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On sentinel method of one-phase Stefan problem

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Abstract

This paper is interested in studying the one-phase Stefan problem. For this purpose, we use the nonlinear sentinel method, which relies typically on the approximate controllability and the Fanchel-Rockafellar duality of the minimization problem, to prove the existence and uniqueness of a solution to this problem. In particular, our research focuses on the application of the nonlinear sentinel method to the single-phase Stefan problem. This approach aids in identifying an unspecified boundary section within the domain undergoing a liquid-solid phase transition. We track the evolution of the temperature profile in the liquid-solid material and the corresponding movement of its interface over time. Eventually, the local convergence used for the iterative numerical scheme is demonstrated.

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1. Introduction

Mathematical modeling is a method that represents and explains a lot of real phenomena using proper mathematical formulas and approaches [1–5]. It can be described by a set of ordinary, partial and fractional differential equations [6–10]. In 1795, Gauss and Legendre elaborated the method of least square, which is the most popular parameter identification technique in mathematical modelling. This technique relies on minimizing the distance between the observed values y_{obs} and the calculated values $y(O)$. After a period of time, particularly in 1992, Lions introduced another method called the sentinel method that can achieve the same objective [11, 12]. This method can provide a sufficient information about a certain parameter or an approximation of the latter (pollution term). To be able to obtain some information, it is necessary to observe "y" from the state on an observatory "O".

The liquid-solid models were studied by Joseph Stefan in his works done in 1889, that is why it is called Stefan's problem later. Stefan's problem has been applied in many typical problems such as sea ice melting and freezing [13], continuous casting of steel [14], crystal growth [15], thermal energy storage systems [16], lithium-ion batteries [17], model population dynamics of the tumor growth process [18], and information diffusion on social networks [19]. Physically, the domain can represent a specific body that could be divided into two parts, liquid and solid, by a position interface. The shape of the first

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side is known and fixed at a given temperature, but the other side must be estimated by measuring the distributed temperature in the middle of the body, where the dynamics of Stefan's problem are influenced by the heat flux. Mathematically, Stefan's problem can be described from two points of view. The first one is the diffusion partial differential equation (PDE) with an unknown boundary [12] in which this PDE represents the heat diffusion equation for the melting or freezing of material. While the second one is an ordinary differential equation (ODE) that can describe the movement of the interface part with the time-varying.

Bodart and Demeestere used the method of the sentinels for identifying a part of an unknown boundary of "Stefan problem" in 1997, and he proved the local convergence of the scheme for this problem in Ref. [20]. Due to the efficiency and accuracy of this method, it was included as an application when the groundwater pollution is addressed whereby several numerical tests were conducted on the Rheman aquifer in the South of Strasbourg. Herein, the classical situation could be explained via Meteorology, where the initial data are never completely known. On the other hand, José and Gianni studied the speed of propagation of the free boundary for the Digue problem [21]. They proved that the free boundary of the saturated part has a finite speed of propagation, which implies that the speed of propagation of the pressure is finite when the medium is not saturated. When the medium is saturated, the speed of propagation of the pressure would be intimate. In the same connection and as a final example of the artificial intelligence, the field of control theory can provide several useful concepts and tools for the machine learning. Within the scheme employed in control theory, there is a necessary need to express the relationship between the input and output of the controlled object, and so there is a very important need for an accurate mathematical model. In 1967, the term "intelligent control" was proposed as it can represent the interface between artificial intelligence and control. Autonomous machines are examples on such a kind of control. Learning control, fuzzy control and neural control are some other kinds of control methods. In 1999, El Badia and Moutazaim solved a one-phase inverse Stefan problem based on the method of the least-square approach in Ref. [22]. In 2017, Demarque and Fernández-Cara proved in Ref. [23] that there exist controls that can drive the state to zero at time $t = T$ (i.e., the null controllability for parabolic systems in non-cylindrical domains). In 2019, Koga *et al.* [24] proved the existence of a non-approach to globally stabilizing a class of nonlinear parabolic PDE via a nonlinear back stepping transformation in Ref. [24].

The remaining of this paper is arranged as follows: In section 2, we recall the nonlinear sentinel method. In section 3, we formulate the main problem, the one-phase Stefan problem, and define its deformation of its space. In section 4, we apply the sentinel method to prove the existence and uniqueness of Stefan problem. In section 5, we transform the problem into a fixed point problem, and then prove the local convergence. Finally, we state the some conclusions and future works in the last section.

2. The nonlinear sentinel Method

The sentinel is generally a function which represents a scalar product between the measure y_{obs} and a function u or a certain control. The goal of this method is to obtain some information about the needed parameters [25]. To get a full description about this method, we let y_{α_i} be given by

$$y_{\alpha_i} = \frac{\partial y(\alpha)}{\partial \alpha_i},$$

where $\alpha \in R^n$ such that it satisfies the following linear relation:

$$y(\alpha) = \sum_{i=1}^{n} \alpha_i y_{\alpha_i}.$$

In the sense of the sentinel method, the sentinel item S uses the parameter i_0 to estimate the system, i.e.

$$S(\alpha) = \sum_{i=1}^{n} \alpha_i (u, y_{\alpha_i}) = \alpha_{i_0},$$

where

$$\begin{cases} (u, y_{\alpha_i}) = 0, & \forall i \neq i_0 \\ (u, y_{\alpha_{i_0}}) = 1, & \text{Otherwise.} \end{cases}$$

Afterward, the sentinel item has to be insensitive to all the parameters but one (the i_0 parameter). It should be noticed that there is a further property of the sentinel method, that is the norm of u represents a unique minimum value. Therefore, if the values of all components of the vector α are needed, one function u for each component of α needs to be constructed. Thus, we suppose that u_j and S_j are certain functions, and the sentinel item is associated with the parameter α_j. From this point of view, we assume

$$(u_j, y_{\alpha_i}) = \delta_{ij}, \quad \forall i, j = \overline{1, n}.$$

The new sentinel item, which was defined by S in Ref. [20], contains a complete set of $(u_j(\alpha))_{j=1,2,\cdots,n}$. In other words, it can be defined by

$$S(\alpha) = (u_j, y(\alpha))_{j=1,2,\cdots,n}. \tag{1}$$

Now, with the use of (1), we can have

$$D_\alpha S = Id, \tag{2}$$

where $D_\alpha S$ is the differential of S with respect to the parameter α. As $S(\cdot)$ depends linearly on α via $y(\alpha)$, $D_\alpha S$ is easy to be computed.

Throughout the rest of the paper, we attempt to extend the notion of sentinels to the case of nonlinear identification of a parameter belonging to $l^2(\mathbb{R})$ instead of $\mathbb{R}^n$. It means that we will use the linearization of the state $y(\alpha)$ to compute the function $(u_j)_{j=1,2,\cdots,\infty}$, which will depend on the point where this linearization is made. Based on what has been discussed, we have

$$S(\tilde{\alpha}, \alpha) = (u_j(\tilde{\alpha}), y(\alpha))_{j=1,2,\cdots,\infty},$$

which represents the new sentinel $S(\alpha)$ containing the whole set of $(u_j(\alpha))_{j=1,2,\cdots,\infty}$.

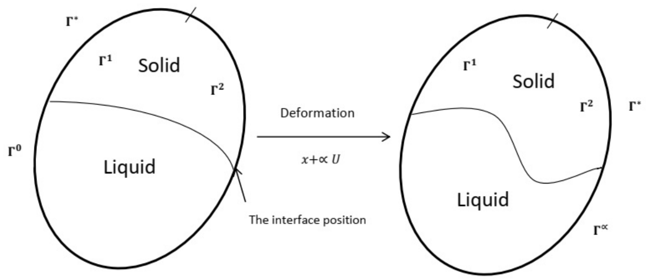

Figure 1. The deformation of the domain Ω_α by vector $x + \alpha(x)U(x)$.

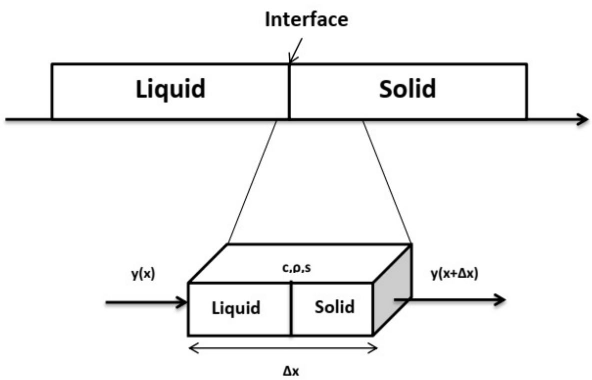

Figure 2. Schematic of one-phase Stefan problem.

3. Setting problem

In this part, we use the notion of deformation reported in Ref. [26]. In fact, the idea of deformation was created by Riemann, and it was applied to Lie algebras by Nijenhuis and Riehardson. In general, a body is said to be deformed, if there has been a variation in the shape and the dimensions of its body from its initial state to the final state. It is usually represented graphically by the change in the shape of a circle of the unit radius or a square of unit dimension, see Figure 1 and Figure 2.

To continue setting the problem, we assume that $\Omega_0 \subset \mathbb{R}^2$ is an open subset with the smooth boundary $\partial\Omega_0 = \Gamma^* \cup \Gamma_0$, and $\Gamma^* \cup \Gamma_0 = \emptyset$. We define the deformation Ω_α of Ω_0 like in Ref. [20] by

$$\Omega_\alpha = \{x + \alpha(x)U(x), x \in \Omega_0\},$$

where U is a known transverse vector field of the class $\mathbb{C}^\infty$, and $\alpha(x)$ is a $\mathbb{C}^2$-function such that Γ^* remains invariant by the deformation αU. The boundary of an open set Ω_α is $\partial\Omega_\alpha = \Gamma^* \cup \Gamma_\alpha$ such that $\Gamma^* \cap \Gamma_\alpha = \emptyset$. Hence, Γ_α is a deformation of Γ_0 such that

$$\Gamma_\alpha = \{x + \alpha(x)U(x),\ x \in \Gamma_0\},$$

where c is a heat coefficient, ρ is volume mass and λ is thermal conductivity. According to Fourier law, we have

$$dQ = dQ_{in} - dQ_{out}, \tag{3}$$

where

$$\frac{dQ_{in}}{dt} = -\lambda s \left.\frac{dy}{dx}\right|_x$$

and

$$\frac{dQ_{out}}{dt} = -\lambda s \left.\frac{dy}{dx}\right|_{x+\Delta x}.$$

By subtracting dQ_{in} and dQ_{out} in (3), we get

$$dQ = -\lambda s \left.\frac{dy}{dx}\right|_x - \left(-\lambda s \left.\frac{dy}{dx}\right|_{x+\Delta x}\right),$$

i.e.

$$dQ = \lambda s dt \left[\Delta x \left.\frac{d^2y}{dx^2}\right|_x\right].$$

This heat equation can be used to heat a small piece of mass dm, that is

$$dQ = c dy dm = c\rho s \Delta x dy.$$

Consequently, we have

$$dm = \rho s \Delta x,$$

$$c\rho s \Delta x dy = \lambda s dt \Delta x \frac{d^2y}{dx^2},$$

and

$$\frac{dy}{dt} = \frac{\lambda}{c\rho}\frac{d^2y}{dx^2}.$$

Let $C = \dfrac{\lambda}{c\rho}$, then we obtain

$$\frac{dy}{dt} = C\frac{d^2y}{dx^2}.$$

Also, let $y = y(x,t;\alpha)$ be the solution of the following problem:

$$\begin{aligned}
&\frac{\partial y}{\partial t} = C\frac{\partial^2 y}{\partial x^2} \ \text{ in } Q_\alpha = \Omega_\alpha \times]0,T[\,,\\
&-k\frac{\partial y}{\partial x}(0,T) = u(t) \quad \text{on } \Sigma_\alpha = \Gamma_\alpha \times]0,T[\,,\\
&y(s(t),t) = y_m \quad \text{on } \Sigma^* = \Gamma^* \times]0,T[\,,\\
&y(x,0) = y_0(x) \quad \text{in } \Omega_\alpha,
\end{aligned} \tag{4}$$

where $u(t) \in L^2\left(]0,T[\,, H^{\frac{3}{2}}(\Gamma)\right)$, k is the thermal conductivity, y_m is the limit of $y(x,t)$ when $t \to \infty$, y_0 is the initial condition, and $s(t)$ is the solution of the following nonlinear differential equation:

$$\dot{s}(t) = -\beta y_x(s(t),t),$$

where $\beta = \frac{k}{\rho\Delta H^*}$ such that ΔH^* is the latent heat of fusion. Hence, we have

$$y \in L^2\left(]0,T[\,, H^2(\Omega_\alpha)\right) \text{and } \left.\frac{\partial y}{\partial n}\right|_{\Gamma_\alpha} \in L^2\left(]0,T[\,, H^{\frac{1}{2}}(\Gamma)\right).$$

In the rest of this work, we consider $\omega \subset \Omega_\alpha$ an open subset for all α such that $O = \omega \times]0,T[$. We suppose that the solution y of the problem observed in O such that

$$y_{obs} = y(x,t;\tilde{\alpha}), \quad \forall (x,t) \in O,$$

where $y(x,t;\tilde{\alpha})$ is the solution of problem (4) in Ω_α.

4. Application of sentinel method

In this part, we intend to apply the sentinel method to prove the existence and uniqueness of the Stefan problem. For this purpose, we should note that the parametrization of Γ_α is given by

$$\Gamma_\alpha = \{x(s) + \alpha(s)U(s), s \in [0,1], x(s) \in \Gamma_0\}, \quad (5)$$

where α is $\mathbb{C}^2$-function over $[0,1]$, i.e. it belongs to $L^2(]0,1[)$. Next, with the decomposition of α over the basis of functions $(b_j)_{j=1,2,\cdots,\infty}$ in $\mathbb{C}^2(0,1)$ and $\alpha \in l^2(\mathbb{R})$, the parametrization (5) can be reexpressed as

$$\Gamma_\alpha = \left\{ x(s) + \sum_{j=1}^{\infty} \alpha_j(s) b_j(s) U(s) \right\}, \quad (6)$$

where $s \in [0,1]$ and $x(s) \in \Gamma_0$.

The following proposition is the main core to prove the main result of this work. In particular, in such a proposition, we aim to define the sentinel item, its existence, and its uniqueness as well.

Proposition 1. *[20] Let $S(\tilde{\alpha}, \alpha)$ be a sentinel item defined as follows:*

$$S : \left| \begin{array}{c} l^2(\mathbb{R}) \times l^2(\mathbb{R}) \longrightarrow l^2(\mathbb{R}) \\ (\tilde{\alpha}, \alpha) \longrightarrow \left(\int_O u_i(\tilde{\alpha}) y(\alpha) dxdt \right)_{i=1,2,\cdots,\infty} \end{array} \right. , \quad (7)$$

where $y(\alpha) = y(x,t;\alpha)$ is the solution of the first problem and $u_i(\tilde{\alpha})_{i=1,2,\cdots,\infty}$ are the functions that need to be found in such a way that

$$D_\alpha S(\hat{\alpha}, \hat{\alpha}) = Id + M, \quad \forall \hat{\alpha} \in l^2(\mathbb{R}), \quad (8)$$

$$u_i(\tilde{\alpha}) = \min \|\phi\|_{L^2(\Theta)}, \quad i = 1,2,\cdots,\infty, \quad (9)$$

where $M \in \pounds(l^2(\mathbb{R}))$ such that

$$\|(M_i)\|_{l^2(\mathbb{R})} = \frac{\epsilon}{i}, \quad \text{for } i = 1,2,\cdots,\infty, \quad (10)$$

in which M_i is the i^{th}-line of M and $D_\alpha S(\hat{\alpha}, \hat{\alpha})$ is the differential of S with respect to its second parameter at the point $(\hat{\alpha}, \hat{\alpha})$. Then, $S(\tilde{\alpha}, \alpha)$, which is defined by (7)-(9), exists and unique.

Proof. The proof of this proposition takes three steps, which are listed below for clarification.
Step 1: The solution $y(x,t;\tilde{\alpha})$ of the problem is differentiable with respect to α, i.e.,

$$y_{\alpha_j} = \frac{\partial y(\tilde{\alpha})}{\partial \alpha_j}, \quad j = 1,2,\cdots,\infty,$$

where y_{α_j} is the solution of

$$\begin{aligned} &\frac{\partial y_{\alpha_j}}{\partial t} = \Delta y_{\alpha_j} \quad \text{in } Q_\alpha = \Omega_\alpha \times]0,T[\\ &y_{\alpha_j}(s(t),t) = 0 \quad \text{on } \Sigma^* = \Gamma^* \times]0,T[, \\ &-k\frac{\partial y_{\alpha_j}}{\partial x} = -b_j(\nabla y(\tilde{\alpha}).U) \quad \text{on } \Sigma_\alpha = \Gamma_\alpha \times]0,T[, \\ &y_{\alpha_j}(x,0) = 0 \quad \text{in } \Omega_\alpha, \end{aligned} \quad (11)$$

where $y(\tilde{\alpha}) = y(x,t;\tilde{\alpha})$ solves (4) with data $\tilde{\alpha}$ and y_m that is dependent of α_j. Thus the general element of the infinite matrix $D_\alpha S(\tilde{\alpha}, \tilde{\alpha})$ is

$$\begin{aligned} (D_\alpha S(\tilde{\alpha}, \tilde{\alpha}))_{ij} &= \left(\int_O u_i(\tilde{\alpha}) y_{\alpha_j} dxdt \right)_{i=1,2,\cdots,\infty} && (12) \\ &= \delta_{ij} + (M)_{ij}, \ j = 1,2,\cdots,\infty, && (13) \end{aligned}$$

where $\|M\| = \frac{\epsilon}{i}$. Now, (8) reads

$$\int_O u_i(\tilde{\alpha}) y_{\alpha_j} dxdt = \delta_{ij} + (M)_{ij}, \quad j = 1,2,\cdots,\infty, \quad (14)$$

where i is fixed and the matrix M is defined as in this proposition. Based on the previous discussion, the adjoint problem is given by

$$\begin{aligned} &-\frac{\partial q_i}{\partial t} - \Delta q_i = u_i(\tilde{\alpha})|_O \quad \text{in } Q_{\tilde{\alpha}} = \Omega_{\tilde{\alpha}} \times]0,T[\\ &q_i = 0 \quad \text{on } \Sigma_{\tilde{\alpha}} = \Gamma_{\tilde{\alpha}} \times]0,T[, \\ &q_i(\cdot,T) = 0 \quad \text{in } \Omega_{\tilde{\alpha}}, \end{aligned} \quad (15)$$

where $q_i \in L^2\left(]0,T[, H_0^1(\Omega_{\tilde{\alpha}}) \cap H^2(\Omega_{\tilde{\alpha}})\right)$ is the solution of the adjoint problem. Multiplying the first equation of (15) by y_{α_j}, and then applying Green's formula yield

$$\int_O u_i(\tilde{\alpha}) \frac{\partial y_{\alpha_j}}{\partial x} dxdt = \int_O \frac{b_j}{k} (\nabla y(\tilde{\alpha}).U) \frac{\partial q_i}{\partial n} d\Sigma. \quad (16)$$

Consequently, we define a linear continuous operator $B \in \pounds(L^2(O); l^2(\mathbb{R}))$ by

$$B : \left| \begin{array}{c} L^2(O) \longrightarrow l^2(\mathbb{R}) \\ u_i(\tilde{\alpha}) \longrightarrow \left(\int_{\Sigma_{\tilde{\alpha}}} \frac{b_j}{k} (\nabla y(\tilde{\alpha}).U) \frac{\partial q_i}{\partial n} d\Sigma \right)_{j=1,2,\cdots,\infty} \end{array} \right. . \quad (17)$$

Equation (16) allows rewriting (12) as

$$(D_\alpha S(\tilde{\alpha}, \tilde{\alpha}))_{ij} = (Bu_i(\tilde{\alpha}))_j. \quad (18)$$

As a result, we have an exact controllability to the considered problem. Thus, the main goal of this part is to find $u_i(\tilde{\alpha}) \in L^2(O)$ of the minimal norm such that $Bu_i(\tilde{\alpha}) = z$ with $z \in l^2(\mathbb{R})$. For the numerical applications, we need to show that there is a proper approximate controllability of the problem [27].
Step 2: In this step, we aim to prove that $\overline{Im(B)} = l^2(\mathbb{R})$ (i.e. $\ker(B^*) = \{0\}$). To do so, we should observe that $B^* \in \pounds(l^2(\mathbb{R}); L^2(O)$ can be given by

$$B^* : \left| \begin{array}{c} l^2(\mathbb{R}) \longrightarrow L^2(O) \\ (\sigma_j)_{j=1,2,\cdots,\infty} \longrightarrow \phi|_\Theta \end{array} \right. , \quad (19)$$

where ϕ solves the following problem:

$$\left\{ \begin{array}{c} \frac{\partial \phi}{\partial t} = \Delta \phi \\ \phi(s(t),t) = 0 \\ -k\frac{\partial \phi}{\partial x} = -(\nabla y(\tilde{\alpha}).U) \sum_{j=1}^{\infty} \sigma_j b_j \\ \phi(x,0) = 0 \end{array} \right. . \quad (20)$$

Now, in view of (20), we can have

$$(u,\phi)_{L^2(\Theta)} = \sum_{j=1}^{\infty}\sigma_j \int_{\Sigma_{\tilde{\alpha}}} b_j(\nabla y(\tilde{\alpha}).U)\frac{\partial q_i}{\partial n}d\Sigma \quad (21)$$
$$= (\sigma, Bu)_{l^2(\mathbb{R})}, \quad (22)$$

i.e. $\phi|_\Theta = B^*\sigma$. Suppose that $B^*\sigma = \phi|_O = 0$, i.e. $\phi = 0$ in O. By the unique continuation theorem, we have

$$-(\nabla y(\tilde{\alpha}).U)\sum_{j=1}^{\infty}\sigma_j b_j = 0. \quad (23)$$

Note that $(b_j)_{j=1,2,\cdots,\infty}$ is a basis of $l^2(\mathbb{R})$ whenever either $\{(\nabla y(\tilde{\alpha}).U) = 0\}$ or $\left\{\sigma_j = 0, j = 1,2,\cdots,\infty\right\}$. Now, we decompose the field U on the $\nu_{\tilde{\alpha}}$ and the tangent vectors $\tau_{\tilde{\alpha}}$ on $\Gamma_{\tilde{\alpha}}$ to obtain

$$\nabla y(\tilde{\alpha}).U(x) = \nabla y(\tilde{\alpha}).(a\nu_{\tilde{\alpha}}(x)) + \nabla y(\tilde{\alpha}).(b\tau_{\tilde{\alpha}}(x)),\ \forall x \in \Gamma_{\tilde{\alpha}}.$$

In other words, since $y(\tilde{\alpha}) = 0$ on $\Gamma_{\tilde{\alpha}}$, we have

$$\nabla y(\tilde{\alpha}).U(x) = a\frac{\partial y(\tilde{\alpha})}{\partial \nu_{\tilde{\alpha}}}(x), \forall x \in \Gamma_{\tilde{\alpha}}. \quad (24)$$

From the Cauchy uniqueness, we can have

$$\left|\frac{\partial y(\tilde{\alpha})}{\partial \nu_{\tilde{\alpha}}}\right._{\Gamma_{\tilde{\alpha}}} \neq 0,$$

otherwise, we have

$$y(\tilde{\alpha}) = 0 \text{ in } Q_{\tilde{\alpha}}.$$

Thus $(\nabla y(\tilde{\alpha}).U) \neq 0$ and B^* is injective, which proves that $\overline{Im(B)} = l^2(\mathbb{R})$, i.e. $\forall\rho > 0$, $\forall z \in l^2(\mathbb{R})$, $\exists u_i(\tilde{\alpha}) \in L^2(O)$ such that

$$\|Bu_i(\tilde{\alpha}) - z\|_{l^2(\mathbb{R})} \leq \rho. \quad (25)$$

Step 3: In this step, we use the Fenchel-Rockafellar duality method to prove the second condition of the proposition at hand. For this purpose, we let

$$U_{ad} = \left\{u \in L^2(O) : \|Bu - z\|_{l^2(\mathbb{R})} \leq \rho, z \in l^2(\mathbb{R})\right\},$$

which represents, from (25), a nonempty convex and closed set in $L^2(O)$. Thus, there exists a unique $u_i(\tilde{\alpha})$ satisfying (9) and the solutions of the following minimization problem:

$$\min_{\omega\in U_{ad}}\frac{1}{2}\|u\|^2_{L^2(O)}. \quad (26)$$

Now, let F and G be two functions defined as

$$F(u) = \frac{1}{2}\|u\|^2_{L^2(O)} \quad (27)$$

and

$$G(w) = \left\{\begin{array}{ll} 0 & \text{if } \|w - z\|_{l^2(\mathbb{R})} \leq \rho, \\ +\infty & \text{otherwise} \end{array}\right. . \quad (28)$$

So, problem (26) can be now rewritten as

$$\min_{\omega\in L^2(O)} F(u) + G(w). \quad (29)$$

By the duality theorem of Fenchel-Rockafellar, we get

$$u_i(\tilde{\alpha}) = B^*\sigma^*, \quad (30)$$

where σ^* is the solution of the dual of (26), i.e.

$$\min_{\sigma\in l^2(\mathbb{R})} F^*(B^*\sigma) + G^*(-\sigma), \quad (31)$$

where F^* and G^* are the Fenchel-Rockafellar conjugates of F and G respectively such that $F^* = F$ and G^* is given by

$$\begin{aligned} G^*(\sigma) &= \sup_{u\in l^2(\mathbb{R})}(u,\sigma)_{l^2(\mathbb{R})} - G(w) \\ &= (z,\sigma)_{l^2(\mathbb{R})} + \rho\|\sigma\|_{l^2(\mathbb{R})}, \end{aligned} \quad (32)$$

where $\overline{B(0,\rho)}$ is $l^2(\mathbb{R})$, which is a closed ball with center 0 and radius ρ. This would immediately turn (20) to be as

$$\min_{\sigma\in l^2(\mathbb{R})} J(\sigma) = F(\phi) + \rho\|\sigma\|_{l^2(\mathbb{R})} - (z,\sigma)_{l^2(\mathbb{R})}, \quad (33)$$

where ϕ is the solution of (20).

Lemma 1. *$\sigma^* = 0$ is the solution of (33) if and only if $\|z\|_{l^2(\mathbb{R})} \leq \rho$.*

Proof.
$\Rightarrow$) Suppose that $\sigma^* = 0$. Then, with the use of (30), we can have $u_i(\tilde{\alpha}) = 0$, and hence we obtain

$$Bu_i(\tilde{\alpha}) - z = -z.$$

This means

$$\|Bu_i(\tilde{\alpha}) - z\|_{l^2(\mathbb{R})} \leq \rho,$$

i.e.

$$\|z\|_{l^2(\mathbb{R})} \leq \rho.$$

$\Leftarrow$) Suppose that $\|z\|_{l^2(\mathbb{R})} \leq \rho$. Then, we have

$$u_i(\tilde{\alpha}) = 0,$$

which represents the solution of (26). Consequently, due to B^* is injective, then (30) yields that $\sigma^* = 0$.

In light of the previous discussion and to show the existence and uniqueness of (8) and (9), we can have

$$\begin{aligned} &\left(\frac{\partial J}{\partial \sigma}, \delta\sigma\right)_{l^2(\mathbb{R})} \\ &= (B^*\sigma, B^*\delta\sigma)_{l^2(\mathbb{R})} + \rho\left(\frac{\sigma}{\|\sigma\|_{l^2(\mathbb{R})}}, \delta\sigma\right)_{l^2(\mathbb{R})} \\ &\qquad - (z,\delta\sigma)_{l^2(\mathbb{R})} \\ &= \left(BB^*\sigma + \rho\frac{\sigma}{\|\sigma\|_{l^2(\mathbb{R})}} - z, \delta\sigma\right)_{l^2(\mathbb{R})}, \end{aligned} \quad (34)$$

for any $\delta\sigma \in l^2(\mathbb{R})$ and $\sigma \neq 0$, where σ^* is such that

$$BB^*\sigma - z = -\rho\frac{\sigma}{\|\sigma\|_{l^2(\mathbb{R})}}. \quad (35)$$

Since $u_i(\tilde{\alpha}) = B^*\sigma^*$, we have

$$\|Bu_i(\tilde{\alpha}) - z\|_{l^2(\mathbb{R})} = \rho. \quad (36)$$

Now, we choose

$$(z)_j = \delta_{ij}, \quad j = 1, 2, \cdots, \infty, \tag{37}$$

where $(z)_j$ is the generic coordinate of z on the canonical basis of $l^2(\mathbb{R})$ and

$$\rho = \frac{\epsilon}{i}, \quad \text{with } \epsilon > 0 \text{ sufficiently small.} \tag{38}$$

Consequently, we can obtain $\|z\|_{l^2(\mathbb{R})} > \rho$. Eventually, (35) gives

$$Bu_i(\tilde{\alpha}) = z - \rho \frac{(\sigma^*)}{\|\sigma^*\|_{l^2(\mathbb{R})}}, \tag{39}$$

which means

$$\begin{aligned} (Bu_i(\tilde{\alpha}))_j &= (z)_j - \rho \frac{(\sigma^*)_j}{\|\sigma^*\|_{l^2(\mathbb{R})}} \\ &= \delta_{ij} - \frac{\epsilon}{i} \frac{(\sigma^*)_j}{\|\sigma^*\|_{l^2(\mathbb{R})}}. \end{aligned} \tag{40}$$

So, we have

$$D_\alpha(S(\tilde{\alpha}, \tilde{\alpha})) = Bu_i(\tilde{\alpha}), \tag{41}$$

which gives

$$D_\alpha(S(\tilde{\alpha}, \tilde{\alpha})) = Id + M, \quad \forall \tilde{\alpha} \in l^2(\mathbb{R}). \tag{42}$$

By combining the above assertion with (18), we obtain (8). Thus, the existence and uniqueness of a family of functions $u_i(\tilde{\alpha})$ are hold for (8) and (9), for $i = 1, 2, \cdots, \infty$.

5. The numerical scheme

In this section, we transform the considered problem into a fixed point problem, and then we prove the local convergence. However, to get an overview about the fixed point problem, the reader may refer to the references [28–30]. Now, in order to carry such a transformation, we differentiae $S(\alpha) = S(\tilde{\alpha}, \alpha)$ to obtain

$$S(\tilde{\alpha}, \alpha) = S(\tilde{\alpha}, \tilde{\alpha}) + D_\alpha S(\tilde{\alpha}, \tilde{\alpha}).(\alpha - \tilde{\alpha}) + o(|\alpha - \tilde{\alpha}|).$$

Accordingly, for $\bar{\alpha} = \alpha$, we have

$$S(\tilde{\alpha}, \bar{\alpha}) = S(\tilde{\alpha}, \tilde{\alpha}) + \bar{\alpha} - \tilde{\alpha} + M(\alpha - \tilde{\alpha}) + o(|\bar{\alpha} - \tilde{\alpha}|).$$

By neglecting the last two terms of the above equality with taking $\tilde{\alpha} = \alpha^k$ and $\bar{\alpha} = \alpha^{k+1}$, we obtain

$$\alpha^{k+1} = \alpha^k + S(\alpha^k, \bar{\alpha}) - S(\alpha^k, \alpha^k),$$

where

$$S(\alpha^k, \bar{\alpha}) = \left(\int_O w_i(\alpha^k) y_{obs} dx dt \right)_{i=1,2,\cdots,\infty},$$

and

$$S(\alpha^k, \alpha^k) = \left(\int_O w_i(\alpha^k) y(x, t, \alpha^k) dx dt \right)_{i=1,2,\cdots,\infty}.$$

Theorem 1. *The sequence* $\left(\alpha^k\right)_{k=0,1,\cdots,\infty}$ *in which*

$$\begin{cases} \alpha^0 \in l^2(\mathbb{R}) \\ \alpha^{k+1} = \alpha^k + S(\alpha^k, \bar{\alpha}) - S(\alpha^k, \alpha^k) \end{cases}$$

locally converges in $l^2(\mathbb{R})$*.*

Proof. The numerical scheme here is a method to solve the fixed point problem

$$\alpha^{k+1} = g(\alpha^k),$$

where g is an operator defined from $l^2(\mathbb{R})$ to itself. For $\mu \in l^2(\mathbb{R})$, we obtain

$$g'(\mu) = Id + D_{\tilde{\alpha}} S(\mu, \tilde{\alpha}) - D_{\tilde{\alpha}} S(\mu, \mu) - D_\alpha S(\mu, \mu),$$

which implies $\mu = \bar{\alpha}$ and so we have

$$g'(\bar{\alpha}) = Id - D_\alpha S(\bar{\alpha}, \bar{\alpha}).$$

Now, in light of Proposition 1, we can get

$$g'(\bar{\alpha}) = -M, \; M \in \mathscr{L}(l^2(\mathbb{R})).$$

This consequently yields

$$\begin{aligned} \left\| g'(\bar{\alpha}) \right\|_{HS}^2 &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} (g'(\bar{\alpha}))_{ij}^2 \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (g'(\bar{\alpha}))_{ij}^2 \\ &= \sum_{i=1}^{\infty} \|(M_i)\|_{l^2(\mathbb{R})}^2 \\ &= \epsilon^2 \sum_{i=1}^{\infty} \frac{1}{i^2}. \end{aligned}$$

This means

$$\left\| g'(\bar{\alpha}) \right\|_{HS} \leq 1.$$

Hence, the local convergence of the sequence is satisfied.

In the following content, we provide an illustrative example with the aim of explaining our results. In particular, the interest of this example is to show the deformation of the square $]0, 1[\times]0, 1[$ numerically. The reader may refer to the article of Bodart and Demeestere that can, with the use of such a deformation, show the error between the exact and calculated boundary [20].

Example 1. *Bodart and Demeestere discretized* 100 *elements, and the observatory was composed of* 36 *mesh center elements. They supposed the initial state and the boundary of the square as* $]0, 1[\times]0, 1[$*. Herein, after eight iterations, we observe that the boundary* Γ_α *deforms on one side and the other three sides are* Γ^**. For more illustration, Figure 3 shows more explanation about the meant deformation, i.e. it can show the clearly deformation of the boundary* Γ_α *after some iterations.*

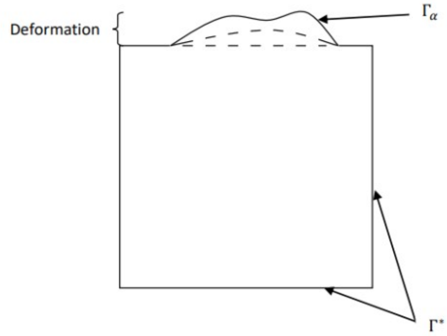

Figure 3. Deformation of the boundary after certain steps.

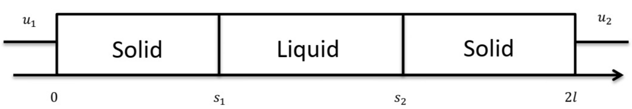

Figure 4. Schematic of the two-sided Stefan problem.

6. Conclusion and future work

In this paper, the one-phase Stefan problem has been studied with the use of the nonlinear sentinel method. Accordingly, the existence and uniqueness of a solution of this problem have been theoretically shown, and the local convergence used for the iterative numerical scheme has been also demonstrated. In this connection, the material related to the two-sided Stefan problem contains two moving solid boundaries with liquid in the middle (between s_1 and s_2), see Figure 4.

The control law can provide the convergence rate in the two-sided case, prove that the observed problem converges asymptotically to the first problem, and also present the results by simulation of exponential convergence. Our future work is to create a problem by changing the place of the liquid and the solid materials and controlling the solid phase by the notion of spray control (spreadability).

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