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One-step block scheme with optimal hybrid points for numerical integration of second-order ordinary differential equations

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Abstract

In this paper, a one-step block of optimized hybrid schemes for the numerical integration of second-order initial value problems (IVP) of ordinary differential equations (ODE) is constructed via collocation techniques. The developed scheme is obtained by considering two intra-step nodal points as hybrid points, which are chosen in order to achieve optimized errors of the main formulae approximating the solution such that $0 < v_1 < v_2 < 1$ where v_1 and v_2 are defined as hybrid points. The characteristics of the developed scheme are analyzed. Application of the new scheme on some second-order IVPs shows the accuracy and effectiveness of the scheme compared with some existing methods.

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1. Introduction

The use of collocation techniques that are based on power series as basis function for the formation of multi-step schemes has gained research interests over the years [1–7]. The development of optimized hybrid schemes based on the selection of hybrid points, in such a way that the approximated solution of the given higher order ODEs is close to the exact solution of the said ODEs, is a recent technique that should be upheld by numerical analysts [5, 6]. This paper is on the numerical solution of the second-order IVPs of the form:

$$\mu'' = g(t,\mu,\mu'), \quad t \in [t_0, t_N], \quad \mu(t_0) = \mu_0, \quad \mu'(t_0) = \mu_0', \qquad (1)$$

where t_0 and t_N represent the start and end points of the integration interval, respectively, μ_0 and μ_0' are real constants and $g(t,\mu,\mu')$ is a continuous real function. Equation (1) often arises in Science and Engineering fields such as molecular dynamics, electronics, celestial mechanics, astrophysics, mathematical modeling and semi-discretization of wave equations. The IVP in equation (1) can be solved directly or solved by reducing it to a system of first-order ODEs before applying different numerical methods to solve the resulting system of first-order ODEs [8–10].

Various researchers have developed numerical schemes for the solution of equation (1) such as Adeyefa [3, 11], Anake [12], Obarhua and Adegboro [13], Bilesanmi *et al.* [14], Kuboye [15] and Mohammed [16, 17] to mention a few. Among the first methods developed are the first derivative methods that are implemented in predictor-corrector mode and Taylor series

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expansion are used to provide the starting value. Predictor-corrector schemes are very costly to implement and have reduced order of accuracy of the predictor.

This study proposes an optimized one-step hybrid block method for the direct solution of second-order initial value problems of ordinary differential equations of the form given in equation (1). The developed scheme is capable of proffering numerical solution to linear and non-linear second-order initial value problems.

2. Construction of the optimized one-step block hybrid scheme

The solution of equation (1) is assumed on an interval $[t_n, t_{n+1}]$ which is locally approximated by a polynomial of the form

$$\mu(t) = \sum_{r=0}^{p+q-1} \phi_r t^r, \tag{2}$$

with corresponding derivative as

$$\mu'(t) = \sum_{r=1}^{p+q-1} r\phi_r t^{r-1}, \tag{3}$$

where p and q are the interpolation and collocation points, respectively. The two hybrid points (v_1 and v_2) are considered in such a way that $0 < v_1 < v_2 < 1$ holds [5]. Interpolating equation (2) and collocating equation (3) at given grid points give

$$\mu_{n+j} = \mu(t_{n+j}), \quad j = 0, \tag{4}$$

$$\mu'_{n+j} = \mu'(t_{n+j}), \quad j = 0, \tag{5}$$

$$\mu''_{n+j} = g(t_{n+j}), \quad j = 0, v_1, v_2, 1, \tag{6}$$

where μ_{n+j} and g_{n+j} are approximations for $\mu(t_{n+j})$ and $\mu''(t_{n+j})$, respectively. The system of six equations in equations (4), (5) and (6) is written in compact form as

$$AB = G, \tag{7}$$

where

$$A = \begin{bmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 \\ 0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 \\ 0 & 0 & 2 & 6t_{n+v_1} & 12t_{n+v_1}^2 & 20t_{n+v_1}^3 \\ 0 & 0 & 2 & 6t_{n+v_2} & 12t_{n+v_2}^2 & 20t_{n+v_2}^3 \\ 0 & 0 & 2 & 6t_{n+1} & 12t_{n+1}^2 & 20t_{n+1}^3 \end{bmatrix},$$

$$B = [\phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5]^T,$$

and

$$G = [\mu_n, \mu'_n, g_n, g_{n+v_1}, g_{n+v_2}, g_{n+1}]^T.$$

Solving equation (7) simultaneously gives the corresponding coefficients of $\phi_r, r = 0(1)5$. Substituting the resulting coefficients $\phi_r, r = 0(1)5$ into equation (2) and its derivatives yields a continuous implicit scheme of the form

$$\alpha_z\mu_{n+z} = \alpha_0\mu_n + h\beta_{10}\mu'_n + h^2 \sum_{j=0}^{1} \rho_j g_{n+j} + h^2 \sum_{j=1}^{2} \rho_{v_j} g_{n+v_j}, \quad z = v_1, v_2, 1. \tag{8}$$

To obtain the approximate values of v_1 and v_2 hybrid points, one optimizes the local truncation errors of one of the schemes in equation (8) and ensures that the hybrid points satisfy the interval $0 < v_1 < v_2 < 1$ [5], specifically we chose μ_{n+1}. To get the local truncation error, one expands the Taylor series about the point t_n of the scheme to obtain

$$L[y(t_{n+1}), h] = \frac{h^6\mu^{(6)}(t_n)Q_1}{1440} + O(h^7), \tag{9}$$

where

$$Q_1 = -5v_1v_2 + 2v_1 + 2v_2 - 1 = 0, \quad 0 < v_1 < v_2 < 1. \tag{10}$$

Imposing that the principal term (Q_1) in the local truncation error of (9) is zero. We use equation (10) and its constraint to scan for v_1 and v_2 such that the scheme attains order five and yields the solution as $v_1 = \frac{1}{4}$ and $v_2 = \frac{2}{3}$ as possible solution.

The discrete scheme and its derivative derived by evaluating equation (7) as well as its derivative at grid and non-grid points $\left(\frac{1}{4}, \frac{2}{3}, 1\right)$ are given in equations (11) and (12). These schemes are used to form a block of hybrid method and its derivative method as:

$$\begin{bmatrix} \mu_{n+\frac{1}{4}} \\ \mu_{n+\frac{2}{3}} \\ \mu_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \mu_n \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{2}{3} \\ 1 \end{bmatrix} \begin{bmatrix} h\mu'_n \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{4095}{230400} \\ 0 & 0 & \frac{930}{18225} \\ 0 & 0 & \frac{135}{1800} \end{bmatrix} \begin{bmatrix} h^2 g_{n-\frac{2}{3}} \\ h^2 g_{n-\frac{1}{4}} \\ h^2 g_n \end{bmatrix} + \begin{bmatrix} \frac{3664}{230400} & -\frac{729}{230400} & \frac{170}{230400} \\ \frac{2816}{18225} & \frac{324}{18225} & -\frac{20}{18225} \\ \frac{512}{1800} & \frac{243}{1800} & \frac{10}{1800} \end{bmatrix} \begin{bmatrix} h^2 g_{n+\frac{1}{4}} \\ h^2 g_{n+\frac{2}{3}} \\ h^2 g_{n+1} \end{bmatrix}. \tag{11}$$

Note that $h^2 g_{n-\frac{2}{3}}$ and $h^2 g_{n-\frac{1}{4}}$ are multiplying the value zero in the 3×3 matrix in the second line of equation (11) and both functions will vanish. This is the reason both functions are not seen in equation (8). The first derivative block methods are given in equation (12).

$$\begin{bmatrix} h\mu'_{n+\frac{1}{4}} \\ h\mu'_{n+\frac{2}{3}} \\ h\mu'_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} h\mu'_n \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{2325}{23040} \\ 0 & 0 & \frac{75}{1215} \\ 0 & 0 & \frac{15}{180} \end{bmatrix} \begin{bmatrix} h^2 g_{n-\frac{2}{3}} \\ h^2 g_{n-\frac{1}{4}} \\ h^2 g_n \end{bmatrix} + \begin{bmatrix} \frac{3872}{23040} & -\frac{567}{23040} & \frac{130}{23040} \\ \frac{512}{1215} & \frac{243}{1215} & -\frac{20}{1215} \\ \frac{64}{180} & \frac{81}{180} & \frac{20}{180} \end{bmatrix} \begin{bmatrix} h^2 g_{n+\frac{1}{4}} \\ h^2 g_{n+\frac{2}{3}} \\ h^2 g_{n+1} \end{bmatrix}. \quad (12)$$

Equations (11) and (12) form the One-step Block Scheme with Optimal Hybrid points (OBSOH) developed for the direct approximation of linear and non-linear second-order IVPs (1).

3. Analysis of OBSOH

In this section, the properties of the derived method are examined.

3.1. The error constant and order of the method

The local truncation error associated with the derived methods can be defined as the linear difference operator [18]:

$$\begin{aligned} L[\mu(t_n);h] &= \textstyle\sum_{j=0}^{1} \alpha_j \mu(t_n+jh) - h\beta_{10}\mu^{(\prime)}(t_n) \\ &-h^2 \textstyle\sum_{j=0}^{1} \rho_j \mu''(t_n+jh) - h^2 \sum_{j=1}^{2} \rho_{v_j}\mu''(t_n+(v_j)h). \end{aligned} \quad (13)$$

Assuming that $\mu(t_n)$ is sufficiently differentiable, then using Taylor series expansion on $\mu(t_n+jh)$,$\mu'(t_n+jh)$ and $\mu''(t_n+jh)$ about t_n, we have

$$\mu(t_n+jh) = \sum_{m=0}^{\infty} \frac{(jh)^m}{m!}\mu^{(m)}(t_n),$$

$$\mu'(t_n+jh) = \sum_{m=1}^{\infty} \frac{(jh)^m}{m!}\mu^{(m+1)}(t_n),$$

$$\mu''(t_n+jh) = \sum_{m=2}^{\infty} \frac{(jh)^m}{m!}\mu^{(m+2)}(t_n).$$

Substituting $\mu(t_n+jh)$, $\mu'(t_n+jh)$ and $\mu''(t_n+jh)$ in equation (13) we obtain

$$\begin{aligned} L[\mu(t_n);h] &= C_0\mu(t_n) + C_1 h\mu'(t_n) + C_2 h^2\mu''(t_n) \\ &+C_3 h^3 \mu'''(t_n) + ... + C_{m+2}h^{m+2}\mu^{(m+2)}(t_n) + \dots \end{aligned} \quad (14)$$

where $C_m, m = 0, 1, 2, \dots$ are constants given as:

$$\begin{aligned} C_0 &= \textstyle\sum_{j=0}^{1} \alpha_j + \sum_{j=1}^{2} \alpha_{v_j}, \\ C_1 &= \left[\textstyle\sum_{j=0}^{1} j\alpha_j + \sum_{j=1}^{2} v_j\alpha_{v_j}\right] - \beta_{10}, \\ &\vdots \\ C_{m+2} &= \tfrac{1}{(m+2)!}\left[\textstyle\sum_{j=0}^{1} j^{m+2}\alpha_j + \sum_{j=1}^{2}(v_j)^{m+2}\alpha_{v_j}\right] \\ &-\tfrac{1}{(m+1)!}\left[\textstyle\sum_{j=0}^{1} j^{m+1}\beta_{1j} + \sum_{j=1}^{2}(v_j)^{m+2}\beta_{1v_j}\right] \\ &-\tfrac{1}{m!}\left[\textstyle\sum_{j=0}^{1} j^{m}\rho_j + \sum_{j=1}^{2}(v_j)^{m}\rho_{v_j}\right]. \end{aligned} \quad (15)$$

The error constants and order of OBSOH are shown in Table 1.

3.2. Zero-stability

The block method is said to be zero-stable if the roots R_u, $u = 1, 2, \dots, 6$ of the first characteristic polynomial $\gamma(R)$ satisfy $|R_u| \le 1, u = 1, \dots, 6$ multiplicity not exceeding the order of the differential equation. The first characteristic polynomial $\gamma(R) = 0$ of the derived method is calculated as

$$\gamma(R) = \det(RB_{(1)} - B_{(0)}),$$

where $B_{(1)}$ is a 6×6 identity matrix and

$$B_{(0)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and $\gamma(R) = R^{w-e}(R-1)^e$, where e is the order of the differential equation and w is the order of the matrices $B_{(1)}$ and $B_{(0)}$ [18]. The OBSOH can be shown to be zero-stable since the first characteristic polynomial $\gamma(R) = R^4(R-1)^2$ satisfies $|R_u| \le 1, u = 1(1)6$.

3.3. Consistency

The developed method is concluded to be consistent since according to Lambert [18], the necessary and sufficient condition for a numerical scheme to be consistent is for it to have order of at least one ($m \ge 1$). The derived method is of at least order 4 since the least order of the block method is of order 4.

3.4. Convergence

A numerical method converges if it is consistent and zero-stable [18, 19]. This implies that OBSOH converges since the method is of order $m = 4 > 1$ and it satisfies the conditions for zero-stability.

4. Numerical results

The following problems are considered in order to examine the accuracy and computational efficiency of the new block method (OBSOH). The Rate of Convergence (ROC) of the scheme on problems 2 to 4 confirms that the order of the derived scheme OBSOH is of order 4. The efficiency curves of the schemes on problems 1 and 6 are shown in Figures 1 and 2, respectively. All computations were done using MATHEMATICA 13.0.

The notations used in representing the existing methods and the derived method in the result Tables are:

Table 1. Order and error constants of OBSOH.

S/N	Scheme	Order(m)	Error Constant (C_{m+2})
1	$\mu_{n+\frac{1}{4}}$	4	$-\frac{41}{5898240}$
2	$\mu_{n+\frac{2}{3}}$	4	$-\frac{1}{262440}$
3	μ_{n+1}	5	$\frac{1}{604800}$
4	$\mu'_{n+\frac{1}{4}}$	4	$-\frac{229}{4423680}$
5	$\mu'_{n+\frac{2}{3}}$	4	$\frac{1}{14580}$
6	μ'_{n+1}	4	$-\frac{1}{17280}$

- EIAK = Error in Adeyefa and Kuboye [3].
- EIAD = Error in Adeyefa [11].
- EIAN = Error in Anake [12].
- EIOA = Error in Obarhua and Adegboro [13].
- EIBWO = Error in Bilesanya, Wusu and Olutimo [14].
- EIRKN = Well Known Runge-Kutta Nystrom Method [14].
- EIK = Error in Kuboye [15].
- EIM = Error in Mohammed [16].
- EIMA = Error in Mohammed and Adeniyi [17].
- EIOBSOH = Error in One-step Block Scheme with Optimal Hybrid points.

where,

$$Error^{(i)} = |y^{(i)}_{exact} - y^{(i)}_{approx.}|.$$

and

$$\text{ROC} = \log_2 \frac{(ME)^{2h}}{(ME)^h},$$

where $(ME)^{2h}$ is Max Error using $2h$ as step size, while $(ME)^h$ is Max Error using h as step size.

The following problems are chosen for numerical computation to aid comparison with other existing methods in literature.

Problem 1 [3]

$$\mu''(t) = t(\mu')^2, \quad h = 0.003125;$$

$$\mu(0) = 1, \quad \mu'(0) = 0.5,$$

with the exact solution

$$\mu(t) = 1 + \frac{1}{2} In \left[\frac{2+t}{2-t} \right].$$

Problem 2 [3]

$$\mu''(t) = \mu', \quad h = 0.01;$$

$$\mu(0) = 0, \quad \mu'(0) = -1,$$

Table 2. Comparison of errors for problem 1.

t	EIK [15], k = 6	EIAN [12], k = 1	EIAK [3], k = 1	EIOBSOH, k = 1
0.1	9.577668E-10	4.98272E-11	6.743939E-12	6.66134E-16
0.2	2.368709E-09	4.10430E-10	5.572787E-11	2.22045E-16
0.3	3.732243E-09	1.42858E-09	1.965739E-10	6.66134E-16
0.4	5.475119E-09	3.52426E-09	4.947556E-10	1.77636E-15
0.5	1.142189E-08	7.24353E-09	1.043623E-09	2.22045E-15
0.6	4.567944E-08	1.33355E-08	1.982763E-09	3.55271E-15
0.7	2.055838E-06	2,28728E-08	3.527785E-09	6.21725E-15
0.8	4.248299E-06	3.74470E-08	6.020838E-09	6.66134E-15
0.9	6.660458E-06	5.95037E-08	1.001993E-08	7.10543E-15
1.0	9.445166E-06	9.29404E-08	1.646376E-08	4.66294E-15

Table 3. Comparison of log of maximum errors and number of iterations for problem 1.

N	Max Error (OBSOH)	log (Max Error)	N	Max Error(EIAN [12])	log(Max Error)
160	2.22933E-13	-29.1319	160	6.2983E-12	-25.7907
320	4.66294E-15	-32.9991	320	4.03455E-13	-28.5387
640	2.53131E-14	-31.3075	640	5.9952E-14	-30.4452

Table 4. Comparison of errors for problem 2.

t	EIM [16], k = 5	EIMA [17], k = 5	EIK [15], k = 6	EIAK [3], k = 1	EIOBSOH, k = 1
0.1	2.19800E-06	2.00400E-07	2.508826E-13	2.095826E-10	3.10862E-15
0.2	6.07040E-06	5.38600E-07	6.493175E-11	2.092718E-09	1.39888E-14
0.3	1.00510E-05	8.84000E-07	1.683146E-09	7.842546E-09	3.24185E-14
0.4	1.40253E-05	1.22970E-06	1.700635E-08	2.009500E-08	6.12843E-14
0.5	1.79934E-05	1.57520E-06	1.025454E-07	4.199771E-08	1.01918E-13
0.6	2.16162E-05	1.92040E-06	2.558711E-06	7.728842E-08	1.56541E-13
0.7	2.79930E-05	2.50600E-06	5.273300E-06	1.303844E-07	2.28706E-13
0.8	3.45610E-05	3.10600E-06	8.275935E-06	2.064839E-07	3.19744E-13
0.9	4.11140E-05	3.70500E-06	1.161667E-05	3.116817E-07	4.33875E-13
1.0	4.76560E-05	4.30400E-06	1.542187E-05	4.531001E-07	5.75096E-13

whose exact solution is

$$\mu(t) = 1 - e^t.$$

Problem 3 [3]

The Vanderpol's oscillator problem is also considered. It is given as

$$\mu''(t) = 2\cos(t) - \cos^3 t - \mu' - \mu - \mu'(\mu)^2, \quad h = 0.1;$$

$$\mu(0) = 0, \quad \mu'(0) = 1,$$

with the theoretical solution

$$\mu(t) = \sin(t).$$

Problem 4 [14]

$$\mu''(t) = \mu(t) + t - 1, \quad t \in [0, 1];$$

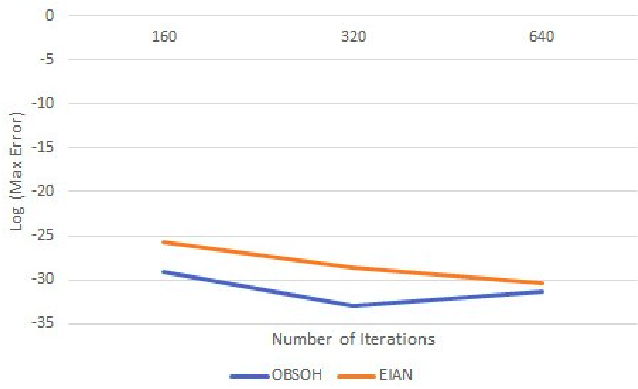

Figure 1. Efficiency curves for problem 1.

Table 5. Comparison of errors for roblem 3.

t	EIAK [3], $k=1, m=6$	EIAD [11], $k=1, m=5$	EIOBSOH, $k=1, m=4$
0.1	3.307291E-10	4.16719627E-13	1.77330E-13
0.2	2.315513E-09	3.54860749E-12	2.84067E-12
0.3	6.161694E-09	9.04722120E-12	1.41904E-11
0.4	1.192381E-08	1.650241042E-11	3.81999E-11
0.5	1.934444E-08	2.544360932E-11	7.82125E-11
0.6	2.775449E-08	3.535590072E-11	1.36439E-10
0.7	3.608020E-08	4.570838971E-11	2.13886E-10
0.8	4.297844E-08	5.598981142E-11	3.10406E-10
0.9	4.708092E-08	6.574464284E-11	4.24862E-10
1.0	4.728457E-08	7.460291389E-11	5.55398E-10

Table 6. Comparison of errors for problem 4.

t	EIRKN [14]	EIBWO [14]	EIOBSOH
0.1	2.973739 E-10	2.591705E-12	3.55271E-15
0.2	7.050944E-10	5.964562E-12	1.22125E-14
0.3	1.217025E-09	9.366508E-12	2.59792E-14
0.4	1.829043E-09	1.286815E-11	4.32987E-14
0.5	2.538910E-09	1.649259E-11	6.55032E-14
0.6	3.346159E-09	2.029099E-11	9.19265E-14
0.7	4.252021E-09	2.428346E-11	1.22236E-13
0.8	5.259365E-09	2.852540E-11	1.56430E-13
0.9	6.372666E-09	3.304634E-11	1.94844E-13
1.0	7.597991E-09	3.792694E-11	2.37643E-13

$$\mu(0) = 2, \quad \mu'(0) = -2, h = 0.01.$$

Exact solution is

$$\mu(t) = 1 - t + e^{-t}.$$

Note that the simulation annealing scheme of Bilesanmi *et al.* [14], an existing method for comparison in problems 4 and 5, is just an algorithm and not a block method that involves steps so the step k does not apply in this scheme.

Problem 5 [14]

$$\mu''(t) = (1 + t^2)\mu(t),$$

$$\mu(0) = 1, \mu'(0) = 0, h = 0.01,$$

Table 7. Comparison of errors for problem 5.

t	EIRKN [14]	EIBWO [14]	EIOBSOH
0.2	1.266432E-07	1.76241E-08	1.69642E-13
0.4	2.923595E-07	3.804534E-08	7.52509E-13
0.6	5.418410E-07	6.027792E-08	1.95022E-12
0.8	9.469284E-07	8.585760E-08	4.19753E-12
1.0	1.627915E-06	1.171072E-07	8.36908E-12

Table 8. Comparison of errors for problem 6.

t	EIAN [12], k=1	EIOA [13], k=3	EIOBSOH, k=1
0.1	0.10E-09	2.00E-10	1.85296E-13
0.2	0.20E-09	3.15E-10	3.35731E-13
0.3	0.28E-09	2.74E-10	4.54636E-13
0.4	0.34E-09	5.44E-10	5.48339E-13
0.5	0.39E-09	7.53E-10	6.19726E-13
0.6	0.43E-09	2.76E-10	6.72684E-13
0.7	0.45E-09	1.18E-10	7.09821E-13
0.8	0.40E-09	1.76E-10	7.34135E-13

Table 9. Comparison of log of maximum errors and number of iterations for problem 6.

N	Max Error (OBSOH)	log(Max Error)	N	Max Error(EIAN [12])	log(Max Error)
10	7.68698E-11	-23.2889	10	2.08868E-08	-17.6841
20	7.51066E-13	-27.9173	20	1.0018E-09	-20.7215
40	3.41394E-14	-31.0083	40	5.3168E-11	-23.6576
80	4.65430E-15	-33.0010	80	3.16314E-12	-26.4795

with the exact solution

$$\mu(t) = e^{\left[\frac{t^2}{2}\right]}.$$

Problem 6 [13]

$$\mu''(t) = -1001\mu' - 1000\mu, \quad h = 0.05;$$

$$\mu(0) = 1, \quad \mu'(0) = -1.$$

Exact solution is

$$\mu(t) = e^{-t}.$$

Problem 7 [13]

A practical problem on resonance vibration of a machine was modeled as:

$$2000\mu'' + 2 \times 10^5\mu = 2000\sin(10t), \quad h = 0.01;$$

$$\mu(0) = 0, \quad \mu'(0) = 0.1.$$

Exact solution is

$$\mu(t) = \frac{1}{10}\cos(10t) + \frac{1}{200}\sin(10t) - \frac{t}{20}\cos(10t).$$

Problem 8 [3]

We considered the following system of second-order equations:

$$\mu_1'' = -4t^2\mu_1 - \frac{2\mu_2}{\sqrt{\mu_1^2 + \mu_2^2}}, \quad \mu_1\left(\sqrt{\frac{\pi}{2}}\right) = 0, \quad \mu_1'\left(\sqrt{\frac{\pi}{2}}\right) = -2\sqrt{\frac{\pi}{2}}$$

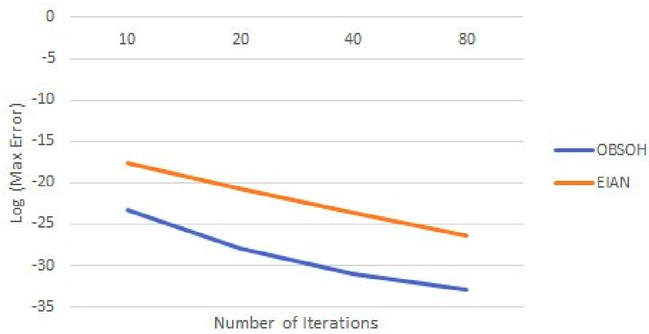

Figure 2. Efficiency curves for problem 6.

Table 10. Comparison of errors for problem 7.

t	EIOA [13], $k = 3$	EIOBSOH, $k = 1$
0.01	3.849660E-10	3.24740E-15
0.02	8.702471E-10	5.66951E-12
0.03	2.929087E-09	1.67078E-11
0.04	4.479915E-09	3.27208E-11
0.05	4.075598E-09	5.32108E-11
0.06	7.575420E-09	7.75898E-11
0.07	1.060883E-09	1.05189E-10
0.08	1.747335E-09	1.35271E-10
0.09	1.836375E-09	1.67042E-10
0.10	2.762184E-08	1.99664E-10

Table 11. Result generated when the new scheme was applied to problem 8 for μ_1.

t	Exact solution μ_1	Numerical solution μ_1	EIOBSOH
$1.253314 = \sqrt{\frac{\pi}{2}}$	0	0	0
1.263314	−0.025163626354015697	−0.025163626354020932	5.23540E-15
1.273314	−0.05051106208227043	−0.05051106208298397	7.135403E-13
1.283314	−0.07602542099130294	−0.07602542100266559	1.13626E-11
1.293314	−0.10168905502407596	−0.10168905510863648	8.45605E-11
1.303314	−0.1274835519786236	−0.12748355238290313	4.04279E-10
1.313314	−0.15338973447367182	−0.15338973592854993	1.45488E-09
1.323314	−0.17938766022292565	−0.17938766452226018	4.29933E-09
1.333314	−0.20545662367994122	−0.20545662367994122	1.09947E-08

$$\mu_2'' = -4t^2\mu_2 - \frac{2\mu_1}{\sqrt{\mu_1^2 + \mu_2^2}}, \quad \mu_2\left(\sqrt{\frac{\pi}{2}}\right) = 1, \quad \mu_2'\left(\sqrt{\frac{\pi}{2}}\right) = 0,$$

$$h = 0.01, \frac{\pi}{2} \leq t \leq 1.4.$$

Exact solution is

$$\mu_1(t) = \cos(t^2), \quad \mu_2(t) = \sin(t^2).$$

5. Discussion of results

In Table 2, the errors from EIK, EIAN and EIAK are compared to that of EIOBSOH and the new method proved superior in solving Problem 1. Table 3 shows the comparison of the Log of Max error and the number of iterations from EIOBSOH and EIAN and the results were used to plot the efficiency curves of both schemes as shown in Figure 1. In Tables 4 and 5, the errors from different numerical schemes were compared to that of EIOBSOH after solving problems 2 and 3, respectively. In Table 5, the results from EIAD is almost as good as that of EIOBSOH because the method is a one-step method of order 5 with three hybrid points while the derived method is a one-step scheme with two hybrid points. Also in Tables 6 - 8, OBSOH had better accuracy when compared to the other existing schemes that solved same set of Initial Value Problems. Results from Table 9, where the Log of Max error and the number of iterations from EIAN and EIOBSOH for problem 6 were compared, are used to plot the efficiency curves of both schemes as displayed in Figure 2. The efficiency curves displayed in Figures 1 and 2 show that OBSOH is efficient in the numerical solution of second-order IVPs. In Table 10, OBSOH is seen to produce smaller error when compared with EIOA. Tables 11 and 12 show that the new method can integrate systems of second-order IVPs directly. The rate of convergence Table shown in Table 13 confirms that the order of OBSOH is of order 4. Thus, all the Tables show that OBSOH can effectively and efficiently Integrate the linear and nonlinear second-order IVPs (1) directly.

Table 12. Result generated when the new scheme was applied to problem 8 for μ_2.

t	Exact solution μ_2	Numerical solution μ_2	EIOBSOH
$1.253314 = \sqrt{\frac{\pi}{2}}$	1	1	0
1.263314	0.9996833458194228	0.9996850201319464	1.67431E-06
1.273314	0.9987235015795518	0.9987369201519218	1.34186E-05
1.283314	0.9971058797154368	0.9971512425384893	4.53628E-05
1.293314	0.9948162323204776	0.9949239221444588	1.07690E-04
1.303314	0.9918406847749862	0.9920513048128204	2.10620E-04
1.313314	0.9881657701813479	0.9885301658317266	3.64396E-04
1.323314	0.983778464574085	0.9843577282034791	5.79264E-04
1.333314	0.9786662228696967	0.9795316807052991	8.65458E-04

Table 13. Rate of convergence of the scheme on some problems.

	h	Max Error (h)	Max Error ($2h$)	ROC
Problem 2	0.01	5.75096×10^{-13}	9.00613×10^{-12}	3.9691
	0.02	9.00613×10^{-12}	1.40129×10^{-10}	3.9597
Problem 3	0.025	2.35367×10^{-12}	3.672×10^{-11}	3.9636
	0.05	3.672×10^{-11}	5.55398×10^{-10}	3.9189
	0.1	5.55398×10^{-10}	7.7488×10^{-09}	3.8024
Problem 4	0.005	1.2601×10^{-14}	2.37643×10^{-13}	4.2372
	0.01	2.37643×10^{-13}	3.66146×10^{-12}	3.9456
	0.02	3.66146×10^{-12}	5.73256×10^{-11}	3.9687

6. Conclusion

A new optimized one-step block hybrid scheme for solving linear and non-linear second-order initial value problems of ordinary differential equations is developed and applied directly without reducing to system of first-order ODEs. The two hybrid points v_1 and v_2 in the derived scheme were introduced in such a way that the points lie in the interval $0 < v_1 < v_2 < 1$. The approximate values of the hybrid points were obtained by optimizing the local truncation error of the derived scheme and its derivatives. The efficiency of the new scheme (OBSOH) is shown in Tables 2 - 12 as its being compared with some existing methods. The analysis of the method were shown to be consistent, zero-stable and convergent. OBSOH has proven effective in the direct integration of second-order initial value problems of ordinary differential equations.

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