Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 6 (2024) 1821

Journal of the Nigerian Society of Physical Sciences

Approximate analytical solution of fractional-order generalized integro-differential equations via fractional derivative of shifted Vieta-Lucas polynomial

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Abstract

In this paper, we extend fractional-order derivative for the shifted Vieta-Lucas polynomial to generalized-fractional integro-differential equations involving non-local boundary conditions using Galerkin method as transformation technique and obtained $N - \lceil \delta \rceil + 1$ system of linear algebraic equations with λ_i, $i = 0, \ldots, N$ unknowns, together with $\lceil \delta \rceil$ non-local boundary conditions, we obtained $(N + 1)-$ linear equations. The accuracy and effectiveness of the scheme was tested on some selected problems from the literature. Judging from the table of results and figures, we observed that the approximate solution corresponding to the problem that has exact solution in polynomial form gives a closed form solution while problem with non-polynomial exact solution gives better accuracy compared to the existing results.

DOI:10.46481/jnsps.2024.1821

Keywords: Vieta-Lucas polynomial, Caputo fractional derivative, Generalized-fractional integro-differential equation, Galerkin method.

Article History :
Received: 01 October 2023
Received in revised form: 24 November 2023
Accepted for publication: 09 December 2023
Published: 21 December 2023

Communicated by: J. Ndam

1. Introduction

Integro-differential equations (IDEs) surface in Mathematical sciences such as modeling some phenomena (science and engineering), financial Mathematics, control theory and many more. In recent decades, various numerical approaches have been developed by various researchers for solving IDEs problems, some of these researchers are Dzhumabaev [1] who developed a new general solutions for solving linear Fredholm integro-differential equations, Issa *et al.* [2–4] employed shifted Chebyshev polynomial of first and fourth kinds to solve IDEs via perturbed Galerkin method, accuracy of Chebyshev Galerkin method was investigated by Biazar & Salehi [5] in the solution of IDEs.

Fractional-order calculus is an important tools in area of applied and computational Mathematics to investigate the myriad of problems that emanate from various aspect of studies, such as physics, mathematical modeling, engineering [6], statistical mechanics, finance, biophysics, hydrology, bio-engineering, control theory, and cosmology. Ionescu & Kelly [7] presented synopsis of fractional calculus tools for characterising respiratory mechanics.

Recently, more attentions are given to the studies of

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fractional-order differential equations (FDE) and fractional-order integro-differential equations (FIE). Different approaches have been studied to solve FIE. Ghosh [8] employed Katugampola fractional operator to investigate analytical approach for the fractional-order Hepatitis B model, Ghosh & Kumar [9] investigated the accuracy of fractional Covid-19 model via spectral collocation method, Jani *et al.* [10] investigated the accuracy of numerical solution of FIE with non-local conditions using Bernstein polynomials, Zaky [11] proposed improved tau method to investigate the accuracy of multi-dimensional fractional Rayleigh-Stokes problem, Wang & Zhu [12] proposed Euler wavelet operational matrix method for solving non-linear Volterra integro-differential equations, Huang *et al.* [13] tested the effectiveness of Taylor expansion method on the solution of FIE, Bayram & Daşcioğlu [14, 15] investigated the accuracy of fractional linear Volterra-Fredholm IDEs via Laguerre polynomials as an approximation, Mittal & Nigam [16] investigated the accuracy of Adomian decomposition method (ADM) in the solution of FIE.

Convergence of Jacobi spectral collocation method was investigated in the solution of FIE by Huang *et al.* [17, 18], FIE with weakly singular kernel was studied using legendre wavelets method by Yi *et al.* [19], spline collocation method was used to solve fractional weakly singular IDEs by Pedas *et al.* [20]. Two-dimensional non-linear Volterra-Fredholm IDEs was investigated using variational ADM by Hendi & Al-Qarni [21], He *et al.* [22, 23] studied a system of linear Fredholm integral equations using Bernstein and improved Block-Pulse functions, collocation method couple with convergence was employed to solve generalized FIE by Sharma *et al.* [24], Turut [25] adopted pade approximation technique to solve FIE with non-local boundary conditions, modified Laplace decomposition method was developed to solve fractional Volterra-Fredholm IDEs by Hamoud & Ghadle [26], Gupta & Pandey [27] proposed adaptive huber method for weakly singular fractional integro-differential equations, analysis of the error involved in 1D Fredholm integro-differential equations was studied by Fairbairn & Kelmanson [28] using Volterra-transformation method, Wang *et al.* [29] proposed a method based on Laplace transform for finding an approximate solution to Fredholm-type integro-differential equation with Atangana-Baleanu fractional derivative in Caputo sense, Toma & Postavaru [30] transformed IDEs to algebraic form using Newton's iterative method then investigate the accuracy of the proposed method.

The focus of this paper is finding approximate solution to the generalized fractional-order integro-differential equations (GFIE) of the form:

$$\begin{aligned} &D^{\delta}y(x) + \Psi_1(x)y(x) + \varsigma_1 \int_0^x S_1(x,t)y(t)dt \\ &+ \varsigma_2 \int_0^1 S_2(x,t)y(t)dt = \Psi_2(x), \\ &r-1 < \delta \le r, \quad a \le x \le b, r \in \mathbb{N} \end{aligned} \tag{1}$$

with non-local boundary conditions given by:

$$\begin{aligned} &\sum_{k=1}^{r} \left(\beta_{nk} y^{(k-1)}(a) + \alpha_{nk} y^{(k-1)}(b) \right) \\ &+ \gamma_n \int_a^b K_n(x) y(x) dx = \rho_n, \ n = 1, 2, \dots, r, \end{aligned} \tag{2}$$

where $K_n(x)$ is a continuous function, $\Psi_m(x), S_m(x), m = 1, 2$ are holomorphic functions, β_{nk}, α_{nk} , γ_n and ρ_n are constants, D^{δ} is the fractional differential operator of order δ and $y(x)$ is the unknown function. In Eq. (1), If $\varsigma_1 = 0$ or $\varsigma_2 = 0$, then the equation becomes fractional Fredholm or Volterra integro-differential equation respectively, via shifted Vieta-Lucas polynomials by employing fractional-order derivative for the shifted Vieta-Lucas polynomial to remove the fractional-order and Galerkin method in transforming the integro-differential equations to algebraic linear equations.

This paper is organized as follows. In section 1 we introduced FIE, in section 2, we discussed the preliminaries of the scheme, which includes review of some frequently used orthogonal polynomials and Caputo fractional differentiation operator. Formulation of scheme is presented in section 3, in section 4 application of the scheme is presented with table of results and figures to show the effectiveness and the accuracy of the scheme, discussion of results and concluding remarks are given in section 5.

2. Preliminaries

2.1. Some frequently used orthogonal polynomials

Some of frequently used orthogonal polynomials $\Phi_i(x)$ are reviewed here: $\Phi_i(x)$ is an orthogonal polynomial with reference to the weight function $\omega(x)$, $x \in [a, b]$, if the inner product of $\Phi_i(x)$ satisfies the following:

$$\langle \Phi_i(x), \Phi_j(x) \rangle = \int_a^b \omega(x)\Phi_i(x)\Phi_j(x)dx = \begin{cases} 0, & i \neq j \\ \lambda_j, & i = j. \end{cases} \tag{3}$$

Some of the prominent orthogonal polynomials $\Phi_i(x)$ are described below.

2.1.1. Shifted Legendre polynomials

Polynomial $P_i(p)$, is an orthogonal with respect to $\omega(p) = 1$ satisfying the recurrence formula:

$$P_{i+1}(p) = \tfrac{1}{i+1} \left[(2i+1)P_1(p)P_i(p) - iP_{i-1}(p) \right], i \ge 1, \tag{4}$$

$$\text{with } P_0(p) = 1, P_1(x) = \frac{2p - (a+b)}{b-a}, \ p \in [a, b] \ [31].$$

2.1.2. Laguerre polynomials

The generalized Laguerre polynomial $L_i^{(\gamma)}(p)$ of degree i in γ with $\omega(p) = p^{\gamma}e^{-p}, \gamma > -1, \ p \in [0, \infty)$ is defined as:

$$L_i^{(\gamma)}(p) = \sum_{j=0}^{i} (-1)^j \binom{i+\gamma}{i-j} \frac{p^j}{j!}. \tag{5}$$

The first two of the polynomials are $L_0^{(\gamma)}(p) = 1$, $L_1^{(\gamma)}(p) = \gamma + 1 - p$ with recurrence relation

$$L_{i+1}^{(\gamma)}(p) = \frac{1}{i+1}\left[(\gamma + 2i + 1 - p)L_i^{(\gamma)}(p) - (\gamma + i)L_{p-1}^{(\gamma)}(p)\right]. \quad (6)$$

The most frequently used Laguerre polynomial is under the condition $\gamma = 0$ defined as:

$$L_i(p) = \sum_{j=0}^{i}(-1)^j \binom{i}{j}\frac{p^j}{j!}, \quad (7)$$

with recurrence relation:

$$L_{i+1}(p) = \frac{1}{i+1}\left[(2i + 1 - p)L_i(p) - iL_{i-1}(p)\right], \quad (8)$$

and the inner product is given as [15, 32]:

$$\langle L_i^{(\gamma)}(p), L_j^{(\gamma)}(p)\rangle = \int_0^{\infty} \omega(p)p^{\gamma}e^{-p}L_i^{(\gamma)}(p)L_j^{(\gamma)}(p)dp = \begin{cases} 0, i \neq j \\ \frac{i!\Gamma(j+\gamma+1)}{\Gamma(\gamma+1)}, \ i = j. \end{cases} \quad (9)$$

2.1.3. χ-th shifted Chebyshev polynomials

Given $p = \left(\frac{2u - (a+b)}{b-a}\right)$, then the χ-th shifted Chebyshev polynomials are orthogonal polynomials reference to their respective weight function $\omega(p)$, $p \in [a, b]$, the prominent ones are given here:

$$\Phi_i(p) = \begin{cases} T_i(p) = \cos(ip), & \omega(p) = \frac{1}{\sqrt{1-p^2}}, \chi = 1 \\ U_i(p) = \frac{\sin(i+1)p}{\sin(p)}, & \omega(p) = \sqrt{1-p^2}, \chi = 2 \\ V_i(u) = \frac{\cos\left(i+\frac{1}{2}\right)p}{\cos\left(\frac{p}{2}\right)}, & \omega(p) = \sqrt{\frac{1+p}{1-p}}, \chi = 3 \\ W_i(p) = \frac{\sin\left(p+\frac{1}{2}\right)p}{\sin\left(\frac{p}{2}\right)}, & \omega(p) = \sqrt{\frac{1-p}{1+p}}, \chi = 4. \end{cases} \quad (10)$$

The inner product for the χ−th kinds Chebyshev polynomials $\Phi_i(p), p \in [-1, 1]$ are given as [33–35]:

$$\langle \Phi_i(p), \Phi_j(p)\rangle = \begin{cases} \langle T_i(p), T_j(p)\rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-p^2}}T_i(p)T_j(p)dp = \begin{cases} 0, i \neq j \\ \frac{\pi}{2}, \ i = j, \end{cases} \\ \langle U_i(p), U_j(p)\rangle = \int_{-1}^{1} \sqrt{1-p^2}U_i(p)U_j(p)dp = \begin{cases} 0, \ i \neq j \\ \frac{\pi}{2}, \ i = j, \end{cases} \\ \langle V_i(p), V_j(p)\rangle = \int_{-1}^{1} \sqrt{\frac{1+p}{1-p}}V_i(p)V_j(p)dp = \begin{cases} 0, \ i \neq j \\ \pi, \ i = j, \end{cases} \\ \langle W_i(p), W_j(p)\rangle = \int_{-1}^{1} \sqrt{\frac{1-p}{1+p}}W_i(p)W_j(p)dp = \begin{cases} 0, \ i \neq j \\ \pi, \ i = j. \end{cases} \end{cases} \quad (11)$$

2.1.4. Shifted Gegenbauer polynomials

The shifted Gegenbauer polynomial $C_m^{(\alpha)*}(p), p \in [a, b]$ is given as:

$$C_{i+1}^{(\alpha)*}(p) = \frac{1}{i+1}\left[2(i+\alpha)\left(\frac{2p-(a+b)}{b-a}\right)C_i^{(\alpha)*}(p) - (i + 2\alpha - 1)C_{i-1}^{(\alpha)*}(p)\right], i \geq 1, \quad (12)$$

where $C_0^{(\alpha)*}(p) = 1$, $C_1^{(\alpha)*}(p) = 2\alpha\left(\frac{2p-(a+b)}{b-a}\right)$.

The corresponding analytical form in the interval $p \in [0, 1]$ is given as:

$$C_i^{(\alpha)*}(p) = \sum_{j=0}^{i} \frac{(-1)^j\Gamma(2\alpha + 2i - j)\Gamma(\alpha + \frac{1}{2})}{(i-j)!\Gamma(j+1)\Gamma(i - j + \alpha + \frac{1}{2})\Gamma(2\alpha)}p^{i-j}. \quad (13)$$

and with the corresponding inner product [36, 37]:

$$\langle C_i^{(\alpha)*}(p), \ C_j^{(\alpha)*}(p)\rangle = \int_0^1 \left(p - p^2\right)^{(\alpha - \frac{1}{2})} C_i^{(\alpha)*}(p)C_j^{(\alpha)*}(p)dp = \begin{cases} 0, & \text{for } i \neq j \\ \frac{\pi 2^{1-4\alpha}\Gamma(j+2\alpha)}{j![\Gamma(\alpha)]^2(j+\alpha)}, & \text{for } i = j \end{cases} \quad (14)$$

2.1.5. Shifted Vieta-Lucas polynomials

The shifted Vieta-Lucas polynomials $VL_i^*(p), p \in [a, b]$ is an orthogonal reference to $\omega(p) = \frac{b-a}{2\left(\sqrt{(b-a)^2-(2p-a-b)^2}\right)}$ satisfying the recurrence relation:

$$VL_{i+1}^*(p) = VL_1^*(p)VL_i^*(p) - VL_{i-1}^*(p), i = 1, 2, \ldots, \quad (15)$$

where $VL_0^*(p) = 2, VL_1^*(p) = \frac{4p - 2(a+b)}{b-a}$.

The analytical form corresponding to Eq. (15) in the interval $p \in [0, 1]$ is [38, 39]:

$$VL_i^*(p) = 2i\sum_{j=0}^{i} \frac{(-1)^j 4^{i-j}\Gamma(2i - j)}{\Gamma(j+1)\Gamma(2i - 2j + 1)}p^{i-j}, \quad (16)$$

the inner product corresponding to Eq. (16) is given as:

$$\langle VL_i^*(p), \ VL_j^*(p)\rangle = \int_0^1 \left(p - p^2\right)^{-\frac{1}{2}} VL_i^*(p)VL_j^*(p)dp = \begin{cases} 0, & i \neq j \neq 0 \\ 2\pi, & i = j \neq 0 \\ 4\pi & i = j = 0. \end{cases} \quad (17)$$

2.2. Caputo fractional differentiation operator (CFDO)

CFDO D^δ, of order δ is defined as:

$$D^\delta f(p) = \frac{1}{\Gamma(k-\delta)}\int_0^p \frac{f^{(k)}(p)}{(p-t)^{\delta+1-k}}dt, \quad \delta > 0, \ k - 1 < \delta < k, k \in \mathbb{N}. \quad (18)$$

with the linearity property:

$$D^\delta(\sigma f(p) + \varsigma g(p)) = \sigma D^\delta f(p) + \varsigma D^\delta g(p), \quad (19)$$

where, σ and ς are constants.

We obtained the following:

$$D^\delta p^i = \begin{cases} 0, \ \ i \in \mathbb{N}_0 \ , \ i < \lceil\delta\rceil \\ \frac{\Gamma(i+1)}{\Gamma(i+1-\delta)}p^{i-\delta}, \ i \in \mathbb{N}_0 \ , \ i \geq \lceil\delta\rceil, \end{cases} \quad (20)$$

where $\lceil\delta\rceil$ is the smallest integer greater than or equal to δ.

Theorem 2.1. *Suppose $VL_j^*(p), p \in [0,1]$ is a shifted Vieta-Lucas polynomial, then the Caputo fractional derivative of $VL_j^*(p)$ in terms of shifted Vieta-Lucas polynomial is:*

$$D^\delta\left(VL_j^*(p)\right)=\sum_{k=0}^{j-\lceil\delta\rceil}\frac{(-1)^k 2i4^{j-k}\Gamma(2j-k)\Gamma(j-k+1)}{\Gamma(2j-2k+1)\Gamma(j-k+1-\delta)\Gamma(k+1)}p^{j-k-\delta}. \quad (21)$$

Proof is given in Ref. [38].

Theorem 2.2. *Let the fractional derivative of $g(p)$ of order N be expressed in terms of shifted Vieta-Lucas polynomials:*

$$D^\delta\left(g_N(p)\right)=\sum_{j=0}^{N}\lambda_j D^\delta\left(VL_j^*(p)\right), \quad (22)$$

then

$$D^\delta\left(g_N(p)\right)=\sum_{j=\lceil\delta\rceil}^{N}\sum_{k=0}^{j-\lceil\delta\rceil}\lambda_j\Upsilon_{j,k}p^{j-k-\delta}, \quad (23)$$

where

$$\Upsilon_{j,k}=\frac{(-1)^k 2i4^{j-k}\Gamma(2j-k)\Gamma(j-k+1)}{\Gamma(2j-2k+1)\Gamma(j-k+1-\delta)\Gamma(k+1)}. \quad (24)$$

Proof is given in Ref. [38].

2.3. He's fractional derivative operator

The He's fractional derivative operator D^δ is defined as [40, 41]:

$$D^\delta f(p)=\frac{1}{\Gamma(k-\delta)}\frac{d^\delta}{dp^\delta}\int_0^p (s-p)^{k-\delta-1}\left[f_0(s)-f(s)\right]ds,$$
$$\delta>0,\ k-1<\delta<k, k\in\mathbb{N}, \quad (25)$$

where $f_0(x,p)$ is the solution of its continuous form of the problem with same initial condition.

3. Formulation of the numerical scheme

In this section, we consider the technique involved in the formulation of the scheme for solution of the generalized FIE via Galerkin method using shifted Vieta-Lucas polynomials as an approximate polynomial.

The approximate solution $y_N(x)$ corresponding to the closed form solution $y(x)$ in Eq. (1) using Galerkin method as transformation technique is derived as follows:

$$y_N(x)=\sum_{i=0}^{N}\lambda_i VL_i^*(x). \quad (26)$$

Substituting Eq. (26) into Eq. (1), and applying theorem 2.1 to the fractional part to the resulting equation, we obtain

$$\sum_{i=\lceil\delta\rceil}^{N}\sum_{j=0}^{i-\lceil\delta\rceil}\lambda_i\Upsilon_{i,j}x^{i-j-\delta}+\Psi_1(x)\sum_{i=0}^{N}\lambda_i VL_i^*(x)$$
$$+\varsigma_1\int_0^x S_1(x,t)\sum_{i=0}^{N}\lambda_i VL_i^*(t)dt$$
$$+\varsigma_2\int_0^1 S_2(x,t)\sum_{i=0}^{N}\lambda_i VL_i^*(t)dt=\Psi_2(x). \quad (27)$$

Multiplying both sides of Eq. (27) with shifted Vieta-Lucas polynomials $VL_j^*(x)$, $j=\lceil\delta\rceil,\lceil\delta\rceil+1,\ldots,N$, and integrating the result obtained in the interval $[a,b]$, we have:

$$\int_a^b\left[\sum_{i=\lceil\delta\rceil}^{N}\sum_{j=0}^{i-\lceil\delta\rceil}\lambda_i\Upsilon_{i,j}x^{i-j-\delta}+\Psi_1(x)\sum_{i=0}^{N}\lambda_i VL_i^*(x)\right.$$
$$+\varsigma_1\int_0^x S_1(x,t)\sum_{i=0}^{N}\lambda_i VL_i^*(t)dt$$
$$\left.+\varsigma_2\int_0^1 S_2(x,t)\sum_{i=0}^{N}\lambda_i VL_i^*(t)dt\right]VL_j^*(x)dx$$
$$=\int_a^b\Psi(x)VL_j^*(x)dx. \quad (28)$$

From Eq.(28), we obtain $(N-\lceil\delta\rceil+1)$ algebraic equations with $N+1$ unknowns and the remaining equations are obtain from non-local boundary conditions (2), given as:

$$\sum_{k=1}^{r}\left(\beta_{nk}\left[\sum_{i=0}^{N}\lambda_i\frac{d^{k-1}}{dx^{k-1}}VL_i^*(x)\right]_{x=a}+\alpha_{nk}\left[\sum_{i=0}^{N}\lambda_i\frac{d^{k-1}}{dx^{k-1}}VL_i^*(x)\right]_{x=b}\right)$$
$$+\gamma_n\int_a^b K_n(x)\sum_{i=0}^{N}\lambda_i VL_i^*(x)dx=\rho_n,\ n=1,2,\ldots,r, \quad (29)$$

to have $(N+1)$ equations, which will be solve to obtain the unknowns $\lambda_i, i=0,1,\ldots,N$ and subsequently, the approximant $y_N(x)$. See Refs. [2, 5] for more explanation on the application of Galerkin method to integro-differential equation.

4. Application to GFIE

The implementation of the technique involved in the formulation of the scheme for the numerical solution of the generalized FIE via Galerkin method using shifted Vieta-Lucas polynomials as an approximate polynomial is consider herein. The scheme was implemented on some selected problems from the literature. We compute the maximum absolute error κ_N for each problem and compare the results of the present scheme (PS) with the results in the literature. Maximum absolute error κ_N is given as:

$$\kappa_N=\max_{0\le\ell\le100}|y(x_\ell)-y_N(x_\ell)|,\quad x_\ell=a+ih \quad (30)$$

Example 4.1

Consider FIE [10, 25]:

$$D^{(0.5)}y(x)-e^x\int_0^x ty(t)dt+\frac{x^2}{3}e^x y(x)$$
$$-\int_0^1 x^2y(t)dt=\frac{\sqrt{x}}{\Gamma(1.5)}-\frac{1}{2}x^2, \quad (31)$$

with non-local condition:

$$y(0) = 3\int_0^1 ty(t)dt - y(1), \tag{32}$$

and exact solution $y(x) = x$.

Comparing Eqs. (31) and (1), we have $\delta = \frac{1}{2}$, seeking a solution of the form (27), Eq. (31) becomes:

$$\begin{aligned}&\sum_{i=\lceil\delta\rceil}^{N}\sum_{j=0}^{i-\lceil\delta\rceil}\lambda_i\Upsilon_{i,j}x^{i-j-\delta} + \frac{x^2}{3}e^x\left(\sum_{i=0}^{N}\lambda_i VL_i^*(x)\right)\\ &- e^x\int_0^x t\left(\sum_{i=0}^{N}\lambda_i VL_i^*(t)\right)dt\\ &- \int_0^1 x^2\left(\sum_{i=0}^{N}\lambda_i VL_i^*(t)\right)dt = \frac{\sqrt{x}}{\Gamma(1.5)} - \frac{1}{2}x^2.\end{aligned} \tag{33}$$

Multiplying Eq. (33) by $VL_\iota^*(x), \iota = \lceil\delta\rceil, \lceil\delta\rceil + 1, \ldots, N$, then integrating the obtained equation in the interval $[0, 1]$, we have:

$$\begin{aligned}&\int_0^1\left[\sum_{i=\lceil\delta\rceil}^{N}\sum_{j=0}^{i-\lceil\delta\rceil}\lambda_i\Upsilon_{i,j}x^{i-j-\delta} + \frac{x^2}{3}e^x\left(\sum_{i=0}^{N}\lambda_i VL_i^*(x)\right)\right. \\ &\left.-e^x\int_0^x t\left(\sum_{i=0}^{N}\lambda_i VL_i^*(t)\right)dt - \int_0^1 x^2\left(\sum_{i=0}^{N}\lambda_i VL_i^*(t)\right)dt\right]VL_\iota^*(x)dx\\ &= \int_0^1\left[\frac{\sqrt{x}}{\Gamma(1.5)} - \frac{1}{2}x^2\right]VL_\iota^*(x)dx,\end{aligned} \tag{34}$$

Eq. (34) gives:

$$\begin{aligned}&-\frac{470}{1001}\lambda_0 + \frac{3659}{2479}\lambda_1 + \frac{2259}{596}\lambda_2 - \frac{735}{1562}\lambda_3 + 2525472\lambda_4 = \frac{576}{4291}\\ &-\frac{431}{1459}\lambda_0 + \frac{626}{649}\lambda_1 + \frac{6922}{3941}\lambda_2 + \frac{5321}{885}\lambda_3 - 17741695\lambda_4 = \frac{568}{6079}\\ &-\frac{204}{1031}\lambda_0 + \frac{540}{769}\lambda_1 + \frac{431}{357}\lambda_2 + \frac{147}{46}\lambda_3 + \frac{1796}{275}\lambda_4 = \frac{369}{4816}\\ &-\frac{431}{2806}\lambda_0 + \frac{687}{1241}\lambda_1 + \frac{668}{743}\lambda_2 + \frac{243}{104}\lambda_3 + \frac{2099}{614}\lambda_4 = \frac{179}{2906}\end{aligned} \tag{35}$$

From non-local boundary condition, we have:

$$\begin{aligned}&y(0) + y(1) - 3\int_0^1 ty(t)dt = \sum_{i=0}^{N}\lambda_i VL_i^*(0)\\ &+\sum_{i=0}^{N}\lambda_i VL_i^*(1) - 3\int_0^1 t\left(\sum_{i=0}^{N}\lambda_i VL_i^*(t)\right)dt = 0\\ &\Rightarrow \frac{1}{2}\lambda_0 - \lambda_1 + \frac{11}{2}\lambda_2 - \frac{2}{5}\lambda_3 + \frac{97}{10}\lambda_4 = 0.\end{aligned} \tag{36}$$

Soving Eqs. (35) and (36) to obtain the values of the unknowns $\lambda_i,\ i = 0, \ldots, 4$, then substitute the values of $\lambda_i,\ i = 0, \ldots, 4$ in Eq. (26) to gives approximate analytical solution.

The same problem was solved in [25] and obtained approximate solution using Pade approximations with maximum absolute error of 8.69×10^{-5}, likewise [10] used Bernstein polynomials as an approximating polynomial and obtained 4.90×10^{-11} as maximum absolute error at $N = 4$ while in the proposed scheme, we obtain exact solution at $N = 1$.

Example 4.2

Consider FIE [10]:

$$D^{\frac{1}{3}}y(x) = \int_0^x x^2\exp(xt)y(t)dt + \frac{3}{2}\frac{x^{\frac{2}{3}}}{\Gamma\left(\frac{2}{3}\right)} - 1 + \exp(x^2) - x^2\exp(x^2),$$

subject to non-local boundary condition

$$y(0) = 3 - 2y(1) - 3\int_0^1 ty(t)dt,$$

with the exact solution $y(x) = x$.

The same problem was solved in Ref. [10] using Bernstein polynomial as an approximating polynomial and obtained maximum error of 3.31×10^{-7} while in the present scheme, exact solution is obtain at $N = 1$.

Example 4.3

Consider FIE [10]:

$$D^{\frac{5}{4}}y(x) = g(x) + (\cos x - \sin x)\, y(x) + \int_0^x \sin(t)y(t)dt,$$

subject to non-local boundary conditions

$$\begin{aligned}&y(0) + y(1) = \int_0^1 ty(t)dt - \frac{1}{2}y'(1) - \left(\frac{e+1}{e+2}\right)y'(0),\\ &2\,(y(0) + y(1)) = y'(1) - \left(\frac{e}{e+1}\right)y'(0).\end{aligned}$$

The closed form solution is $y(x) = x^2$, where $g(x) = \frac{8}{3}\frac{x^{\frac{3}{4}}}{\Gamma\left(\frac{3}{4}\right)} - 2x\sin x + x^2\sin x - \cos x + 2$.

The same problem was solved in Ref. [10] using Bernstein polynomial as an approximating polynomial and obtained maximum error of 3.51×10^{-8} while in the present scheme, exact solution is obtain at $N = 4$.

Example 4.4

Consider the following FIE [10]:

$$\begin{aligned}D^{\frac{1}{2}}y(x) + \int_0^x ty(t)dt + \int_0^1 t^2y(t)dt\\ = (\mathrm{erf}(\sqrt{x}) + x - 1)\exp(x) + \exp(1) - 1.\end{aligned}$$

with non-local boundary condition : $y(0) - \int_0^1 ty(t)dt = 0$.

The exact solution is $y(x) = \exp(x)$.

Table 1. Maximum absolute errors for Example 4.3 at various values of N.

N	Ref. [10]	PS
4	1.05×10^{-4}	9.24×10^{-5}
6	1.57×10^{-7}	9.78×10^{-7}
12	1.78×10^{-16}	1.66×10^{-16}

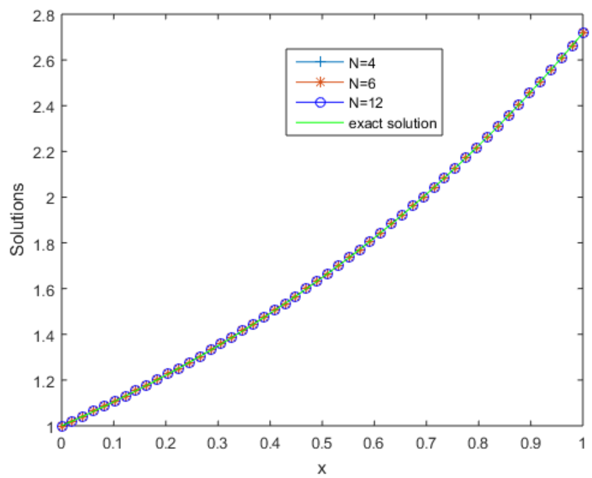

Figure 1. Exact and its corresponding approximants at various values of N for example 4.4.

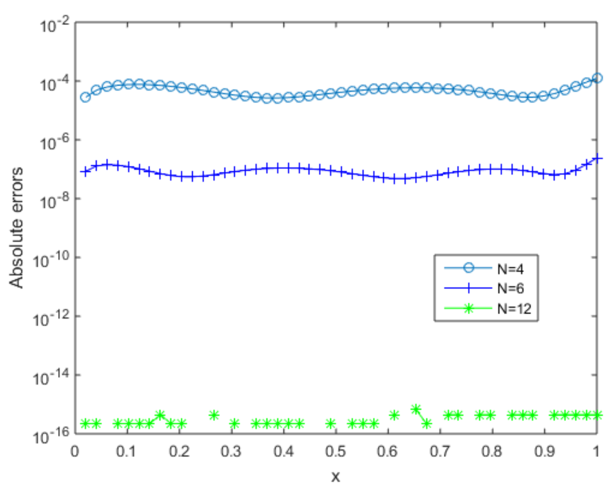

Figure 2. Asolute errors for example 4.4.

5. Discussion of results and conclusion

5.1. Discussion of Results

Table 1 depicts the absolute maximum errors obtained for example 4.4 and Figure 1 is the corresponding figure while Figure 2 displays the absolute errors at various degree of approximation. From Table 1, the proposed method is accurate and effective as it compares favourably with the results obtained in Ref. [10]. Figure 1 shows relationship between the exact and its approximate while Figure 2 displays the accuracy at various values of N. It was observed that all the numerical solutions to examples 4.1 – 4.3 give closed form solutions while example 4.4 give very good accuracy of results compare to the results in the literature.

5.2. Conclusion

In this paper, we proposed an extension of fractional-order derivative for the shifted Vieta-Lucas polynomial to solve generalized FIE involving non-local boundary conditions using Galerkin method to transform the IDEs to a system of algebraic equations and the fractional part of the IDE was removed using Caputo properties. The equations were solved together with the non-local boundary conditions to obtain the approximate solutions. For experiment, we implement the scheme on existing problems selected from the literature. We obtained exact results for problems with polynomial exact solutions and obviously, from the numerical results obtained for the problem with non-polynomial exact solution, shows the accuracy and effectiveness of the proposed method.

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