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Approximate bound state solutions of the fractional Schrödinger equation under the spin-spin-dependent Cornell potential

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Abstract

In this work, the approximate bound state solutions of the fractional Schrödinger equation under a spin-spin-dependent Cornell potential are obtained via the convectional Nikiforov-Uvarov approach. The energy spectra are applied to obtain the mass spectra of the heavy mesons such as bottomonium, charmonium and bottom-charm. The masses for the singlet and triplet spin numbers increase as the quantum numbers increase. The fractional Schrödinger equation improves the mass spectra compared to other theoretic masses. The bottomonium masses agree with the experimental results where percentage errors for fractional parameters of $\beta = 1, \alpha = 0.97$ and $\beta = 1, \alpha = 0.50$ were found to be 0.67% and 0.49% respectively. The respective percentage errors of 1.97% and 1.62% for fractional parameters of $\beta = 1, \alpha = 0.97$ and $\beta = 1, \alpha = 0.50$ were obtained for charmonium meson. The results indicate that the potential curves coupled with the fractional parameters account for the short-range gluon exchange between the quark-antiquark interactions and the linear confinement phenomena which is associated with the quantum chromo-dynamic and phenomenological potential models in particle and high-energy physics.

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1. Introduction

Particles and their interactions are the fundamental building blocks of nature and may be produced or annihilated in high-energy accelerators. The elementary particles interact via the four forces of nature such as the strong nuclear force, weak, electromagnetic and the gravitational force. Of these particles, the quarks are the smallest which mediate the strong force and their interaction can be understood with a theory built on quantum chromo-dynamics and phenomenology.

The quarks are composite structure, spin $1/2$ fermions and possess anti-particles. Two quarks constitute a meson, while three quark pairs combine to form baryons. The mesons and baryons are further classed as hadrons. Owing to the vital roles quarks play in understanding the evolvement of the universe, many particle and high-energy physicists have contributed significant research to study their properties.

The rate of decay and spin average masses are among the properties of hadrons studied where the mass spectra is directly

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related to the energy of the hadrons. The bound state solutions of the wave equation with a quark anti-quark interaction and a phenomenological potential function have been used as primary properties in determining other properties of hadrons. The nature of the potential consists of a linear confinement function and a short-range Coulomb potential which is responsible for the gluon exchange between the quark and its anti-particle.

In contrast, molecular potential functions have been used to investigate the mass spectra [1, 2]. Douglas *et al.* [1] used the rovibrational model with the molecular Morse potential to obtain the resonances and masses of hadrons. Their findings were in strong accord with those of experiments and theoretical research. With the addition of the spin-spin, spin-orbit, and tensor components, Fang *et al.* [2] utilized the rovibrational model to derive the s-wave mass splitting between the singlet and triplet spin states as well as the triplet spin states for orbital quantum number larger than zero.

The quarkonia energy spectrum is a crucial source of knowledge on the quarks interaction [3]. The Schrödinger equation (SE) could be used to describe many aspects of heavy quarkonia, such as mass spectra (MS), and decay properties. It is generally accepted that the non-relativistic method is suitable in describing the heavy quarks and the level spacing between the energy levels is less than the component masses [4, 5].

Different types of the inter-quark potential have been applied in heavy quarkonia mass spectroscopy [6, 7]. Energy-dependent potentials have been applied to study of properties of mesons within the context of the SE [8]. It is possible to examine the properties of hadrons using potential models [9–13].

The use of the non-relativistic method delivers a substantial description of the characteristics of the heavy mesons, including their MS, rates of decay, radius, and other characteristics. We are thus driven by fundamental features of quantum chromo-dynamics (QCD), specifically the spin dependency of the strong interaction fields in this study.

The investigation of the properties of mesons such as the decay constant and the radius are not possible to reach across the entire range of separations from the QCD principle. The spin components of the interaction potentials have often been neglected in the literature since it is difficult to obtain an exact solution [14–17]. In these circumstances, numerical solutions are widely applied [18–21].

Recent research has suggested that two different forms of quark interactions can represent different meson sectors [22]. Kher and Rai [23] were able to determine the mass splitting and decay characteristics of the charmonium meson utilizing the spin-dependent Cornell potential (CP). Using the CP, Luz *et al.* [24] solved the SE in the phase space. The Wigner function and related Airy function of the charmonium meson were examined using their derived solutions.

Additionally, Gupta and Mehrotra [25] evaluated the masses, decay rates, and position expectation values of heavy mesons utilizing the energy-dependent potential. Extensive works on the properties of the quarkonia systems can be found in the literature [26–36].

The motivation for this research is as follows:

1. To obtain an analytical solution for the spin-dependent SE utilizing the CP and generalized fractional derivative.
2. To apply the energy spectra to obtain the MS of the heavy mesons. To our knowledge, this research is original and has not been published anywhere else.

The outline of our paper is as follows. In section 2, we give a brief review of the generalized fractional derivative (GFD) and the generalized fractional Nikiforov-Uvarov (NU) method. In the section 3, we solve the SE of the quarkonia under spin-spin interaction. Results and discussion are presented in the section 4. Finally, a brief concluding remarks is given in the section 5.

2. The Generalized Fractional Derivative

The idea of fractional calculus has recently attracted a lot of interest in a number of physics fields that deal with complex nonlinear events [37–41]. In these studies, the conformable fractional derivative was used in conjunction with the NU approach to effectively solve the fractional SE. The generalized fractional derivative is a new concept for the fractional derivative proposed by Abu-Shady and Kaabar [42, 43]. This new definition coincides with the classical fractional derivative and has become a useful tool for solving fractional differential equations. It has been successfully applied in heavy meson and molecular physics, as can be seen in the references [44–48]. The GFD is a novel formula for a fractional derivative. Theorems that have been satisfied in the GFD, which offers a new approach for solving fractional differential equations (see Ref. [43]), include the Rolle(’s) theorem, the mean value theorem, the derivative of two functions, the derivative of the quotient of two functions, and others. It has been argued that these definitions have more benefits than other traditional Caputo and Riemann-Liouville fractional derivative definitions. For a function $\mathrm{Z} : (0,\infty) \rightarrow R$, the generalized fractional derivative of order $0 < \alpha \leq 1$ of $\mathrm{Z}(t)$ at > 0 is defined as

$$D^{GFD}\mathrm{Z}(t) = \lim_{\varepsilon \to 0} \frac{\mathrm{Z}\left(t + \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\varepsilon t^{1-\alpha}\right) - \mathrm{Z}(t)}{\varepsilon} \;;\beta > -1,\ \beta \epsilon R^{+} \tag{1}$$

The properties of the generalized fractional derivative are,

- $D^{\alpha}[\,\mathrm{Z}_{nl}(t)] = k_1 t^{1-\alpha}\grave{\mathrm{Z}}_{nl}(t).$ (2)
- $D^{\alpha}[D^{\alpha}\ \mathrm{Z}(t)] = {k_1}^{2}[(1-\alpha)t^{1-2\ \alpha}\grave{\mathrm{Z}}_{nl}(t)\ + t^{2-2\ \alpha}\ {\mathrm{Z}_{nl}}''(t)],$ (3) where, $k_1 = \frac{\Gamma[\beta]}{\Gamma[\beta-\alpha+1]}$, with $0 < \alpha \leq 1$, $0 < \beta \leq 1$.
- $D^{\alpha}\ D^{\beta}\ t^{m} = D^{\alpha+\beta}t^{m}$ for function derivative of $\mathrm{Z}(t) = t^{m}$, $\mathrm{m}\ \epsilon\ R^{+}$
- $D^{GFD}(\mathrm{X}\Upsilon) = \mathrm{X}D^{GFD}(\Upsilon) + \Upsilon D^{GFD}(\mathrm{X})$, where X, Υ be $\alpha-$ differentiable function
- $D^{GFD}(\ \frac{\mathrm{X}}{\Upsilon}) = \frac{\Upsilon D^{GFD}(\mathrm{X}) - \mathrm{X}\ D^{GFD}(\Upsilon)}{\Upsilon^{2}}$, where X, Υ be α- differentiable function
- $D^{\alpha}\ I_{\alpha}\mathrm{Z}(t) = \mathrm{Z}\,(t)$ for ≥ 0 and Z is any continuous function in the domain.

2.1. The Generalized Fractional Nikiforov-Uvarov (NU) Method

In the fractional form as in Ref. [49], the second-order parametric generalized differential equation is precisely solved.

$$D^{\alpha}[D^{\alpha}\psi(s)] + \frac{\overline{\tau}(s)}{\sigma(s)}D^{\alpha}\psi(s) + \frac{\bar{\sigma}(s)}{\sigma^{2}(s)}\psi(s) = 0, \tag{4}$$

where $\bar{\sigma}(s)$, $\sigma(s)$ and $\overline{\tau}(s)$ are polynomials of 2α-th, 2α-th and α-th degree. Where,

$$\pi(s) = \frac{D^{\alpha}\sigma(s) - \overline{\tau}(s)}{2} \pm \sqrt{\left(\frac{D^{\alpha}\sigma(s) - \overline{\tau}(s)}{2}\right)^{2} - \bar{\sigma}(s) + K\,\sigma(s)}\,, \tag{5}$$

and

$$\lambda = K + D^{\alpha}\pi(s)\,. \tag{6}$$

λ is constant and $\pi(s)$ is α-th degree polynomial. The values of K in the square root of Eq. (5) is possible to determine whether the expression under the square root is square of expression. Replacing K into Eq. (5), we define

$$\tau(s) = \overline{\tau}(s) + 2\pi(s) \tag{7}$$

the derivative of τ should be negative [50], since $\rho(s) > 0$ and $\sigma(s) > 0$ then this is the solution. If λ in Eq.(6) is

$$\lambda = \lambda_{n} = -nD^{\alpha}\,\tau - \frac{n(n-1)}{2}D^{\alpha}[D^{\alpha}\sigma(s)]. \tag{8}$$

The hypergeometric type equation has a particular solution with degree α. Eq. (4) has a solution which is the product of two independent parts

$$\psi(s) = \phi(s)y(s)\,, \tag{9}$$

where,

$$y_{n}(s) = \frac{B_{n}}{\rho(s)}(D^{\alpha})^{n}(\,\sigma(s)^{n}\,\rho_{n}(s)). \tag{10}$$

So, we can determine $\rho(s)$ as follows

$$D^{\alpha}[\sigma(s)\,\rho(s)] = \tau(s)\sigma(s)\,, \tag{11}$$

Also, we can determine ϕ(s) as follows

$$\frac{D^{\alpha}\phi(\mathrm{s})}{\phi(\mathrm{s})} = \frac{\pi(s)}{\sigma(s)}. \tag{12}$$

3. The solutions of the energy and wave function of quarkonia under spin-dependent Cornell potential

The interaction potential takes the form [51]

$$V(r) = \frac{-4\alpha_{s}}{3r} + br + \frac{16\alpha_{s}\pi\left(\frac{\sigma}{\sqrt{\pi}}\right)^{3}e^{-\sigma^{2}r^{2}}\left(s(s+1) - \frac{3}{2}\right)}{9m_{q}m_{\overline{q}}}, \tag{13}$$

where b, α_{s} and σ are constants that will be solved later from the experimental data [52, 53]. The notations $m_{q}, m_{\overline{q}}$ denote the masses of the quark and its anti-particle and s represent the spin quantum number. The colour factor is $-4/3$ multiplying the first term in the potential. The quantum number J^{PC}, is a representation of the hyperfine splitting between the spin quantum states. The symbolisations $P\left((-1)^{L+1}\right)$ and $C\left((-1)^{L+s}\right)$ are the respective parity and charge conjugation. The quantum number J(L+s) is obtained by adding the orbital and spin components. The singlet ($s = 0$) and triplet($s = 1$) states are derived using the states $n^{2s+1}L_{J}$. The SE with energy eigenvalue (E_{nl}), wave function ($R_{nl}(r)$), and the reduced mass μ takes the form

$$\frac{\hbar^{2}}{2\mu}\frac{d^{2}\psi}{dr^{2}} + V_{eff}(r)\,\psi_{nl}(r) = E_{nl}\psi_{nl}(r)\,. \tag{14}$$

With the potential in Eq. (13) and the Gaussian function present, Eq. (14) cannot be analytically solved. To solves Eq. (14), the Gaussian function is approximated by Taylor expanding it to second-order in r.

$$e^{-\sigma^{2}r^{2}} \approx 1 - \sigma^{2}r^{2}, \sigma r \ll 1. \tag{15}$$

Inserting Eq.(15) into (14) we obtained

$$\frac{d^{2}\psi}{dr^{2}} + \left(\epsilon_{nl} + \frac{a}{r} - cr + dr^{2} - \frac{L}{r^{2}}\right)\psi(r) = 0, \tag{16}$$

where

$$a = \frac{8\mu\alpha_{s}}{3\hbar^{2}}, c = \frac{2\mu b}{\hbar^{2}}, d = \frac{32\mu\sigma^{2}\alpha_{s}\pi\left(\frac{\sigma}{\sqrt{\pi}}\right)^{3}\left(s(s+1) - \frac{3}{2}\right)}{9\hbar^{2}m_{q}m_{\overline{q}}}.$$

$$\epsilon_{nl} = \frac{2\mu E_{nl}}{\hbar^{2}} - \frac{d}{\sigma^{2}}, L = l(l+1)\,.$$

Using coordinate transformation, $= \frac{1}{q}$, Eq. (16) reduces to

$$\frac{d^{2}\psi}{dq^{2}} + \frac{2q}{q^{2}}\frac{d\psi}{dq} + \frac{1}{q^{4}}\left(\epsilon_{nl} + aq - \frac{c}{q} + \frac{d}{q^{2}} - Lq^{2}\right)\psi(q) = 0. \tag{17}$$

The wave function is given by

$$\psi_{n}(q) = N_{nl}q^{\frac{B}{2\sqrt{C}}}e^{\frac{\sqrt{C}}{q}}\frac{d^{n}}{dq^{n}}\left(q^{2n-\frac{B}{\sqrt{C}}}e^{-2\frac{\sqrt{C}}{q}}\right), q = \frac{1}{r} \tag{18}$$

To calculate the mass spectra for the heavy mesons, we used the energy mass relations

$$M_{nl} = m_{q} + m_{\overline{q}} + E_{nl}. \tag{19}$$

It is possible to solve Eq. (17) using the fractional Nikiforov-Uvarov method shown in Section (2). However, we must expand $\frac{c}{q}$ and $\frac{d}{q^{2}}$ in a power series to second-order around r_{0} $\left(\delta = \frac{1}{r_{0}}\right)$ in q-space which is assume to be the characteristics radius of the mesons and use the approximation method to convert the equation to a solvable form. This is comparable to the Pekeris approximation [54], which aids in deforming the

Table 1: Calculated potential parameters for the mesons.

Parameters	Bottomonium	$m_b = m_{\bar{b}} = 4.890$	Charmonium	$m_c = m_{\bar{c}} = 1.27$		Bottom-charm
Mass(GeV)	α=0.97, β=1	α=0.5, β=1	α=0.97, β=1	α=0.5, β=1	α=0.97, β=1	α=0.5, β=1
α_s	1.6854	1.2918	2.8391	1.8164	2.1208	1.8289
$\delta\,(GeV)$	0.7610	0.7153	0.5910	0.4882	0.5910	0.4882
$b\left(GeV^2\right)$	0.3200	0.3884	0.4281	0.4175	0.2931	0.3711
$\sigma\,(GeV)$	0.5384	0.6586	0.3021	-0.2969	0.3021	-0.2969

Table 2: Bottomonium mass spectra in GeV under spin-spin interaction potential.

State		[52]	Present		[34]	[28]	[36]	[27]	[33]	[35]	Expt.[53]
$n^{2s+1}L_j$	J^{PC}		α=0.97, β=1	α=0.5, β=1							
$\Upsilon\left(1^3S_1\right)^*$	1^{--}	9.460	9.460	9.497	9.463	9.465	9.460	9.525	9.460	9.460	$9.460 \pm (2.6\times10^{-4})$
$\eta_b\left(1^1S_0\right)^*$	0^{-+}	9.399	9.437	9.419	9.423	9.402	9.398	9.472	9.390	9.428	$9.399 \pm \left(2\times10^{-3}\right)$
$\Upsilon\left(2^3S_1\right)$	1^{--}	10.064	10.084	10.023	10.001	10.003	10.023	10.049	10.015	9.979	$10.023 \pm \left(3.1\times10^{-4}\right)$
$\eta_b\left(2^1S_0\right)$	0^{-+}	10.074	10.084	9.993	9.983	9.976	9.990	10.028	9.990	9.955	
$\Upsilon\left(3^3S_1\right)^*$	1^{--}	10.355	10.399	10.355	10.354	10.354	10.355	10.371	10.343	10.359	$10.355 \pm \left(5\times10^{-4}\right)$
$\eta_b\left(3^1S_0\right)$	0^{-+}	10.408	10.412	10.357	10.342	10.336	10.329	10.360	10.326	10.338	
$\Upsilon\left(4^3S_1\right)$	1^{--}	10.517	10.579	10.579	10.650	10.635	10.586	10.598	10.597	10.683	$10.579 \pm \left(1.2\times10^{-3}\right)$
$\eta_b\left(4^1S_0\right)$	0^{-+}	10.596	10.600	10.606	10.638	10.623	10.573	10.592	10.584	10.663	
$\Upsilon\left(5^3S_1\right)$	1^{--}	10.617	10.692	10.736	10.912	10.878	10.851	10.870	10.811	10.975	$10.885^{+2.6\times10^{-3}}_{-1.6\times10^{-3}}$
$\eta_b\left(5^1S_0\right)$	0^{-+}	10.713	10.718	10.781	10.901	10.869	10.869	10.790	10.800	10.956	
$\Upsilon\left(6^3S_1\right)$	1^{--}	10.682	10.767	10.852	11.150	11.102	11.061	11.022	10.997	11.243	$11.020 \pm \left(4\times10^{-3}\right)$
$\eta_b\left(6^1S_0\right)$	0^{-+}	10.791	10.797	10.910	11.140	11.097	11.088	10.961	10.988	11.226	11.014
$\chi_{b_1}\left(1^3P_1\right)$	1^{++}	9.827	9.810	9.893	9.894	9.876	9.892	9.875	9.903	9.819	$9.893 \pm \left(2.6\times10^{-4}\right) \pm \left(3.1\times10^{-4}\right)$
$h_b\left(1^1P_1\right)$	1^{+-}	9.789	9.796	9.842	9.899	9.882	9.900	9.884	9.909	9.821	$9.899 \pm \left(8\times10^{-4}\right)$
$\chi_{b_1}\left(2^3P_1\right)$	1^{++}	10.235	10.255	10.271	10.265	10.246	10.255	10.229	10249	10.2170	$10.255 \pm \left(2.2\times10^{-4}\right) \pm \left(5\times10^{-4}\right)$
$h_b\left(2^1P_1\right)^*$	1^{+-}	10.260	10.260	10.260	10.268	10.250	10.260	10.237	10254	10.220	$10.260 \pm \left(1.2\times10^{-3}\right)$
$\chi_{b_1}\left(3^3P_1\right)$	1^{++}	10.448	10.494	10.521	10.567	10.538	10.541	10.339	10.515	10.553	$10.5135 \pm \left(7\times10^{-4}\right)$
$h_b\left(3^1P_1\right)$	1^{+-}	10.510	10.510	10.538	10.570	10.541	10.544	10.362	10.519	10.556	
$\chi_{b_1}\left(4^3P_1\right)$	1^{++}	10.573	10.637	10.695		10.788	10.802	10.571			
$h_b\left(4^1P_1\right)$	1^{+-}	10.659	10.661	10.732		10.790	10.804	10.594			
$\Upsilon_2\left(1^3D_2\right)^*$	2^{--}	10.164	10.164	10.319	10.149	10.147	10.161	10.096	10.153	10.075	$10.164 \pm \left(1.4\times10^{-3}\right)$
$\eta_{b_2}\left(1^1D_2\right)$	2^{-+}	10.167	10.163	10.307	10.150	10.148	10.163	9.767	10.153	10.074	
$\Upsilon_2\left(2^3D_2\right)$	2^{--}	10.295	10.443	10.554	10.465	10.449	10.443	10.071	10.432	10.424	
$\eta_{b_2}\left(2^1D_2\right)$	2^{-+}	10.407	10.455	10.570	10.465	10.450	10.445	10.093	10.432	10.424	
$\Upsilon_2\left(3^3D_2\right)$	2^{--}	10.549	10.606	10.718	10.740	10.705	10.711	10.345		10.733	
$\eta_{b_2}\left(3^1D_2\right)$	2^{-+}	10.627	10.627	10.756	10.740	10.706	10.713	10.368		10.733	
$\chi_{b_3}\left(1^3F_3\right)$	3^{++}	10.386	10.412	10.628		10.355	10.346	9.754	10.340	10.287	
$\eta_{b_3}\left(1^1F_3\right)$	3^{+-}	10.427	10422	10.651		10.355	10.347	9.778	10.339	10.288	
$\chi_{b_3}\left(2^3F_3\right)$	3^{++}	10.535	10.587	10.772		10.619	10.614	10.081		10.607	
$\eta_{b_3}\left(2^1F_3\right)$	3^{+-}	10.608	10.606	10.814		10.619	10.615	10.104		10.607	
$\chi_{b_4}\left(1^3G_4\right)$	4^{--}	10.527	10.576	10.833		10.531	10.512				
$\eta_{b_4}\left(1^1G_4\right)$	4^{-+}	10.595	10.594	10.882		10.530	10.513				
%Error		0.75	0.67	0.49	0.33	0.24	0.13	0.40	0.15	0.72	

centrifugal potential such that the changed potential may be calculated using the NU technique. Putting $y = q - \delta$ and around y = 0, so that we can write

$$\frac{c}{q} = \frac{c}{(y+\delta)} \approx \frac{c}{\delta}\left(1 - \frac{y}{\delta} + \frac{y^2}{\delta^2}\right) = c\left(\frac{3}{\delta} - \frac{3q}{\delta^2} + \frac{q^2}{\delta^3}\right). \tag{20}$$

Table 3: Charmonium mass spectra in *GeV* under spin-spin interaction potential.

State		Present		[52]	[35]	[32] LP(SP)	[23]	[36]	[27]	Expt.[53]
$n^{2s+1}L_j$	J^{PC}	α=0.97, β=1	α=0.5, β=1							
$J/\psi\left(1^3S_1\right)^*$	1^{--}	2.950	3.097	3.081(3.097)	3.094	3.097(3.097)	3.094	3.096	3.126	$3.097 \pm \left(6\times10^{-6}\right)$
$\eta_c\left(1^1S_0\right)^*$	0^{-+}	2.984	3.045	3.061(2.979)	2.989	2.983(2.984)	2.995	2.981	3.033	2.984 ± 0.0004
$\psi\left(2^3S_1\right)$	1^{--}	3.686	3.677	3.717(3.736)	3.681	3.679(3.679)	3.649	3.685	3.701	$3.686 \pm \left(6\times10^{-5}\right)$
$\eta_c\left(2^1S_0\right)$	0^{-+}	3.709	3.638	3.696(3.640)	3.602	3.635(3.637)	3.606	3.635	3.666	$3.638 \pm \left(1.1\times10^{-3}\right)$
$\psi\left(3^3S_1\right)^*$	1^{--}	4.039	4.039	4.003(4.039)	4.129	4.078(4.030)	4.036	4.039	4.055	4.039 ± 10^{-3}
$\eta_c\left(3^1S_0\right)$	0^{-+}	4.066	4.002	3.983(3.937)	4.058	4.048(4.004)	4.000	3.989	4.158	
$\psi\left(4^3S_1\right)$	1^{--}	4.235	4.280	4.156(4.206)	4.514	4.412(4.281)	4.362	4.427	4.415	$4.421 \pm \left(4\times10^{-3}\right)$
$\eta_c\left(4^1S_0\right)$	0^{-+}	4.267	4.242	4.137(4.096)	4.448	4.388(4..264)	4.328	4.401	4.415	
$\psi\left(5^3S_1\right)$	1^{--}	4.355	4.449	4.247(4.308)	4.863	4.711(4.472)	4.654	4.837	4.585	$4.63 \pm \left(6\times10^{-3}\right)$
$\eta_c\left(5^1S_0\right)$	0^{-+}	4.391	4.408	4.228(4.190)	4.799	4.690(4.459)	4.622	4.811	4.607	
$\psi\left(6^3S_1\right)$	1^{--}	4.434	4.572	4.305(4.374)	5.185		4.925	5.167	4.733	
$\eta_c\left(6^1S_0\right)$	0^{-+}	4.473	4.527	4.287(4.250)	5.124		4.893	5.155	4.754	
$\chi_{c_1}\left(1^3P_1\right)$	1^{++}	3.412	3.547	3.528(3.501)	3.468	3.516(3.521)	3.523	3.511	3.487	$3.511 \pm \left(5\times10^{-5}\right)$
$h_c\left(1^1P_1\right)^*$	1^{+-}	3.416	3.525	3.503(3.446)	3.470	3.522(3.526)	3.534	3.525	3.502	$3.525 \pm \left(1.1\times10^{-4}\right)$
$\chi_{c_1}\left(2^3P_1\right)^*$	1^{++}	3.900	3.955	3.912(3.921)	3.938	3.937(3.914)	3.925	3.906	3.786	$3.872 \pm \left(6\times10^{-5}\right)$
$h_c\left(2^1P_1\right)$	1^{+-}	3.915	3.931	3.889(3.843)	3.943	3.940(3.916)	3.936	3.926	3.821	
$\chi_{c_1}\left(3^3P_1\right)$	1^{++}	4.155	4.224	4.105(4.139)	4.338	4.284(4.192)	4.257	4.319	4.123	$4.147 \pm \left(3\times10^{-3}\right)$
$h_c\left(3^1P_1\right)$	1^{+-}	4.179	4.194	4.084(4.043)	4.344	4.285(4.193)	4.269	4.337	4.164	
$\chi_{c_1}\left(4^3P_1\right)$	1^{++}	4.305	4.409	4.215(4.266)	4.696			4.728	4.373	
$h_c\left(4^1P_1\right)$	1^{+-}	4.335	4.374	4.196(4.158)	4.704			4.744	4.420	
$\psi_2\left(1^3D_2\right)$	2^{--}	3.823	4.015	3.868(3.855)	3.772	3.807(3.807)	3.805	3.795	3.348	$3.823 \pm \left(5\times10^{-4}\right)$
$\eta_{c_2}\left(1^1D_2\right)$	2^{-+}	3.825	4.001	3.844(3.798)	3.765	3.806(3.805)	3.802	3.807	3.376	
$\psi_2\left(2^3D_2\right)$	2^{--}	4.113	4.264	4.081(4.102)	4.188	4.165(4.109)	4.152	4.190	3.801	
$\psi_2\left(2^1D_2\right)$	2^{-+}	4.129	4.241	4.059(4.019)	4.182	4.164(4.108)	4.150	4.196	3.836	
$\psi_2\left(3^3D_2\right)$	2^{--}	4.279	4.438	4.201(4.243)	4.557	4.478(4.337)	4.456	4.544	4.135	
$\eta_{c_2}\left(3^1D_2\right)$	2^{-+}	4.304	4.407	4.181(4.143)	4.553	4.478(4.336)	4.455	4.549	4.176	
$\chi_{c_3}\left(1^3F_3\right)$	3^{++}	4.089	4.346	4.068(4.081)	4.012			4.068	3.375	
$\eta_{c_3}\left(1^1F_3\right)$	3^{+-}	4.100	4.325	4.046(4.006)	4.017			4.071	3.403	
$\chi_{c_3}\left(2^3F_3\right)$	3^{++}	4.265	4.497	4.194(4.231)	4.396			4.400	3.823	
$\eta_{c_3}\left(2^1F_3\right)$	3^{+-}	4.287	4.467	4.173(4.135)	4.400			4.406	3.858	
$\chi_{c_4}\left(1^3G_4\right)$	4^{--}	4.257	4.562	4.189(4.223)				4.343		
$\eta_{c_4}\left(1^1G_4\right)$	4^{-+}	4.277	4.531	4.168(4.131)				4.345		
% Error		1.97%	1.62%	2.09(1.52)	1.77	0.74(0.83)	0.81	0.90	0.85	

$$\frac{d}{q^2} = \frac{d}{(y+\delta)^2} \approx \frac{d}{\delta^2}\left(1 - \frac{2y}{\delta} + \frac{3y^2}{\delta^2}\right) = d\left(\frac{6}{\delta^2} - \frac{8q}{\delta^3} + \frac{3q^2}{\delta^4}\right). \tag{21}$$

We note that Eq. (20) and (21) give a good accuracy when y tends to zero means q ≈ δ as seen Figure 2. Inserting (20) and (21) into (17) results to

$$\frac{d^2\psi}{dq^2} + \frac{2q}{q^2}\frac{d\psi}{dq} + \frac{1}{q^4}\left(-Aq^2 + Bq - C\right)\psi(q) = 0, \tag{22}$$

where

$$A = \frac{c}{\delta^3} - \frac{3d}{\delta^4} + L \tag{23}$$

$$B = a + \frac{3c}{\delta^2} - \frac{8d}{\delta^3} \tag{24}$$

$$C = -\epsilon_{nl} + \frac{3c}{\delta} - \frac{6d}{\delta^2} \tag{25}$$

Eq. (22) is written in the fractional form of dimensionless units by taking q=sμ_1 where s is the dimensionless variable and $\mu$is scale parameter equals 1 GeV, so we can write Eq. (22)

$$D^\alpha D^\alpha \psi(s) + \frac{\widetilde{\tau}(s)}{\sigma(s)} D^\alpha \psi(s) + \frac{\widetilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{26}$$

where

$$\widetilde{\tau}(s) = 2s^\alpha, \sigma(s) = s^{2\alpha} \text{ and } \widetilde{\sigma}(s) = -As^{2\alpha} + \frac{B}{\mu_1}s^\alpha - \frac{C}{\mu_1} \tag{27}$$

Table 4: Bottom-charm mass spectra in GeV under spin-spin interaction potential.

State		Present		[52]	[36]	[1]	[9]	Expt.[54]
$n^{2s+1}L_j$	J^{PC}	α=0.97, β=1	α=0.5, β=1					
1^3S_1	1^{--}	6.278	6.272	6.2808	6.3330	6.313	6.320	
$\left(1^1S_0\right)^*$	0^{-+}	6.275	6.275	6.2750	6.2720	6.276	6.721	6.275
2^3S_1	1^{--}	6.842	6.841	6.8523	6.8820	6.867	6.900	
$\left(2^1S_0\right)^*$	0^{-+}	6.842	6.842	6.8420	6.8420	6.841	6.864	6.842
3^3S_1	1^{--}	7.118	7.213	7.1179	7.2580	7.308	7.338	
3^1S_0	0^{-+}	7.120	7.211	7.1027	7.2260	7.281	7.306	
4^3S_1	1^{--}	7.272	7.469	7.2626	7.6090	7.660	7.714	
4^1S_0	0^{-+}	7.276	7.465	7.2437	7.5850	7.634	7.684	
5^3S_1	1^{--}	7.368	7.653	7.3500	7.9470	7.941	8.054	
5^1S_0	0^{-+}	7.373	7.647	7.3284	7.9280	7.917	8.025	
6^3S_1	1^{--}	7.431	7.789	7.4068		8.168	8.368	
6^1S_0	0^{-+}	7.437	7.782	7.3833		8.144	8.340	
1^3P_1	1^{++}	6.619	6.664	6.6570	6.743	6.281	6.705	
1^1P_1	1^{+-}	6.614	6.669	6.6567	6.750	6.290	6.706	
2^3P_1	1^{++}	7.003	7.095	7.0213	7.134	6.836	7.165	
2^1P_1	1^{+-}	7.003	7.096	7.0117	7.147	6.846	7.168	
3^3P_1	1^{++}	7.206	7.386	7.2079	7.500	7.278	7.555	
3^1P_1	1^{+-}	7.208	7.385	7.1924	7.510	7.287	7.559	
4^3P_1	1^{++}	7.326	7.593	7.3161	7.844	7.631	7.905	
4^1P_1	1^{+-}	7.330	7.589	7.2968	7.853	7.640	7.908	
1^3D_2	2^{--}	6.936	7.125	6.9697	7.025	6.299	6.997	
1^1D_2	2^{-+}	6.934	7.128	6.9646	7.026	6.308	6.994	
2^3D_2	2^{--}	7.169	7.407	7.1797	7.399	6.852	7.403	
2^1D_2	2^{-+}	7.170	7.407	7.1668	7.400	6.861	7.401	
3^3D_2	2^{--}	7.303	7.608	7.2991	7.741	7.290	7.764	
3^1D_2	2^{-+}	7.306	7.605	7.2813	7.743	7.302	7.762	
1^3F_3	3^{++}	7.148	7.484	7.1639	7.269	6.326	7.242	
1^1F_3	3^{+-}	7.148	7.484	7.1527	7.268	6.335	7.241	
2^3F_3	3^{++}	7.290	7.664	7.2896	7.616	6.876	7.615	
2^1F_3	3^{+-}	7.293	7.661	7.2729	7.615	6.885	7.614	
1^3G_4	4^{--}	7.283	7.734	7.3635	7.489			
1^1G_4	4^{-+}	7.285	7.731	7.3437	7.487			

From eq. (5), we have $\pi(s)$ as

$$\pi(q) = \frac{(2\alpha k_1 - 2)\, s^{\alpha}}{2} \pm \sqrt{\left(\frac{(2\alpha k_1 - 2)^2}{4} + A + k\right) s^{2\alpha} - \frac{B}{\mu_1} s^{\alpha} + \frac{C}{\mu_1}}. \tag{28}$$

We note that $\pi(s)$is a first-order polynomial, consequently the square root's terms must be equal to $(Xq+Y)^2$. With this, we have $(Xs^{\alpha}+Y)^2 = \left(\frac{(2\alpha k_1-2)^2}{4} + A + k\right) s^{2\alpha} - \frac{B}{\mu_1} s^{\alpha} + \frac{C}{\mu_1}$ and solving it gives

$$k = \frac{B^2}{4\mu C} - \frac{(2\alpha k_1 - 2)^2}{4} - A \tag{29}$$

Inserting Eq. (29) into Eq. (28) gives

$$\pi(s) = \frac{(2\alpha k_1 - 2)^2}{2} s^{\alpha} \pm \left[\pm\left(\frac{Bs^{\alpha}}{2\sqrt{\mu C}} - \sqrt{\frac{C}{\mu}}\right)\right]. \tag{30}$$

In Eq. (30), $\pi(s) = \frac{(2\alpha k_1-2)^2}{2} s^{\alpha} - \left[\left(\frac{Bs^{\alpha}}{2\sqrt{\mu C}} - \sqrt{\frac{C}{\mu_1}}\right)\right]$ that will make $\frac{d\tau(q)}{dq} < 0$

The energy eigenvalue in (8) is

$$\sqrt{\mu_1 C} = \frac{B}{2\left(n\alpha k_1 + \frac{\alpha k_1}{2} \pm \sqrt{\left(n\alpha k_1 + \frac{\alpha k_1}{2}\right)^2 - F}\right)}, \tag{31}$$

where

$$\begin{aligned} F &= n(n-1)k_1^2\alpha^2 + 2nk_1\alpha + nk_1\alpha(2k_1\alpha - 2) \\ &- \frac{(2k_1\alpha - 2)^2}{4} - A + \frac{k_1\alpha(2k_1\alpha - 2)}{2}. \end{aligned} \tag{32}$$

Injecting the constants A, B and C in Eqs. (23)-(25) into Eq.(31) yields

$$|\epsilon_{nl} = \varDelta_0 - \frac{1}{4}\left(\frac{\varDelta_1}{(n\alpha k_1 + \varDelta_2)}\right)^2, \tag{33}$$

where

$$\varDelta_0 = \frac{3c}{\delta} - \frac{6d}{\delta^2} \tag{34}$$

$$\varDelta_1 = a + \frac{3c}{\delta^2} - \frac{8d}{\delta^3} \tag{35}$$

$$\varDelta_2 = n\alpha k_1 + \frac{\alpha k_1}{2} \pm \sqrt{\left(n\alpha k_1 + \frac{\alpha k_1}{2}\right)^2 - F}. \tag{36}$$

Using the parameters for a, c, d and L in (16), the ES of the HM system is obtained as

$$E_{nl} = \varGamma_0 - \frac{\mu}{2\hbar^2}\left(\frac{\varGamma_1}{n\alpha k_1 + \varGamma_2}\right)^2, \tag{37}$$

where

$$\varGamma_0 = \frac{3b}{\delta} + \frac{16\alpha_s\pi\left(\frac{\sigma}{\sqrt{\pi}}\right)^3\left(s\left(s+1\right) - \frac{3}{2}\right)}{9m_q m_{\bar{q}}}\left[1 - \frac{6\sigma^2}{\delta^2}\right],$$

$$\varGamma_1 = \frac{4\alpha_s}{3} + \frac{3b}{\delta^2} - \frac{128\alpha_s\sigma^2\pi\left(\frac{\sigma}{\sqrt{\pi}}\right)^3\left(s\left(s+1\right) - \frac{3}{2}\right)}{9m_q m_{\bar{q}}\delta^3},$$

$$\varGamma_2 = \frac{\alpha k_1}{2} \pm \sqrt{\left(n\alpha k_1 + \frac{\alpha k_1}{2}\right)^2 - \Gamma_3},$$

where

$$\begin{aligned}\varGamma_3 \;=\;& n\left(n-1\right)k_1^2\alpha^2 + 2nk_1\alpha + nk_1\alpha\left(2k_1\alpha - 2\right)\\ &- \frac{\left(2k_1\alpha - 2\right)^2}{4}\\ &- \frac{2\mu b}{\hbar^2\delta^3} + \frac{96\alpha_s\sigma^2\pi\left(\frac{\sigma}{\sqrt{\pi}}\right)^3\left(s\left(s+1\right) - \frac{3}{2}\right)}{9m_q m_{\bar{q}}\delta^4}\\ &- \left(l + \frac{1}{2}\right)^2 + \frac{1}{4}.\end{aligned}$$

The wave function (WF) is determined from weight functions as

$$\rho\left(s\right) = s^{s_1}e^{-\frac{\sqrt[2]{\frac{c}{\mu_1}}}{\alpha}s^{-\alpha}}, \tag{38}$$

where

$$s_1 = -2\alpha + \frac{2}{k_1} + \frac{\left(2\alpha k_1 - 2\right)}{2} - \frac{B}{k_1\sqrt{C}}$$

$$\chi_n\left(s\right) = B_{nl}s^{-s_1}e^{\frac{\sqrt[2]{\frac{C}{\mu_1}}}{\alpha}s^{-\alpha}}\left(D^\alpha\right)^n\left(s^{2n\alpha+S_1}e^{-\frac{\sqrt[2]{\frac{C}{\mu_1}}}{\alpha}s^{-\alpha}}\right). \tag{39}$$

The other part of the WF is obtained using Eq. (12)

$$\phi\left(s\right) = s^{s_2}e^{-\frac{\sqrt{\frac{C}{\mu}}}{\alpha k_1}s^{-\alpha}}, \tag{40}$$

where

$$s_2 = \frac{\left(2\alpha k_1 - 2\right)}{2k_1} - \frac{B}{2k_1\sqrt{\mu C}}$$

Using Eq. (39) and Eq. (40) the total WF is gotten as

$$\begin{gathered}\psi_n\left(s\right) = B_{nl}s^{-s_1+s_2}e^{\frac{\sqrt[2]{\frac{C}{\mu_1}}}{\alpha}s^{-\alpha} - \frac{\sqrt{\frac{C}{\mu_1}}}{\alpha k_1}s^{-\alpha}}\left(D^\alpha\right)^n\\ \left(s^{2n\alpha+S_1}e^{-\frac{\sqrt[2]{\frac{C}{\mu_1}}}{\alpha}s^{-\alpha}}\right)s = \frac{1}{\mu_1 r}\end{gathered} \tag{41}$$

The normalization constant can be obtain using the relation

$$\int_0^\infty \left|\psi_{nl}\left(r\right)\right|^2 dr = 1 \tag{42}$$

4. Numerical results and discussion

In section 3, we have obtained the energy eigenvalue and wave function in the fractional forms as in Eqs. (37) and (41). We have obtained the results of Ref. [51] at $\alpha = \beta = 1 \rightarrow k_1 = 1$ and $\mu_1 = 1$

$$E_{nl} = \varGamma_0 - \frac{\mu}{2\hbar^2}\left(\frac{\varGamma_1}{n + \varGamma_2}\right)^2, \tag{43}$$

where

$$\varGamma_0 = \frac{3b}{\delta} + \frac{16\alpha_s\pi\left(\frac{\sigma}{\sqrt{\pi}}\right)^3\left(s\left(s+1\right) - \frac{3}{2}\right)}{9m_q m_{\bar{q}}}\left[1 - \frac{6\sigma^2}{\delta^2}\right],$$

$$\varGamma_1 = \frac{4\alpha_a}{3} + \frac{3b}{\delta^2} - \frac{128\alpha_s\sigma^2\pi\left(\frac{\sigma}{\sqrt{\pi}}\right)^3\left(s\left(s+1\right) - \frac{3}{2}\right)}{9m_q m_{\bar{q}}\delta^3},$$

$$\varGamma_2 = \frac{1}{2} \pm \sqrt{\frac{2\mu b}{\hbar^2\delta^3} - \frac{96\alpha_s\sigma^2\pi\left(\frac{\sigma}{\sqrt{\pi}}\right)^3\left(s\left(s+1\right) - \frac{3}{2}\right)}{9m_q m_{\bar{q}}\delta^4} + \left(l + \frac{1}{2}\right)^2}.$$

The potential free parameters for arbitrary fractional values were obtained by fitting the analytical equation with the experimental data of the particle Data Group [52]. We used fixed values of bottom and charm quark masses which had been measured as $1.2 < m_c < 1.8GeV$ and $4.5 < m_b < 5.4GeV$ respectively [24]. The free parameters of the spin-dependent potential with fixed values of fractional terms and masses are given in Table 1. For the bottom-charm meson due to lack of available experimental data, we have assumed a constant characteristic radius (δ) and smearing constant σ to obtain other unknown constants α_s and b. In addition, the parameters were used to compute the masses of the heavy mesons given in Table 2, Table 3 and Table 4. The masses for the triplet (s=1) and singlet states (s=0) have been obtained for different quantum S, P, D, F and G states.

In Table 2, the bottomonium mass spectra with different values of fractional parameters $\beta = 1, \alpha = 0.97$ and $\beta = 1, \alpha = 0.50$ have been obtained for the s-wave and $l > 0$ quantum states. Generally, the bottomonium masses increase with the

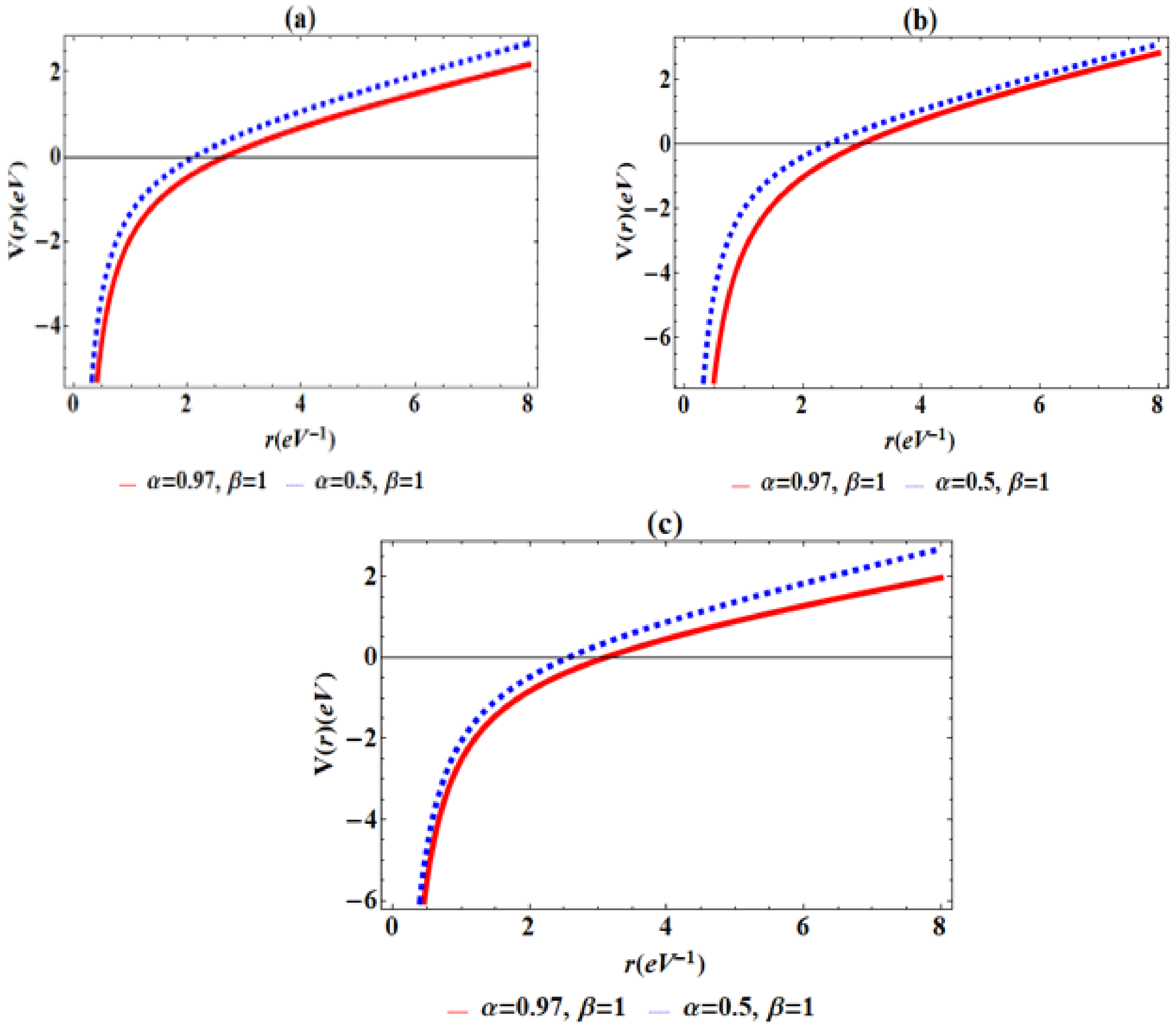

Figure 1: Potential curves for (a) bottomonium. (b) charmonium. (c) bottom-charm mesons.

increase in the quantum number from $n = 1-6$ for both the singlet ($s = 0$) and triplet ($s = 1$) quantum states. Also, a similar trend is observed for the $l > 0$ quantum states. The bottomonium masses are in good agreement with the masses obtained in existing literature [26, 27, 32–35] and available experimental data [52]. The percentage errors deviation from experimental data for $\beta = 1, \alpha = 0.97$ and $\beta = 1, \alpha = 0.50$ were found to be 0.67% and 0.49% respectively. The deviations give improvement over the previous results obtained in [51] where the authors solved the SE without considering the fractional parameters.

Also, the results for bottomonium mass spectra give improvement over the results reported in Ref. [34]. It is observed that the smaller the value of α in the fractional Schrodinger equation the better the results. This is evident when $\alpha = 0.50$ yielding a 0.49% error which is comparable to the percentage errors of 0.33% [33] and 0.40% [36] obtained in previous works.

Furthermore, Using potential parameters for charmonium meson given in Table 1, we obtained the mass spectra for different quantum states as tabulated in Table 3. The charmonium masses increase as the quantum number increases for spin-singlet ($s = 0$) and triplet (s=1) states. In comparison to experimental data [52], a percentage error of 1.97% and 1.62% were obtained for fractional values of $\beta = 1, \alpha = 0.97$ and $\beta = 1, \alpha = 0.50$ respectively.

For small fractional parameters, the results give improvements over previous studies [51] and literature data [35]. The results further indicated that the potential parameters coupled with the adjustment of the fractional parameters in the

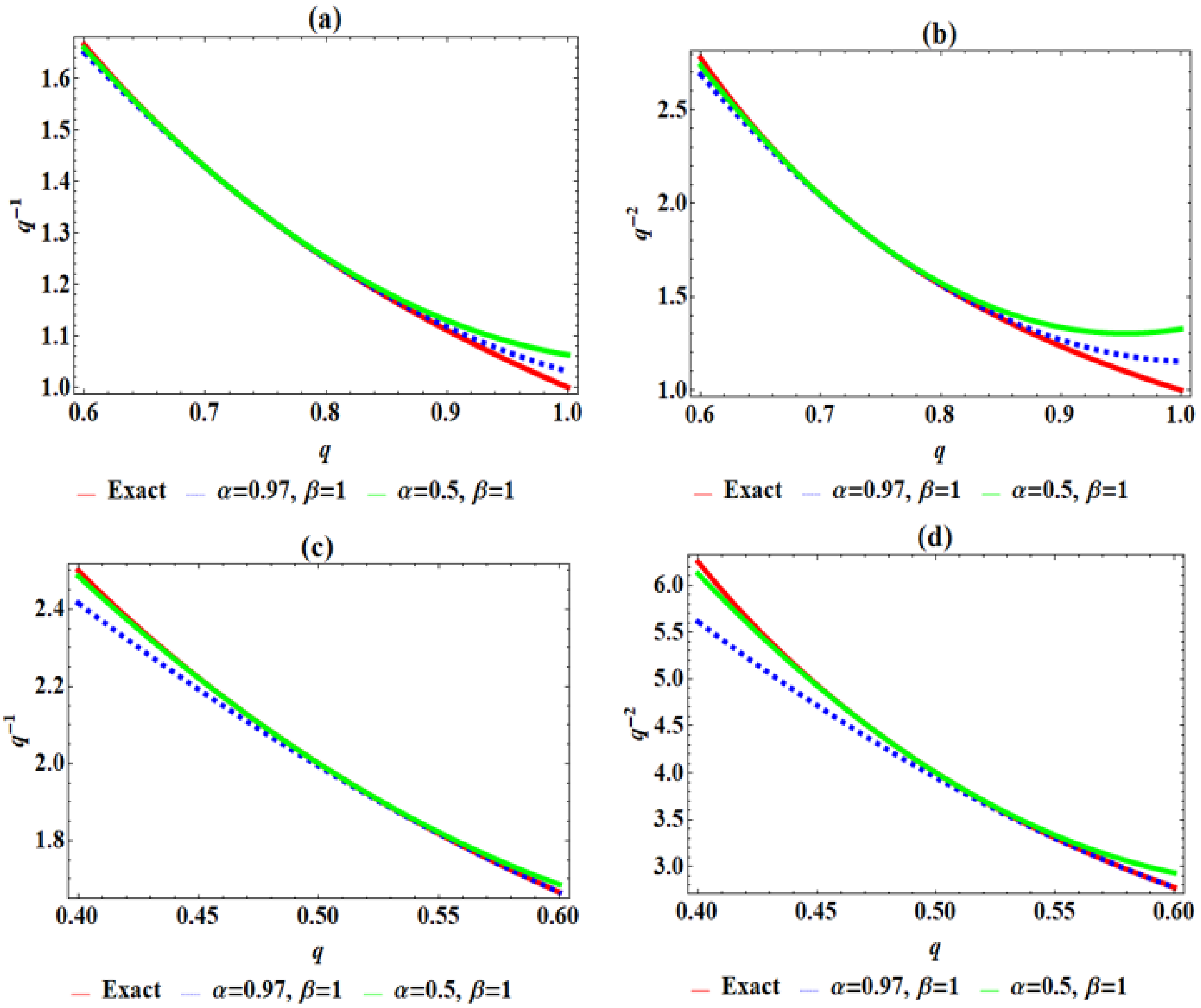

Figure 2: Pekeris approximation around $q \sim \delta$. $(a,\ b)$ Bottomonium. $(c,\ d)$ Charmonium.

Schrödinger equation strongly correlate with the accuracy of the heavy meson mass spectra.

In Table 4, we obtained the mass spectra of the bottom-charm meson for the s-wave and $l > 0$ states for the spin number $s = 0$ and $s = 1$. The masses also follow the trends of the bottomonium and charmonium and conform to the results reported in existing literature [26, 34, 35]. In Figure 1a-c, we plotted the potential against radial distance using the calculated parameters for the bottomonium and charmonium mesons in Table 1. It can be seen that the potential curves account for the attractive Coulomb and linear confinement which have been used to describe the non-relativistic quarkonium interaction within the quantum chromo-dynamic and phenomenology approach. In Figure 2a-d, we graphed the Pekeris-type approximation given in Eq. (20) and (21) in the region $q \sim \delta$ which gives a fair approximation compared to the exact curve.

5. Conclusions

The bound state solutions of the fractional SE under a spin-dependent Cornell potential has been investigated. By using the convectional Nikiforov-Uvarov method, we obtained the energy eigenvalues and associated wave functions in the fractional forms. At fractional parameters of α=β= 1, we obtained the results of Ref. [51] which represents a special case from the present results.

The energy spectra were applied to obtain the masses of the heavy mesons such as the bottomonium, charmonium and bottom charm. The obtained masses were found to be in good agreement with existing works and available experimental data. The results indicated that the fractional Schrodinger equation improves the mass spectra and the presence of the fractional parameters accounts for the short-range gluon exchange between the quark-antiquark interactions and the linear confinement phenomena which is associated with the quantum chromo-dynamic and phenomenological inspired potential models in particle and high-energy physics. We hope to extend this work to extreme conditions as in Refs. [55, 56].

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