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Some theorems on fixed points in bi-complex valued metric spaces with an application to integral equations

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Abstract

Recent studies have highlighted fixed point theorems in the context of bicomplex valued metric spaces, utilizing rational type contractions with coefficients defined by two-variable control functions. In our research, we extend these findings by proposing new theorems for identifying common fixed points within bicomplex valued metric spaces, employing rational type contractions characterized by three-variable control functions as coefficients. We have refined the contraction conditions presented in numerous existing theorems by substituting constants with a limited number of control functions for more versatility in bicomplex valued metric spaces. This advancement broadens the scope of several significant findings in the literature. To demonstrate the efficacy of our results, we offer compelling examples that validate our theorems. Furthermore, we apply our primary findings to effectively address the Urysohn integral equation system, showcasing the practical application of our research.

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1. Introduction

Fixed point theory is the most important branch of functional analysis and its recent developments in different fields of mathematics show its importance and applicability. Fixed point theory has grown in importance as a tool for studying theoretical problems that are directly applicable in a variety of scientific fields. Optimization issues, control theory, economics, and various other applications are just a few examples. It is especially useful in determining whether solutions to differential and integral equations exist because these equations govern the behavior of various real-world problems. At present, the fixed point notion is used to study mathematical models of diseases such as COVID-19, in which such points play a vital role in finding solutions to a given system. Banach has established the Banach contraction principle, which ensures the existence and uniqueness of a fixed point on a complete metric space, in 1922. Despite its remarkable simplicity, it is one of the most frequently used fixed point theorems in all of the analyses. Several authors have extended, generalized, and improved it in many ways and in various spaces.

In mathematics, metric space is one of the most useful and important space. Its broad scope makes it an effective tool for studying variational inequalities, optimization and approximation theory, computer sciences, and other topics. Biology, medicine, physics, and computer science are just a few of the

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pure and applied science fields that have emphasized the value of metric spaces (see Refs. [1, 2]). The concept of metric spaces has been expanded upon by Perov [3] by introducing vector-valued metric spaces. Mustafa and Sims [4] have introduced G-metric spaces and Huang and Zhang [5], the cone metric spaces, and Chistyakov [6], modular metric spaces and more real valued fixed points on different spaces [7, 8]. Many articles have been published about fixed point theory in various metric spaces.

Sir Carl Fredrich Gauss has established the emergence of complex numbers in the 17th century, but his work was not documented. Later, in 1840, Augustin Louis Cauchy, known as the effective founder of complex analysis, began analysis of complex numbers. Segre has initialized the study of bi-complex numbers by providing a commutative substitute to the skew field of quaternions. These numbers are a more precise generalization of complex numbers to quaternions. For a more in-depth look at bi-complex numbers, see Ref. [9].

Azam *et al.* [10] have introduced complex-valued metric spaces as a generalization of the metric spaces in 2011. They have found some fixed point solutions for a pair of mappings that satisfy a rational expression for the contraction condition. As a result, many analytical results could not be generalized to cone metric spaces and could be applied to complex-valued metric spaces. The authors of [11] have established common fixed point theorems that are more general than those of Klineam and Suanoom [12], Rouzkard and Imdad [13], and *et al.* [10] on complex valued metric spaces and expanded and refined the conditions of contraction from the entire space to closed ball. More results on complex valued metric spaces are discussed in [14–18]. Numerous branches of mathematics as well as physics, including hydrodynamics, mechanical engineering, and electrical engineering, benefit from complex-valued metric space.

In 2017, Choi *et al.* [19] have introduced the idea of bi-complex valued metric spaces by combining the concepts of bi-complex numbers and complex-valued metric spaces, and have proved common fixed point results for weakly compatible mappings. In order to define the max function for the partial order in bi-complex valued metric space and to obtain common fixed point results for a pair of mappings, Jebril *et al.* [20] have used the recently introduced space concept. A. J. Gnanaprakasam *et al.* [21] have proved some common fixed point theorems on bi-complex metric space and using this, they have solved the linear system of equations. Recently A.Tassaddiq *et al.* in [22], have used the idea of bi-complex valued metric spaces to get common fixed point results for rational type contractions involving two-variable control functions. For more details about bi-complex valued metric spaces refer [23, 24].

Motivated by these authors, we have proved a number of results on a common fixed point using more general rational type contraction conditions involving three variable control functions as coefficients on bi-complex valued metric space. We have extended and improved the conditions of contraction of many existing theorems by using control functions as coefficients instead of the constants of contraction on bi-complex valued metric spaces. In addition, we have provided a valid example to show the validity of the proven results. As an application, in the context of bi-complex valued metric space, we have developed common fixed point results for rational contractions involving control functions of two variables to a system of Urysohn integral equations involving control functions of three variables. Following are the symbols, notations, definitions, and lemmas relevant to this study.

Let $\mathbb{C}_0, \mathbb{C}_1$ and $\mathbb{C}_2$ be the sets of real, complex, and bi-complex numbers, respectively. According to Segre, a bi-complex number is defined as follows $h = b_1 + i_1 b_2 + i_2 b_3 + b_4 i_1 i_2$ where $b_1, b_2, b_3, b_4 \in \mathbb{C}_0$, and i_1, i_2 are the independent units such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. Here the set $\mathbb{C}_2$ is defined as $\mathbb{C}_2 = \{h : h = b_1 + i_1 b_2 + i_2 b_3 + i_1 i_2 b_4 : b_1, b_2, b_3, b_4 \in \mathbb{C}_0\}$. $(i.e.)$ $\mathbb{C}_2 = \{h : h = z_1 + i_2 z_2 : z_1, z_2, \in \mathbb{C}_1\}$ where $z_1 = b_1 + i_1 b_2 \in \mathbb{C}_1$ and $z_2 = b_3 + i_1 b_4 \in \mathbb{C}_1$. If there exits h, k such that $hk = 1$, then $h = z_1 + i_2 z_2 \in \mathbb{C}_2$ is said to be invertible, and k is called the inverse (multiplicative) of h. As a result, h is known as the inverse of k. An element $h = z_1 + i_2 z_2 \in \mathbb{C}_2$ is nonsingular if and only if $|z_1^2 + z_2^2| \neq 0$ and singular if and only if $|z_1^2 + z_2^2| = 0$. In this order, we represent the set of singular members of $\mathbb{C}_0$ and $\mathbb{C}_1$ by $\aleph_0$ and $\aleph_1$. Many members in $\mathbb{C}_2$ do not have a multiplicative inverse. A bi-complex number $h = b_1 + b_2 i_1 + b_3 i_2 + b_4 i_1 i_2 \in \mathbb{C}_2$ is said to be degenerated if the 2×2 matrix of h is degenerated. At the same time inverse of h exists and it is generated too and the norm $\|.\| : \mathbb{C}_2 \to \mathbb{C}_0^+$ is defined as $\|h\| = \|z_1 + i_2 z_2\| = \sqrt{\frac{|z_1 - i_1 z_2|^2 + |z_1 + i_1 z_2|^2}{2}} = \sqrt{b_1^2 + b_2^2 + b_3^2 + b_4^2}$. A Banach space is the space $\mathbb{C}_2$ with respect to the aforementioned norm. If $h, k \in \mathbb{C}_2$, then $\|hk\| \leq \sqrt{2}\,\|h\|\,\|k\|$ holds instead of $\|hk\| \leq \|h\|\,\|k\|$. Let $h, k \in \mathbb{C}_2$. A partial order relation on $\mathbb{C}_2$ is defined as follows. $h \preceq_{i_2} k$ iff $\mathfrak{Rea}(z_1) \leq \mathfrak{Rea}(w_1)$and $\mathfrak{Ima}(z_2) \leq \mathfrak{Ima}(w_2)$

Consequently, we can say that $h \preceq_{i_2} k$ if any one of the following cases exists

$$(a_1)\ z_1 = w_1 \text{ and } z_2 = w_2$$
$$(a_2)\ z_1 = w_1 \text{ and } z_2 < w_2$$
$$(a_3)\ z_1 < w_1 \text{ and } z_2 = w_2$$
$$(a_4)\ z_1 < w_1 \text{ and } z_2 < w_2.$$

Azam *et al.* [10] have defined the complex-valued metric spaces as follows.

Definition 1.1. *Let P be a nonempty set. A mapping $d_{\mathbb{C}_1} : P \times P \to \mathbb{C}_1$ is said to be a complex-valued metric if the following conditions hold.*

- (a_1) $0 \preceq d_{\mathbb{C}_1}(h, k), \forall h, k \in P$ *and* $d_{\mathbb{C}_1}(h, k) = 0 \Leftrightarrow h = k$
- (a_2) $d_{\mathbb{C}_1}(h, k) = d_{\mathbb{C}_1}(k, h), \forall h, k \in P$
- (a_3) $d_{\mathbb{C}_1}(h, b) \preceq d_{\mathbb{C}_1}(h, k) + d_{\mathbb{C}_1}(k, b), \forall h, k, b \in P$. *Then* $(P, d_{\mathbb{C}_1})$ *is called a complex-valued metric space.*

A bi-complex valued metric space has been defined by Choi *et al.* [19] as follows.

Definition 1.2. *Let P be a nonempty set. A mapping $d_{\mathbb{C}_2} : P \times P \to \mathbb{C}_2$ is said to be a bi-complex valued metric if the following conditions hold.*

(a_1) $0 \preceq_{i_2} d_{\mathbb{C}_2}(h,k), \forall h,k \in P$ and $d_{\mathbb{C}_2}(h,k) = 0 \Leftrightarrow h = k$

(a_2) $d_{\mathbb{C}_2}(h,k) = d_{\mathbb{C}_2}(k,h), \forall h,k \in P$

(a_3) $d_{\mathbb{C}_2}(h,b) \preceq_{i_2} d_{\mathbb{C}_2}(h,k) + d_{\mathbb{C}_2}(k,b), \forall h,k,b \in P$.
Then $(P, d_{\mathbb{C}_2})$ *is called a bi-complex valued metric space.*

Lemma 1.1. *[23] Let* $(P, d_{\mathbb{C}_2})$ *be a bi-complex-valued metric space and let* $\{h_m\}$ *be a sequence in* P. *Then* $\{h_m\}$ *converges to* $h \Leftrightarrow |d_{\mathbb{C}_2}(h_m, h)| \to 0$ *as* $n \to \infty$.

Lemma 1.2. *[23] Let* $(P, d_{\mathbb{C}_2})$ *be a bi-complex-valued metric space and let a sequence in* P *be* $\{h_m\}$. *Then* $\{h_m\}$*is a Cauchy sequence* $\Leftrightarrow |d_{\mathbb{C}_2}(h_m, h_{m+n})| \to 0$ *as* $n \to \infty$.

2. Some Theorems on Fixed Points

This section begins with the following observation.

Proposition 2.1. *Let* $(P, d_{\mathbb{C}_2})$ *be a complete bi-complex valued metric space and* $S_1, S_2 : P \to P$. *Let* $h_0 \in P$ *and the sequence* $\{h_m\}$ *be defined by* $h_{2m+1} = S_1 h_{2m}$ *and* $h_{2m+2} = S_2 h_{2m+1}$. *Assume that there exists* $F : P \times P \times P \to [0,1)$ *satisfying* $F(S_2S_1h,k,b) \leq F(h,k,b)$ *and* $F(h, S_1S_2k, b) \leq F(h,k,b)$ *for all* $h,k \in P$ *and for a fixed element* $b \in P$. *Then* $F(h_{2m},k,b) \leq F(h_0,k,b)$ *and* $F(h, h_{2m+1}, b) \leq F(h, h_1, b)$ $\forall m = 0,1,2,...$ *and* $h,k \in P$.

Proof. Let $h,k \in P$ and $m = 0,1,2,...$ Then we have

$$\begin{aligned} F(h_{2m},k,b) &= F(S_2S_1h_{2m-2},k,b) \leq F(h_{2m-2},k,b) \\ &= F(S_2S_1h_{2m-4},k,b) \leq F(h_{2m-4},k,b) \\ &\leq ... \leq F(h_0,k,b). \end{aligned}$$

Similarly, we have

$$\begin{aligned} F(h,h_{2m+1},b) &= F(h,S_1S_2h_{2m-1},b) \leq F(h,h_{2m-1},b) \\ &= F(h,S_1S_2h_{2m-3},b) \leq F(h,h_{2m-3},b) \\ &\leq ... \leq F(h,h_1,b). \end{aligned}$$

□

Following is a theorem on contractive condition in which the coefficient is a control function with three variables out of which one variable is fixed.

Theorem 2.1. *Let* $(P, d_{\mathbb{C}_2})$ *be a complete bi-complex valued metric space and* $S_1, S_2 : P \to P$. *If there exists mappings* $F_1, F_2, F_3, F_4 : P \times P \times P \to [0,1)$ *such that for i = 1, 2, 3, 4.*

(a) $F_i(S_2S_1h,k,b) \leq F_i(h,k,b)$ *and*
$F_i(h,S_1S_2k,b) \leq F_i(h,k,b)$,

(b)

$$\begin{aligned} d_{\mathbb{C}_2}(S_1h, S_2k) \preceq_{i_2} & F_1(h,k,b) d_{\mathbb{C}_2}(h,k) \\ &+ F_2(h,k,b) \frac{d_{\mathbb{C}_2}(h,S_1h) d_{\mathbb{C}_2}(k,S_2k)}{1 + d_{\mathbb{C}_2}(h,k)} \\ &+ F_3(h,k,b) \frac{d_{\mathbb{C}_2}(k,S_1h) d_{\mathbb{C}_2}(h,S_2k)}{1 + d_{\mathbb{C}_2}(h,k)} \\ &+ F_4(h,k,b) \Bigg(\frac{d_{\mathbb{C}_2}(h,S_1h) d_{\mathbb{C}_2}(h,S_2k)}{1 + d_{\mathbb{C}_2}(h,S_2k) + d_{\mathbb{C}_2}(k,S_1h)} \\ &+ \frac{d_{\mathbb{C}_2}(k,S_2k) d_{\mathbb{C}_2}(k,S_1h)}{1 + d_{\mathbb{C}_2}(h,S_2k) + d_{\mathbb{C}_2}(k,S_1h)} \Bigg), \end{aligned} \tag{1}$$

(c) $F_1(h,k,b) + \sqrt{2}\, F_2(h,k,b) + \sqrt{2}\, F_3(h,k,b) + \sqrt{2}\, F_4(h,k,b) < 1, \forall h,k \in P$ *and for a fixed* $b. \in P$

Then S_1 *and* S_2 *have a unique common fixed point.*

Proof. Let $h,k \in P$. From (1), we have

$$\begin{aligned} d_{\mathbb{C}_2}(S_1h, S_2S_1h) \preceq_{i_2} & F_1(h,S_1h,b) d_{\mathbb{C}_2}(h,S_1h) \\ &+ F_2(h,S_1h,b) \frac{d_{\mathbb{C}_2}(h,S_1h) d_{\mathbb{C}_2}(S_1h,S_2S_1h)}{1 + d_{\mathbb{C}_2}(h,S_1h)} \\ &+ F_3(h,S_1h,b) \frac{d_{\mathbb{C}_2}(S_1h,S_1h) d_{\mathbb{C}_2}(h,S_2S_1h)}{1 + d_{\mathbb{C}_2}(h,S_1h)} \\ &+ F_4(h,S_1h,b) \\ & \Bigg(\frac{d_{\mathbb{C}_2}(h,S_1h) d_{\mathbb{C}_2}(h,S_2S_1h)}{1 + d_{\mathbb{C}_2}(h,S_2S_1h) + d_{\mathbb{C}_2}(S_1h,S_1h)} \\ &+ \frac{d_{\mathbb{C}_2}(S_1h,S_2S_1h) d_{\mathbb{C}_2}(S_1h,S_1h)}{1 + d_{\mathbb{C}_2}(h,S_2S_1h) + d_{\mathbb{C}_2}(S_1h,S_1h)} \Bigg). \end{aligned}$$

$$\begin{aligned} \left\| d_{\mathbb{C}_2}(S_1h, S_2S_1h) \right\| \leq & F_1(h,S_1h,b) \left\| d_{\mathbb{C}_2}(h,S_1h) \right\| \\ &+ F_2(h,S_1h,b) \left\| \frac{d_{\mathbb{C}_2}(h,S_1h) d_{\mathbb{C}_2}(S_1h,S_2S_1h)}{1 + d_{\mathbb{C}_2}(h,S_1h)} \right\| \\ &+ F_3(h,S_1h,b) \left\| \frac{d_{\mathbb{C}_2}(S_1h,S_1h) d_{\mathbb{C}_2}(h,S_2S_1h)}{1 + d_{\mathbb{C}_2}(h,S_1h)} \right\| \\ &+ F_4(h,S_1h,b) \\ & \Bigg(\left\| \frac{d_{\mathbb{C}_2}(h,S_1h) d_{\mathbb{C}_2}(h,S_2S_1h)}{1 + d_{\mathbb{C}_2}(h,S_2S_1h) + d_{\mathbb{C}_2}(S_1h,S_1h)} \right\| \\ &+ \left\| \frac{d_{\mathbb{C}_2}(S_1h,S_2S_1h) d_{\mathbb{C}_2}(S_1h,S_1h)}{1 + d_{\mathbb{C}_2}(h,S_2S_1h) + d_{\mathbb{C}_2}(S_1h,S_1h)} \right\| \Bigg) \\ &\leq F_1(h,S_1h,b) \left\| d_{\mathbb{C}_2}(h,S_1h) \right\| \\ &+ \sqrt{2}\, F_2(h,S_1h,b) \left\| \frac{d_{\mathbb{C}_2}(h,S_1h)}{1 + d_{\mathbb{C}_2}(h,S_1h)} \right\| \\ &\times \left\| d_{\mathbb{C}_2}(S_1h,S_2S_1h) \right\| + \sqrt{2}\, F_4(h,S_1h,b) \\ &\times \left\| \frac{d_{\mathbb{C}_2}(h,S_2S_1h)}{1 + d_{\mathbb{C}_2}(h,S_2S_1h)} \right\| \left\| d_{\mathbb{C}_2}(h,S_1h) \right\| \\ &\leq F_1(h,S_1h,b) \left\| d_{\mathbb{C}_2}(h,S_1h) \right\| \\ &+ \sqrt{2}\, F_2(h,S_1h,b) \left\| d_{\mathbb{C}_2}(S_1h,S_2S_1h) \right\| \\ &+ \sqrt{2}\, F_4(h,S_1h,b) \left\| d_{\mathbb{C}_2}(h,S_1h) \right\|. \end{aligned}$$

$$\begin{aligned} \left\| d_{\mathbb{C}_2}(S_1h, S_2S_1h) \right\| \leq & F_1(h,S_1h,b) \left\| d_{\mathbb{C}_2}(h,S_1h) \right\| \\ &+ \sqrt{2}\, F_2(h,S_1h,b) \left\| d_{\mathbb{C}_2}(S_1h,S_2S_1h) \right\| \\ &+ \sqrt{2}\, F_4(h,S_1h,b) \left\| d_{\mathbb{C}_2}(h,S_1h) \right\|. \end{aligned} \tag{2}$$

Similarly, from(1), we have

$$\begin{aligned} \left\| d_{\mathbb{C}_2}(S_1S_2k, S_2k) \right\| \leq & F_1(S_2k,k,b) \left\| d_{\mathbb{C}_2}(S_2k,k) \right\| \\ &+ \sqrt{2}\, F_2(S_2k,k,b) \left\| d_{\mathbb{C}_2}(S_2k,S_1S_2k) \right\| \\ &+ \sqrt{2}\, F_4(S_2k,k,b) \left\| d_{\mathbb{C}_2}(k,S_2k) \right\|. \end{aligned} \tag{3}$$

Let $h_0 \in P$ and the sequence $\{h_m\}$ be defined by $h_{2m+1} = S_1 h_{2m}$ and $h_{2m+2} = S_2 h_{2m+1}$. Using (2) and (3) in Proposition(2.1),

$$\left\| d_{\mathbb{C}_2}(h_{2m+1}, h_{2m}) \right\| \leq F_1(h_{2m}, h_{2m-1}, b) \left\| d_{\mathbb{C}_2}(h_{2m-1}, h_{2m}) \right\|$$

$$
\begin{aligned}
&+ \sqrt{2}\, F_2(h_{2m}, h_{2m-1}, b) \left\| d_{\mathbb{C}_2}(h_{2m}, h_{2m+1}) \right\| \\
&+ \sqrt{2}\, F_4(h_{2m}, h_{2m-1}, b) \left\| d_{\mathbb{C}_2}(h_{2m-1}, h_{2m}) \right\| \\
&\leq F_1(h_0, h_{2m-1}, b) \left\| d_{\mathbb{C}_2}(h_{2m-1}, h_{2m}) \right\| \\
&+ \sqrt{2}\, F_2(h_0, h_{2m-1}, b) \left\| d_{\mathbb{C}_2}(h_{2m}, h_{2m+1}) \right\| \\
&+ \sqrt{2}\, F_4(h_0, h_{2m-1}, b) \left\| d_{\mathbb{C}_2}(h_{2m-1}, h_{2m}) \right\| \\
&\leq F_1(h_0, h_1, b) \left\| d_{\mathbb{C}_2}(h_{2m-1}, h_{2m}) \right\| \\
&+ \sqrt{2}\, F_2(h_0, h_1, b) \left\| d_{\mathbb{C}_2}(h_{2m}, h_{2m+1}) \right\| \\
&+ \sqrt{2}\, F_4(h_0, h_1, b) \left\| d_{\mathbb{C}_2}(h_{2m-1}, h_{2m}) \right\| \\
&\leq \frac{(F_1(h_0, h_1, b) + \sqrt{2}\, F_4(h_0, h_1, b))}{1 - \sqrt{2}\, F_2(h_0, h_1, b)} \\
&\times \left\| d_{\mathbb{C}_2}(h_{2m}, h_{2m-1}) \right\| .
\end{aligned}
$$

$$
\left\| d_{\mathbb{C}_2}(h_{2m+1}, h_{2m}) \right\| \leq \frac{(F_1(h_0, h_1, b) + \sqrt{2}\, F_4(h_0, h_1, b))}{1 - \sqrt{2}\, F_2(h_0, h_1, b)} \times \left\| d_{\mathbb{C}_2}(h_{2m}, h_{2m-1}) \right\| . \tag{4}
$$

Similarly, we obtain,

$$
\left\| d_{\mathbb{C}_2}(h_{2m+2}, h_{2m+1}) \right\| \leq \frac{(F_1(h_0, h_1, b) + \sqrt{2}\, F_4(h_0, h_1, b))}{1 - \sqrt{2}\, F_2(h_0, h_1, b)} \times \left\| d_{\mathbb{C}_2}(h_{2m+1}, h_{2m}) \right\| . \tag{5}
$$

Let $k = \dfrac{(F_1(h_0, h_1, b) + \sqrt{2}\, F_4(h_0, h_1, b))}{1 - \sqrt{2}\, F_2(h_0, h_1, b)}$. Then from (4) and (5), we have $\left\| d_{\mathbb{C}_2}(h_m, h_{m+1}) \right\| \leq k \left\| d_{\mathbb{C}_2}(h_{m-1}, h_m) \right\| \leq k^2 \left\| d_{\mathbb{C}_2}(h_{m-2}, h_{m-1}) \right\| \leq \ldots \leq k^m \left\| d_{\mathbb{C}_2}(h_0, h_1) \right\| \forall m \in \mathbb{N}$. Now for $n > m$, we find,

$$
\begin{aligned}
\left\| d_{\mathbb{C}_2}(h_m, h_n) \right\| &\leq k \left\| d_{\mathbb{C}_2}(h_0, h_1) \right\| + k^2 \left\| d_{\mathbb{C}_2}(h_0, h_1) \right\| + \ldots \\
&+ k^{n-1} \left\| d_{\mathbb{C}_2}(h_0, h_1) \right\| \\
&\leq \frac{k^m}{1-k} \left\| d_{\mathbb{C}_2}(h_0, h_1) \right\| .
\end{aligned}
$$

By allowing $m, n \to \infty$, we have $\left\| d_{\mathbb{C}_2}(h_m, h_m) \right\| \to 0$. Therefore $\{h_m\}$ is a Cauchy sequence by Lemma (1.2). As P is complete, the sequence converges to ξ as $n \to \infty$. Now to show that ξ is a fixed point of S_1 consider

$$
\begin{aligned}
d_{\mathbb{C}_2}(\xi, S_1\xi) &\preceq_{i_2} d_{\mathbb{C}_2}(\xi, S_2 h_{2m+1}) + d_{\mathbb{C}_2}(S_2 h_{2m+1}, S_1\xi) \\
&= d_{\mathbb{C}_2}(\xi, h_{2m+2}) + d_{\mathbb{C}_2}(S_2 h_{2m+1}, S_1\xi) \\
&\preceq_{i_2} d_{\mathbb{C}_2}(\xi, h_{2m+2}) + F_1(\xi, h_{2m+1}, b) d_{\mathbb{C}_2}(\xi, h_{2m+1}) \\
&+ F_2(\xi, h_{2m+1}, b) \frac{d_{\mathbb{C}_2}(\xi, S_1\xi) d_{\mathbb{C}_2}(h_{2m+1}, S_2 h_{2m+1})}{1 + d_{\mathbb{C}_2}(\xi, h_{2m+1})} \\
&+ F_3(\xi, h_{2m+1}, b) \frac{d_{\mathbb{C}_2}(h_{2m+1}, S_1\xi) d_{\mathbb{C}_2}(\xi, S_2 h_{2m+1})}{1 + d_{\mathbb{C}_2}(\xi, h_{2m+1})} \\
&+ F_4(\xi, h_{2m+1}, b) \Bigg(\frac{d_{\mathbb{C}_2}(\xi, S_1\xi) d_{\mathbb{C}_2}(\xi, S_2 h_{2m+1})}{1 + d_{\mathbb{C}_2}(\xi, S_2 h_{2m+1}) + d_{\mathbb{C}_2}(h_{2m+1}, S_1\xi)} \\
&+ \frac{d_{\mathbb{C}_2}(h_{2m+1}, S_2 h_{2m+1}) d_{\mathbb{C}_2}(h_{2m+1}, S_1\xi)}{1 + d_{\mathbb{C}_2}(\xi, S_2 h_{2m+1}) + d_{\mathbb{C}_2}(h_{2m+1}, S_1\xi)} \Bigg).
\end{aligned}
$$

$$
\begin{aligned}
\left\| d_{\mathbb{C}_2}(\xi, S_1\xi) \right\| &\leq \left\| d_{\mathbb{C}_2}(\xi, h_{2m+2}) \right\| + F_1(\xi, h_1, b) \left\| d_{\mathbb{C}_2}(\xi, h_{2m+1}) \right\| \\
&+ F_2(\xi, h_1, b) \left\| \frac{d_{\mathbb{C}_2}(\xi, S_1\xi) d_{\mathbb{C}_2}(h_{2m+1}, h_{2m+2})}{1 + d_{\mathbb{C}_2}(\xi, h_{2m+1})} \right\| \\
&+ F_3(\xi, h_1, b) \left\| \frac{d_{\mathbb{C}_2}(h_{2m+1}, S_1\xi) d_{\mathbb{C}_2}(\xi, h_{2m+2})}{1 + d_{\mathbb{C}_2}(\xi, h_{2m+1})} \right\| \\
&+ F_4(\xi, h_1, b) \Bigg(\left\| \frac{d_{\mathbb{C}_2}(\xi, S_1\xi) d_{\mathbb{C}_2}(\xi, h_{2m+2})}{1 + d_{\mathbb{C}_2}(\xi, h_{2m+2}) + d_{\mathbb{C}_2}(h_{2m+1}, S_1\xi)} \right\| \\
&+ \left\| \frac{d_{\mathbb{C}_2}(h_{2m+1}, h_{2m+2}) d_{\mathbb{C}_2}(h_{2m+1}, S_1\xi)}{1 + d_{\mathbb{C}_2}(\xi, h_{2m+2}) + d_{\mathbb{C}_2}(h_{2m+1}, S_1\xi)} \right\| \Bigg).
\end{aligned}
$$

Letting $m \to \infty$, we have $\|\xi, S_1\xi\| = 0$ and hence $S_1\xi = \xi$. In same line, one can prove that $\|\xi, S_2\xi\| = 0$ and hence $S_2\xi = \xi$. Next, we have to show that ξ is a unique common fixed point of mappings S_1 and S_2. Let ξ^* be another common fixed point of the mappings such that $S_1\xi^* = S_2\xi^* = \xi^*$ with $\xi^* \neq \xi$. From (1), we have

$$
\begin{aligned}
\left\| d_{\mathbb{C}_2}(\xi, \xi^*) \right\| &\leq F_1(\xi, \xi^*, b) \left\| d_{\mathbb{C}_2}(\xi, \xi^*) \right\| \\
&+ F_2(\xi, \xi^*, b) \left\| \frac{d_{\mathbb{C}_2}(\xi, S_1\xi) d_{\mathbb{C}_2}(\xi^*, S_2\xi^*)}{1 + d_{\mathbb{C}_2}(\xi, \xi^*)} \right\| \\
&+ F_3(\xi, \xi^*, b) \left\| \frac{d_{\mathbb{C}_2}(\xi^*, S_1\xi) d_{\mathbb{C}_2}(\xi, S_2\xi^*)}{1 + d_{\mathbb{C}_2}(\xi, \xi^*)} \right\| \\
&+ F_4(\xi, \xi^*, b) \Bigg(\left\| \frac{d_{\mathbb{C}_2}(\xi, S_1\xi) d_{\mathbb{C}_2}(\xi, S_2\xi^*)}{1 + d_{\mathbb{C}_2}(\xi, S_2\xi^*) + d_{\mathbb{C}_2}(\xi^*, S_1\xi)} \right\| \\
&+ \left\| \frac{d_{\mathbb{C}_2}(\xi^*, S_2\xi^*) d_{\mathbb{C}_2}(\xi^*, S_1\xi)}{1 + d_{\mathbb{C}_2}(\xi, S_2\xi^*) + d_{\mathbb{C}_2}(\xi^*, S_1\xi)} \right\| \Bigg) \\
&\leq F_1(\xi, \xi^*, b) \left\| d_{\mathbb{C}_2}(\xi, \xi^*) \right\| \\
&+ \sqrt{2}\, F_3(\xi, \xi^*, b) \left\| \frac{d_{\mathbb{C}_2}(\xi, \xi^*)}{1 + d_{\mathbb{C}_2}(\xi, \xi^*)} \right\| \left\| d_{\mathbb{C}_2}(\xi^*, \xi) \right\| \\
&\leq F_1(\xi, \xi^*, b) \left\| d_{\mathbb{C}_2}(\xi, \xi^*) \right\| \\
&+ \sqrt{2}\, F_3(\xi, \xi^*, b) \left\| \frac{d_{\mathbb{C}_2}(\xi, \xi^*)}{1 + d_{\mathbb{C}_2}(\xi, \xi^*)} \right\| \left\| d_{\mathbb{C}_2}(\xi^*, \xi) \right\| \\
1 &\leq F_1(\xi, \xi^*, b) + \sqrt{2}\, F_3(\xi, \xi^*, b).
\end{aligned}
$$

Since $F_1(\xi, \xi^*, b) + \sqrt{2}\, F_3(\xi, \xi^*, b) < 1$, we get $\left\| d_{\mathbb{C}_2}(\xi, \xi^*) \right\| = 0$. Therefore ξ is a unique common fixed point of S_1 and S_2. □

Following example illustrates the correctness of our Theorem (2.1).

Example 1. *Let $d_{\mathbb{C}_2} : [0,1] \times [0,1] \to \mathbb{C}_2$ be defined by $d_{\mathbb{C}_2}(h,k) = (1 + i_2)|h - k|$ for all $h, k \in [0,1]$. Clearly it is a bi-complex valued metric space. Define $S_1, S_2 : P \to P$ by $S_1(h) = \frac{h}{6}$ and $S_2(k) = \frac{k}{6}$ and consider $F_1, F_2, F_3, F_4 : P \times P \times P \to [0,1)$ by $F_1(h,k,b) = (\frac{h}{4} + \frac{k}{6} + b)$, $F_2(h,k,b) = (\frac{h^2}{16} + \frac{k^2}{36} + b^2)$, $F_3(h,k,b) = (\frac{hkb}{12})$, $F_3(h,k,b) = (\frac{h^2k^2b^2}{12})$, $\forall h,k \in P$ and for $b = \frac{1}{5}$. Then*

(a) $F_1(S_2S_1(h), k, b) = F_1(S_2(\frac{h}{6}), k, b) = F_1(\frac{h}{36}, k, b) = \frac{h}{144} + \frac{k}{6} + b \leq (\frac{h}{4} + \frac{k}{6} + b) = F_1(h,k,b)$ and $F_1(h, S_1S_2(k), b) = F_1(h, S_1(\frac{k}{6}), b) = F_1(h, \frac{k}{36}, b) = \frac{h}{4} + \frac{k}{216} + b \leq (\frac{h}{4} + \frac{k}{6} + b) = F_1(h,k,b)$.

That is $F_1(S_2S_1h,k,b) \leq F_1(h,k,b)$ and $F_1(h,S_1S_2k,b) \leq F_1(h,k,b)$. In the same line, we can show that for F_2, F_3, F_4.

$$\begin{aligned} d_{\mathbb{C}_2}(S_1h,S_2k) &= d_{\mathbb{C}_2}(\frac{h}{6},\frac{k}{6}) \\ &= \frac{1}{6}(1+i_2)|h-k| \\ &\preceq_{i_2} \frac{1}{5}(1+i_2)|h-k| \\ &\preceq_{i_2} (\frac{h}{4}+\frac{k}{6}+\frac{1}{5})(1+i_2)|h-k| \\ &= F_1(h,k,b)d_{\mathbb{C}_2}(h,k). \end{aligned}$$

It is easy to check the other conditions of the Theorem (2.1) and thus 0 is a common fixed point of mappings S_1 and S_2.

We arrive at the following Corollaries on complete bi-complex valued metric space $(P, d_{\mathbb{C}_2})$ by assigning values to the control functions in Theorem (2.1).

If we fix $F_2(h,k,b)$, $F_3(h,k,b)$ and $F_4(h,k,b)$ at zero in Theorem (2.1), then we have the following.

Corollary 2.1. *Let S_1, S_2 be two self-mappings from P to itself. If there exists a map $F_1 : P\times P\times P \to [0,1)$ such that $\forall\ h,k \in P$ and for a fixed $b \in P$,*

(a) $F_1(S_2S_1h,k,b) \leq F_1(h,k,b)$ and $F_1(h,S_1S_2k,b) \leq F_1(h,k,b)$,
(b) $d_{\mathbb{C}_2}(S_1h,S_2k) \preceq_{i_2} F_1(h,k,b)d_{\mathbb{C}_2}(h,k)$,
(c) $F_1(h,k,b) < 1$.

Then S_1 and S_2 have a unique common fixed point.

If we replace $F_i(h,k,b)$ by $F_i(h,k)$ for $i = 1,2,3,4$ in Theorem (2.1) we switch over from $F_1,F_2,F_3,F_4 : P\times P\times P \to [0,1)$ to $F_1,F_2,F_3,F_4 : P\times P \to [0,1)$ and we have the following corollary.

Corollary 2.2. *Let $S_1, S_2 : P \to P$. If there exists mappings $F_1,F_2,F_3,F_4 : P\times P \to [0,1)$ such that for all $h,k \in P$, for i = 1, 2, 3, 4.*

(a) $F_i(S_2S_1h,k) \leq F_i(h,k)$ and $F_i(h,S_1S_2k) \leq F_i(h,k)$,

(b)
$$\begin{aligned} d_{\mathbb{C}_2}(S_1h,S_2k) \preceq_{i_2} &F_1(h,k)d_{\mathbb{C}_2}(h,k) \\ &+ F_2(h,k)\frac{d_{\mathbb{C}_2}(h,S_1h)d_{\mathbb{C}_2}(k,S_2k)}{1+d_{\mathbb{C}_2}(h,k)} \\ &+ F_3(h,k)\frac{d_{\mathbb{C}_2}(k,S_1h)d_{\mathbb{C}_2}(h,S_2k)}{1+d_{\mathbb{C}_2}(h,k)} \\ &+ F_4(h,k)\Bigg(\frac{d_{\mathbb{C}_2}(h,S_1h)d_{\mathbb{C}_2}(h,S_2k)}{1+d_{\mathbb{C}_2}(h,S_2k)+d_{\mathbb{C}_2}(k,S_1h)} \\ &+ \frac{d_{\mathbb{C}_2}(k,S_2k)d_{\mathbb{C}_2}(k,S_1h)}{1+d_{\mathbb{C}_2}(h,S_2k)+d_{\mathbb{C}_2}(k,S_1h)}\Bigg), \end{aligned}$$

(c) $F_1(h,k)+\sqrt{2}\,F_2(h,k)+\sqrt{2}\,F_3(h,k)+\sqrt{2}\,F_4(h,k) < 1$.

Then S_1 and S_2 have a unique common fixed point.

Example 2. *Let $d_{\mathbb{C}_2} : [0,1]\times[0,1] \to \mathbb{C}_2$ be defined by $d_{\mathbb{C}_2}(h,k) = |h-k| + i_2|h-k|$ for all $h,k \in [0,1]$. Define $S_1,S_2 : P \to P$ by $S_1(h) = \frac{h}{8}$ and $S_2(k) = \frac{k}{8}$ and consider $F_1,F_2,F_3,F_4 : P\times P \to [0,1)$ by $F_1(h,k) = (\frac{h+1}{6}+\frac{k}{7})$, $F_2(h,k) = (\frac{hk}{10})$, $F_3(h,k) = (\frac{h^2k^2}{9})$, $F_3(h,k) = (\frac{h^3k^3}{8})$, for all h,k in P. Then*

(a) $F_1(S_2S_1(h),k) = F_1(S_2(\frac{h}{8}),k) = F_1(\frac{h}{64},k) = \frac{h+64}{384}+\frac{k}{7} \leq (\frac{h+1}{6}+\frac{k}{7}) = F_1(h,k)$ and
$F_1(h,S_1S_2(k)) = F_1(h,S_1(\frac{k}{8})) = F_1(h,\frac{k}{64}) = \frac{h}{6}+\frac{k}{448} \leq (\frac{h}{6}+\frac{k}{7}) = F_1(h,k)$. Similarly, we have
(b) $F_2(S_2S_1h,k) \leq F_2(h,k)$ and $F_2(h,S_1S_2k) \leq F_2(h,k)$.
(c) $F_3(S_2S_1h,k) \leq F_3(h,k)$ and $F_3(h,S_1S_2k) \leq F_3(h,k)$.
(d) $F_4(S_2S_1h,k) \leq F_4(h,k)$ and $F_4(h,S_1S_2k) \leq F_4(h,k)$.

$$\begin{aligned} d_{\mathbb{C}_2}(S_1h,S_2k) &= d_{\mathbb{C}_2}(\frac{h}{8},\frac{k}{8}) \\ &= \frac{1}{8}(|h-k|+i_2|h-k|) \\ &\preceq_{i_2} \frac{h+1}{6}(|h-k|+i_2|h-k|) \\ &\preceq_{i_2} (\frac{h+1}{6}+\frac{k}{7})(|h-k|+i_2|h-k|) \\ &= F_1(h,k)d_{\mathbb{C}_2}(h,k). \end{aligned}$$

All the conditions of the above Corollary 2.2 are satisfied. Therefore 0 is a common fixed point of the mappings S_1 and S_2.

If we replace the mappings $F_1,F_2,F_3,F_4 : P\times P\times P \to [0,1)$ by the mappings $F_1,F_2,F_3,F_4 : P \to [0,1)$ with $F_i(h,k,b) = F_i(h)$ for $i = 1,2,3,4.$ in Theorem (2.1), then we have the following .

Corollary 2.3. *Let $S_1,S_2 : P \to P$. If there exists mappings $F_1,F_2,F_3,F_4 : P \to [0,1)$ such that $\forall\ h,k$ in P and for a fixed $b \in P$ with*

(a) $F_i(S_1h) \leq F_i(h)$ and $F_i(S_2h) \leq F_i(h)$, for i = 1, 2, 3, 4.

(b)
$$\begin{aligned} d_{\mathbb{C}_2}(S_1h,S_2k) \preceq_{i_2} &F_1(h)d_{\mathbb{C}_2}(h,k) \\ &+ F_2(h)\frac{d_{\mathbb{C}_2}(h,S_1h)d_{\mathbb{C}_2}(k,S_2k)}{1+d_{\mathbb{C}_2}(h,k)} \\ &+ F_3(h)\frac{d_{\mathbb{C}_2}(k,S_1h)d_{\mathbb{C}_2}(h,S_2k)}{1+d_{\mathbb{C}_2}(h,k)} \\ &+ F_4(h)\Bigg(\frac{d_{\mathbb{C}_2}(h,S_1h)d_{\mathbb{C}_2}(h,S_2k)}{1+d_{\mathbb{C}_2}(h,S_2k)+d_{\mathbb{C}_2}(k,S_1h)} \\ &+ \frac{d_{\mathbb{C}_2}(k,S_2k)d_{\mathbb{C}_2}(k,S_1h)}{1+d_{\mathbb{C}_2}(h,S_2k)+d_{\mathbb{C}_2}(k,S_1h)}\Bigg), \end{aligned}$$

(c) $F_1(h)+\sqrt{2}\,F_2(h)+\sqrt{2}\,F_3(h)+\sqrt{2}\,F_4(h) < 1$.

Then S_1 and S_2 have a unique common fixed point.

Example 3. *Let $P = [0,1]$ and $d_{\mathbb{C}_2} : P \times P \to \mathbb{C}_2$ be defined by $d_{\mathbb{C}_2}(h,k) = (1+i_2)|h-k|$. Clearly it is a bi-complex valued metric space. Define $S_1, S_2 : P \to P$ by $S_1(h) = \frac{h}{4}$ and $S_2(k) = \frac{k}{3}$ and consider $F_1, F_2, F_3, F_4 : P \to [0,1)$ by $F_1(h) = \frac{h+1}{3}$, $F_2(h) = \frac{h^2}{9}$, $F_3(h) = \frac{h^3}{27}$, $F_4(h) = \frac{h^3}{81}$, $\forall h,k \in P$. Then, as usual, by simple calculation, we can check the conditions of above Corollary 2.3.*

Corollary 2.4. *Let $S_1, S_2 : P \to P$. If there exists $F_1, F_2, F_3, F_4 \in [0,1)$ such that for all $h,k \in P$,*

(a)

$$\begin{aligned} d_{\mathbb{C}_2}(S_1h, S_2k) \preceq_{i_2}\ & F_1 d_{\mathbb{C}_2}(h,k) \\ & + F_2 \frac{d_{\mathbb{C}_2}(h,S_1h) d_{\mathbb{C}_2}(k,S_2k)}{1+d_{\mathbb{C}_2}(h,k)} \\ & + F_3 \frac{d_{\mathbb{C}_2}(k,S_1h) d_{\mathbb{C}_2}(h,S_2k)}{1+d_{\mathbb{C}_2}(h,k)} \\ & + F_4 \bigg(\frac{d_{\mathbb{C}_2}(h,S_1h) d_{\mathbb{C}_2}(h,S_2k)}{1+d_{\mathbb{C}_2}(h,S_2k)+d_{\mathbb{C}_2}(k,S_1h)} \\ & + \frac{d_{\mathbb{C}_2}(k,S_2k) d_{\mathbb{C}_2}(k,S_1h)}{1+d_{\mathbb{C}_2}(h,S_2k)+d_{\mathbb{C}_2}(k,S_1h)} \bigg), \end{aligned}$$

(b) $F_1 + \sqrt{2}\,F_2 + \sqrt{2}\,F_3 + \sqrt{2}\,F_4 < 1,$

then S_1 and S_2 have a unique common fixed point.

We have proved the following fixed point theorem for a single mapping using a different condition.

Theorem 2.2. *Let $(P, d_{\mathbb{C}_2})$ be a complete bi-complex valued metric space and $S_1 : P \to P$. If there exists mappings $F_1, F_2, F_3 : P \times P \times P \to [0,1) \ni \forall\ h,k$ in P, for a fixed b in P, for i = 1, 2, 3.*

(a) $F_i(S_1h,k,b) \le F_i(h,k,b)$ *and* $F_i(h,S_1k,b) \le F_i(h,k,b)$,

(b)

$$\begin{aligned} d_{\mathbb{C}_2}(S_1h, S_1k) \preceq_{i_2}\ & F_1(h,k,b) d_{\mathbb{C}_2}(h,k) \\ & + F_2(h,k,b) \frac{d_{\mathbb{C}_2}(k,S_1k)(1+d_{\mathbb{C}_2}(h,S_1h))}{1+d_{\mathbb{C}_2}(h,k)} \\ & + F_3(h,k,b) \frac{d_{\mathbb{C}_2}(h,S_1k) d_{\mathbb{C}_2}(h,S_1h)}{1+d_{\mathbb{C}_2}(h,k)}, \end{aligned} \quad (6)$$

(c) $F_1(h,k,b) + \sqrt{2}\,F_2(h,k,b) + 2\sqrt{2}\,F_3(h,k,b) < 1.$

Then S_1 has a unique fixed point.

Proof. Let $h_0 \in P$ and the sequence $\{h_r\}$ be defined as $h_{r+1} = S_1 h_r$, where $n = 0,1,2,....$ Using (a) for $i = 1,2,3.$ we have

$$\begin{aligned} F_i(h_r, h_{r+1}, b) &= F_i(S_2h_{r-1}, h_{r+1}, b) \\ &\le F_i(h_{r-1}, h_{r+1}, b) = F_i(S_2h_{r-2}, h_{r+1}, b) \\ &\le F_i(h_{r-2}, h_{r+1}, b) = F_i(S_2h_{r-3}, h_{r+1}, b) \\ & . \\ & . \\ & . \\ &\le F_i(h_0, h_{r+1}, b) \end{aligned}$$

$$\begin{aligned} F_i(h_0, h_{r+1}, b) &= F_i(h_0, S_2h_r, b) \\ &\le F_i(h_0, h_r, b) = F_i(h_0, S_2h_{r-1}, b) \\ &\le F_i(h_0, h_{r-1}, b) \\ & . \\ & . \\ & . \\ &\le F_i(h_0, h_0, b). \end{aligned}$$

$$\begin{aligned} d_{\mathbb{C}_2}(h_{r+1}, h_{r+2}) &= d_{\mathbb{C}_2}(S_1h_r, S_1h_{r+1}) \\ &\preceq_{i_2} F_1(h_r,h_{r+1},b) d_{\mathbb{C}_2}(h_r,h_{r+1}) \\ &+ F_2(h_r,h_{r+1},b) \frac{d_{\mathbb{C}_2}(h_{r+1},S_1h_{r+1})(1+d_{\mathbb{C}_2}(h_r,S_1h_r))}{1+d_{\mathbb{C}_2}(h_r,h_{r+1})} \\ &+ F_3(h_r,h_{r+1},b) \frac{d_{\mathbb{C}_2}(h_r,S_1h_{r+1}) d_{\mathbb{C}_2}(h_r,S_1h_r)}{1+d_{\mathbb{C}_2}(h_r,h_{r+1})} \\ &\preceq_{i_2} F_1(h_r,h_{r+1},b) d_{\mathbb{C}_2}(h_r,h_{r+1}) \\ &+ F_2(h_r,h_{r+1},b) \frac{d_{\mathbb{C}_2}(h_{r+1},h_{r+2})(1+d_{\mathbb{C}_2}(h_r,h_{r+1}))}{1+d_{\mathbb{C}_2}(h_r,h_{r+1})} \\ &+ F_3(h_r,h_{r+1},b) \frac{d_{\mathbb{C}_2}(h_r,h_{r+2}) d_{\mathbb{C}_2}(h_r,h_{r+1})}{1+d_{\mathbb{C}_2}(h_r,h_{r+1})}. \end{aligned}$$

$$\begin{aligned} \left\|d_{\mathbb{C}_2}(h_{r+1},h_{r+2})\right\| &\le F_1(h_r,h_{r+1},b) \left\|d_{\mathbb{C}_2}(h_r,h_{r+1})\right\| \\ &+ \sqrt{2}\,F_2(h_r,h_{r+1},b) \left\|d_{\mathbb{C}_2}(h_{r+1},h_{r+2})\right\| \\ &+ \sqrt{2}\,F_3(h_r,h_{r+1},b) \left\|d_{\mathbb{C}_2}(h_r,h_{r+1})\right\| \\ &+ \sqrt{2}\,F_3(h_r,h_{r+1},b) \left\|d_{\mathbb{C}_2}(h_{r+1},h_{r+2})\right\| \\ &\le F_1(h_0,h_0,b) \left\|d_{\mathbb{C}_2}(h_r,h_{r+1})\right\| \\ &+ \sqrt{2}\,F_2(h_0,h_0,b) \left\|d_{\mathbb{C}_2}(h_{r+1},h_{r+2})\right\| \\ &+ \sqrt{2}\,F_3(h_0,h_0,b) \left\|d_{\mathbb{C}_2}(h_r,h_{r+1})\right\| \\ &+ \sqrt{2}\,F_3(h_0,h_0,b) \left\|d_{\mathbb{C}_2}(h_{r+1},h_{r+2})\right\| \\ &\le k \left\|d_{\mathbb{C}_2}(h_r,h_{r+1})\right\|, \end{aligned}$$

where $k = \dfrac{F_1(h_0,h_0,b) + \sqrt{2}\,F_3(h_0,h_0,b)}{1-\sqrt{2}\,(F_2(h_0,h_0,b)+F_3(h_0,h_0,b))}$. It is easy to prove that $\{h_r\}$ is a Cauchy sequence. As P is complete, the sequence converges to ζ as $n \to \infty$. Now we have to prove that ζ is a fixed point of S_1.

$$\begin{aligned} d_{\mathbb{C}_2}(\zeta, S_1\zeta) &\preceq_{i_2} d_{\mathbb{C}_2}(\zeta, S_1h_r) + d_{\mathbb{C}_2}(S_1h_r, S_1\zeta) \\ &\preceq_{i_2} d_{\mathbb{C}_2}(\zeta, S_1h_r) + F_1(h_r,\zeta,b) d_{\mathbb{C}_2}(h_r,\zeta) \\ &+ F_2(h_r,\zeta,b) \frac{d_{\mathbb{C}_2}(\zeta,S_1\zeta)(1+d_{\mathbb{C}_2}(h_r,S_1h_r))}{1+d_{\mathbb{C}_2}(h_r,\zeta)} \\ &+ F_3(h_r,\zeta,b) \frac{d_{\mathbb{C}_2}(h_r,S_1\zeta) d_{\mathbb{C}_2}(h_r,S_1h_r)}{1+d_{\mathbb{C}_2}(h_r,\zeta)} \\ &\preceq_{i_2} d_{\mathbb{C}_2}(\zeta, h_{r+1}) + F_1(h_r,\zeta,b) d_{\mathbb{C}_2}(h_r,\zeta) \\ &+ F_2(h_r,\zeta,b) \frac{d_{\mathbb{C}_2}(\zeta,S_1\zeta)(1+d_{\mathbb{C}_2}(h_r,h_{r+1}))}{1+d_{\mathbb{C}_2}(h_r,\zeta)} \\ &+ F_3(h_r,\zeta,b) \frac{d_{\mathbb{C}_2}(h_r,S_1\zeta) d_{\mathbb{C}_2}(h_r,h_{r+1})}{1+d_{\mathbb{C}_2}(h_r,\zeta)} \end{aligned}$$

$$\preceq_{i_2} F_2(h_0,\zeta,b)d_{\mathbb{C}_2}(\zeta,S_1\zeta)$$
$$1 \preceq_{i_2} F_2(h_0,\zeta,b),$$

which is a contradiction. Therefore ζ is a fixed point of S_1. Also it is easy to check the uniqueness of the fixed point. □

Example 4. *Let $P=[0,1]$ and $d_{\mathbb{C}_2}: P\times P\to\mathbb{C}_2$ be defined as $d_{\mathbb{C}_2}(h,k)=(1+i_2)|h-k|$. Define the single mapping $S_1: P\to P$ by $S_1(h)=\frac{h}{5}$ and consider $F_1,F_2,F_3: P\times P\times P\to[0,1)$ by $F_1(h,k,b)=(\frac{h}{6}+\frac{k}{5}+b)$, $F_2(h,k,b)=(\frac{hkb}{10})$, $F_4(h,k,b)=(\frac{h^2k^2b^2}{14})$, $\forall h,k\in P$ and for $b=\frac{1}{4}$. Then,*

(a) $F_1(S_1(h),k,b)=F_1(\frac{h}{5},k,b)=\frac{h}{30}+\frac{k}{5}+b\leq(\frac{h}{6}+\frac{k}{5}+b)=F_1(h,k,b)$ and $F_1(h,S_1(k),b)=F_1(h,\frac{k}{5},b)=\frac{h}{6}+\frac{k}{25}+b\leq(\frac{h}{6}+\frac{k}{5}+b)=F_1(h,k,b)$. In the same line, we can show that

(b) $F_2(S_1h,k,b)\leq F_2(h,k,b)$ and $F_2(h,S_1k,b)\leq F_2(h,k,b)$.

(c) $F_3(S_1h,k,b)\leq F_3(h,k,b)$ and $F_3(h,S_1k,b)\leq F_3(h,k,b)$.

$$\begin{aligned}
d_{\mathbb{C}_2}(S_1h,S_1k) &= d_{\mathbb{C}_2}(\frac{h}{5},\frac{k}{5})\\
&= \frac{1}{5}(1+i_2)|h-k|\\
&\preceq_{i_2} \frac{1}{4}(1+i_2)|h-k|\\
&\preceq_{i_2} (\frac{h}{6}+\frac{5}{6}+\frac{1}{4})(1+i_2)|h-k|\\
&= F_1(h,k,b)d_{\mathbb{C}_2}(h,k).
\end{aligned}$$

As a result, the requirements of Theorem (2.2) are all fulfilled. Then, S_1 has a fixed point at 0, which is unique.

Replacement of $F_i(h,k,b)$ by using $F_i(h,k)$ for $i=1,2,3$ leads to $F_1,F_2,F_3: P\times P\to[0,1)$ from $F_1,F_2,F_3: P\times P\times P\to[0,1)$ and then the corollary is as follows..

Corollary 2.5. *Let $(P,d_{\mathbb{C}_2})$ be a complete bi-complex valued metric space and $S_1: P\to P$. If there exists mappings $F_1,F_2,F_3: P\times P\to[0,1)$ such that $\forall$ h, k in P, for i = 1, 2, 3.*

(a) $F_i(S_1h,k)\leq F_i(h,k)$ and $F_i(h,S_1k)\leq F_i(h,k)$,

(b)
$$\begin{aligned}
d_{\mathbb{C}_2}(S_1h,S_1k) \preceq_{i_2}\ & F_1(h,k)d_{\mathbb{C}_2}(h,k)\\
&+F_2(h,k)\frac{d_{\mathbb{C}_2}(k,S_1k)(1+d_{\mathbb{C}_2}(h,S_1h))}{1+d_{\mathbb{C}_2}(h,k)}\\
&+F_3(h,k)\frac{d_{\mathbb{C}_2}(h,S_1k)d_{\mathbb{C}_2}(h,S_1h)}{1+d_{\mathbb{C}_2}(h,k)},
\end{aligned}$$

(c) $F_1(h,k)+\sqrt{2}\,F_2(h,k)+2\sqrt{2}\,F_3(h,k)<1$.

Then S_1 has a unique fixed point.

If we replace the mappings $F_1,F_2,F_3: P\times P\times P\to[0,1)$ by $F_1,F_2,F_3: P\to[0,1)$ with $F_i(h,k,b)=F_i(h)$ for $i=1,2,3$. in Theorem (2.2), then we have the following .

Corollary 2.6. *Let $(P,d_{\mathbb{C}_2})$ be a complete bi-complex valued metric space and $S_1: P\to P$. If there exists mappings $F_1,F_2,F_3: P\to[0,1)$ such that $\forall$ h, k in P,*

(a) $F_i(S_1h)\leq F_i(h)$ and $F_i(S_1k)\leq F_i(k)$, for i = 1, 2, 3.

(b)
$$\begin{aligned}
d_{\mathbb{C}_2}(S_1h,S_1k) \preceq_{i_2}\ & F_1(h)d_{\mathbb{C}_2}(h,k)\\
&+F_2(h)\frac{d_{\mathbb{C}_2}(k,S_1k)(1+d_{\mathbb{C}_2}(h,S_1h))}{1+d_{\mathbb{C}_2}(h,k)}\\
&+F_3(h)\frac{d_{\mathbb{C}_2}(h,S_1k)d_{\mathbb{C}_2}(h,S_1h)}{1+d_{\mathbb{C}_2}(h,k)},
\end{aligned}$$

(c) $F_1(h)+\sqrt{2}\,F_2(h)+2\sqrt{2}\,F_3(h)<1$.

Then S_1 has a unique fixed point.

Remark 2.1. *By using the appropriate point-dependent control functions with constant coefficients F_1,F_2,F_3,F_4 and mappings S_1 and S_2 in Theorem (2.1), Theorem (2.2) and Corollaries, one can derive numerous results from the existing literature, including the well-known Banach fixed point theorem in bi-complex valued complete metric spaces, complete complex valued metric spaces, and complete metric spaces.*

3. Applications

In this section, we have shown how Theorem (2.1) can be used to prove the existence of a common solution to the Urysohn integral equation system.

Theorem 3.1. *Let $P=C([a_1,a_2],\mathbb{R}^n)$, where $[a_1,a_2]\subset\mathbb{R}^+$ and $d_{\mathbb{C}_2}: P\times P\to\mathbb{C}_2$ be defined by $d_{\mathbb{C}_2}(h,k)=\max\limits_{t\in[a_1,a_2]}(|h(t)-k(t)|+i_2|h(t)-k(t)|)$ for all $h,k\in P$ and $t\in[a_1,a_2]$, where $|.|$ is the usual real modulus. Consider the following Urysohn type integral equations*

$$h(t)=\int_{a_1}^{a_2}K_1(t,u,h(u))du+f(t), \tag{7}$$
$$h(t)=\int_{a_1}^{a_2}K_2(t,u,h(u))du+g(t), \tag{8}$$

where $\forall h,k\in P$ and $t\in[a_1,a_2]$. Suppose that $K_1,K_2: [a_1,a_2]\times[a_1,a_2]\times\mathbb{R}^n\to\mathbb{R}^n$ are $\ni$

$$F_h(t)=\int_a^b K_1(t,u,h(u))du,$$
$$G_h(t)=\int_a^b K_2(t,u,h(u))du,$$

which belong to $P,\forall t\in[a_1,a_2]$ and if there exist mappings $F_1,F_2,F_3,F_4: P\times P\times P\to[0,1)\ni\forall h,k\in P$ and for a fixed $b\in P$, satisfying the following conditions

(a) $F_i(S_2(F_h+f),k,b)\leq F_i(h,k,b)$ and $F_i(h,S_1(G_h+g),b)\leq F_i(h,k,b)$, for i = 1, 2, 3, 4.

(b) $F_1(h,k,b)+\sqrt{2}\ F_2(h,k,b)+\sqrt{2}\ F_3(h,k,b)+\sqrt{2}\,F_4(h,k,b)<1$.

(c) $|F_h(t) - G_k(t) + f(t) - g(t)| \leq F_1(h,k,b)A(h,k)(t) + F_2(h,k,b)B(h,k)(t) + F_3(h,k,b)C(h,k)(t) + F_4(h,k,b)D(h,k)(t),$

where

$$A(h,k)(t) = |h(t) - k(t)|,$$
$$B(h,k)(t) = \frac{|h(t) - F_h(t) - f(t)||k(t) - G_k(t) - g(t)|}{1 + |h(t) - k(t)|}$$

$$C(h,k)(t) = \frac{|k(t) - F_h(t) - f(t)||h(t) - G_k(t) - g(t)|}{1 + |h(t) - k(t)|},$$
$$D(h,k)(t) = \left(\frac{|h(t) - F_h(t) - f(t)||h(t) - G_k(t) - g(t)|}{1 + |h(t) - G_k(t) - g(t)| + |k(t) - F_h(t) - f(t)|} + \frac{|k(t) - G_k(t) - g(t)||k(t) - F_h(t) - f(t)|}{1 + |h(t) - G_k(t) - g(t)| + |k(t) - F_h(t) - f(t)|}\right)$$

then the system of the integral equations (7) and (8) have a unique common solution.

Proof. Define continuous mappings $S_1, S_2 : P \to P$ by $S_1 h = F_h + f$ and $S_2 h = G_h + g$. Then, we have

$$d_{\mathbb{C}_2}(S_1 h, S_2 k) = \max_{t\in[a,b]}(1 + i_2)(|F_h(t) + f(t) - G_k(t) + g(t)|),$$
$$d_{\mathbb{C}_2}(h, S_1 h) = \max_{t\in[a,b]}(1 + i_2)(|h(t) - F_h(t) - f(t)|),$$
$$d_{\mathbb{C}_2}(k, S_2 k) = \max_{t\in[a,b]}(1 + i_2)(|k(t) - G_k(t) - g(t)|),$$
$$d_{\mathbb{C}_2}(h, S_2 k) = \max_{t\in[a,b]}(1 + i_2)(|h(t) - G_k(t) - g(t)|)$$
$$\text{and } d_{\mathbb{C}_2}(k, S_1 h) = \max_{t\in[a,b]}(1 + i_2)(|k(t) - F_h(t) - f(t)|).$$

Clearly condition (1) of Theorem (2.1) is satisfied for all $h, k \in P$ and so by Theorem (2.1), the equations (7) and (8) have a unique common solution. □

4. Conclusion

In our study, we have introduced several findings related to common fixed points, achieved by broadening the scope of rational type contraction conditions with the integration of three-variable control functions as coefficients within the framework of bicomplex valued metric spaces. By replacing contraction constants with control functions as coefficients, we have enhanced and expanded upon the contraction conditions of a wide array of pre-existing theorems, thereby enriching the field of bicomplex valued metric spaces with our contributions.

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