right]^{-1}X'\left(\Sigma^{-1}\otimes I\right)X\left[X'\left(\Sigma^{-1}\otimes I\right)X+kI\right.$$
$$=\left(\Lambda+kI_{p}\right)^{-1}\Lambda\left(\Lambda+kI_{p}\right)^{-1}$$

where $\Lambda = X'(\Sigma^{-1}\otimes I)X$

$$TMSE\left(\hat{\alpha}^{*}_{R_{SUR}}\right) = \sigma^{2}\left(I_{p}+k\Lambda^{-1}\right)^{-1}\Lambda^{-1}\left(I_{p}+k\Lambda^{-1}\right)^{-1}$$
$$+\left(-k\left(\Lambda+kI_{p}\right)^{-1}\right)\alpha^{*}\alpha^{*}\left(-k\left(\Lambda+kI_{k}\right)^{-1}\right)' \quad (13)$$

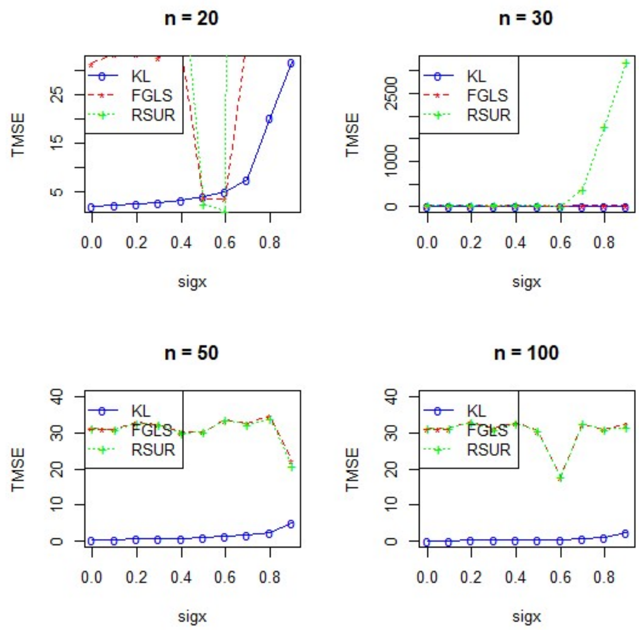

Figure 1. Estimated TMSE when $\rho_{\varepsilon_M} = 0.1$

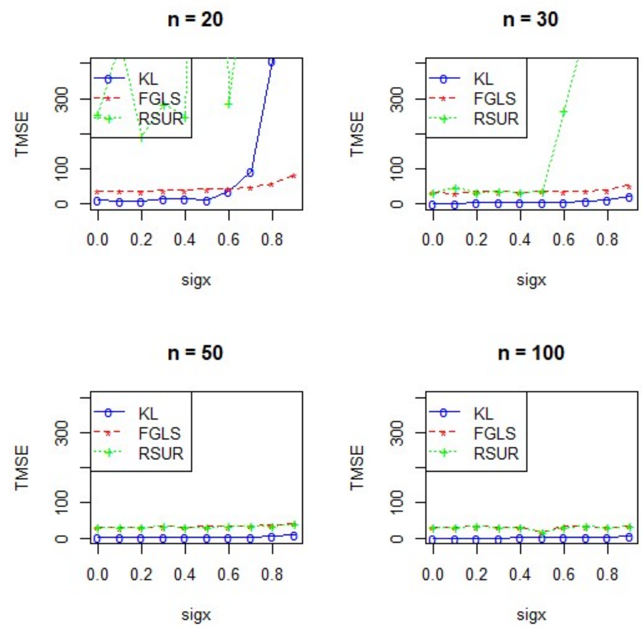

Figure 2. Estimated TMSE when $\rho_{\varepsilon_M} = 0.4$

Following [19], we define the K-LSUR estimator as follows:

$$\hat{\alpha}^*_{KL_{SUR}} = \left[\left(I_p + k\Lambda^{-1}\right)^{-1}\left(I_p - k\Lambda^{-1}\right)\right]\hat{\alpha}^* \tag{14}$$

$$\begin{aligned} E\left(\hat{\alpha}^*_{KL_{SUR}}\right) &= \left(I_p + k\Lambda^{-1}\right)^{-1}\left(I_p - k\Lambda^{-1}\right)E\left(\hat{\alpha}^*\right) \\ &= \left(I_p + k\Lambda^{-1}\right)^{-1}\left(I_p - k\Lambda^{-1}\right)\hat{\alpha}^* \end{aligned} \tag{15}$$

The K-L estimator is an unbiased estimator when k = 0

$$\begin{aligned} TMSE\left(\hat{\alpha}^*_{R_{SUR}}\right) &= \sigma^2\left(I_p + k\Lambda^{-1}\right)^{-1}\left(I_p - k\Lambda^{-1}\right)\Lambda^{-1}\left(I_p - k\Lambda^{-1}\right)' \\ &\left(\left(I_p + k\Lambda^{-1}\right)^{-1}\right)' + \left[\left(I_p + k\Lambda^{-1}\right)^{-1}\left(I_p - k\Lambda^{-1}\right) - I_p\right] \\ &\alpha^*\alpha^{*\prime}\left[\left(I_p + k\Lambda^{-1}\right)^{-1}\left(I_p - k\Lambda^{-1}\right) - I_p\right]' \end{aligned} \tag{16}$$

3. Numerical Analysis

A simulation study is considered to compare the performance of the estimators in this section. It consists of two parts (i) Simulation study (ii) Discussion of Results.

3.1. Simulation Study

The Monte Carlo experiment was performed by generating data according to the following algorithm.

1. Generate the explanatory variables from $MVN_3(0, \Sigma_x)$
2. Set the true values of β to $(1, 1, 1, 1)'$
3. The variance-covariance is given as

$$\Sigma = \begin{bmatrix} \sigma^2_{11} & \sigma^2_{12} & \sigma^2_{13} \\ \sigma^2_{21} & \sigma^2_{22} & \sigma^2_{13} \\ \sigma^2_{31} & \sigma^2_{32} & \sigma^2_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0.6 & 0.8 \\ & 1 & 0.7 \\ & & 1 \end{bmatrix}$$

such that $\begin{matrix} Y_1 : N\left(X_1\beta_1, \sigma^2_{11}\right) \\ Y_2 : N\left(X_2\beta_2, \sigma^2_{22}\right) \\ Y_3 : N\left(X_3\beta_3, \sigma^2_{33}\right) \end{matrix}$

4. Simulate the vector random error from $MVN_3(0, \Sigma_e)$
5. For a given X structure, transform the original model to the canonical form.
6. Compute the trace mean squared error of $\beta_{KL_{SUR}}, \beta_{RFGLSE}$, and $\beta_{R_{SUR}}$
7. Repeat the above step 10000 times.

Table 1. Description of Variables, Equations, Observations and Contemporaneous Correlations

Factors	Symbol	Design
No of equations	M	3
No of observations	T	20, 30, 50, 100
Correlation among the explanatory variables	ρ_x	0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9
Contemporaneous correlation between corresponding errors among the equaions	ρ_Σ	0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9

Table 2. Estimated TMSEs for the Different Methods when ρ_{ε_M} = 0.1, 0.2 and 0.3 at n = 20

$n = 20$									
	$\rho_{\varepsilon_m} = 0.1$			$\rho_{\varepsilon_m} = 0.2$			$\rho_{\varepsilon_m} = 0.3$		
$\rho_{x_i x_j}$	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}
0.0	1.9671	31.4793	1078.181	2.9993	34.9221	34.9869	4.3360	33.2312	210.5745
0.1	2.2637	33.5409	40.7652	3.3615	34.4638	39.1302	4.7297	33.9507	179.0863
0.2	2.5411	33.5427	38.5592	3.7682	34.7496	47.0992	5.2687	36.4858	71.1576
0.3	2.8784	32.8007	49.5184	4.2931	35.8924	42.8646	7.7218	37.2722	53717.79
0.4	3.4011	33.7350	53.2065	5.2779	35.8924	75.3289	64.0115	37.3879	113.8371
0.5	4.0056	3.6415	2.5955	6.0753	37.7050	216.6385	8.4646	39.2840	208.3292
0.6	4.9863	3.5757	1.0886	214.7618	38.7659	295.7213	14.9161	40.7934	741.0087
0.7	7.5773	36.6885	1137.117	11.5006	40.7506	462.7717	345.8575	43.9871	1399.8543
0.8	20.4227	42.1282	1948.9526	45.0750	45.1791	16809.71	55.2401	51.3216	3303.2105
0.9	31.9145	51.3059	441.2463	979.4674	59.5731	27157.19	822.7517	70.6499	4580.6889

Table 3. Estimated TMSEs for the Different Methods when ρ_{ε_M} = 0.1, 0.2 and 0.3 at n = 30

$n = 30$									
	$\rho_{\varepsilon_m} = 0.1$			$\rho_{\varepsilon_m} = 0.2$			$\rho_{\varepsilon_m} = 0.3$		
$\rho_{x_i x_j}$	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}
0.1	1.1817	32.6906	32.3140	1.4956	32.0123	31.5930	2.3545	33.3188	32.6339
0.2	1.1929	29.9211	29.4511	1.6637	32.7880	32.3673	2.1419	30.0010	32.0582
0.3	1.3553	22.4464	33.2031	1.8878	32.9291	32.5181	2.4342	34.7959	34.2291
0.4	1.5684	30.1094	29.5395	2.1817	31.1496	30.5211	2.8013	33.1929	32.2543
0.5	1.8712	20.1180	31.9485	2.6034	32.1032	31.3159	3.3439	33.6351	32.5898
0.6	2.3131	17.2247	16.8764	3.2240	32.7284	47.2943	4.1401	34.5030	36.8375
0.7	6.9799	31.2066	382.9524	4.3324	33.5406	47.9402	84.9530	35.5565	42.7001
0.8	4.6595	33.2122	1786.0465	6.7134	35.9082	166.2044	19.7855	36.5740	1303.1226
0.9	9.9985	37.0700	3186.9899	16.5210	42.0564	13172.89	528.8485	45.9672	3383.9887

Table 4. Estimated TMSEs for the Different Methods when ρ_{ε_M} = 0.1, 0.2 and 0.3 at n = 50

$n = 50$									
	$\rho_{\varepsilon_m} = 0.1$			$\rho_{\varepsilon_m} = 0.2$			$\rho_{\varepsilon_m} = 0.3$		
$\rho_{x_i x_j}$	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}
0.1	0.6101	30.9197	30.7837	0.8323	17.4569	17.3667	1.0599	29.7809	29.5481
0.2	0.6797	32.7346	32.6091	0.9276	32.2903	32.1370	1.1776	33.7307	33.5492
0.3	0.7701	32.2188	32.0525	1.0484	31.8266	31.6524	1.3189	18.5846	18.3629
0.4	0.8906	30.1703	29.9848	1.2107	31.6182	31.4187	1.5282	31.8069	31.5339
0.5	1.0613	30.2766	30.0612	1.4362	19.7875	19.4246	1.8204	34.1597	33.8706
0.6	1.3208	33.5028	33.3809	1.7953	33.5506	33.2716	2.2618	34.4465	34.0654
0.7	1.7521	32.6755	32.2909	2.3778	34.2552	33.8549	3.0007	34.2756	33.6832
0.8	2.6126	34.5632	33.9872	3.5386	35.1982	34.4567	4.4997	36.0475	34.9669
0.9	5.1513	22.4060	20.8302	7.1461	38.2472	36.3301	9.1181	40.2009	38.1831

The simulation results were presented in Tables 2 to 13 for $\rho_{\varepsilon_M} = (0.1, 0.2, 0.3)$,$\rho_{\varepsilon_M} = (0.4, 0.5, 0.6)$ and $\rho_{\varepsilon_M} = (0.7, 0.8, 0.9)$ respectively. In addition, some results are illustrated graphically in Figures 1 to 4 for ρ_{ε_M}= 0.1, 0.4, 0.7 and 0.9 respectively. We considered R software to conduct the simulation study [27], [28].

4. Discussion of Results

From Tables 2 to 13 and Figures 1 to 4, we can see that the proposed KL_{SUR} estimator uniformly dominate the RFGLS and R_{SUR} estimator except when the sample size is small (n = 20). Also, an increase in the $\rho_{x_i x_i}$ increases the estimated TMSE values of the estimators. Obviously, the TMSE values were significantly large at sample size n = 20 when ρ_{ε_M}= 0.7, 0.8 and 0.9 than other corresponding TMSE values with n = 30, 50, 100 for $\rho_{\varepsilon_M} = \{0.0, 0.1, 0.2, \cdots, 0.9\}$ and n = 20

Table 5. Estimated TMSEs for the Different Methods when ρ_{ε_M} = 0.1, 0.2 and 0.3 at n = 100

$n = 100$									
	$\rho_{\varepsilon_m} = 0.1$			$\rho_{\varepsilon_m} = 0.2$			$\rho_{\varepsilon_m} = 0.3$		
$\rho_{x_i x_j}$	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}
0.1	0.2958	31.1069	31.0652	0.4001	32.4019	32.3509	0.5034	32.0166	31.9597
0.2	0.3294	32.9002	32.8577	0.4439	31.5425	31.4946	0.5560	31.9488	31.8862
0.3	0.3730	31.0733	31.0145	0.5013	31.7451	31.6767	0.5259	30.9504	30.8669
0.4	0.4317	32.7167	32.6697	0.5793	32.3633	32.3027	0.68903	31.7897	31.7127
0.5	0.5141	30.5062	30.4347	0.6888	30.4318	30.3485	0.85712	30.8029	30.6942
0.6	0.6373	17.8189	17.7757	0.8544	32.3865	32.3051	1.06185	32.3417	32.2352
0.7	0.8467	32.4540	32.3723	1.1325	32.6137	32.4948	1.4077	33.2381	33.0755
0.8	1.2616	30.9976	30.7480	1.6856	31.4613	31.1143	2.0931	32.4275	31.9626
0.9	2.5207	32.4311	31.6849	3.3763	33.1247	32.1534	4.1961	33.8402	32.4072

Table 6. Estimated TMSEs for the Different Methods when ρ_{ε_M} = 0.4, 0.5 and 0.6 at n = 20

$n = 20$									
	$\rho_{\varepsilon_m} = 0.4$			$\rho_{\varepsilon_m} = 0.5$			$\rho_{\varepsilon_m} = 0.6$		
$\rho_{x_i x_j}$	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}
0.0	9.7560	36.2392	256.4695	716.2386	35.4935	406.7923	12.0147	40.1432	308.325
0.1	6.4118	36.2199	459.8962	10.6846	37.8609	658.1879	12.6002	38.8413	253.003
0.2	7.7878	36.0814	193.1198	9.9320	39.0815	1772.816	739.9870	42.2980	171.349
0.3	14.6262	39.5822	284.7212	11.7001	40.6640	251.5625	27.9862	43.0574	692.950
0.4	16.2656	39.2694	248.8817	68.8660	42.2592	860.8801	21.2453	45.0407	477.386
0.5	12.6930	42.5478	2629.349	20.1253	45.4033	22608.84	25.9884	47.7564	4292.16
0.6	37.4753	43.3222	286.5887	78.1266	47.4747	252.3527	76.0018	51.3518	581.977
0.7	91.8338	48.2507	769.2762	186.599	53.0107	1751.206	12291.26	57.8010	7708.11
0.8	408.0553	57.7729	6626.982	482.442	65.6280	94973.63	7459.204	72.5054	1255.63
0.9	27135.45	82.4109	3004.294	8687.57	96.2975	18763.57	98272.26	96.1662	2682.17

Table 7. Estimated TMSEs for the Different Methods when ρ_{ε_M} = 0.4, 0.5 and 0.6 at n = 30

$n = 30$									
	$\rho_{\varepsilon_m} = 0.4$			$\rho_{\varepsilon_m} = 0.5$			$\rho_{\varepsilon_m} = 0.6$		
$\rho_{x_i x_j}$	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}
0.0	2.1482	33.8490	33.4394	2.58122	33.6536	32.9963	3.0486	34.1725	33.4501
0.1	2.4351	29.8788	47.1569	2.83488	30.7532	29.9543	3.3416	31.2885	32.5218
0.2	2.2597	34.6346	33.5944	3.1033	31.2482	30.7619	3.5688	32.5834	31.1876
0.3	2.9817	35.3758	34.6432	3.5219	36.6019	35.7699	4.0414	36.7851	35.7306
0.4	3.4258	32.9304	32.1480	4.0364	33.0953	32.6963	4.6313	33.6982	48.0887
0.5	4.1175	37.4169	36.4355	4.7903	34.6699	35.0898	5.5344	34.1078	1134.971
0.6	5.0357	35.1816	266.3296	5.9218	35.9237	35.6257	7.0024	36.8212	38.8586
0.7	6.8262	37.1196	504.2674	8.0604	38.0785	683.5997	9.21488	39.7283	2209.564
0.8	12.1846	39.4336	26492.285	13.0759	41.5627	775.9910	16.8529	43.1572	330.2718
0.9	23.4235	52.9490	1302.4404	36.8351	56.4241	6609.2648	44.5576	57.7298	2350.237

for $\rho_{\varepsilon_M} = \{0.0, 0.1, 0.2, \cdots, 0.7\}$. The TMSE values become larger at n=20 as ρ_{ε_M} increases from 0.7 to 0.9. The tables and figures consistently showed that the proposed KL_{SUR} estimator performs better than the RFGLS and R_{SUR} estimators when there exist moderate to high correlation among the regressors. The plots of TMSE against various ρ_{ε_M} in Figures 1 to 4 showed that as the value of ρ increases, TMSE also increase, while as the sample size increases, the TMSE value decreases.

The preferred TMSE values were at n =100 for ρ_{ε_M} = 0.1 for KL_{SUR} estimator. KL_{SUR} produced the smallest TMSE values when $\rho_{x_i x_i}$ range from 0.1 to 0.9, such that the corresponding TMSE values for KL_{SUR} estimator increase as the $\rho_{x_i x_i}$ increase. However, it was noted that the ideal and smallest TMSE value for KL_{SUR} estimator occurred at ρ_{ε_M} = 0.1 and $\rho_{x_i x_i}$ = 0.1. Concisely, KL_{SUR} also produced small TMSE values at ρ_{ε_M} = 0.1 and $\rho_{x_i x_i}$ = 0.1, but not smaller in comparison to the one produced by KL_{SUR} estimator. Con-

Table 8. Estimated TMSEs for the Different Methods when ρ_{ε_M} = 0.4, 0.5 and 0.6 at n = 50

$n = 50$									
	$\rho_{\varepsilon_m} = 0.4$			$\rho_{\varepsilon_m} = 0.5$			$\rho_{\varepsilon_m} = 0.6$		
$\rho_{x_i x_j}$	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}
0.0	1.2231	31.8356	31.6180	1.4567	33.0326	32.7876	1.6850	30.8215	30.4527
0.1	1.2845	29.9504	29.6901	1.51162	30.2718	29.9303	1.7267	31.5870	31.1975
0.2	1.4195	32.2654	32.0162	1.6617	31.4414	31.1212	1.8969	31.7036	31.3102
0.3	1.5985	33.1219	32.8367	1.8689	33.8384	33.5117	2.1299	33.8781	33.4787
0.4	1.8389	32.0315	31.6715	2.1469	33.3783	32.9510	2.4424	32.6150	32.0562
0.5	2.1859	32.7642	32.3262	2.5474	32.0638	31.4425	2.9026	33.2641	32.5604
0.6	2.7172	34.1213	33.5881	3.1823	34.3878	33.7234	3.6293	35.1303	34.3261
0.7	3.6180	34.9418	34.1512	4.1749	35.4996	34.5134	4.7830	36.0520	34.80777
0.8	5.4192	36.9573	35.5262	6.3468	38.1684	36.3989	7.2398	38.6846	36.4323
0.9	11.0051	41.8592	41.3861	12.9492	43.3830	53.5260	15.1569	44.8944	47.0988

Table 9. Estimated TMSEs for the Different Methods when ρ_{ε_M} = 0.4, 0.5 and 0.6 at n = 100

$n = 100$									
	$\rho_{\varepsilon_m} = 0.4$			$\rho_{\varepsilon_m} = 0.5$			$\rho_{\varepsilon_m} = 0.6$		
$\rho_{x_i x_j}$	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}
0.0	0.5570	31.1110	31.0429	0.65712	31.2067	31.1265	0.7562	32.5784	32.4877
0.1	0.6064	31.5343	31.4591	0.7069	31.4484	31.3653	0.80516	31.0438	30.9381
0.2	0.6666	32.7558	32.6821	0.7744	32.4927	32.4042	0.88786	32.4789	32.3799
0.3	0.7477	31.0310	30.9287	0.8670	30.8659	30.7475	0.9813	30.4286	30.3006
0.4	0.8616	32.4530	32.3617	0.9960	32.7987	32.6845	1.1274	32.8786	32.7485
0.5	1.0175	17.1745	17.1026	1.18127	31.1289	30.9470	1.3339	31.6014	31.3899
0.6	1.2645	32.7726	32.6332	1.4586	32.5574	32.3776	1.6516	32.0510	31.7438
0.7	1.6723	33.5980	33.3833	1.9303	33.4664	33.1906	2.1856	33.9971	33.6736
0.8	2.4884	32.3812	31.7566	2.8805	32.7172	31.9339	3.2131	19.5499	18.9774
0.9	4.9937	34.7593	32.9295	5.7754	35.4817	33.2136	6.5181	36.4971	33.78511

Table 10. Estimated TMSEs for the Different Methods when ρ_{ε_M} = 0.7, 0.8 and 0.9 at n = 20

$n = 20$									
	$\rho_{\varepsilon_m} = 0.7$			$\rho_{\varepsilon_m} = 0.8$			$\rho_{\varepsilon_m} = 0.9$		
$\rho_{x_i x_j}$	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}
0.0	1.5970	4.0118	1.5448	27.0078	41.4233	161.6998	20.9443	44.5509	2102.610
0.1	13.3597	41.6366	7274.501	19.8636	43.4407	1096.485	45.4412	44.9649	366.5848
0.2	14.5438	42.4626	301.5057	21.2077	45.1824	402.2686	22.9364	46.0099	133.9393
0.3	15.8899	45.4626	125.4716	30.3843	47.2784	878.4280	109.9613	49.0023	644.4463
0.4	953.5130	47.2117	1997.0092	577.5404	49.5397	939.3170	1.1322	5.2701	2.0398
0.5	30.2281	51.3641	6113.3947	34.0842	53.2688	1525.4125	4787.2699	56.5041	2354.758
0.6	96.7198	53.3738	8236.2489	1440.019	57.9711	403.6016	481.5750	61.4689	10141.04
0.7	44014.384	62.7783	864.9972	9420.9595	67.5711	1731.6622	7184.6716	71.3247	34749.18
0.8	21669.976	79.6336	26152.861	47515.817	84.3908	36052.59	5962.1447	91.7812	717.1957
0.9	75800.340	127.1244	1039.1268	12340.321	137.575	1158.7834	50556.851	148.5709	9359.699

sequently, this implies that KL_{SUR} is the preferred estimator when considering SUR model when the regressors of the model are collinear among the three considered estimators. The strength of collinearity among the considered variables for $MVN_3(0, \Sigma_e)$, m = 3 with varied for sample sizes 20 to 100 for KL_{SUR} estimator relies on when n = 100 with ρ_{ε_M} = 0.1 and $\rho_{x_i x_i}$ = 0.1. This implies that the strength and type of dependency among the explanatory variables affect the performance of each of the three estimators of KL_{SUR}, RFGLS and R_{SUR}

As T increases, such that ρ_{ε_M} and $\rho_{x_i x_i}$ decreases (or relatively low), the TMSE values of KL_{SUR} decreases. This implies that there is a gain in the efficiency of the KL_{SUR} estimator.

Table 11. Estimated TMSEs for the Different Methods when ρ_{ε_M} = 0.7, 0.8 and 0.9 at n = 30

$n = 30$									
	$\rho_{\varepsilon_m} = 0.7$			$\rho_{\varepsilon_m} = 0.8$			$\rho_{\varepsilon_m} = 0.9$		
$\rho_{x_i x_j}$	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}
0.1	3.6831	32.4952	31.1870	4.0739	32.6618	30.9791	4.4489	32.9211	31.8359
0.2	4.0185	34.1073	32.5706	4.4305	33.9127	31.9412	4.8275	34.5264	32.4991
0.3	4.5332	37.4486	36.2957	5.0095	37.3701	35.8343	5.4564	38.6045	37.1221
0.4	5.1828	33.9950	40.1823	5.8834	34.7873	35.0340	6.2366	34.9937	36.4678
0.5	6.1682	36.0594	35.1923	6.8139	36.9741	98.5351	7.4014	37.2651	653.2791
0.6	7.6710	36.6365	59.8780	8.4420	38.3879	42.2726	9.1107	3.9449	1.0753
0.7	41.4184	40.5272	92.5242	11.4445	42.0848	4231.299	12.5689	42.5401	83.8831
0.8	28.8063	44.9286	137.4677	1023.891	46.0243	332.3192	23.1223	47.7619	11443.84
0.9	60.4966	62.9849	156.5118	701.8199	65.4741	234.6494	7.8273	6.8693	1.3251

Table 12. Estimated TMSEs for the Different Methods when ρ_{ε_M} = 0.7, 0.8 and 0.9 at n = 50

$n = 50$									
	$\rho_{\varepsilon_m} = 0.7$			$\rho_{\varepsilon_m} = 0.8$			$\rho_{\varepsilon_m} = 0.9$		
$\rho_{x_i x_j}$	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}
0.1	1.9390	30.7288	30.2195	2.1375	32.1671	31.6753	2.3270	32.5277	31.9061
0.2	2.1207	31.4594	30.9896	2.7655	32.6984	31.8318	2.5422	34.9202	34.2994
0.3	2.3833	34.6569	34.1233	2.6170	34.4649	33.9328	2.8471	35.1956	34.5160
0.4	2.7287	33.9790	33.2775	2.9985	33.0041	32.2088	3.2564	33.0878	32.1194
0.5	3.2429	33.4776	32.5970	3.5620	33.9245	32.9419	3.8698	34.2694	33.1379
0.6	4.0373	34.8706	33.8102	4.4264	34.1091	32.6144	4.8349	36.7864	35.4188
0.7	5.3456	36.8501	35.2405	5.8171	37.0795	35.3813	6.3794	38.0211	35.8226
0.8	8.0694	39.0196	36.2068	8.8507	39.8438	36.5279	9.5831	39.7160	35.8735
0.9	16.8842	46.8999	215.3601	29.3585	48.1159	97.7010	51.5108	49.3468	56.2438

Table 13. Estimated TMSEs for the Different Methods when ρ_{ε_M} = 0.7, 0.8 and 0.9 at n = 100

$n = 100$									
	$\rho_{\varepsilon_m} = 0.7$			$\rho_{\varepsilon_m} = 0.8$			$\rho_{\varepsilon_m} = 0.9$		
$\rho_{x_i x_j}$	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}	KL_{SUR}	$RFGLS$	R_{SUR}
0.1	0.8994	31.9839	31.8691	0.9882	31.6234	31.4959	1.0733	31.7436	31.5857
0.2	0.9785	33.1136	32.9996	1.0742	32.6911	32.5621	1.1653	32.2650	32.1150
0.3	1.0918	31.3878	31.2122	1.1969	31.2668	31.0811	1.2964	31.9378	31.7250
0.4	1.2536	32.6015	32.4556	1.3732	32.8539	32.6776	1.4871	32.2467	32.0410
0.5	1.4785	31.6940	31.4353	1.6218	31.1869	30.8941	1.7562	32.0679	31.7361
0.6	1.8292	31.9719	31.6237	2.0088	32.4433	32.0269	2.1661	32.3290	31.8782
0.7	2.4185	34.1733	33.7924	2.6589	34.4667	34.0165	2.8793	34.3942	33.8972
0.8	3.5981	33.4160	32.3067	3.9354	33.7854	32.4953	4.2864	34.0114	32.6416
0.9	7.2537	36.9970	33.7463	7.9236	37.7464	34.1527	8.6199	38.7249	34.4544

Table 14. Autocorrelation Test (Durbin-Watson Test)

Equation (model)	Test Statistic	p-Value
Brazil	2.3800	0.2846
India	1.9160	0.0658
Indonesia	2.4776	0.4547
South Africa	1.9669	0.1225
Turkey	1.5431	0.0032

Table 15. Heteroscedasticity Test (Breusch-Pagan Test)

Equation (model)	Test Statistic	p-Value
Brazil	12.4250	0.1332
India	11.0220	0.2005
Indonesia	6.0900	0.6372
South Africa	5.6808	0.6829
Turkey	3.8404	0.8712

5. Application

To further illustrate the results of the theoretical part of this paper, we consider the dataset and structural model by [12]. The study considered the foreign direct investment on

Table 16. Multicollinearity Test (Variance Inflation Factor)

Equation	Deflator	Current Account	Capital Growth	Final Consumption	Import of Goods	Personal Remittance	Total Reserves	Export of Goods
Brazil	56.241	23.344	2.144	6.652	11.143	20.487	46.461	24.510
India	23.798	13.750	1.741	2.497	101.771	2.438	23.185	66.900
Indonesia	118.769	12.219	3.552	4.588	42.317	3.214	72.550	124.779
South Africa	29.423	39.671	4.278	10.284	109.568	10.157	49.739	45.688
Turkey	21.052	206.371	3.548	5.611	258.126	4.557	15.662	224.422

Table 17. Specification Test (Regression Equation Specification Error Test)

Equation (model)	Test Statistic	p-Value
Brazil	0.0591	0.943
India	1.0576	0.3913
Indonesia	0.98417	0.4148
South Africa	0.52381	0.6113
Turkey	1.0114	0.4059

Table 18. Shrinkage Parameter Estimators for the Life Study

	KL_{SUR}	RFGLS	R_{SUR}
TMSE	35.00	40.52786	40.5278

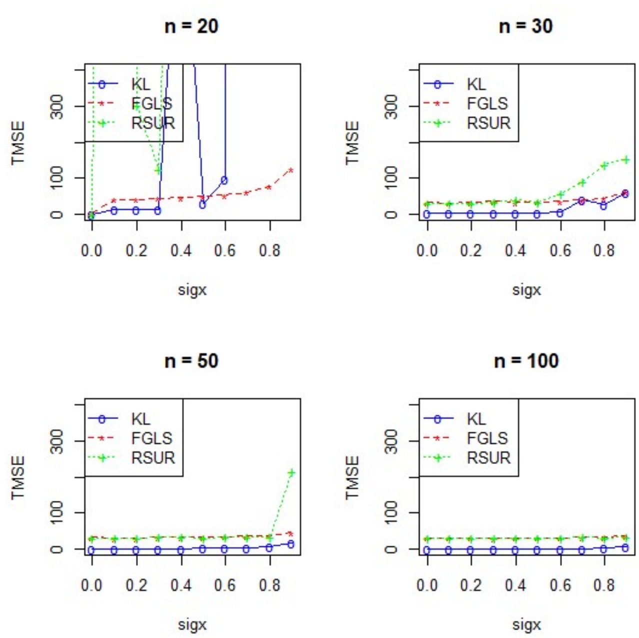

Figure 3. Estimated TMSE when $\rho_{\varepsilon_M} = 0.7$

some economic and financial variables over the period of twenty years between 2001 and 2019. The "Fragile Five" countries (cited in [12]) include Turkey (TUR), South Africa (ZAF), Brazil (BRA), India (IND), and Indonesia (IDN). Hence, M = 5 blocks, with measurements of T = 20 years per equation. The raw data were extracted online from the World Bank Indicators. The structural SUR model adopted is given as:

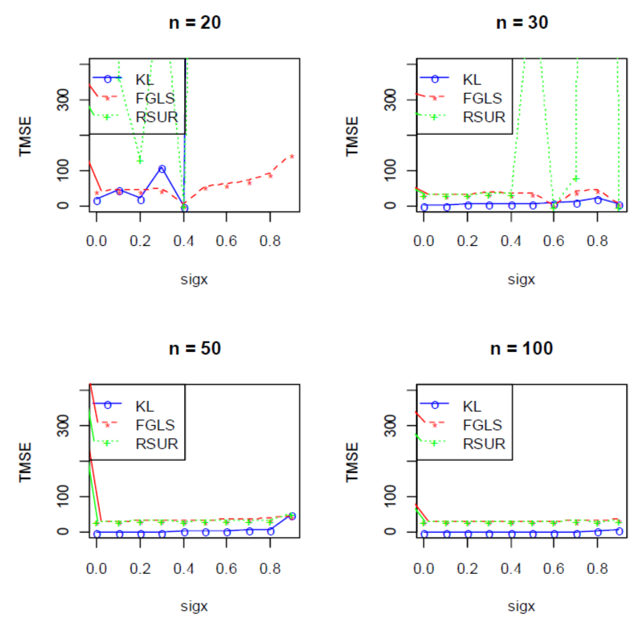

Figure 4. Estimated TMSE when $\rho_{\varepsilon_M} = 0.9$

$$FDI_{it} = f(GRO, DF, CAB, FC, IPS, PRR, TR, EGS)$$

$$\begin{aligned} Fdi_i = \beta_{0i} + \beta_{1i}Gro + \beta_{2i}Df + \beta_{3i}Cab + \beta_{4i}Fc + \beta_{5i}Ips \\ +\beta_{6i}Pr + \beta_{7i}Tr + \beta_{8i}Egs + e_i \end{aligned} \tag{17}$$

i denotes countries (i= TUR, ZAF, BRA, IND, IDN).

Fdi_i is the Foreign Direct Investment of the "fragile five" countries, Gro is the GDP per Capital growth, Df is the GDP deflator, Cab is the Current Account Balance, Fc is the General Government Final Consumption Expenditure, Ips is the Imports of Goods and Services, Prr is the Personal Remittances Received, Tr is the Total Reserves and Egs is the Exports of Goods and Services. The assumptions in each equation and joint model were put into consideration. Normality, homoscedasticity, multicollinearity and serial correlation of the error term were examined. The results are available in Tables 14 to 18.

The null hypothesis for the Durbin-Watson test is that errors are random and independent. A significant p-value in this test rejects the null hypothesis that the time series is not autocorrelated. Table 5 suggests a rejection of the null hypothesis for Turkey at the significance level, that is, p-value = 0.0032 <0.05. This implies that each equation satisfied the assumption of non-autocorrelation.

The null hypothesis for the Breusch-Pagan Test is that there is no homoscedasticity (that is, there is presence of heteroscedasticity). Since, the p-value for each of the ”fragile five” country is greater than 0.05. We did not reject the null hypothesis in each equation, so the assertion of homoscedasticity in each equation is satisfied.

The Variance Inflation Factor (VIF) makes it possible to measure how many times the variance of the regression coefficients will be for multicollinear data than for orthogonal/ canonical data. If VIF>10 this indicates multicollinearity. Concisely, deflator, current account, total reserves; import and export of goods posed to be problematic (that is, multicollinearity problem exist) as their VIF values were strictly greater than 10, while others also call for little concern. This implies that the problem of multicollinearity exists in the equations. Therefore, the KL_{SUR} estimator will be ideal to solve the problem.

The regression equation specification error test is meant for testing the exogeneity of explanatory variables. The null hypothesis for the test is that there is no correlation between the error term and the explanatory variables or that linearity exist in the functional form of the regression model, that is, $E[\varepsilon_i \mid X_i] = 0$. Since, the p-value for each of the ”Fragile Five” country is greater than 0.05 in table 14, this suggests that there is no correlation between the error term and the explanatory variables for all the countries (Turkey, Brazil, India, Indonesia, and South Africa).

The shrinkage parameter estimator is designed for model reduction for each of the M equations in order to absolve only significant explanatory variables in line with their number of observations. The shrinkage estimator makes it possible to standardize each reduced equation. From Table 18, the shrinkage parameters of the KL_{SUR} estimator via the TMSE produced the smallest TMSE value of 35.00 compared to RFGLS and R_{SUR} that gave higher TMSE of 40.52786 and 40.5278 respectively. This makes the KL_{SUR} estimator to possess a higher efficiency gain than the two other estimators. Summarily, $KL_{SUR} = 35.00$ indicates that the estimator outperforms RFGLS and R_{SUR}.

6. Conclusion

The seemingly unrelated regression model is an exciting and celebrated model when the error structure of joint models are correlated. We considered the ridge feasible generalized least squares estimator, ridge seemingly unrelated regression and the Kibria-Lukman KL_{SUR} estimator for estimating the parameters of the seemingly unrelated regression model when the regressors of the model are collinear. Findings from this comparative study revealed that KL_{SUR} estimator is a better alternative when compared with ridge and ridge feasible generalized least squares, which are often used to tackle the problem of multicollinearity in single and joint equations respectively. Simulation and real-life data were used for assessment. Note that we have considered one of many possible estimators of the ridge parameter k. The conclusion of the paper may change if we consider different values of k and such possibility is under the current investigation.

Acknowledgements

This paper was partially completed while the first author was visiting Prof. B. M. Golam Kibria, Florida international University in the month of April-May, 2021.

Availability of Data and Codes: Available on request.

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