Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 5 (2023) 1749

Journal of the Nigerian Society of Physical Sciences

Analysis of a fractional order climate model due to excessive emission and accumulation of carbon dioxide in the atmosphere

Peter Urane Achimugwu[a,*], Mathew Ngugi Kinyanjui[b], David Mumo Malonza[c]

[a]Pan African University Institute for Basic Sciences, Technology and Innovation, Nairobi, Kenya
[b]Department of Pure and Applied Mathematics, Jomo Kenyatta University of Agriculture and Technology, Juja, Kenya
[c]Department of Mathematics and Actuarial Science, South Eastern Kenya University, Kitui, Kenya

Abstract

The devastating consequences of climate change on our planet cannot be taken lightly. Greenhouse gas emissions due to the various activities of the increasing human population are solely responsible for this change. The chief and most significant of these gases is carbon dioxide. A fractional-order model of five compartments is considered. The uniqueness and existence, the positivity, and the boundedness of the model solution are established. The equilibrium points of the model are given. By formulating different Lyapunov functions, the global stability of the four equilibrium points was determined. The numerical simulation of the model was done using the Predict-Evaluate-Correct-Evaluate method of Adam-Bashforth-Moulton by considering four different orders of 0.7, 0.8, 0.9, and 1.0. According to the results, excessive concentrations of carbon dioxide in the atmosphere can be reduced by the joint employment of mitigation measures.

10.46481/jnsps.2023.1749

Keywords: Climate change, Fractional order, Carbon dioxide, Excessive emission

Article History :
Received: 28 August 2023
Received in revised form: 11 October 2023
Accepted for publication: 13 October 2023
Published: 18 October 2023

Communicated by: B. J. Falaye

1. Introduction

The issue of climate change has become a global problem beside other global challenges such as terrorism, environmental degradation, hunger, poverty, and many others which are highlighted and enshrined in the Sustainable Development Goals (SDGs) projected to be achieved in 2030. The current global increasing trends in carbon dioxide emission and accumulation can be attributed to factors such as per capita GDP, urbanization level, technology level, trade openness, energy consumption intensity, population growth, energy consumption structure and industrial structure [1, 2]. About 50% of the global total carbon emissions by volume in countries in the "Belt and Road Initiative (BRI)" are majorly from transportation industry, which is about 23.96% of the total emissions from all industries in the BRI countries [2]. Global climate change is one of the burning issues posing threats to human existence and the ecosystem primarily caused by the emission of carbon dioxide into the atmosphere from energy generating and transport industries [3]. There is a strong correlation between increase in global temperature and the accumulation of carbon dioxide in the atmosphere, which is contributing greatly to climate change [4].

*Corresponding author: Tel.: +2348024956769 / +254748859647;
Email address: peter.achimugwu@students.jkuat.ac.ke (Peter Urane Achimugwu)

The amount of carbon dioxide in the atmosphere is increased by the combustion of fossil fuels [4].

Carbon dioxide is a colourless and odourless gas that traps heat from the sun and warms the earth through a natural process known as the Greenhouse Effect [4]. The power stations that use coal to generate electricity contribute to about 83% of the current emission of carbon dioxide [4]. Carbon dioxide can exist in three different forms in sea water (solute carbon dioxide, hydrogen carbonate and carbonate) making the seawater capacity to store carbon dioxide superior to that of the atmosphere [4]. Gas pollutants like nitrogen dioxide have negative consequences like depletion of ozone layer, damaging green plants and trees, acid rain, hazardous fog and clouds as well as contamination of coastal water bodies [5]. Climate change due to large amounts of carbon monoxide lead to change in land and sea temperature which can further lead to extreme weather conditions [5]. It is very important to capture and remove about 20 Gt of carbon dioxide annually in compliance with the Paris agreement of keeping the global temperature increase below a value of 2° C [6].

Research have revealed that excessive emission and accumulation of greenhouse gases into our environment is responsible for the current world climate change. Due to the visible consequences of climate change that is affecting global economy, different parties with interest in seeking for solution to this ravaging and burning issue of concern are looking for solutions in forms of policies, programmes and clean technologies that would help in reducing excessive emission of the greenhouse gases [7]. Mathematical modelling is very useful in providing solutions to human problems such as dynamics of global infections [8], toxic effects of cytotoxic and hemotoxic snake venom [9], honeybee farming improvement [10] and many other fields. As part of the global solution-seeking process, there is the need to develop some mathematical models of climate change due to carbon dioxide emission and accumulation by incorporating some mitigation measures into the model. A number of mathematical models on climate change attributable to excessive emission of carbon dioxide have been developed [11–17].

Most of them considered one or two mitigation measure (s) in their model dynamics for the reduction of carbon dioxide. Leveraging on the idea that a combination of more simple, viable and easy to implement mitigation measures could bring a much better considerable reduction of carbon dioxide emission and accumulation in the atmosphere, a mathematical model involving three mitigation measures was developed [18]. In this current work, we extend this model in Ref. [18] using the concept of fractional calculus. Fractional calculus has become very crucial and an integral tool in the last three decades in fields such as biomathematics, economics, physics, finance, engineering and many others as it provides different scenarios and more accurate results compared to other techniques [8]. Almeida [19] in his work presented two reasons for the preference of fractional calculus over the classical (traditional or ordinary) calculus.

Fractional derivatives are nonlocal operators and may be more suitable for long-term behavioural studies. Thus, they contain memory (solution depends on previous instant) unlike integer-order derivatives. Secondly, the consideration of the order of the derivative as an arbitrary real number which is not necessarily an integral value, enables modelling more efficiently real data compared to the theoretical model. Yuan *et al*. [20] in studying long term memory (LTM) in climate variability using fractional integral technique, considered Fractional Integral Statistical Model (FISM) that could estimate the long-lasting effects of historical climate states on the contemporary time quantitatively and also that of climate memory signals. They concluded that one could ascertain the trends in time series change using extracted climate memory signals, which gave a new dimension to the narrative of prediction of climate.

Kumar *et al*. [21] proposed and analyzed a fractional order nonlinear mathematical model using the generalized form of the Caputo fractional derivative to describe the dynamics of the problem. They presented their novel results graphically and recommended the usefulness of the fractional approach in solving real-world phenomena. Eze & Oyesanya [22] presented fractional model on the impact of climate change with dominant earth's fluctuations. Their model was a second order ODE that was solved using modified Laplace Adomian decomposition method. They compared the results of the fractional solutions with the integer solutions. Their findings revealed that the fractional model gives a better situation compared to the integer solution.

Xie *et al.* [23] developed a novel continuous fractional non-linear grey Bernoulli model with Grey Wolf Optimizer for forecasting fuel combustion-related CO_2 emissions in China. Their findings revealed that their developed model produced better results than other available similar models and that about 1×10^{10} tonnes of fuel combustion-related CO_2 will be emitted by 2023. Based on their forecast results, they recommended developing low-carbon technologies, acceleration of the promotion of the national carbon market and strengthening citizens environmental awareness. Ilhan *et al.* [24] found a series solution for a system of fractional ODE describing the atmospheric dynamics of carbon dioxide using the q-homotopy analysis transform method (q-HAM). Their results revealed that the fractional scheme is highly methodical and very effective in analyzing the nature of the system of arbitrary order differential equations in daily life.

Ozarslan & Sekerci [25] developed a fractional order differential model on oxygen-plankton-zooplanktoon system under climate change using the Caputo fractional operator and considering a temperature function to represent the production rate of photosynthesis. Their findings revealed that the effect of global warming on the rate of oxygen production was severe, leading to a decrease in oxygen level which could further cause plankton extinction. Qureshi & Yusuf [26] worked on mathematical modelling for the impacts of deforestation on wildlife species using Caputo differential operator. Their results were displayed graphically and they concluded that the fractional model was better in performance compared to the usual classical model, as it captured all the historical information of the considered system unlike the corresponding classical system.

Sekerci & R. Ozarslan [27] studied a fractional oxygen-plankton-zooplankton mathematical model with climate change effect by employing nonsingular fractional operators, Caputo-Fabrizio (CF) and Atangana-Baleanu (ABC). In conclusion, they averred that increase in global warming adversely affect the rate of oxygen production, leading to depletion in the level of oxygen and the extinction of plankton. Din *et al.* [28] worked on the mathematical study of climate change model under nonlocal fractional derivatives by considering a three-compartment of oxygen concentration, phytoplankton and zooplankton. Their findings revealed that firstly, there was a reduction in the level of oxygen concentration whose value tended to the free equilibrium point. Secondly, there was a decrease in the number of phytoplankton also due to decrease in the concentration of oxygen. Lastly, the number of zooplankton reached its maximum value due to the consumption of the oxygen and then became stable thereafter. In conclusion, they opined that the fractional model gives a better situation for climate control compared to the classical case.

Machado & Lopes [29] studied global warming patterns using the concepts of dynamical systems and fractional calculus. From their findings, an assertive representation of the global warming dynamics and a simpler analysis of its characteristics was revealed by the application of Fourier transforms and power law trend lines. Furthermore, they opined that fractional calculus took into account long range effects usually ignored by classical models.

Xu *et al.* [30] proposed and analyzed the correlation that existed between energy consumption and carbon dioxide emissions using Non-equigap GM(1,1) model with conformable fractional accumulation [CFNGM(1,1)]. From their findings, the carbon dioxide emissions of 30 out of the 53 countries studied had risen to different levels, the top three being China, USA and India. They recommended that particular attention should be paid mostly to emission trends of China. They also showed that conformal fractional order had a better prediction effect when the order was smaller. Considering the current emergency caused by climate change and the need to seek for solutions from all possible sources, there is a motivation to formulate a mathematical model that combines a number of mitigations that are feasible and realistic in mitigating against excessive emission of carbon dioxide. Furthermore,the use of fractional calculus approach gives more realistic scenarios compared to use of integer order ordinary equations.

The rest of the paper is organized as follows: In Section 2, related mathematical preliminaries are presented. In Section 3, the Caputo fractional model equations are given and their corresponding mathematical properties determined. In Section 4, the stability analysis of the model is done. In Section 5, the numerical simulation results of the model are presented. In Section 6, the discussion of the results is done. The conclusion is presented in Section 7.

2. Mathematical Preliminaries

Definition 1 (Lipschitz Condition). *[31].*
A function $g(t, y(t))$ is said to be Lipschitz if

$$|g(t, y(t)) - g(t, w(t))| \leq L|y(t) - w(t)|$$

for some constants $L > 0$ which does not depend on t, y and w.

Lemma 1. *[13, 32, 33].*
Let $p(t) \in C[t_0, +\infty]$. If $p(t)$ satisfies

$$D^r p(t) + hp(t) \leq n, p(t_0) = p_0 \in \Re, \text{for } 0 < r \leq 1, h, n \in \Re \text{ and } h \neq 0, \text{then}$$

$$p(t) \leq \left(p_0 - \frac{n}{h}\right) E_r\left[-h(t - t_0)^r\right] + \frac{n}{h}.$$

Here, $E_r\left[-h(t - t_0)^r\right]$ is the Mittag-Leffler function.

Lemma 2. *[13, 32].*
Let $p(t) \in C(\Re^+)$ and its fractional derivatives of order r exist for any $0 < r \leq 1$. Then, for any $t > 0$:

$$D^r\left[p(t) - \vec{p^*} - \vec{p^*}\ln\frac{p(t)}{\vec{p^*}}\right] \leq \left(1 - \frac{\vec{p^*}}{p(t)}\right) D^r p(t),\ \vec{p^*} \in \Re^+.$$

Lemma 3. *[34, 35]*
Let $p(t) \in \Re$ be a continuous and differentiable function. Then, for any time instant $t \geq t_0$

$$\frac{1}{2} D^r p^2(t) \leq D^r p(t), \forall r \in (0, 1].$$

Lemma 4. *[33].*
Let $D^r p(t) = g(t, p(t)), t > 0,\ p(0) \geq 0,\ 0 < r \leq 1$ be a fractional order system, where
$g : (0, \infty) \times \Phi \to \Re^m$, $\Phi \subseteq \Re^m$. A unique solution exists for the fractional order system on $(0, \infty) \times \Phi$ if $g(t, p(t))$ satisfies the locally Lipschitz condition with respect to p, that is

$$|g(t, p(t)) - g(t, q(t))| \leq L|p(t) - q(t)|, L \text{ a non-zero positive constant.}$$

Lemma 5. *(**Generalized Lasalle Invariance Principle**). [32].*
Suppose Φ is a bounded closed set and every solution of

$$D^r p(t) = g(p(t))$$

begins from a point in Φ and stays in Φ for all time. If there exists $W(p) : \Phi \to \Re$ having continuous first partial derivatives satisfying

$$D^r W|_{D^r p(t)} = g(p(t)) \leq 0, \text{ let}$$

$$F = \{p | D^r W|_{D^r p(t)} = g(p(t)) = 0\}$$

and Q be the greatest invariant set of F. Then every solution $p(t)$ starting in Φ approaches Q as $t \to \infty$.

3. The Fractional Order Model and its Properties

Using the Caputo derivative operator, the fractional model of order r (the order not visibly written on the RHS for the sake of ease of writing, reading and simplicity), where $0 < r \leq 1$ of the model equations in Ref. [18] become:

$$D^r\ C(t) = \beta^r C\left(1 - \frac{C}{C_m}\right) - d_1^r CP - (d_2^r + d_3^r + d_4^r + \mu_0^r)C. \quad (1)$$

Table 1: Descriptions of Model Variables and Parameters [18]

Symbols	Descriptions
t	Time
$C(t)$	Concentration of CO_2 in the atmosphere
$P(t)$	Photosynthetic biomass density
$R(t)$	Recycling and good conservation policies density
$E(t)$	Enlightenment/ awareness programmes density
$T(t)$	Direct air capture technology density
β	Intrinsic rate of accumulation of CO_2 in the atmosphere
C_m	Maximum tolerated concentration of CO_2 beyond which the emission becomes dangerous
d_1	Rate of decrease in CO_2 concentration due to interaction between CO_2 and the photosynthetic biomass
d_2	Rate of decrease in CO_2 concentration due to implementation of recycling and good conservation policies
d_3	Rate of decrease in CO_2 concentration due to implementation of enlightenment/awareness programmes
d_4	Rate of decrease in CO_2 concentration due to implementation of direct air capture technology
μ_0	Natural rate of depletion in concentration of CO_2
ω	Intrinsic rate of growth of the photosynthetic biomass
N	Carrying capacity for the photosynthetic biomass
ϕ	Rate of increase in the photosynthetic biomass due to the interaction between this biomass and CO_2
τ	Rate of increase in photosynthetic biomass due to interaction between good conservation policies and the photosynthetic biomass
μ_1	Rate of decrease in photosynthetic biomass due to natural phenomena
μ_2	Rate of decrease in photosynthetic biomass due to human activities
a_1	Rate of success of recycling and good conservation policies
a_0	Rate of negligence or evasion of recycling and good conservation policies
b_1	Rate of success of enlightenment/ awareness programmes
b_0	Rate of Ignorance, negligence and evasion of the enlightenment programmes
m_1	Rate of success of direct air capture technology
m_0	Rate of decline in the implementation of direct air capture technology

$$D^r\ P(t) = \omega^r P\left(1 - \frac{P}{N}\right) + \phi^r CP + \tau^r PR - (\mu_1^r + \mu_2^r)P. \tag{2}$$
$$D^r\ R(t) = a_1^r C - a_0^r R. \tag{3}$$
$$D^r\ E(t) = b_1^r C - b_0^r E. \tag{4}$$
$$D^r\ T(t) = m_1^r C - m_0^r T. \tag{5}$$

The initial conditions of these equations are: $C(0) = C_0, P(0) = P_0, R(0) = R_0, E(0) = E_0, T(0) = T_0$. In subsequent writings, the order, r, on the right hand side of equations (1)-(5) would be omitted for ease of writing and simplification. Hence, it would be implied.

3.1. Existence and Uniqueness

Lemma 6. *Let* $\Phi = \{(C, P, R, E, T) \in \Re^5_+ : max\ (|C|, |P|, |R|, |E|, |T| \leq \delta,\ \delta > 0)\}$, t_∞ *and* $\Re^5_+ = \{(C, P, R, E, T) : C \geq 0, P \geq 0, R \geq 0, E \geq 0, T \geq 0\}$. *The model system given by equations (1) to (5) with initial value in* $\Re^5_+ \subseteq \Phi$ *has a unique solution in* $\Phi \times [0, t_\infty]$

Proof: The proof of this lemma is done in like manner as presented in the works of Refs. [13, 31–33] and Refs. [36–38].

Let $A = (C, P, R, E, T)$ and $\bar{A} = (\bar{C}, \bar{P}, \bar{R}, \bar{E}, \bar{T})$ for any $A, \bar{A} \in \Phi$.

By considering a mapping

$$F(A) = (F_1(A), F_2(A), F_3(A), F_4(A), F_5(A)) :$$

$$\begin{aligned}\|F(A)-F(\bar{A})\| = &|F_1(A)-F_1(\bar{A})|+|F_2(A)-F_2(\bar{A})|\\ &+|F_3(A)-F_3(\bar{A})|+|F_4(A)-F_4(\bar{A})|\\ &+|F_5(A)-F_5(\bar{A})|. \end{aligned} \tag{6}$$

From equations (1)-(5), let

$$F_1(A)=\beta C\left(1-\frac{C}{C_m}\right)-d_1CP-(d_2+d_3+d_4+\mu_0)C. \tag{7}$$

$$F_2(A)=\omega P\left(1-\frac{P}{N}\right)+\phi CP+\tau PR-(\mu_1+\mu_2)P. \tag{8}$$

$$F_3(A)=a_1C-a_0R. \tag{9}$$

$$F_4(A)=b_1C-b_0E. \tag{10}$$

$$F_5(A)=m_1C-m_0T. \tag{11}$$

Evaluating each term on the RHS of equation (6) using equations (7) to (11):

$$\begin{aligned}F_1(A)-F_1(\bar{A}) = &-\frac{\beta}{C_m}(C+\bar{C})(C-\bar{C})-d_1P(C-\bar{C})\\ &-d_1\bar{C}(P-\bar{P})+\beta(C-\bar{C})\\ &-(d_2+d_3+d_4+\mu_0)(C-\bar{C}). \end{aligned} \tag{12}$$

$$\begin{aligned}F_2(A)-F_2(\bar{A}) = &(\omega+\mu_1+\mu_2)(P-\bar{P})-\frac{\omega}{N}(P+\bar{P})(P-\bar{P})\\ &-\phi P(C-\bar{C})-\phi\bar{C}(P-\bar{P})+\tau R(P-\bar{P})\\ &+\tau\bar{P}(R-\bar{R}). \end{aligned} \tag{13}$$

$$\begin{aligned}F_3(A)-F_3(\bar{A}) &= (a_1C-a_0R)-\left(a_1\bar{C}-a_0\bar{R}\right)\\ &= a_1(C-\bar{C})-a_0(R-\bar{R}). \end{aligned} \tag{14}$$

$$\begin{aligned}F_4(A)-F_4(\bar{A}) &= (b_1C-b_0E)-\left(b_1\bar{C}-b_0\bar{E}\right)\\ &= b_1(C-\bar{C})-b_0(E-\bar{E}). \end{aligned} \tag{15}$$

$$\begin{aligned}F_5(A)-F_5(\bar{A}) &= (m_1C-m_0T)-\left(m_1\bar{C}-m_0\bar{T}\right)\\ &= m_1(C-\bar{C})-m_0(T-\bar{T}). \end{aligned} \tag{16}$$

Substituting equations (12), (13), (14), (15) and (16) into equation (6):

$$\begin{aligned}\|F(A)-F(\bar{A})\| = &\Big|\frac{\beta}{C_m}(C+\bar{C})(C-\bar{C})-d_1P(C-\bar{C})-d_1\bar{C}(P-\bar{P})\\ &+\beta(C-\bar{C})-(d_2+d_3+d_4+\mu_0)(C-\bar{C})|\\ &+\Big|(\omega+\mu_1+\mu_2)(P-\bar{P})-\frac{\omega}{N}(P+\bar{P})(P-\bar{P})\\ &-\phi P(C-\bar{C})-\phi\bar{C}(P-\bar{P})+\tau R(P-\bar{P})\\ &+\tau\bar{P}(R-\bar{R})|+\Big|a_1(C-\bar{C})-a_0(R-\bar{R})|\\ &+\Big|b_1(C-\bar{C})-b_0(E-\bar{E})\Big|\\ &+\Big|m_1(C-\bar{C})-m_0(T-\bar{T})\Big|\\ \le &\frac{\beta}{C_m}|C+\bar{C}||C-\bar{C}|+d_1|P||C-\bar{C}|\\ &+d_1|\bar{C}||(P-\bar{P})|+\beta|C-\bar{C}|\\ &+(d_2+d_3+d_4+\mu_0)|C-\bar{C}|+(\omega+\mu_1+\mu_2)|P-\bar{P}|\\ &+\frac{\omega}{N}|P+\bar{P}||(P-\bar{P})|+\phi|P||C-\bar{C}|+\phi|\bar{C}||P-\bar{P}|\\ &+\tau|R||P-\bar{P}|+\tau|\bar{P}||R-\bar{R}|+a_1|C-\bar{C}|+a_0|R-\bar{R}|\\ &+b_1|C-\bar{C}|+b_0|E-\bar{E}|+m_1|C-\bar{C}|+m_0|T-\bar{T}|. \end{aligned}$$

From the statement of lemma 6;

$$(|C|,|P|,|R|,|E|,|T|\le\delta)\Rightarrow(|\bar{C}|,|\bar{P}|,|\bar{R}|,|\bar{E}|,|\bar{T}|\le\delta).$$

$$\begin{aligned}\therefore\ \|F(A)-F(\bar{A})\| \le &\frac{2\beta\delta}{C_m}|C-\bar{C}|+d_1\delta|C-\bar{C}|+d_1\delta|P-\bar{P}|\\ &+\beta|C-\bar{C}|+(d_2+d_3+d_4+\mu_0)|C-\bar{C}|\\ &+(\omega+\mu_1+\mu_2)|P-\bar{P}|\\ &+\frac{2\delta\omega}{N}|(P-\bar{P})|+\phi\delta|C-\bar{C}|+\phi\delta|P-\bar{P}|\\ &+\tau\delta|P-\bar{P}|+\tau\delta|R-\bar{R}|+a_1|C-\bar{C}|\\ &+a_0|R-\bar{R}|+b_1|C-\bar{C}|+b_0|E-\bar{E}|\\ &+m_1|C-\bar{C}|+m_0|T-\bar{T}| \end{aligned}$$

$$\begin{aligned}&=(\frac{2\beta\delta}{C_m}+d_1\delta+d_2+d_3+d_4+\mu_0+\beta+\phi\delta+a_1+b_1+m_1)|C-\bar{C}|\\ &+(\omega+\mu_1+\mu_2+\frac{2\delta\omega}{N}+\phi\delta+\tau\delta)|P-\bar{P}|+(\tau\delta+a_0)|R-\bar{R}|\\ &+b_0|E-\bar{E}|+m_0|T-\bar{T}|. \end{aligned}$$

Let

$$\begin{aligned}L_1 &= (\frac{2\beta\delta}{C_m}+d_1\delta+d_2+d_3+d_4+\mu_0+\beta+\phi\delta+a_1+b_1+m_1),\\ L_2 &= (\omega+\mu_1+\mu_2+\frac{2\delta\omega}{N}+\phi\delta+\tau\delta),\\ L_3 &= (\tau\delta+d_5\delta+a_0),\\ L_4 &= b_0,\\ L_5 &= m_0. \end{aligned}$$

$$\begin{aligned}\therefore\ \|F(A)-F(\bar{A})\| &\le L_1|C-\bar{C}|+L_2|P-\bar{P}|+L_3|R-\bar{R}|+L_4|E-\bar{E}|\\ &+L_5|T-\bar{T}|.\\ &\le L\left|A-\bar{A}\right|. \end{aligned} \tag{17}$$

Where L = max $\{L_1, L_2, L_3, L_4, L_5\}$. Therefore, the inequality (17) shows that $F(A)$ satisfies the Lipschitz Condition. Hence, using Lemma 4 the model system given by equations (1) to (5) has a unique solution $A(t)$ for non-negative initial condition.

3.2. *Positivity of the Model Solution*

The biological meaningfulness of the model system given by equations (1) to (5) is dependent importantly on establishing that all its state variables are non-negative for all time, (t). In other words, it suffices to proving that for positive initial data, the solutions of the model system will remain positive for all time $t > 0$.

Lemma 7. *Let the initial data be* $C(0) = C_0 > 0$, $P(0) = P_0 > 0$, $R(0) = R_0 > 0$, $E(0) = E_0 > 0$, $T(0) = T_0 > 0$, *then the solutions* (C, P, R, E, T) *of the model are positive for all time* $t > 0$.

Proof 1. *The proof of this lemma is done by considering similar approaches as presented in the works of Refs. [32, 39–42].* *Let* $t_1 = sup\{t > 0 : C(0) = C_0 > 0, P(0) = P_0 > 0, R(0) = R_0 > 0, E(0) = E_0 > 0, T(0) = T_0 > 0\}$.
Since the system represents a biological system and there is continuous emission and accumulation of carbon dioxide into the atmosphere, we assume that all derivatives are positive. Thus, for $t_1 > 0$:

$$D^r\ C(t) = \beta C\left(1 - \frac{C}{C_m}\right) - d_1 CP - (d_2 + d_3 + d_4 + \mu_0)C$$
$$\Rightarrow D^r\ C(t) \geq -(d_2 + d_3 + d_4 + \mu_0)C. \tag{18}$$

Taking the Laplace transform of both sides of the inequality in (18):

$$s^r C(s) - s^{r-1}C(0) \geq -(d_2 + d_3 + d_4 + \mu_0)C(s)$$
$$\Rightarrow C(s) \geq \frac{s^{r-1}}{s^r + (d_2 + d_3 + d_4 + \mu_0)}C_0. \tag{19}$$

Taking the inverse Laplace transform of the inequality in (19):

$$C(t) \geq C_0 E_{r,1}\left(-(d_2 + d_3 + d_4 + \mu_0)t^r\right). \tag{20}$$

Since
$C_0 > 0$ *and the Mittag-Leffler function*
$E_{r,1}\left(-(d_2 + d_3 + d_4 + \mu_0 + \beta)t^r\right) > 0$, *then* $C(t) > 0$, $\forall t > 0$.
Taking the second equation given by equation (2):

$$D^r\ P(t) = \omega P\left(1 - \frac{P}{N}\right) + \phi CP + \tau PR - (\mu_1 + \mu_2)P$$
$$\Rightarrow D^r\ P(t) \geq -(\mu_1 + \mu_2)P. \tag{21}$$

Taking the Laplace transform of both sides of the inequality in (21):

$$s^r P(s) - s^{r-1}P_0 \geq -(\mu_1 + \mu_2)P(s)$$
$$\Rightarrow P(s) \geq \frac{s^{r-1}}{s^r + (\mu_1 + \mu_2)}P_0. \tag{22}$$

Taking the inverse Laplace transform of both sides of the inequality in (22):

$$\mathcal{L}^{-1}\{P(s)\} \geq \mathcal{L}^{-1}\left\{\frac{s^{r-1}}{s^r + (\mu_1 + \mu_2)}P_0\right\}$$
$$\Rightarrow P(t) \geq P_0 E_{r,1}\left(-(\mu_1 + \mu_2)t^r\right). \tag{23}$$

Since
$P_0 > 0$ *and the Mittag-Leffler function* $E_{r,1}\left(-(\mu_1 + \mu_2)t^r\right) > 0$, *then* $P(t) > 0$, $\forall t > 0$.
Taking the third equation given by equation (3):

$$D^r\ R(t) = a_1 C - a_0 R$$
$$\Rightarrow D^r\ R(t) \geq -a_0 R. \tag{24}$$

Taking the Laplace transform of both sides of the inequality in (24):

$$s^r R(s) - s^{r-1}R_0 \geq -a_0 R(s)$$
$$\Rightarrow R(s) \geq \frac{s^{r-1}}{s^r + a_0}R_0. \tag{25}$$

Taking the inverse Laplace transform of both sides of the inequality in (25):

$$\mathcal{L}^{-1}\{R(s)\} \geq \mathcal{L}^{-1}\left\{\frac{s^{r-1}}{s^r + a_0}R_0\right\}$$
$$\Rightarrow R(t) \geq R_0 E_{r,1}\left(-a_0 t^r\right). \tag{26}$$

Since $R_0 > 0$ *and the Mittag-Leffler function* $E_{r,1}\left(-a_0 t^r\right) > 0$, *then* $R(t) > 0$, $\forall t > 0$.
Taking the fourth equation given by equation (4):

$$D^r\ E(t) = b_1 C - b_0 E$$
$$\Rightarrow D^r\ E(t) \geq -b_0 E. \tag{27}$$

Taking the Laplace transform of both sides of (27):

$$s^r E(s) - s^{r-1}E_0 \geq -b_0 E(s)$$
$$\Rightarrow E(s) \geq \frac{s^{r-1}}{s^r + b_0}E_0. \tag{28}$$

Taking the inverse Laplace transform of both sides of the inequality (28):

$$\mathcal{L}^{-1}\{E(s)\} \geq \mathcal{L}^{-1}\left\{\frac{s^{r-1}}{s^r + b_0}E_0\right\}$$
$$\Rightarrow E(t) \geq E_0 E_{r,1}\left(-b_0 t^r\right). \tag{29}$$

Since $E_0 > 0$ *and the Mittag-Leffler function* $E_{r,1}\left(-b_0 t^r\right) > 0$, *then* $E(t) > 0$, $\forall t > 0$.
Taking the fifth equation given by equation (5):

$$D^r\ T(t) = m_1 C - m_0 T$$
$$\Rightarrow D^r\ T(t) \geq -m_0 T. \tag{30}$$

Taking the Laplace transform of both sides of the inequality (30):

$$s^r T(s) - s^{r-1}T_0 \geq -m_0 T(s)$$
$$\Rightarrow T(s) \geq \frac{s^{r-1}}{s^r + m_0}T_0. \tag{31}$$

Taking the inverse Laplace transform of both sides of the inequality (31):

$$\mathcal{L}^{-1}\{T(s)\} \geq \mathcal{L}^{-1}\left\{\frac{s^{r-1}}{s^r + m_0}T_0\right\}$$
$$\Rightarrow T(t) \geq T_0 E_{r,1}\left(-m_0 t^r\right). \tag{32}$$

Since $T_0 > 0$ *and the Mittag-Leffler function* $E_{r,1}\left(-m_0 t^r\right) > 0$, *then* $T(t) > 0$, $\forall t > 0$.
Therefore, the solution of the model system given by equations (1) to (5) as seen from equations (20), (23), (26), (29) and (32) is positive provided that the initial data are positive.

3.3. Invariant Region

The formulated fractional climate mathematical model given by equations (1), (2), (3), (4) and (5) would be bounded and biologically meaningful if all the solutions with non-negative initial data remain non-negative for $t \in (0, \infty)$.

Lemma 8. *The region* Φ=$\{(C, P, R, E, T) : 0 < C(t) < L_C, 0 < P(t) < L_P, 0 < R(t) < L_R, 0 < E(t) < L_E, 0 < T(t) < L_T\} \subset \Re^5_+$ *is positively-invariant for the model system given by equations (1) to (5) with initial conditions in* $\Re^5_+$*, that is* $(C_0, P_0, R_0, E_0, T_0) \in \Re^5_+$.

Proof 2. *Using the approach of getting boundedness of a model as presented in the works of Refs. [15, 17, 43, 44], the regions or domains of positive attraction (invariant regions) are obtained as follows:*
Since the model must be positive for the biological meaningfulness of the model which has been established in the proof of lemma 7, then we have:
Taking equation (1):

$$D^r\ C(t) = \beta C\left(1 - \frac{C}{C_m}\right) - d_1CP - (d_2 + d_3 + d_4 + \mu_0)C > 0$$

Let

$$\beta C\left(1 - \frac{C}{C_m}\right) - (d_2 + d_3 + d_4 + \mu_0)C = 0$$
$$\Rightarrow C = 0,\ \beta\left(1 - \frac{C}{C_m}\right) - (d_2 + d_3 + d_4 + \mu_0) = 0.$$

$C = 0$ is invalid in this case. Hence,

$$C = \frac{C_m}{\beta}[\beta - (d_2 + d_3 + d_4 + \mu_0)] = L_C,\ \beta > (d_2 + d_3 + d_4 + \mu_0). \tag{33}$$

From the definition or statement of lemma 8, $C_0 > 0$ implies that

$$\begin{aligned} &0 < C_0 \leq C \\ &\Rightarrow 0 < C(t) \leq \frac{C_m}{\beta}[\beta - (d_2 + d_3 + d_4 + \mu_0)] = L_C, \\ &\beta > (d_2 + d_3 + d_4 + \mu_0). \end{aligned} \tag{34}$$

Taking equation (4):

$$D^r\ E(t) > 0 \Rightarrow b_1C - b_0E > 0 \Rightarrow E < \frac{b_1}{b_0}C.$$

From equation (33):

$$\begin{aligned} E &< \frac{b_1}{b_0}C \Rightarrow E < \frac{b_1C_m}{b_0\beta}[\beta - (d_2 + d_3 + d_4 + \mu_0)] \\ &= \frac{b_1}{b_0}L_C = L_E, \beta > (d_2 + d_3 + d_4 + \mu_0) \end{aligned} \tag{35}$$

$$\Rightarrow 0 < E(t) \leq \frac{b_1C_m}{b_0\beta}[\beta - (d_2 + d_3 + d_4 + \mu_0)] = \frac{b_1}{b_0}L_C = L_E, \tag{36}$$

$$\beta > (d_2 + d_3 + d_4 + \mu_0).$$

Taking equation (5):

$$D^r\ T(t) > 0 \Rightarrow m_1C - m_0T > 0$$
$$\therefore T < \frac{m_1}{m_0}C \Rightarrow T < \frac{m_1C_m}{\beta m_0}[\beta - (d_2 + d_3 + d_4 + \mu_0)],$$

since from equation (33), $C = \dfrac{C_m}{\beta}[\beta - (d_2 + d_3 + d_4 + \mu_0)]$.

$$\therefore 0 < T(t) \leq \frac{m_1C_m}{\beta m_0}[\beta - (d_2 + d_3 + d_4 + \mu_0)] = \frac{m_1}{m_0}L_C = L_T. \tag{37}$$

Considering equation (3):

$$D^r\ R(t) > 0 \Rightarrow R < \frac{a_1}{a_0}C.$$

From the inequality results in (33) and (35):

$$R < \frac{a_1}{a_0}L_C \Rightarrow R < \frac{a_1C_m}{a_0\beta}[\beta - (d_2 + d_3 + d_4 + \mu_0)] = L_R. \tag{38}$$

$$\therefore 0 < R(t) \leq \frac{a_1C_m}{a_0\beta}[\beta - (d_2 + d_3 + d_4 + \mu_0)] = \frac{a_1}{a_0}L_C = L_R. \tag{39}$$

Lastly, taking equation (2):

$$D^r\ P(t) > 0 \Rightarrow \omega P\left(1 - \frac{P}{N}\right) + \phi CP + \tau PR - (\mu_1 + \mu_2)P > 0$$
$$\therefore R < \frac{N}{\omega}\left[\phi C + \tau R + \omega - (\mu_1 + \mu_2)\right].$$

Substituting the inequality results in (33) and (38) into this, we have:

$$\begin{aligned} P &< \frac{N}{\omega}\left[\phi C + \tau R + \omega - (\mu_1 + \mu_2)\right] \\ &= \frac{N}{\omega}\left[\phi L_C + \tau\frac{a_1}{a_0}L_C + \omega - (\mu_1 + \mu_2)\right] \\ P &< \frac{N}{\omega}\left\{\frac{C_m}{\beta}\left(\phi + \frac{a_1\tau}{a_0}\right)[\beta - (d_2 + d_3 + d_4 + \mu_0)] + [\omega - (\mu_1 + \mu_2)]\right\} \\ &= L_P \\ 0 < P &\leq \frac{N}{\omega}\left\{\frac{C_m}{\beta}\left(\phi + \frac{a_1\tau}{a_0}\right)[\beta - (d_2 + d_3 + d_4 + \mu_0)] + [\omega - (\mu_1 + \mu_2)]\right\} \\ &= L_P, \omega > \mu_1 + \mu_2. \end{aligned} \tag{40}$$

From the results got in equations (34),(40), (39), (36) and (37), the region or domain of attraction of the model solution with given positive initial data is Φ=$\{(C, P, R, E, T) : 0 < C(t) < L_C, 0 < P(t) < L_P, 0 < R(t) < L_R,$
$0 < E(t) < L_E, 0 < T(t) < L_T\} \subset \Re^5_+$ *and it is positively-invariant, where*

$$L_C = \frac{C_m}{\beta}[\beta - \Theta],$$

$\beta > \Theta$.

$$L_P = \frac{N}{\omega}\left\{\frac{C_m}{\beta}\left(\phi + \frac{a_1\tau}{a_0}\right)[\beta - \Theta] + [\omega - (\mu_1 + \mu_2)]\right\},$$

$\omega > \mu_1 + \mu_2$.

$$L_R = \frac{a_1 C_m}{a_0\beta}[\beta - \Theta] = \frac{a_1}{a_0}L_C,\ \beta > \Theta.$$

$$L_E = \frac{b_1 C_m}{b_0\beta}[\beta - \Theta] = \frac{b_1}{b_0}L_C,\ \beta > \Theta.$$

$$L_T = \frac{m_1 C_m}{\beta m_0}[\beta - \Theta] = \frac{m_1}{m_0}L_C,\ \beta > \Theta.$$

$$\Theta = (d_2 + d_3 + d_4 + \mu_0).$$

4. Stability Analysis of the Model

4.1. Equilibrium Points

The equilibrium points are obtained by putting each of the expressions on the RHS of equations (1) to (5) to zero and solving. That is, $D^r\ C(t) = 0$, $D^r\ P(t) = 0$, $D^r\ R(t) = 0$, $D^r\ E(t) = 0$ and $D^r\ T(t) = 0$ are solved to obtain the equilibrium points. The equilibrium points of the model (as also obtained in Ref. [18]) are:

$$\varepsilon_0 = (C^0, P^0, R^0, E^0, T^0) = (0, 0, 0, 0, 0).$$

$$\varepsilon_1 = (C^*, P^*, R^*, E^*, T^*).$$

$$\varepsilon_2 = (C^{**}, P^{**}, R^{**}, E^{**}, T^{**}) = \left(0, \frac{N}{\omega}[\omega - (\mu_1 + \mu_2)], 0, 0, 0\right).$$

$$\varepsilon_3 = (C^{***}, P^{***}, R^{***}, E^{***}, T^{***}).$$

Where,

$$C^* = \frac{C_m}{\beta}[\beta - \Theta], \beta > \Theta.$$

$$P^* = 0.$$

$$R^* = \frac{a_1 C_m}{a_0\beta}[\beta - \Theta], \beta > \Theta.$$

$$E^* = \frac{b_1 C_m}{b_0\beta}[\beta - \Theta], \beta > \Theta.$$

$$T^* = \frac{m_1 C_m}{m_0\beta}[\beta - \Theta], \beta > \Theta.$$

$$C^{***} = \frac{a_0\beta C_m\omega - a_0 C_m\{d_1 N[\omega - (\mu_1 + \mu_2)] + \omega\Theta\}}{a_0\beta\omega + C_m d_1 N(a_0\phi + \tau a_1)},$$

$$\beta > \{d_1 N[\omega - (\mu_1 + \mu_2)] + \omega\Theta\}.$$

$$P^{***} = N\left[\frac{C_m(a_0\phi + \tau a_1)[\beta - \Theta] + a_0\beta[\omega - (\mu_1 + \mu_2)]}{a_0\beta\omega + C_m d_1 N(a_0\phi + \tau a_1)}\right], \beta > \Theta.$$

$$R^{***} = \frac{a_1}{a_0}C^{***}.$$

$$E^{***} = \frac{b_1}{b_0}C^{***}.$$

$$T^{***} = \frac{m_1}{m_0}C^{***}.$$

$$\Theta = (d_2 + d_3 + d_4 + \mu_0).$$

4.2. Global Stability Analysis

We establish the global stability of the equilibrium points obtained using the procedures as presented in Refs. [45–52] .

Theorem 1. *The equilibrium point $\varepsilon_0 = (C^0, P^0, R^0, E^0, T^0) = (0, 0, 0, 0, 0)$ is globally asymptotically stable (GAS) if $d_1 \geq \phi$, $(d_2 + d_3 + d_4 + \mu_0) \geq (\beta + a_1 + b_1 + m_1)$ and $(\mu_1 + \mu_2) \geq \omega$ in the region Φ.*

Proof 3. *Let $V_1(C, P, R, E, T) \equiv V_1 = \frac{1}{2}C^2 + \frac{1}{2}P^2 + \frac{1}{2}R^2 + \frac{1}{2}E^2 + \frac{1}{2}T^2$ be a quadratic Lyapunov function. Then we establish that it satisfies all the axioms that guarantees the global stability of $\varepsilon_0 = (0, 0, 0, 0, 0)$.*
Obviously, $V_1 > 0$ for $C \neq 0, P \neq 0, R \neq 0, E \neq 0, T \neq 0$. Thus, the first axiom is satisfied.
Also,

$$V_1(C, P, R, E, T) \equiv V_1 = \frac{1}{2}C^2 + \frac{1}{2}P^2 + \frac{1}{2}R^2 + \frac{1}{2}E^2 + \frac{1}{2}T^2$$
$$\Rightarrow V_1(\varepsilon_0) = 0.$$

Hence, the second axiom for V_1 being a Lyapunov function is satisfied.
Next, we establish the last axiom as given thus:

$$D^r V_1 = \frac{1}{2}D^r C^2 + \frac{1}{2}D^r P^2 + \frac{1}{2}D^r R^2 + \frac{1}{2}D^r E^2 + \frac{1}{2}D^r T^2. \quad (41)$$

Using lemma 3, equation (41) becomes:

$$D^r V_1 \leq D^r C + D^r P + D^r R + D^r E + D^r T. \quad (42)$$

Substituting equations (1)-(5) into the inequality (42) and simplifying;

$$\begin{aligned} D^r V_1 &\leq D^r C + D^r P + D^r R + D^r E + D^r T \\ &= -\frac{\beta}{C_m}C^2 - \frac{\omega}{N}P^2 - (d_1 - \phi)CP + \tau PR - [(d_2 + d_3 + d_4 + \mu_0) \\ &\quad - (\beta + a_1 + b_1 + m_1)]C \\ &\quad - [(\mu_1 + \mu_2 - \omega)]P - a_0 R - b_0 E - m_0 T \end{aligned}$$

By neglecting τPR, obviously

$$\begin{aligned} D^r V_1 &\leq -\frac{\beta}{C_m}C^2 - \frac{\omega}{N}P^2 - (d_1 - \phi)CP - [(d_2 + d_3 + d_4 + \mu_0) \\ &\quad - (\beta + a_1 + b_1 + m_1)]C \\ &\quad - [(\mu_1 + \mu_2) - \omega]P - a_0 R - b_0 E - m_0 T \\ &= -\Big[\frac{\beta}{C_m}C^2 + \frac{\omega}{N}P^2 + (d_1 - \phi)CP + [(d_2 + d_3 + d_4 + \mu_0) \\ &\quad - (\beta + a_1 + b_1 + m_1)]C \\ &\quad + [(\mu_1 + \mu_2) - \omega]P + a_0 R + b_0 E + m_0 T\Big] \leq 0. \end{aligned}$$

$D^r V_1 < 0$ if and only if $d_1 \geq \phi$ and $(d_2 + d_3 + d_4 + \mu_0) \geq (\beta + a_1 + b_1 + m_1)$.
$D^r V_1 = 0$ if $C = C^0 = 0, P = P^0 = 0, R = R^0 = 0, E = E^0 = 0, T = T^0 = 0$.

Hence, V_1 is a Lyapunov function in Φ and the largest compact invariant set in $\{C, P, R, E, T : D^r V_1 \leq 0\}$ is the singleton $\{\varepsilon_0\}$. Employing the Generalized LaSalle Invariance Principle as given by lemma 5, every solution of the model system given by equations (1)-(5) with initial conditions in Φ, tends to ε_0 as $t \to \infty$ provided $d_1 \geq \phi$, $(d_2 + d_3 + d_4 + \mu_0) \geq (\beta + a_1 + b_1 + m_1)$ and $(\mu_1 + \mu_2) \geq \omega$. Hence, the equilibrium point, ε_0, is globally asymptotically stable.

Theorem 2. *The equilibrium point $\varepsilon_1 = (C^*, P^*, R^*, E^*, T^*)$ is globally asymptotically stable (GAS) if $(\mu_1 + \mu_2) \geq \omega$ in the region Φ.*

Proof 4. *Let $V_2(C, P, R, E, T) \equiv V_2 = \left(C - C^* - C^* \ln\left(\frac{C}{C^*}\right)\right) + \frac{1}{2}P^2 + \left(R - R^* - R^* \ln\left(\frac{R}{R^*}\right)\right) + \left(E - E^* - E^* \ln\left(\frac{E}{E^*}\right)\right) + \left(T - T^* - T^* \ln\left(\frac{T}{T^*}\right)\right)$ be a hybrid Lyapunov function. Then we establish that it satisfies all the axioms that guarantees the global stability of $\varepsilon_1 = (C^*, P^*, R^*, E^*, T^*)$ using it.*
Obviously, $V_2 > 0$ for $C \neq C^*, P \neq P^*, R \neq R^*, E \neq E^*, T \neq T^*$. Thus, the first axiom is satisfied.
Also,

$$V_2(\varepsilon_1) = V_1(C^*, 0, R^*, E^*, T^*) = \left(C^* - C^* - C^* \ln\left(\frac{C^*}{C^*}\right)\right) + \frac{1}{2}(0)^2 + \left(R^* - R^* - R^* \ln\left(\frac{R^*}{R^*}\right)\right) + \left(E^* - E^* - E^* \ln\left(\frac{E^*}{E^*}\right)\right) + \left(T^* - T^* - T^* \ln\left(\frac{T^*}{T^*}\right)\right) = 0.$$

Hence, the second axiom for V_2 being a Lyapunov function is satisfied.
Next, we establish the last axiom as given thus:

$$D^r V_2 = D^r\left(C - C^* - C^* \ln\left(\frac{C}{C^*}\right)\right) + \frac{1}{2}D^r P^2 + D^r\left(R - R^* - R^* \ln\left(\frac{R}{R^*}\right)\right) + D^r\left(E - E^* - E^* \ln\left(\frac{E}{E^*}\right)\right) + D^r\left(T - T^* - T^* \ln\left(\frac{T}{T^*}\right)\right)$$

Using the results given by lemmas 2 and 3:

$$D^r V_2 \leq \left(\frac{C - C^*}{C}\right) D^r C(t) + D^r P(t) + \left(\frac{R - R^*}{R}\right) D^r R(t) + \left(\frac{E - E^*}{E}\right) D^r E(t) + \left(\frac{T - T^*}{T}\right) D^r T(t).$$

Substituting equations (1),(2), (3), (4) and (5) into this and simplifying:

$$D^r V_2 \leq -\frac{\beta}{C_m}(C - C^*)^2 - d_1 CP + d_1 C^* P - [(\mu_1 + \mu_2) - \omega]P - \frac{\omega}{N}P^2 + \phi CP + \tau PR + a_1 C - \frac{a_1 C^*}{R^*}R - a_1 R^* \frac{C}{R} + a_1 C^* + b_1 C - \frac{b_1 C^*}{E^*}E - b_1 E^* \frac{C}{E} + b_1 C^* + m_1 C - \frac{m_1 C^*}{T^*}T - m_1 T^* \frac{C}{T} + m_1 C^*,$$

Neglecting the positive terms;

$$\begin{aligned} D^r V_2 &\leq -\frac{\beta}{C_m}(C - C^*)^2 - d_1 CP - [(\mu_1 + \mu_2) - \omega]P - \frac{\omega}{N}P^2 - \frac{a_1 C^*}{R^*}R - a_1 R^* \frac{C}{R} - \frac{b_1 C^*}{E^*}E - b_1 E^* \frac{C}{E} - \frac{m_1 C^*}{T^*}T - m_1 T^* \frac{C}{T} \\ &= -\left[\frac{\beta}{C_m}(C - C^*)^2 + \frac{\omega}{N}P^2 + d_1 CP + [(\mu_1 + \mu_2) - \omega]P + \frac{a_1 C^*}{R^*}R + a_1 R^* \frac{C}{R} + \frac{b_1 C^*}{E^*}E + b_1 E^* \frac{C}{E} + \frac{m_1 C^*}{T^*}T + m_1 T^* \frac{C}{T}\right] \\ &\leq 0, \end{aligned}$$

provided that $(\mu_1 + \mu_2) \geq \omega$.
$D^r V_2 < 0$ if and only if $(\mu_1 + \mu_2) \geq \omega$. Therefore, V_2 is a Lyapunov function in Φ and the largest compact invariant set in $\{C, P, R, E, T : D^r V_2 \leq 0\}$ is the singleton $\{\varepsilon_1\}$. Employing the Generalized LaSalle Invariance Principle as given by lemma 5, every solution of the model system given by equations (1)-(5) with initial conditions in Φ, tends to ε_1 as $t \to \infty$ provided $(\mu_1 + \mu_2) \geq \omega$. Hence, the equilibrium point, ε_1, is globally asymptotically stable.

Theorem 3. *The equilibrium point $\varepsilon_2 = (C^{**}, P^{**}, R^{**}, E^{**}, T^{**}) = \left(0, \frac{N}{\omega}[\omega - (\mu_1 + \mu_2)], 0, 0, 0\right)$ is globally asymptotically stable (GAS) if $d_1 \geq \phi$ and $(\phi P^{**} + d_2 + d_3 + d_4 + \mu_0) \geq (\beta + a_1 + b_1 + m_1)$ in the region Φ.*

Proof 5. *Let $V_3 = \frac{1}{2}C^2 + \left(P - P^{**} - P^{**} \ln \frac{P}{P^{**}}\right) + \frac{1}{2}R^2 + \frac{1}{2}E^2 + \frac{1}{2}T^2$ be a hybrid Lyapunov function. Then we establish that it satisfies all the axioms that guarantee the global stability of $\varepsilon_2 = (C^{**}, P^{**}, R^{**}, E^{**}, T^{**})$ using it.*
Obviously, $V_3 > 0$ for $C \neq C^{**}, P \neq P^{**}, R \neq R^{**}, E \neq E^{**}, T \neq T^{**}$. Thus, the first axiom is satisfied.
Also,

$$V_3(\varepsilon_2) = V_3(C^{**}, P^{**}, R^{**}, E^{**}, T^{**}) = \frac{1}{2}(0)^2 + \left(P^{**} - P^{**} - P^{**} \ln \frac{P^{**}}{P^{**}}\right) + \frac{1}{2}(0)^2 + \frac{1}{2}(0)^2 + \frac{1}{2}(0)^2 = 0.$$

Hence, the second axiom for V_3 being a Lyapunov function is satisfied. Next, we establish the last axiom.

$$D^r V_3 = \frac{1}{2}D^r C^2 + D^r\left(P - P^{**} - P^{**} \ln \frac{P}{P^{**}}\right) + \frac{1}{2}D^r R^2 + \frac{1}{2}D^r E^2 + \frac{1}{2}D^r T^2. \tag{43}$$

Using lemmas 2 and 3, equation (43) becomes:

$$D^r V_3 \leq D^r C(t) + \left(\frac{P - P^{**}}{P}\right) D^r P(t) + D^r R(t) + D^r E(t) + D^r T(t). \tag{44}$$

Substituting equations (1)-(5) into the inequality (44) and simplifying using the equilibrium point:

$$D^r V_3 \leq -\Bigg(\frac{\beta}{C_m}C^2 + \frac{\omega}{N}(P-P^{**})^2 + (d_1-\phi)CP + [(\phi P^{**}+d_2+d_3+d_4+\mu_0)-(\beta+a_1+b_1+m_1)]C + [(\tau P^{**}+a_0)R - b_0E - m_0T]\Bigg) \leq 0$$

$D^r V_3 < 0$ if and only if $d_1 \geq \phi$ and $(\phi P^{**}+d_2+d_3+d_4+\mu_0) \geq (\beta+a_1+b_1+m_1)$. Thus, V_3 is a Lyapunov function in Φ and the largest compact invariant set in $\{C,P,R,E,T : D^r V_3 \leq 0\}$ is the singleton $\{\varepsilon_2\}$. Employing the Generalized LaSalle Invariance Principle as given by lemma 5, every solution of the model system given by equations (1)-(5) with initial conditions in Φ, approaches ε_1 as $t\to\infty$ provided $d_1 \geq \phi$ and $(\phi P^{**}+d_2+d_3+d_4+\mu_0) \geq (\beta+a_1+b_1+m_1)$. Hence, the equilibrium point, ε_2, is globally asymptotically stable.

Theorem 4. *The equilibrium point $\varepsilon_3 = (C^{***},P^{***},R^{***},E^{***},T^{***})$ is globally asymptotically stable (GAS) in the region Φ.*

Proof 6. *Let*

$$V = \left(C - C^{***} - C^{***}\ln\left(\frac{C}{C^{***}}\right)\right) + \left(P - P^{***} - P^{***}\ln\left(\frac{P}{P^{***}}\right)\right) + \left(R - R^{***} - R^{***}\ln\left(\frac{R}{R^{***}}\right)\right) + \left(E - E^{***} - E^{***}\ln\left(\frac{E}{E^{***}}\right)\right) + \left(T - T^{***} - T^{***}\ln\left(\frac{T}{T^{***}}\right)\right)$$

be a Lyapunov function as similarly presented in Ref. [10]. Then we verify that it satisfies the axioms of Global Asymptotic Stability for $\varepsilon_3 = (C^{***},P^{***},R^{***},E^{***},T^{***})$ as follows:
Obviously, $V>0$ for $C\neq C^{***}, P\neq P^{***}, R\neq R^{***}, E\neq E^{***}, T\neq T^{***}$. Thus, the first axiom is satisfied.
Also,

$$\begin{aligned} V(\varepsilon_3) &= V(C^{***},P^{***},R^{***},E^{***},T^{***}) \\ &= \left(C^{***} - C^{***} - C^{***}\ln\left(\frac{C^{***}}{C^{***}}\right)\right) \\ &+ \left(P^{***} - P^{***} - P^{***}\ln\left(\frac{P^{***}}{P^{***}}\right)\right) \\ &+ \left(R^{***} - R^{***} - R^{***}\ln\left(\frac{R^{***}}{R^{***}}\right)\right) \\ &+ \left(E^{***} - E^{***} - E^{***}\ln\left(\frac{E^{***}}{E^{***}}\right)\right) \\ &+ \left(T^{***} - T^{***} - T^{***}\ln\left(\frac{T^{***}}{T^{***}}\right)\right) = 0. \end{aligned}$$

Hence, the second condition of the Lyapunov Stability Theorem is satisfied.
Next, we verify the third axiom thus:

$$\begin{aligned} D^r V = D^r &\left(C - C^{***} - C^{***}\ln\left(\frac{C}{C^{***}}\right)\right) \\ + D^r &\left(P - P^{***} - P^{***}\ln\left(\frac{P}{P^{***}}\right)\right) \\ + D^r &\left(R - R^{***} - R^{***}\ln\left(\frac{R}{R^{***}}\right)\right) \\ + D^r &\left(E - E^{***} - E^{***}\ln\left(\frac{E}{E^{***}}\right)\right) \\ + D^r &\left(T - T^{***} - T^{***}\ln\left(\frac{T}{T^{***}}\right)\right) \end{aligned}$$

Using the result given by lemma 2:

$$D^r V \leq \left(\frac{C-C^{***}}{C}\right)D^r C(t) + \left(\frac{P-P^{***}}{P}\right)D^r P(t) + \left(\frac{R-R^{***}}{R}\right)D^r R(t) + \left(\frac{E-E^{***}}{E}\right)D^r E(t) + \left(\frac{T-T^{***}}{T}\right)D^r T(t).$$

Substituting equations (1),(2), (3), (4) and (5) into this and simplifying:

$$\begin{aligned} D^r V \leq &\Bigg[d_1(CP^{***}+C^{***}P) + \phi(CP + C^{***}P^{***}) \\ &+ \tau(PR + P^{***}R^{***}) + a_1\Big(C + \frac{C^{***}R^{***}}{C}\Big) \\ &+ b_1\Big(C + \frac{C^{***}E^{***}}{E}\Big) + m_1\Big(C + \frac{C^{***}T^{***}}{T}\Big)\Bigg] \\ &- \Bigg[\frac{\beta}{C_m}(C-C^{***})^2 + \frac{\omega}{N}(P-P^{***})^2 + \frac{a_0}{R}(R-R^{***})^2 \\ &+ \frac{b_0}{E}(E-E^{***})^2 + \frac{m_0}{T}(T-T^{***})^2 \\ &+ d_1(CP + C^{***}P^{***}) + \phi(CP^{***} + C^{***}P) \\ &+ \tau(PR^{***} + P^{***}R) + a_1\Big(C^{***} + \frac{CR^{***}}{C}\Big) \\ &+ b_1\Big(C^{***} + \frac{CE^{***}}{E}\Big) + m_1\Big(C^{***} + \frac{CT^{***}}{T}\Big)\Bigg] \\ &= G - H. \end{aligned}$$

Where,

$$\begin{aligned} G = &\Bigg[d_1(CP^{***}+C^{***}P) + \phi(CP + C^{***}P^{***}) + \tau(PR + P^{***}R^{***}) \\ &+ a_1\Big(C + \frac{C^{***}R^{***}}{C}\Big) + b_1\Big(C + \frac{C^{***}E^{***}}{E}\Big) + m_1\Big(C + \frac{C^{***}T^{***}}{T}\Big)\Bigg] \\ H = &\Bigg[\frac{\beta}{C_m}(C-C^{***})^2 + \frac{\omega}{N}(P-P^{***})^2 + \frac{a_0}{R}(R-R^{***})^2 + \frac{b_0}{E}(E-E^{***})^2 \\ &+ \frac{m_0}{T}(T-T^{***})^2 + d_1(CP + C^{***}P^{***}) + \phi(CP^{***} + C^{***}P) \\ &+ \tau(PR^{***} + P^{***}R) + a_1\Big(C^{***} + \frac{CR^{***}}{C}\Big) + b_1\Big(C^{***} + \frac{CE^{***}}{E}\Big) \\ &+ m_1\Big(C^{***} + \frac{CT^{***}}{T}\Big)\Bigg]. \end{aligned}$$

Thus, if $G<H$, then $D^r V \leq 0$. Therefore, the largest compact invariant set in $\{(C^{***},P^{***},R^{***},E^{***},T^{***}) \subset \Phi : D^r V = 0\}$

is the singleton $\{\varepsilon_3\}$, where ε_3 is the fourth equilibrium point of the model system given by equations (1) to (5). Hence, using the Lasalle's Invariance Principle given in lemma 5, we conclude that ε_3 is globally asymptotically stable in Φ if $G < H$.

5. Results

5.1. Parameters and their Values

The following parameter values (as also presented in Ref. [18] for the classical case) were used for the numerical simulation of the model in MATLAB using the Predict-Evaluate-Correct- Evaluate method of Adams-Bashforth-Moulton [53]. The initial conditions of the model were set as: $C_0 = 1$, $P_0 = 4$, $R_0 = 2$, $E_0 = 5$ and $T_0 = 7$.

6. Discussion

Three cases between the accumulation rate of carbon dioxide, β, and the intrinsic growth rate of the photosynthetic biomass, ω, for four different orders of the model are depicted in Figure 1. By varying both the intrinsic accumulation rate of carbon dioxide, β, and the intrinsic growth rate of the photosynthetic biomass, ω, for different orders of the model of $r = 1.0, r = 0.9, r = 0.8$ and $r = 0.7$, the maximum excessive concentration of carbon dioxide corresponding to these orders obtained were $(2.2759, 1.9971, 1.5504)$, $(1.6862, 1.554, 1.3069)$, $(1.3311, 1.2759, 1.1475)$ and $(1.1281, 1.1098, 1.0507)$ respectively. From these results, lower values of the excessive concentration were obtained as the order reduces. The excessive concentration of carbon dioxide was less for $\beta < \omega$ compared to when it was $\beta > \omega$. $\beta < \omega$ is supposed to be the idea situation, as the expectation would be that emitted carbon dioxide should be removed by the natural mechanism in place (photosynthetic biomass). However, reality is represented by $\beta > \omega$, as the natural mechanism is overwhelmed and thereby leading to excessive accumulation of this greenhouse gas. Hence, other mitigation measures are needed.

In Figure 2, the effect of both accumulation rate of carbon dioxide and the photosynthetic rate for different orders of $r = 1.0, 0.9, 0.8, 0.7$ are simulated. The natural mechanism (sink) in place to regulate the concentration of carbon dioxide are the photosynthetic biomass. For four different orders of $r = 1.0$, $r = 0.9$, $r = 0.8$ and $r = 0.7$, three cases of relationships between the intrinsic accumulation rate of carbon dioxide, β and the rate of reduction of carbon dioxide by the photosynthetic biomass, d_1, were shown. Corresponding to these model orders of $r = 1.0$, $r = 0.9$, $r = 0.8$ and $r = 0.7$, the values of the excessive concentration obtained were $(6.347, 1.9971, 1.5504)$, $(1.554, 1, 1)$, $(1.2759, 1, 1)$ and $(1.1098, 1, 1)$ respectively. As seen in Figure 2, $\beta > d_1$ means that the rate of accumulation of carbon dioxide surpasses the rate at which the same gas is removed from the atmosphere. Hence, something needs to be done to reduce this accumulation into the atmosphere. $\beta < d_1$ represents a scenario where the excessive concentration of carbon dioxide is effectively checked by the photosynthetic biomass. Unfortunately, this scenario does not represent the current reality.

The variation of the accumulation rate of carbon dioxide for different orders of the model are shown in Figure 3. Three different values of $\beta = 6, \beta = 3$ and $\beta = 1.5$ for the model orders $r = 1.0$, $r = 0.9$, $r = 0.8$ and $r = 0.7$ correspondingly produced values of the maximum excessive concentration of carbon dioxide as $(19.651, 11.625, 3.8053)$, $(15.055, 7.3906, 2.5096)$, $(9.948, 4.2791, 1.747)$ and $(5.5851, 2.4337, 1.3241)$ respectively. These represent reduction in the maximum excessive concentration of carbon dioxide in the atmosphere to about $(40.84\%, 80.03\%)$, $(50.91\%, 83.33\%)$, $(56.99\%, 82.44\%)$ and $(56.43\%, 76.29\%)$ respectively. The lesser the value of the accumulation rate, β, the lower the concentration of carbon dioxide in the atmosphere. This can be achieved by putting effective and efficient mitigation measures in place to reduce the excessive concentration. Such mitigations could include: good conservations policies, enlightenment programmes and the use of technology that can capture and store the carbon dioxide.

In Figure 4, three different maximum tolerated concentration values, $C_m = 25, C_m = 15, C_m = 10$, of carbon dioxide are shown for four different values of $r = 1.0$, $r = 0.9$, $r = 0.8$ and $r = 0.7$. The values of the maximum excessive concentration of carbon dioxide obtained corresponding to these values and the four orders of $r = 1.0$, $r = 0.9$, $r = 0.8$ and $r = 0.7$ were $(19.651, 12.569, 8.6625)$, $(15.055, 10.239, 7.3234)$, $(9.948, 7.3578, 5.5722)$ and $(5.5851, 4.5472, 3.7141)$ respectively . Th percentage reduction of the excessive concentration equivalent to these values are $(36.04\%, 55.92\%)$, $(31.99\%, 51.36\%)$, $(26.04\%, 43.99\%)$ and $(18.58\%, 33.50\%)$ respectively. From the simulated results, the lower the value of the maximum tolerated concentration, C_m, the lower the excessive concentration of carbon dioxide in the atmosphere. $C_m = 10$ gave least values for excessive concentration of carbon dioxide compared to $C_m = 25$ which gave highest values for the various orders simulated.

The effect of varying both the natural and artificial depletion rates of the photosynthetic biomass on the concentration of carbon dioxide in the atmosphere are shown for different orders of $r = 1.0$, $r = 0.9$, $r = 0.8$ and $r = 0.7$ in Figure 5. Different values of $\mu_1 = (0.02, 0.002, 0.0001)$ and $\mu_2 = (0.04, 0.0088, 0.0002)$ simulated simultaneously for different orders generated values for the maximum excessive concentration of carbon dioxide available in the atmosphere as $(19.432, 10.096, 3.8807)$, $(14.646, 5.9705, 2.3118)$, $(9.3417, 3.2197, 1.5201)$ and $(4.9224, 1.8162, 1.1653)$ respectively. Corresponding to these values, $(48.04\%, 80.03\%)$, $(59.23\%, 84.23\%)$, $(65.53\%, 83.73\%)$ and $(63.10\%, 76.33\%)$ are the percentage reduction in the maximum excessive concentration of carbon dioxide for the different orders of $r = 1.0$, $r = 0.9$, $r = 0.8$ and $r = 0.7$. Putting measures in place to reduce the activities that contribute to these depletion can reduce the excessive concentration (as seen in figure 5), as reducing $\mu_1 = 0.02$ to $\mu_1 = 0.0001$ and $\mu_2 = 0.04$ to $\mu_2 = 0.0002$ showed.

By keeping the natural depletion rate of the photosynthetic biomass constant in Figure 6, the effect of variation of only the artificial depletion rate of the photosyn-

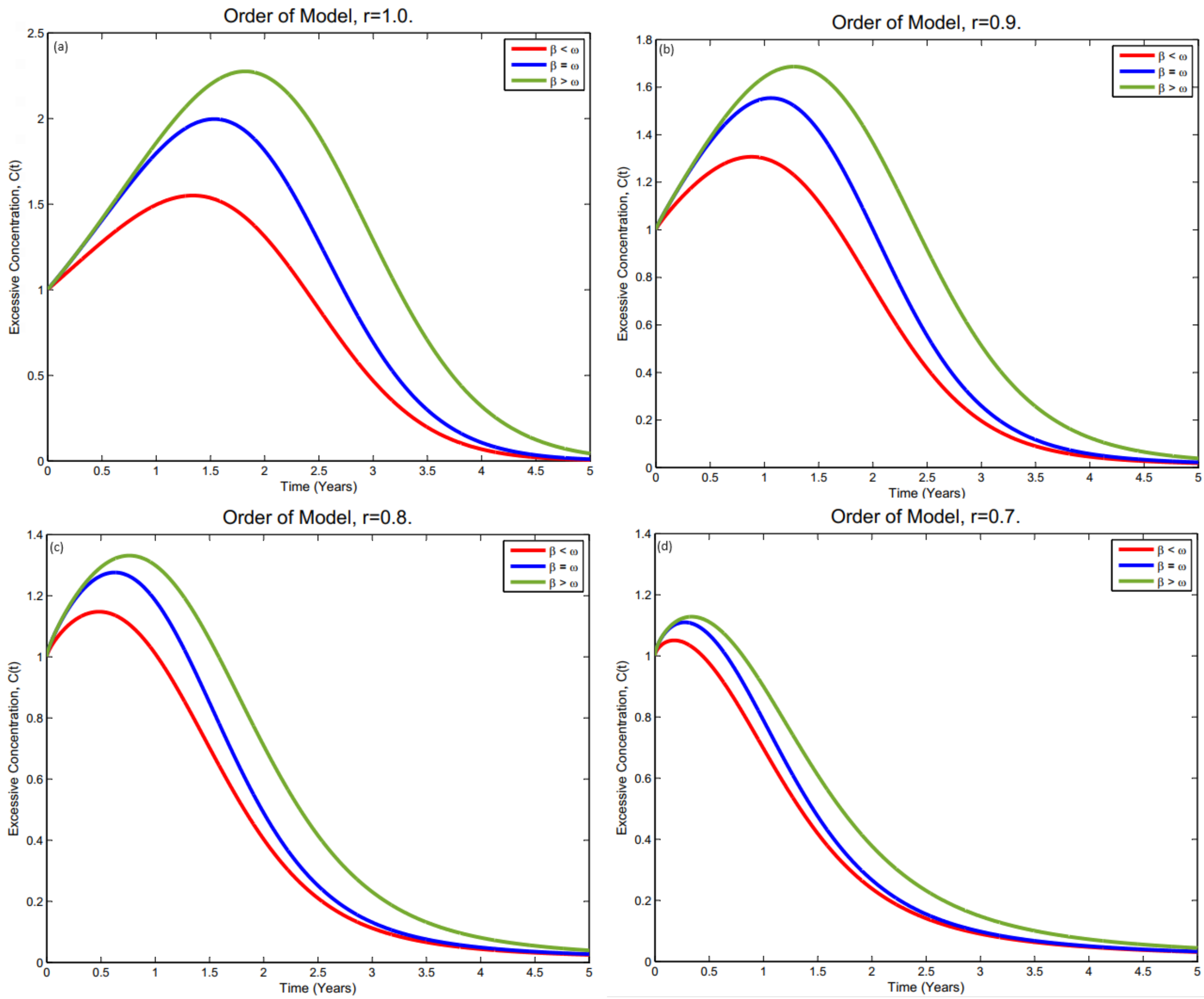

Figure 1: Effect of comparing accumulation rate of carbon dioxide with intrinsic growth rate of photosynthetic biomass for different orders of the model.

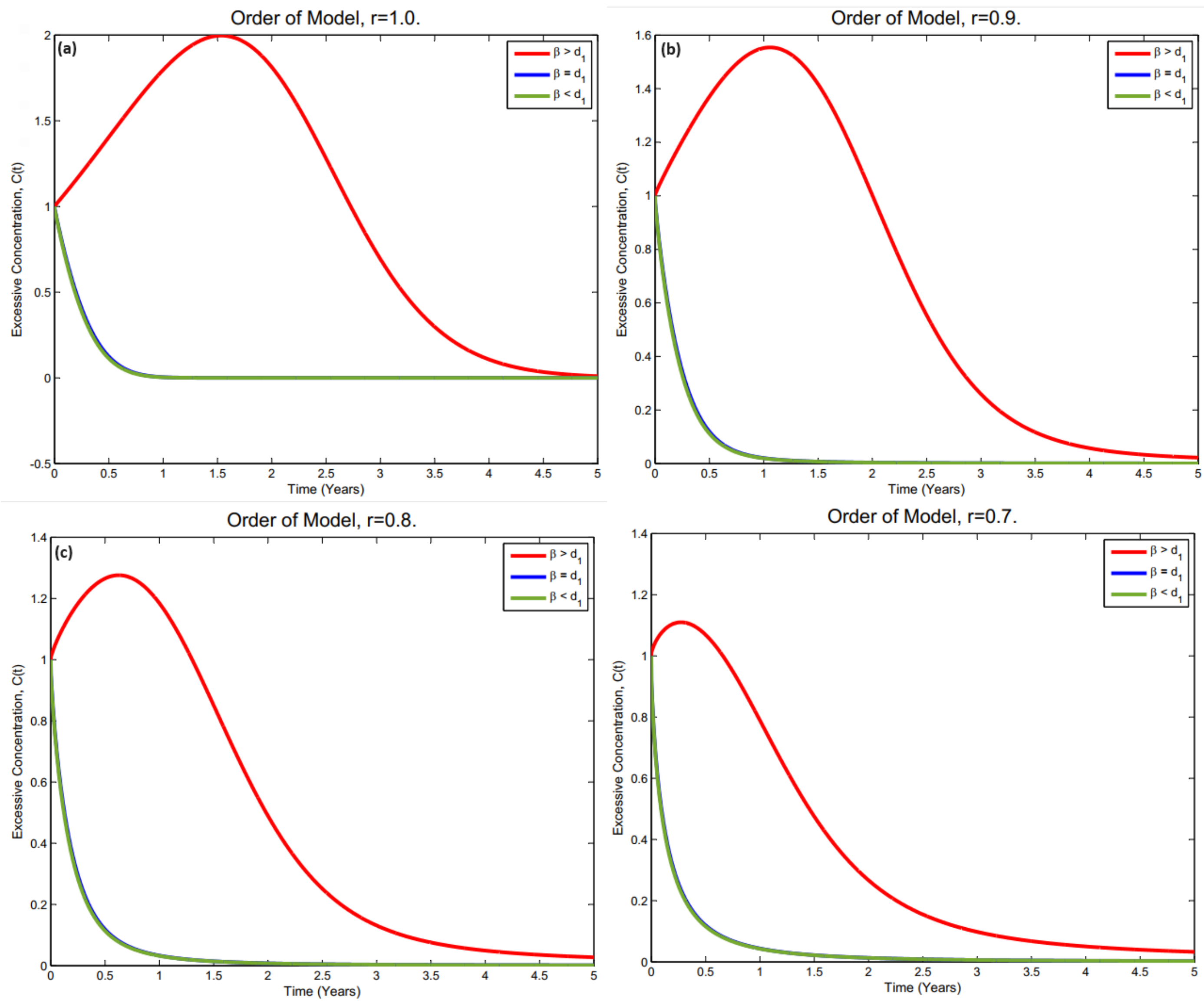

Figure 2: Effect of comparing the accumulation rate of carbon dioxide with the photosynthetic rate of the photosynthetic biomass for a variation of the order of the model

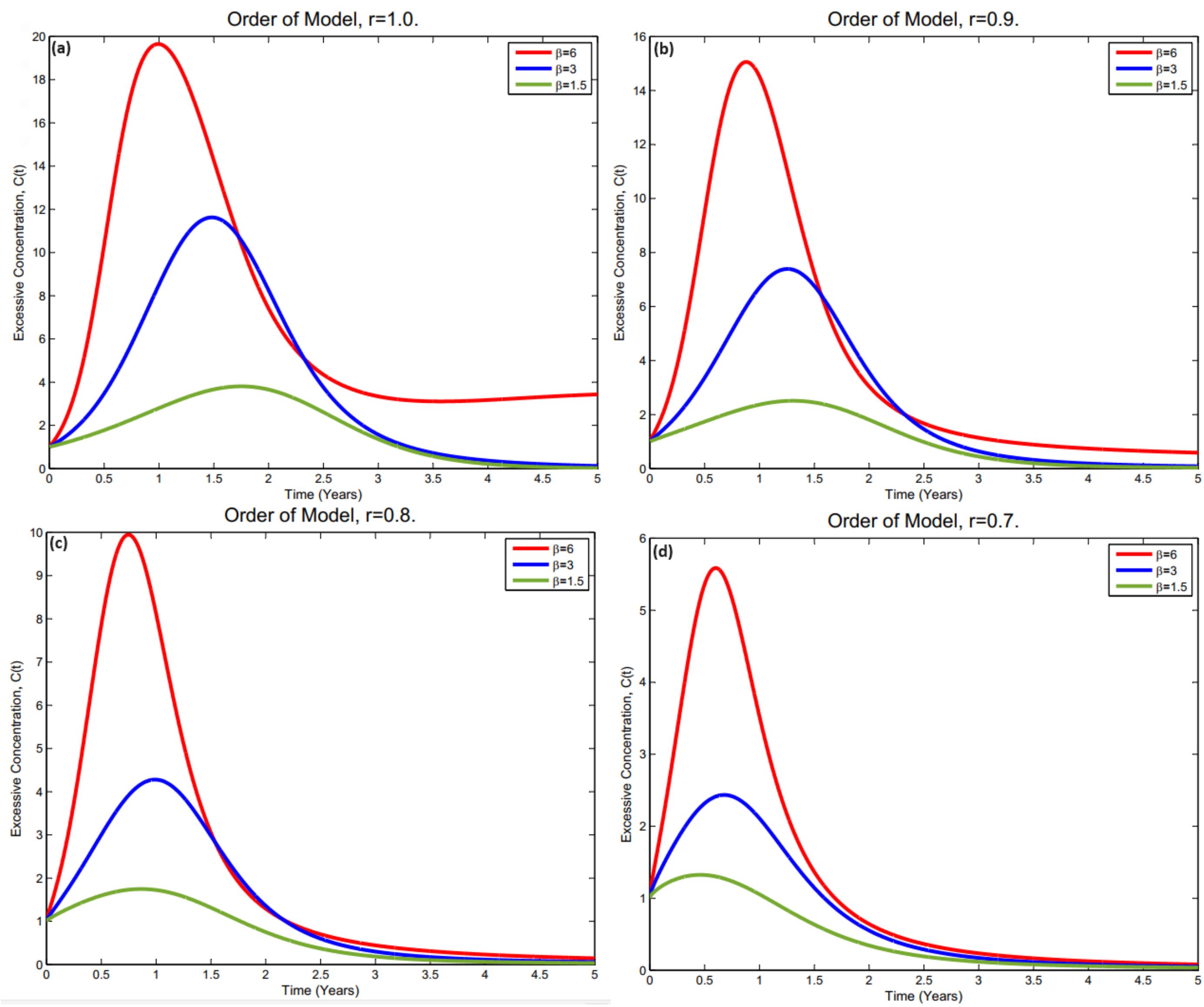

Figure 3: Effect of the accumulation rate of carbon dioxide on the excessive concentration of carbon dioxide for a variation of the order of the model

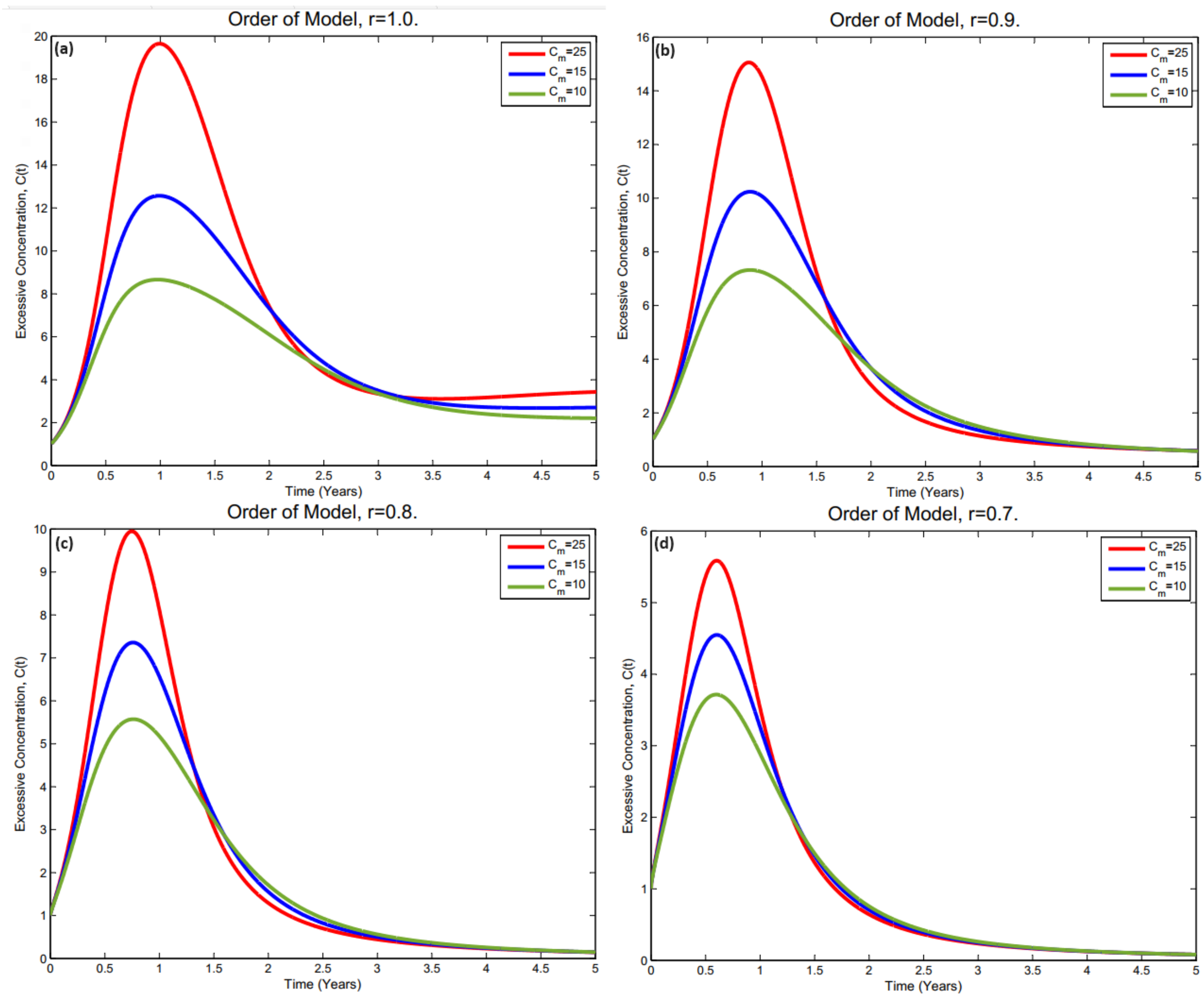

Figure 4: Effect of variation of maximum tolerated concentration of carbon dioxide on the excessive concentration of carbon dioxide for different orders of the model

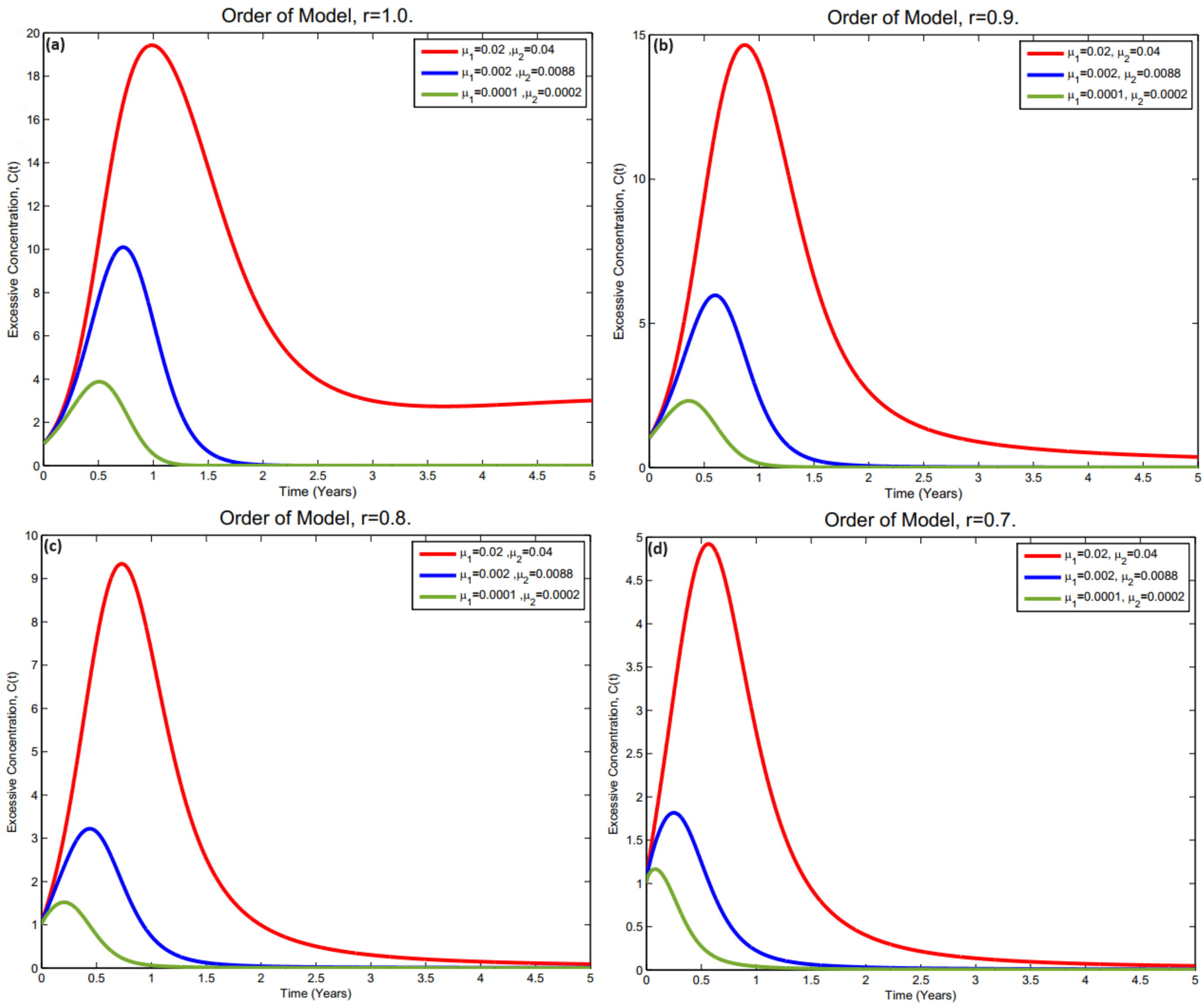

Figure 5: Effect of variation of both the natural and artificial depletion of the photosynthetic biomass on the excessive concentration of carbon dioxide for variation of the order of the model

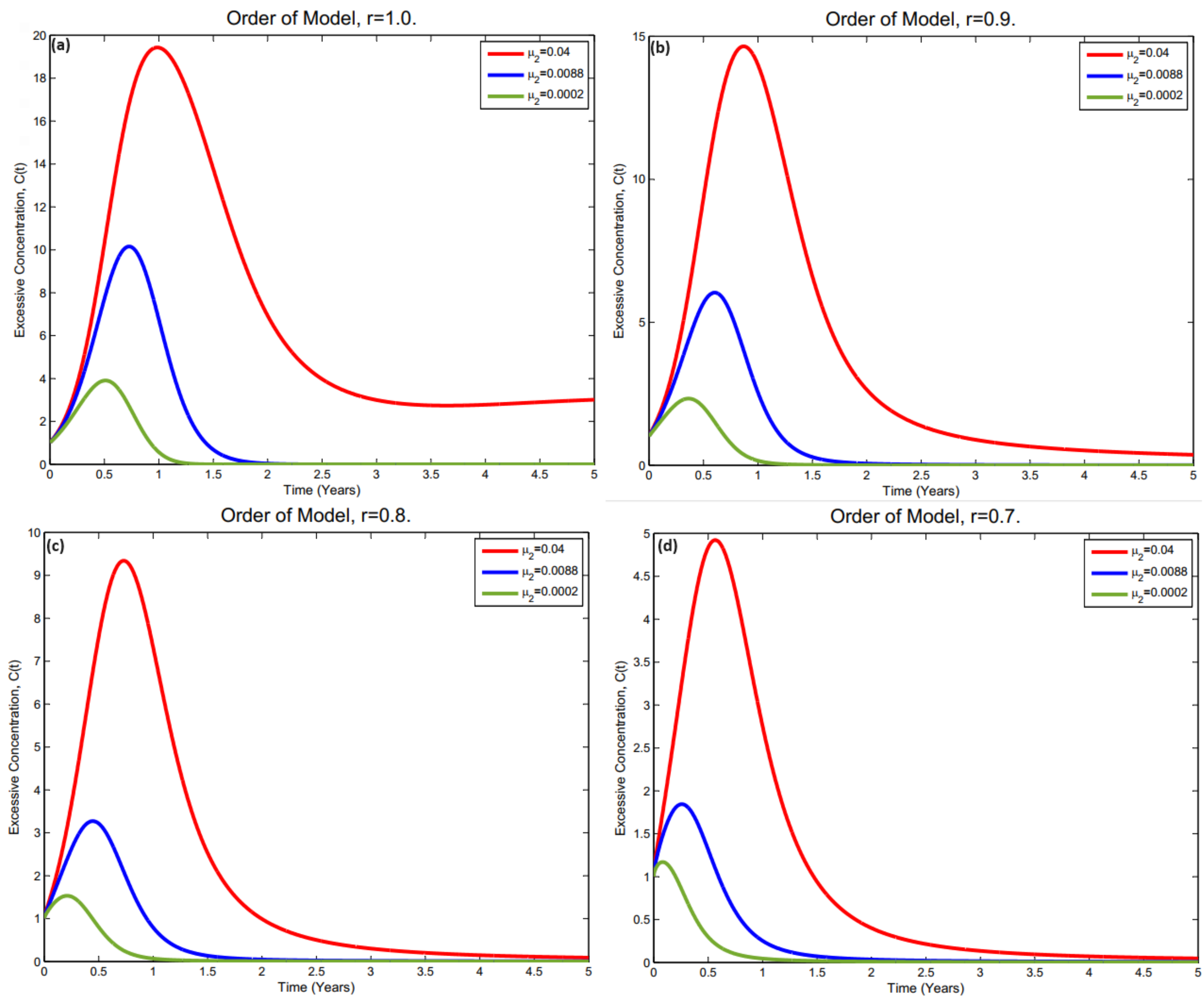

Figure 6: Effect of variation of only the artificial depletion of the photosynthetic biomass on excessive concentration of carbon dioxide for a variation of the order of the model

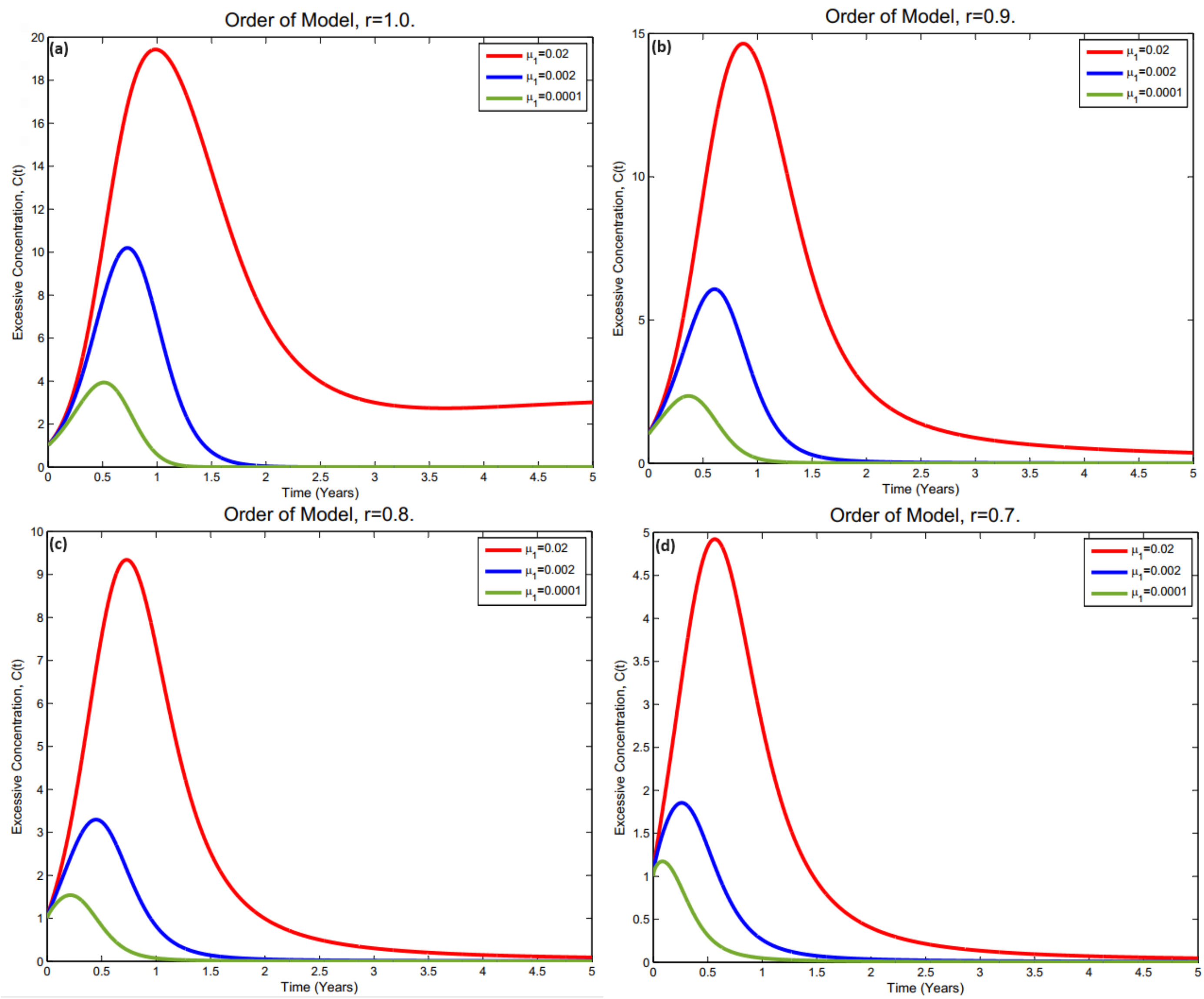

Figure 7: Effect of varying only the natural depletion rate of the photosynthetic biomass on excessive concentration of carbon dioxide for a variation of the model order

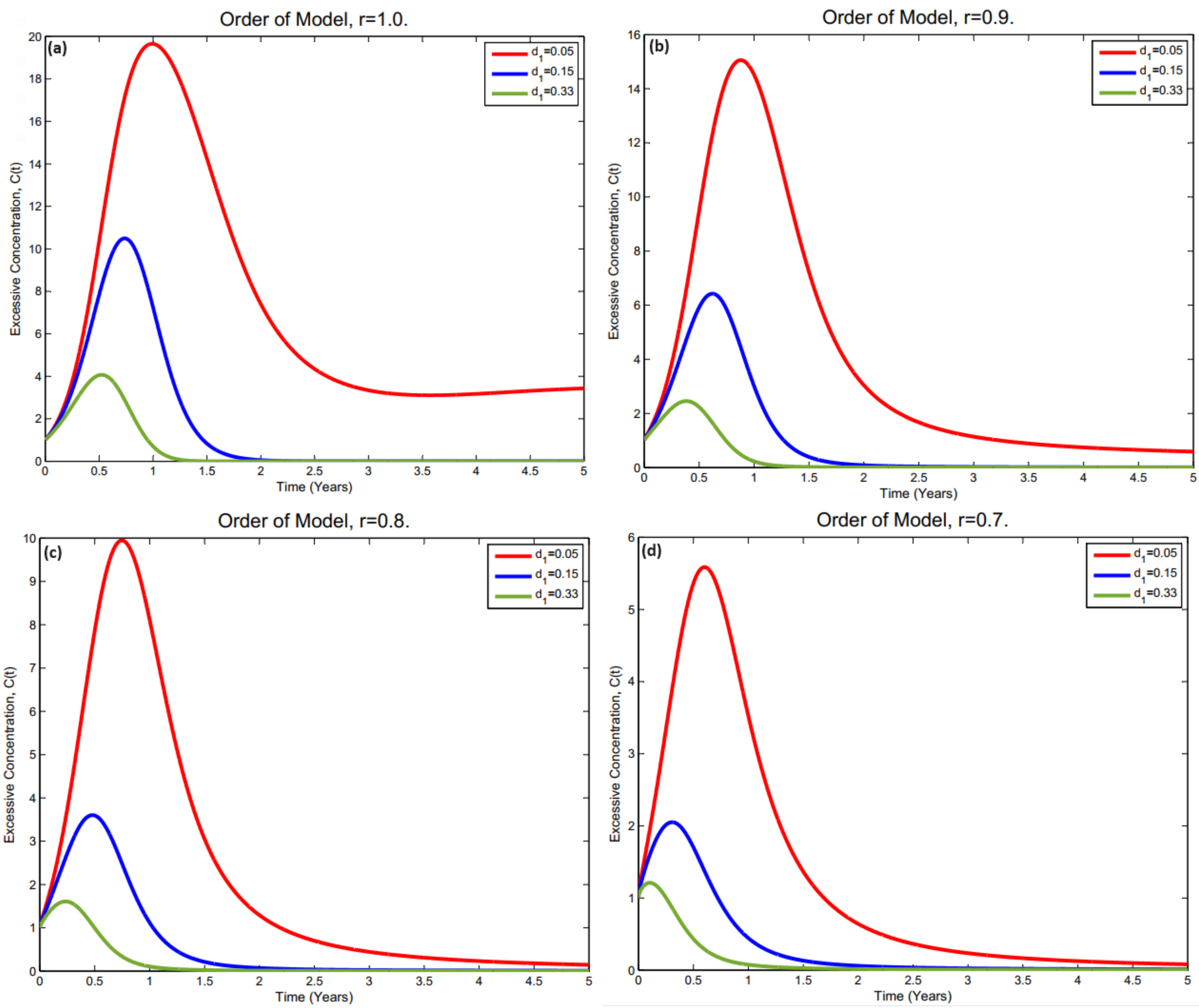

Figure 8: Effect of the photosynthetic biomass on excessive concentration of carbon dioxide for a variation of the model order

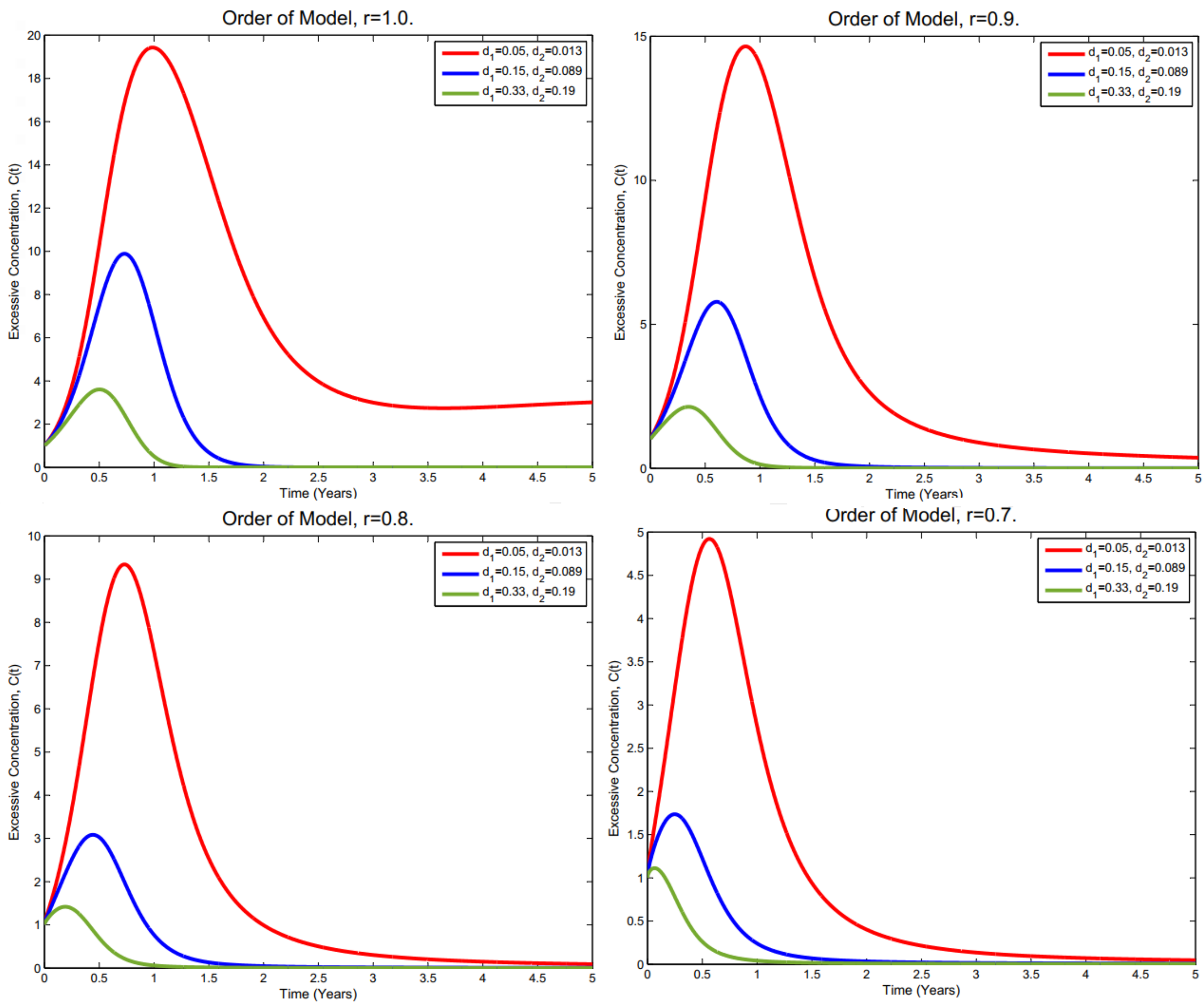

Figure 9: Effect of photosynthetic biomass and good conservation policies on excessive concentration of carbon dioxide for a variation of the order of the model

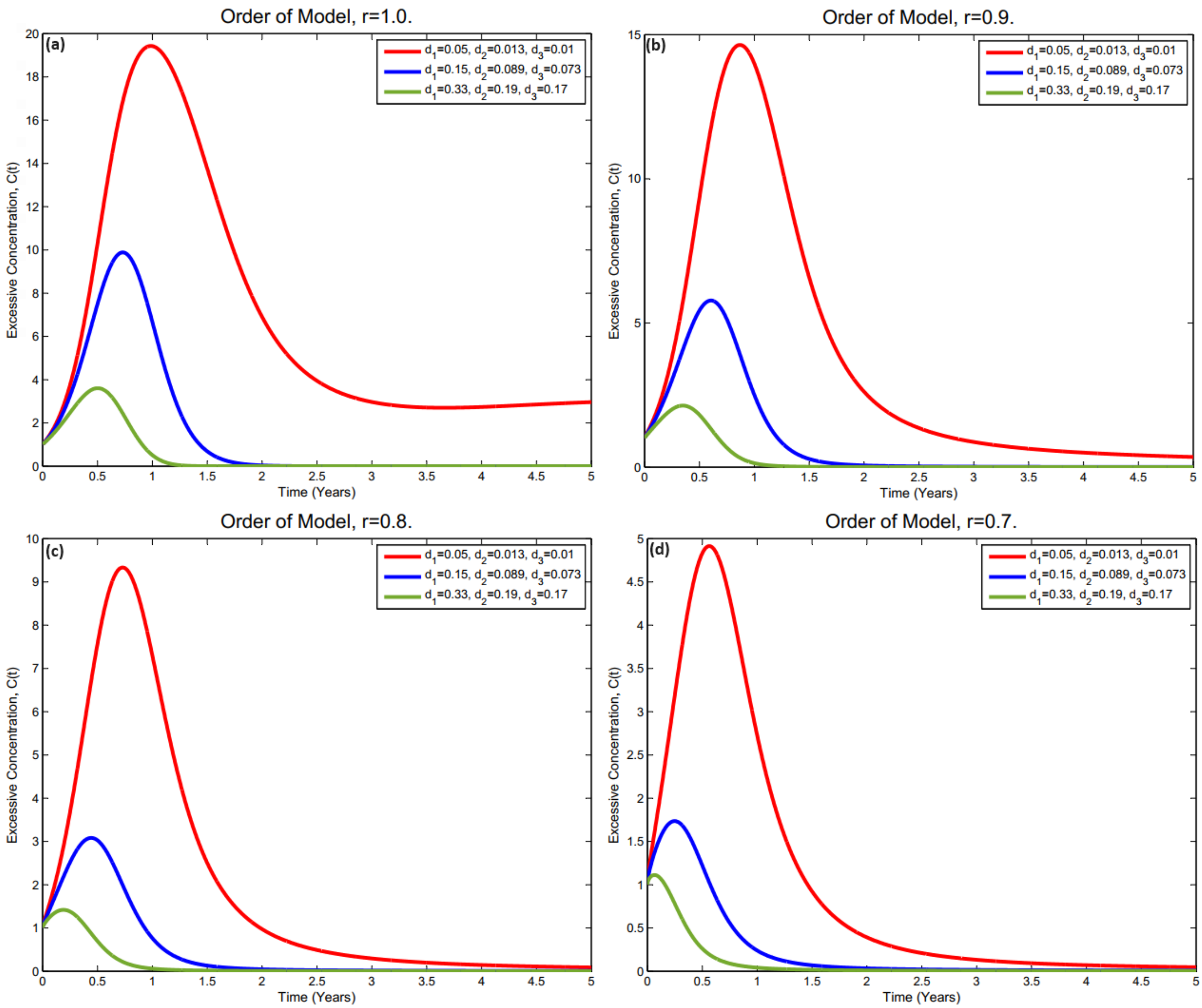

Figure 10: Effect of photosynthetic biomass, good conservation policies and enlightenment programmes on excessive concentration of carbon dioxide for a variation of the order of the model

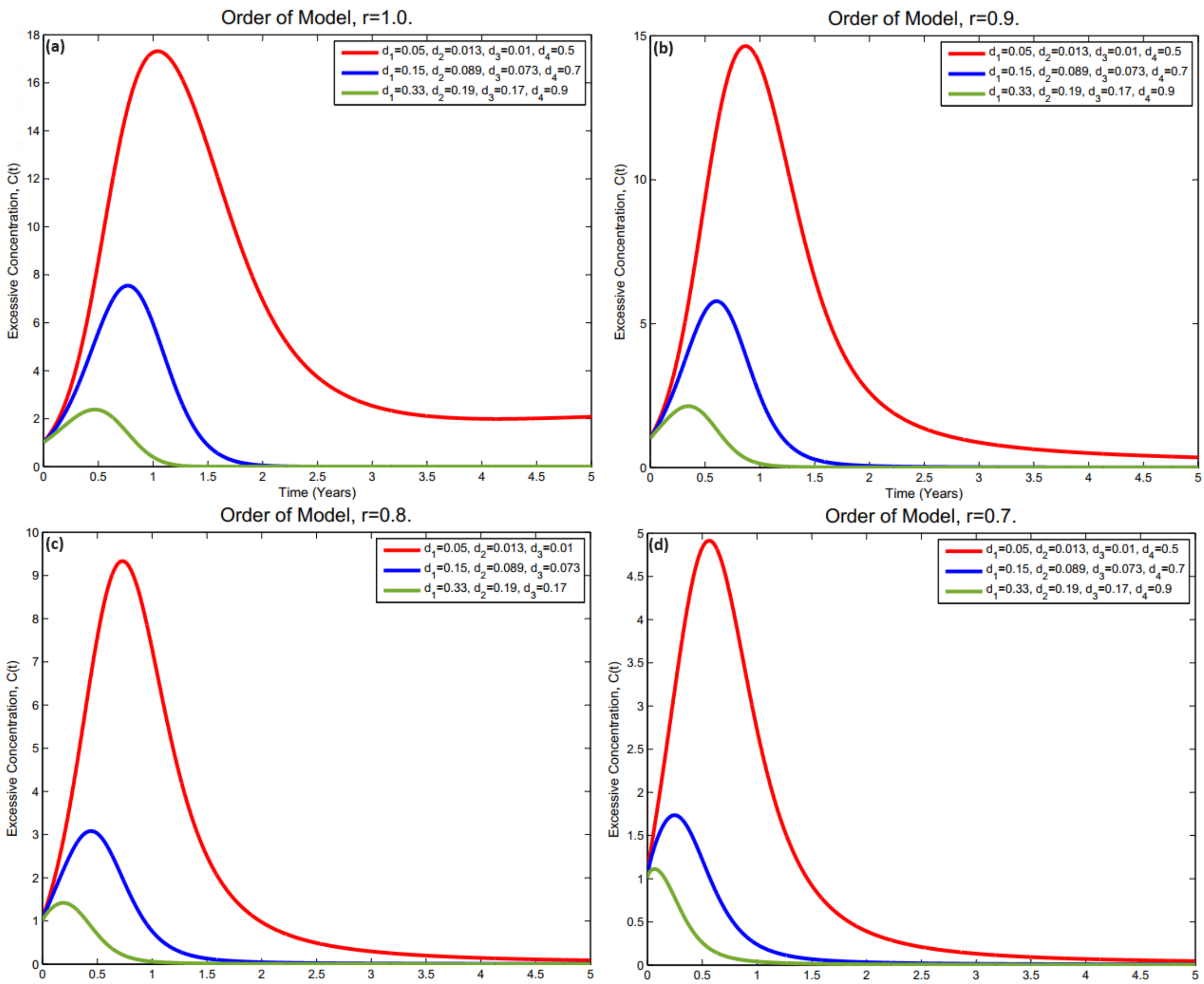

Figure 11: Effect of photosynthetic biomass, good conservation policies, enlightenment programmes and direct air capture technology on excessive concentration of carbon dioxide for a variation of the order of the model

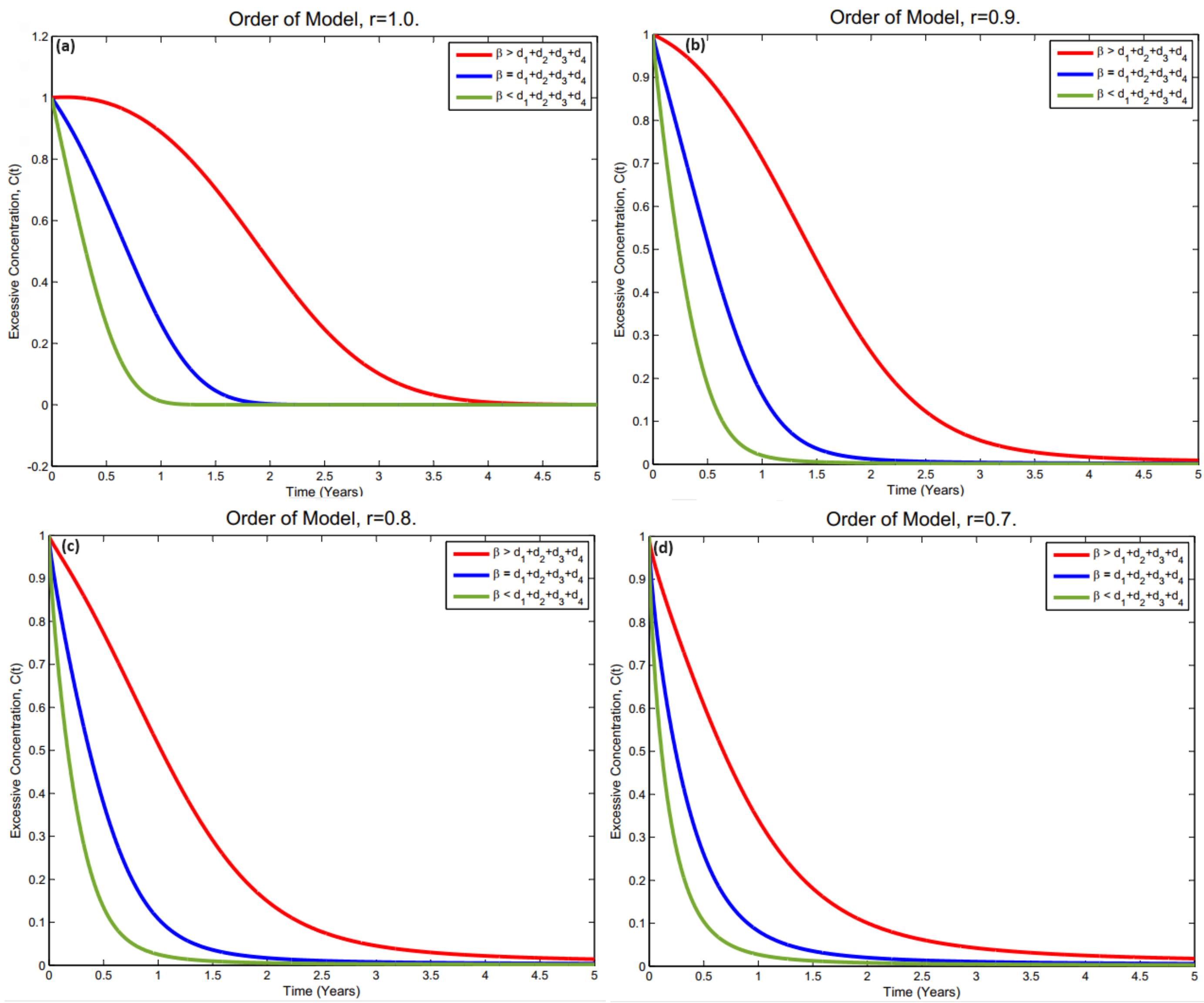

Figure 12: Effect of comparing the accumulation rate of CO_2 with combined proportions of success of photosynthetic biomass, good conservation policies, enlightenment programmes and direct air capture technology on excessive concentration of carbon dioxide for a variation of the order of the model

Table 2: Model Parameters and Their Corresponding Values [18]

Parameters	Values	Units	References
β	1 and 6	μmol per mol m^{-2}	[14], [44]
C_m	25	m^{-2}	Fixed
d_1	0.05	m^{-2}	[15]
d_2	0.013	m^{-2}	Fixed
d_3	0.01	m^{-2}	Fixed
d_4	0.5	m^{-2}	[54]
μ_0	0.016	$year^{-1}$	Fixed
ω	0.8 and 1.0	$year^{-1}$	[15, 17, 44]
N	80	kg m^{-2}	[17]
ϕ	0.1	m^{-2}	[15]
τ	0.03	m^{-2}	[15]
μ_1	0.02	$year^{-1}$	[44]
μ_2	0.04	$year^{-1}$	[17]
a_1	0.008	m^{-2}	Assumed
a_0	0.001	m^{-2}	Assumed
b_1	0.0078	m^{-2}	Assumed
b_0	0.0019	m^{-2}	Assumed
m_1	0.0068	m^{-2}	Assumed
m_0	0.0012	m^{-2}	Assumed

thesis biomass are shown for different orders of the model. Three different values of the artificial depletion rate $\mu_2 = 0.04$, $\mu = 0.0088$, $\mu_2 = 0.0002$ gave results for the maximum excessive concentration of carbon dioxide in the atmosphere as $(19.432, 10.154, 3.9079)$, $(14.646, 6.0364, 2.3319)$, $(9.3417, 3.2725, 1.5318)$ and $(4.9224, 1.8448, 1.1704)$ respectively. These results equivalently represent $(47.75\%, 79.89\%)$, $(58.78\%, 84.08\%)$, $(64.97\%, 83.60\%)$ and $(62.52\%, 76.22\%)$ decrease in the values of the maximum excessive concentration of carbon dioxide. The artificial depletion are mainly as a result of human activities in the quest for survival, civilization, industrialization and urbanization. A reduction of $\mu_2 = 0.04$ to $\mu_2 = 0.0002$ brought about a corresponding decrease in the excessive concentration of carbon dioxide. This can be done by putting in good measures such as regulating deforestation.

Three different values of the natural depletion rate $\mu_1 = 0.02$, $\mu_1 = 0.002$ and $\mu_1 = 0.0001$ were simulated for the maximum excessive concentration of carbon dioxide while keeping the artificial depletion rate at a fixed value of $\mu_2 = 0.04$. The corresponding values of the maximum excessive concentration obtained for the different orders $r = 1.0$, $r = 0.9$, $r = 0.8$ and $r = 0.7$ were $(19.432, 10.197, 3.9355)$, $(14.646, 6.0756, 2.3498)$, $(9.3417, 3.297, 1.5408)$ and $(4.9224, 1.8545, 1.1738)$. By calculation, these values correspondingly represent $(47.75\%, 79.75\%)$, $(58.52\%, 83.96\%)$, $(64.71\%, 83.51\%)$ and $(62.33\%, 76.15\%)$ reduction in the maximum concentration values respectively. A variation of the natural depletion rate of the photosynthetic biomass from $\mu_1 = 0.02$ to $\mu_1 = 0.0001$ as depicted in Figure 7, led to a proportional reduction in excessive concentration of carbon dioxide in the atmosphere. Hence, the higher the rate of natural depletion of the photosynthetic biomass, the more the concentration of carbon dioxide that would be present in the atmosphere and vice versa.

In Figure 8, the effect of photosynthetic biomass density on the dynamics of reducing the excessive concentration of carbon dioxide in the atmosphere is depicted. By simulating for the different orders $r = 1.0$, $r = 0.9$, $r = 0.8$ and $r = 0.7$, the maximum excessive concentration values for three different parameter values of $d_1 = 0.05$, $d_1 = 0.015$ and $d_1 = 0.33$ obtained were $(19.651, 10.494, 4.0699)$, $(15.055, 6.4239, 2.4546)$, $(9.948, 3.6025, 1.6085)$ and $(5.5851, 2.0498, 1.202)$ respectively. Expressing these results in percentage, they are equivalent to $(46.60\%, 79.29\%)$, $(57.33\%, 83.70\%)$, $(63.79\%, 83.83\%)$ and $(63.30\%, 78.33\%)$ reduction in the values of the maximum excessive concentration of carbon dioxide respectively. Improving the density of the photosynthetic biomass can lead to a reasonable reduction of the concentration of carbon dioxide as seen in Figure 8 by increasing $d_1 = 0.05$ to $d_1 = 0.15$ and $d_1 = 0.33$ respectively.

The effect of the combination of the photosynthetic biomass and good conservation policies rates are presented in Figure 9. A variation of three different values of $(d_1, d_2) = \{(0.05, 0.013), (0.15, 0.089), (0.33, 0.19)\}$ for the different orders $r = 1.0$, $r = 0.9$, $r = 0.8$ and $r = 0.7$ were simulated. The maximum excessive concentration of carbon dioxide corresponding to these parameter variations and different orders respectively were $(19.432, 9.8859, 3.6062)$, $(14.646, 5.7804, 2.1335)$, $(9.3417, 3.0832, 1.4184)$ and $(4.9224, 1.7359, 1.1126)$. The percentage reduction in the values of the maximum excessive concentration of carbon dioxide were equivalent to $(49.13\%, 81.44\%)$, $(60.53\%, 85.43\%)$, $(67.00\%, 84.82\%)$ and $(64.73\%, 77.40\%)$ respectively. The results obtained are lower compared to those obtained in Figure 8. This means that combining two mitigation measures have better effect than using only one.

The effect of combining the photosynthetic biomass, good conservation policies and enlightenment programmes are presented in Figure 10 for four different orders of the model. For the different orders of $r = 1.0$, $r = 0.9$, $r = 0.8$ and $r = 0.7$, a simulation of three different values of the parameters $(d_1, d_2, d_3) = \{(0.05, 0.013, 0.01), (0.15, 0.089, 0.073), (0.33, 0.19, 0.17)\}$ was done. The corresponding values of the maximum excessive concentration of carbon dioxide got were $(19.4300, 9.8848, 3.6060)$, $(14.642, 5.779, 2.1334)$, $9.3342, 3.0822, 1.4183$ and $(4.9148, 1.7355, 1.1126)$ respectively. A calculation of the percentage decrease in the maximum excessive concentration values of carbon dioxide gave $(49.13\%, 81.44\%)$, $(60.53\%, 85.43\%)$, $(66.98\%, 84.81\%)$ and $(64.69\%, 77.36\%)$ respectively. The results obtained were better compared to those obtained for a single measure (Figure 8) as well as a combination of two measures as well (Figure 9).

In Figure 11, the effect of the photosynthetic biomass, good conservation policies, enlightenment programmes and direct air capture technology are shown for different values of the associated parameters and order of the model. Different variations of the parameters involved, $(d_1, d_2, d_3, d_4) = \{(0.05, 0.013, 0.01, 0.5), (0.15, 0.089, 0.073, 0.7), (0.33, 0.19, 0.17, 0.9)\}$ for the different orders $r = 1.0$, $r = 0.9$, $r = 0.8$ and $r = 0.7$ gave the results $(17.316, 7.5432, 2.3793)$, $(14.642, 5.779, 2.1334)$, $(9.3342, 3.0822, 1.4183)$ and $(4.9148, 1.7355, 1.1126)$ respectively. Calculating the percentage decrease in the maximum excessive concentration of carbon dioxide from these results gave $(56.44\%, 86.26\%)$, $(60.53\%, 85.43\%)$, $(66.98\%, 84.81\%)$ and $(64.69\%, 77.36\%)$ respectively. This combination gave the best result as the excessive concentration of carbon dioxide was lowest compared to the results in Figures 8, 9 and 10.

In Figure 12, we look at different scenarios involving the excessive accumulation of carbon dioxide into the atmosphere and the collective combination of the mitigation measures. By taking different parameter values for $\beta > d_1 + d_2 + d_3 + d_4$, $\beta = d_1 + d_2 + d_3 + d_4$, $\beta < d_1 + d_2 + d_3 + d_4$ and for the different orders of $r = 1.0$, $r = 0.9$, $r = 0.8$ and $r = 0.7$, the maximum excessive concentration of carbon dioxide obtained were $(1.0019, 1, 1)$, $(1, 1, 1)$, $(1, 1, 1)$ and $(1, 1, 1)$ respectively. As seen from the figure, more concentration of carbon dioxide was available in the atmosphere when $\beta > d_1 + d_2 + d_3 + d_4$ compared to when $\beta < d_1 + d_2 + d_3 + d_4$ which would represent a logical explanation of what should be expected.

7. Conclusion

A fractional order deterministic model on climate change of five compartments is presented. The compartments comprises: Excessive Concentration of Carbon Dioxide, Photosynthetic Biomass, Good Conservation Policies, Enlightenment Programmes and Direct Air Capture Technology. The proofs of the uniqueness and existence of the model solution, the positivity of the model solution as well as the boundedness of the model solution were shown. By constructing different Lyapunov functions, the global stability analysis of the four equilibrium points was done. The simulation of the model for four different orders of 1.0, 0.9, 0.8 and 0.7 was done in Matlab using the Predict-Evaluate-Correct-Evaluate (PECE) method of Adam-Bashforth-Moulton. The results obtained were presented graphically and discussed. Findings revealed that the excessive concentration of carbon dioxide can be reduced by effective implementation of mitigation measures.

Acknowledgment

Gratitude to the African Union Commission for the scholarship opportunity to contribute to research for growth and development globally and Africa in particular.

References

[1] J. Song, Q. Song, D. Zhang, Y. Lu,& L. Luan, "Study on infuencing factors of carbon emissions from energy consumption of shandong province of china from 1995 to 2012", The Scientific World Journal **2014** (2014) 1. https://doi.org/10.1155/2014/684796

[2] C. Zhu & D. Gao, "A research on the factors influencing carbon emission of transportation industry in "the belt and road initiative" countries based on panel data", Energies **12** (2019) 1. https://doi.org/10.3390/en12122405.

[3] K. Peterson, *Explaining human influences on carbon dioxide emissions across countries*, Honors Projects, 2009. https://digitalcommons.iwu.edu/econ_honproj/100.

[4] F. Vadillo, "On the walker's model for the carbon dioxide in the earth's atmosphere", Journal of Environmental Protection **10** (2019) 48. https://doi.org/10.4236/jep.2019.101004.

[5] S. Banaji, "Nitrogen dioxide, carbon monoxide, natural and anthropomorphic effects, and earth's changing climate", Atmospheric and Climate Sciences, **13** (2023) 62. https://doi.org/10.4236/acs.2023.131005.

[6] O. O. Ayeleru, H. U. Modekwe, O. R. Onisuru, C. R. Ohoro, C. A. Akinnawo, & P. A. Olubambi, "Adsorbent technologies and applications for carbon capture, and direct air capture in environmental perspective and sustainable climate action", Sustainable Chemistry for Climate Action **3** (2023) 100029. https://doi.org/10.1016/j.scca.2023.100029.

[7] V. Moutinho, M. Madaleno, R. Inglisi-Lotz, & E. Dogan, "Factors affecting carbon dioxide emissions in top countries on renewable energies : A LMDI decomposition application", Renewable and Sustainable Energy Reviews **90** (2018) 605. https://doi.org/10.1016/j.rser.2018.02.009.

[8] D. K. Almutairi, M. A. Abdoon, S. Y. M. Salih, S. A. Elsamani, F. E. L. Guma, & M. Berir, "Modeling and analysis of a fractional visceral leishmaniosis with caputo and caputo-fabrizio derivatives", J. Nig. Soc. Phys. Sci. **5** (2023) 1453.https://doi.org/10.46481/jnsps.2023.1453.

[9] S. A. Abdulahi, A. G. Habib & N. Hussaini, "Mathematical model of in-host dynamics of snakebite envenoming", J. Nig. Soc. Phys. Sci. **4** (2022)193. https://doi.org/10.46481/jnsps.2022.548.

[10] A. Umar, S. Musa & A. A. Agada, "Analysis of varroosis model in honeybee colony with interventions", African Scientific Reports, **2** (2023)98. https://doi.org/10.46481/asr.2023.2.2.98.

[11] S. Mandal, M. S. I Sonia Akter, & M. H. A. Biswas, "A mathematical model to investigate the frequent impact of global warming on coastal lives", In Procceedings of the International Conference on Industrial and Mechanical Engineering and Operations Management, Dhaka, Bangladesh, 2020. https://www.google.com/url?sa=t&rct=j&q=&esrc=s&source=web&cd=&ved=2ahUKEwj78IbslemBAxW33AIHHUP4A6YQFnoECA\oQAQ&url=http.

[12] A. K. Misra, M. Verma & E. Venturino, "Modelling the control of atmospheric carbon dioxide through reforestation:effect of time delay. Model", Earth Syst. Environ **1** (2015) 1. https://doi.org/10.1007/s40808-015-0028-z.

[13] S. Mondal, M. Biswas & N. Bairagi, "Local and global dynamics of a fractional-order predator-prey system with habitat complexity and the corresponding discretized fractional-order system", Journal of Applied Mathematics and Computing **63** (2020) 1. https://doi.org/10.1007/s12190-020-01319-6.

[14] B. F. Etin-Osa & I. M. Azubike, "Parametric sensitivity analysis of a mathematical model of the effect of carbon dioxide on the climate change", Applied and Computational Mathematics **9** (2020) 96. https://doi.org/10.11648/j.acm.20200903.16.

[15] K. R. Gandhi & M. R. Tailor, "Solution of problems arising due global warming and climate change in the modelling process", International Journal for Research in Applied Science and Engineering Technology **8** (2020) 1. https://www.ijraset.com/fileserve.php?FID=28154.

[16] S. Sundar, R. Naresh, B. Dubey & J. B. Shukla, "Modeling the removal of cement dust particles from a chimney by water sprays and green plantation", International e-Journal of Mathematical Modeling and Analysis of Complex Systems: Nature and Society **1** (2015) 52.

[17] A. H. Teru & P. R. Koya, "Mathematical modelling of deforestation of forested area due to lack of awareness of human and its conservation", Mathematical Modelling and Applications **5** (2020) 94. http://dx.doi.org/10.11648/j.mma.20200502.15

[18] P. U. Achimugwu, M. N. Kinyanjui & D.M. Malonza, "Mitigation of climate change due to excessive carbon dioxide emission and accumulation: A mathematical model approach", Commun. Math. Biol. Neurosci. **70** (2023) 1. https://doi.org/10.28919/cmbn/8027.

[19] R. Almeida, "Dynamical analysis of a fractional sir model with treatment and quarantine", Chaotic Modelling & Simulation (CMSIM) **2** (2020) 115. https://www.researchgate.net/publication/341407213_Dynamical_analysis_of_a_fractional_SIR_model_with_treatment_and_quarantine.

[20] N. Yuan, Z. Fu, & S. Liu, "Extracting climate memory using fractional integrated statistical model: A new perspective on climate prediction", Scientific Reports **4** (2014) 1. https://doi.org/10.1038/srep06577.

[21] P. Kumar, V. Govindaraj, V. S. Ertuk, & M. S. Mohamed, "Effects of greenhouse gases and hypoxia on the population of aquatic species: A fractional mathematical model", Advances in Continuous and Discrete Models **31** (2022) 1. https://doi.org/10.1186/s13662-022-03679-8.

[22] S. C. Eze & M. O. Oyesanya, "Fractional order on the impact of climate change with dominant earth's fluctuations", Math.Clim.Weather Forecast, **5** (2019) 1. http://dx.doi.org/10.1515/mcwf-2019-0001.

[23] W. Xie, W-Z. Wu, C. Liu, T. Zhang & Z. Dong, "Forecasting fuel combustion-related carbon dioxide emissions by a novel continuous fractional nonlinear grey bernoulli model with grey wolf optimizer", Environmental Science and Pollution Research **28** (2021) 1.https://doi.org/10.1007/s11356-021-12736-w.

[24] E. Ilhan, P. Veeresha & H. M. Baskonus, "Fractional approach for a mathematical model of atmospheric dynamics of carbon dioxide gas with an efficient method", Chaos, Solitons & Fractals **152** (2021) 111347. https://doi.org/10.1016/j.chaos.2021.111347.

[25] R. Ozarslan & Y. Sekerci, "Fractional order oxygen-plankton system under climate change", Chaos: An Interdisciplinary Journal of Nonlinear Science **30** (2020) 1. https://doi.org/10.1063/1.5129766.

[26] S. Qureshi & A. Yusuf, "Mathematical modeling for the impacts of deforestation on wildlife species using caputo differential operator", Chaos, Solitons & Fractals **126** (2019) 32. https://doi.org/10.1016/j.chaos.2019.05.037.

[27] Y. Sekerci & R. Ozarslan, "Oxygen-plankton model under the effect of global warming with nonsingular fractional order", Chaos, Solitons & Fractals **132** (2020) 1. https://doi.org/10.1016/j.chaos.2019.109532.

[28] A. Din, F. M. Khan, Z. U. Khan, A. Yusuf, & T. Munir, "The mathematical study of climate change model under nonlocal fractional derivative", Partial Differential Equations in Applied Mathematics **5** (2022) 1. https://doi.org/10.1016/j.padiff.2021.100204.

[29] J. A. T. Machado & A. M. Lopes, "Dynamical analysis of the global warming", Mathematical Problems in Engineering **2012** (2012) 1. https://doi.org/10.1155/2012/971641.

[30] Z. Xu, L. Liu & L. Wu, "Forecasting the carbon dioxide emissions in 53 countries and regions using a non-equigap grey model", Environmental Science and Pollution Research **28** (2021) 15659. https://doi.org/10.1007/s11356-020-11638-7.

[31] M. F. Elettreby, M. Nabil & A. A. Al-Raezah, "Dynamical analysis of a prey-predator fractional order model", Journal of Fractional Calculus and Applications **8** (2017) 237.https://journals.ekb.eg/article_308437_6d20323e000d0c4621b786befe467f8f.pdf.

[32] A. Suryanto, I. Darti, H. S. Panigoro & A. Kilicman, "A fractional-order predator-prey model with ratio-dependent functional response and linear harvesting", Mathematics **7** (2019) 1.https://doi.org/10.3390/math7111100.

[33] M. F. Carfora & I. Torcicollo, "A fractional-in-time prey-predator model with hunting cooperation: Qualitative analysis, stability and numerical approximations", Axioms **10** (2021) 1. https://doi.org/10.3390/axioms10020078.

[34] C. Vargas-De-Leon, "Volterra-type lyapunov functions for fractional-order epidemic systems", Commun. Nonlinear Sci Numer Simulat **24** (2015) 75. https://doi.org/10.1016/j.cnsns.2014.12.013.

[35] N. Aguila-Camacho, M. A. Duarte-Mermoud, & J. A. Gallegos, " Lyapunov functions for fractional order systems", Commun. Nonlinear Sci. Numer. Simulat. **19** (2014) 2951. https://doi.org/10.1016/j.cnsns.2014.01.022.

[36] I. Ahmed, I. A. Baba, A. Yusuf, P. Kumam, & W. Kumam, " Analysis of caputo fractional-order model for covid-19 with lockdown", Advances in Difference Equations **2020** (2020) 394. https://doi.org/10.1186/s13662-020-02853-0.

[37] M. R. Lemnaouar, M. Khalfaoui, Y. Louartassi & I. Tolaimate, "Fractional order prey-predator model with infected predators in the presence of competition and toxicity", Math. Model. Nat. Phenom. **15** (2020) 1. https://doi.org/10.1051/mmnp/2020002.

[38] H. S. Panigoro, A. Suryanto, W. M. Kusumawinahyu & I. Darti, "Global stability of a fractional-order gause-type predator-prey model with threshold harvesting policy in predator", Commun. Math. Biol. Neurosci. **63** (2021) 1. https://doi.org/10.28919/cmbn/6118.

[39] J. A. Akingbade & B. S. Ogundare, "Boundedness and stability properties of solutions of mathematical model of measles", TAMKANG JOURNAL OF MATHEMATICS **52** (2021) 91. https://doi.org/10.5556/j.tkjm.52.2021.3367.

[40] M. A. Aba Oud, A. Ali, H. Alrabaiah, S. Ullah, M. A. Khan, & S. Islam, "A fractional order mathematical model for covid-19 dynamics with quarantine, isolation, and environmental viral load", Advances in Difference Equations **106** (2021) 1. https://doi.org/10.1186/s13662-021-03265-4.

[41] A. Rehman, R. Singh, T. Abdeljawad, E. Okyere, & L. Guran, "Modeling,analysis and numerical solution to malaria fractional model with temporary immunity and relapse", Advances in Difference Equations **390** (2021) 1.https://doi.org/10.1186/s13662-021-03532-4.

[42] G. T. Tilahun, W. A. Woldegerima & N. Mohammed, "A fractional order model for the transmission dynamics of hepatitis b virus with two-age structure in the presence of vaccination", Arab Journal of Basic and Applied Sciences **28** (2021) 87. https://doi.org/10.1080/25765299.2021.1896423.

[43] S. Sundar, "An ecological type nonlinear model for the removal of carbon dioxide from the atmosphere by introducing liquid species", Combinatorial Ecology and Software **3** (2013) 33. http://www.iaees.org/publications/journals/ces/articles/2013-3.

[44] P. Panja, "Is the forest biomass a key regulator of global warming? A mathematical modelling study", Geology, Ecology, and Landscapes **2020** (2020) 1.https://doi.org/10.1080/24749508.2020.1752021.

[45] M. Kizito & J. Tumwiine, "Mathematical model of treatment and vaccination interventions of pneumococcal pneumonia infection dynamics", Journal of Applied Mathematics **2018** (2018) 1. https://doi.org/10.1155/2018/2539465.

[46] E. Mayanja, L. S. Luboobi, J. Kasozi, & R. N. Nsubuga, "Mathematical modelling of hiv-hcv coinfection dynamics in absence of therapy", Computational and Mathematical Methods in Medicine **2020** (2020) 1. https://doi.org/10.1155/2020/2106570.

[47] I. Zada, M. N. Jan, N. Ali, D. Alrowail, K. S. Nisar, & G. Zaman, " Mathematical analysis of hepatitis B epidemic model with optimal control", Advances in Difference Equations **451** (2021) 1. https://doi.org/10.1186/s13662-021-03607-2.

[48] S. B. Hsu, "A survey of constructing lyapunov functions for mathematical models in population biology", Taiwanese Journal of Mathematics **9** (2005) 151. https://www.jstor.org/stable/43833596.

[49] S. Guo & W. Jiang, "Global stability and hopf bifurcation for gause-type predator-prey system", Journal of Applied Mathematics **2012** (2012) 1. https://doi.org/10.1155/2012/260798.

[50] P. Andayani, "Global stability analysis on a predator-prey model with omnivores", Applied Mathematical Sciences **9** (2015) 1771. http://dx.doi.org/10.12988/ams.2015.5120.

[51] C. Liu, Z. Wang, & B. Weng, "Dynamical analysis of fractional-order holling type-ii food chain model", Mathematical Biosciences and Engineering **18** (2021) 5221. https://doi.org/10.3934/mbe.2021265.

[52] J. Baranowski, M. Zagorowska, W. Bauer, T. Dziwinski & P. Piatek, "Applications of direct lyapunov method in caputo non-integer order systems", Elektronika Ir Elektrotechnika **21** (2015) 1.http://dx.doi.org/10.5755/j01.eee.21.2.11505.

[53] S. Rosa & D. F. M. Torres, "Numerical fractional optimal control of respiratory syncytial virus infection in octave/matlab", Mathematics **11** (2023) 1511. https://doi.org/10.3390/math11061511.

[54] A. Altamirano-Fernandez, A. Rojas-Palma & S. Espinoza-Meza, " Optimal management strategies to maximize carbon capture in forest plantations: A case study with pinus radiata d.don", Forests **14** (2023) 1. https://doi.org/10.3390/f14010082.