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Differential and fuzzy differential sandwich theorems involving quantum calculus operators

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Abstract

The principle of subordination is useful in comparing two holomorphic functions when the range of one holomorphic function is a subset of the other and they comply at a single point. The subordination, when spoken in fuzzy set theory, becomes fuzzy subordination as the comparison between two holomorphic functions is made using the fuzzy membership function. In this article, differential and fuzzy differential Subordination, superordination, and sandwich theorems have been discussed for the classes defined by using q-derivative and symmetric q-derivative operators.

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1. Introduction

Let $\mathcal{H}(\mathbb{U})$ consist of all analytic functions in the open unit disc $\mathbb{U} := \{z : |z| < 1\}$ and for $n \in Z^+, a \in \mathbb{C}$ let $\mathcal{H}[a,n] = \{f(z) \ni f(z) = a + \sum_{t=n}^{\infty} a_t z^t\}$. Clearly $\mathcal{H}[a,n] \subset \mathcal{H}$ and $\mathcal{H}[0,1] = \mathcal{A}$, the class of all normalized analytic functions. According to the principle of subordination, if $f_1, f_2 \in \mathcal{H}$ with $f_1(0) = f_2(0)$, then f_1 is subordinate to f_2, if $\exists$ a Schwarz function $w_1 \ni w_1(0) = 0, |w_1(z)| < 1$ and $f_1(z) = f_2(w_1(z)), \forall z \in \mathbb{U}$ and we write $f_1 \prec f_2$.

Let $h_1 \in \mathcal{S}$, a subclass of $\mathcal{A}$ that comprises of all univalent functions and let $\phi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$. If p_1 and $\phi(p_1(z), zp'_1(z); z) \in \mathcal{S}$ and if p_1 satisfies the (first-order) differential superordination

$$h_1(z) \prec \phi(p_1(z), zp'_1(z); z \in \mathbb{U}) , \quad (1)$$

then p_1 is called an integral of the differential superordination in (1). An analytic function q_1 is called subordinant of the integrals of the differential superordination if $q_1 \prec p_1$. A subordinant $\tilde{q}_1$ with $q_1 \prec \tilde{q}_1$ for all subordinants of q_1 of (1) is said to be the best subordinant.

The principle of subordination, plays a vital part in the theory of univalent functions. Though Lindelof [1] was the first person to initiate this theory, the contributions of Littlewood [2] and Rogosinki [3] were remarkable in this field. Hallenbeck and Ruscheweyh [4] studied the subordinate functions and showed many interesting results for subordination in 1975. Ma and Minda [5] introduced different subclasses of star-like functions using the concept of subordination in 1992.

Differential subordination is the generalisation of differential inequalities with real variables. Goluzin [6] and Robinson [7] gave some important results involving differential implications of order one in 1935 and 1947 respectively. In 1970, Suf-

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fridge [8] extended Goluzin's result. The renowned theory of differential subordination was developed by Miller and Mocanu [9]. They introduced the concept of differential subordination and differential superordination [10].

The results of Miller and Mocanu motivated Bulboaca [11, 12] to study the first-order differential superordination for certain classes and superordination preserving integral operators. Thus, the topics subordination and superordination became the focal point for many researchers. The study of quantum calculus has been captivating the interest of researchers since 1707. Though Leonhard Euler and Carl Gustav Jacobi laid foundation to the study of q-calculus, it was a publication of Albert Einstein that made q- calculus popular because of its applications to quantum Mechanics after 1905. Unlike conventional calculus, q-calculus does not require continuity. In quantum calculus, the notion of limits is not considered.

The q-derivatives and q-integrals were first introduced by Jackson [13, 14]. The contributions of Srivastava to q-calculus for analytic functions and also to q-hypergeometric functions in function theory were remarkable. All his contributions were mentioned in his book [15]. Ismail *et al.* [16] have studied about starlike functions using q-calculus. Abelman *et al.* [17] have made use of fractional q- calculus operators on a class of non-Bazilevic functions to study the subordination conditions. A study on starlike functions in q-calculus and an extension of it to q-starlikeness for particular subclasses of starlike functions have been carried out by Agrawal and Sahoo [18]. For more details one can refer to Refs.[19–22].

On the other hand, Zadeh [23] introduced the concept of fuzzy in 1965. The notions of fuzzy subordination and fuzzy differential subordination were initially introduced by Oros and Oros [24–26]. This new initiative associates fuzzy set theory with geometric function theory. The duplet notion of fuzzy differential superordination was studied in Ref. [27].

Wanas and Majeed [28] have proved certain results on fuzzy differential subordination of analytic functions using generalized differential operator. Fuzzy differential subordinations involving integral operators can be found in the work of El-Deeb and Lupas [29]. Also El-Deeb and Oros [30] have studied fuzzy differential subordinations using linear operator. Lupas and Catas [31] have obtained the results on fuzzy subordination for analytic functions associated with the Atangana-Baleanu fractional integral of Bessel functions. For further study, see Refs.[32–34].

For results on a subclass of analytic and bi-univalent functions, using the symmetric q-derivative operator one can refer to Refs. [35–39]. The theory of symmetric q-calculus has its applications in various fields, specifically in fractional calculus and quantum physics. The importance of symmetric q-calculus in areas like quantum mechanics has been discussed in Ref. [40, 41].

Following are some of the q-analogues in complex (or real) analysis which we need for our work:

Definition 1. *[13] For $0 < q < 1$, the Jackson's q-derivative operator of a function is defined as*

$$(D_q f_1)(z) = \begin{cases} \frac{f_1(z)-f_1(qz)}{(1-q)z} & for \quad z \neq 0 \\ f_1'(0) & for \quad z = 0, \end{cases}$$

where $D_q f_1(0) = f_1'(z)$, if $f_1'(z)$ exists.

It is to be noted that $(D_q f_1(z)) = 1 + \sum_{t=2}^{\infty} [t]_q a_t z^{t-1}$; where $[t]_q = \frac{1-q^t}{1-q}$.

Definition 2. *[41] The symmetric q-derivative operator is defined by $(\tilde{D}_q f_1)(z) = \frac{f_1(qz)-f_1(q^{-1}z)}{(q-q^{-1}z)}$; if $z \neq 0$ and $(\tilde{D}_q f_1)(0) = f_1{}'(0)$ provided $f_1{}'(0)$ exists.*

It is to be observed that $(\tilde{D}_q f_1)(z) = 1 + \sum\limits_{t=2}^{\infty} [t]_q a_t z^{t-1}$, where $[t]_q = \frac{q^t - q^{-t}}{q-q^{-1}}$.

Definition 3. *[23] A pair (K, F_K), with $F_K : Y \to [0,1], K = \{x \in Y : 0 < F_K(x) \leq 1\} = supp(K, F_K)$ is named as a fuzzy set of Y. F_K is called the membership function of (K, F_K).*

Definition 4. *[24] Consider two functions $f_1, g_1 \in \mathcal{H}(\mathcal{D})$, $\mathcal{D} \subseteq \mathbb{C}$ and $z_0 \in \mathcal{D}$ being a fixed point. f_1 is said to be fuzzy subordinate to g_1 (i.e) $f_1 \prec_F g_1$ otherwise $f_1(z) \prec_F g_1(z)$ if the following conditions are satisfied.*

$$f_1(z_0) := g_1(z_0) \,, \tag{2}$$

$$F_{f_1(D)}(f_1(z)) \leq F_{g_1(D)}(g_1(z)), \quad (z \in \mathcal{D}) \,, \tag{3}$$

where

$$f_1(D) = supp(D, F_{f_1}(D) = \Big\{z \in \mathbb{C} : 0 < F_{f_1}(D)(z) \leq 1\Big\},$$
$$g_1(D) = supp(D, F_{g_1}(D) = \Big\{z \in \mathbb{C} : 0 < F_{g_1}(D)(z) \leq 1\Big\}.$$

Definition 5. *[25] If $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ and $h_1 \in \mathcal{S}$ satisfy $\psi(c, 0; 0) = h_1(0) = c$, then p_1 is a fuzzy integral of the fuzzy differential subordination if p_1 is an analytic function in $\mathbb{U}$ such that it satifies the (first-order) fuzzy differential subordination*

$$F_{\psi(\mathbb{C}^2\times\mathbb{U})}(\psi(p_1(z), zp_1'(z)) \leq F_{h_1(\mathbb{U})}(h_1 z), \forall\ p_1, \tag{4}$$

and $p_1(0) = c$. $q_1 \in \mathcal{S}$ is called a fuzzy dominant of the fuzzy integral of the fuzzy differential subordination or simply a fuzzy dominant, if $F_{p_1(\mathbb{U})} p_1(z) \leq F_{q_1(\mathbb{U})} q_1(z)$. A fuzzy dominant $\tilde{q}_1$ which satisfies $F_{\tilde{q}_1(\mathbb{U})}\tilde{q}_1(z) \leq F_{q_1(\mathbb{U})} q_1(z)$ for all fuzzy dominants q_1 of (4) is said to be the fuzzy best dominant of (4).

Definition 6. *[25] Consider $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ and h_1 an univalent function in $\mathbb{U}$. If p_1 is analytic in $\mathbb{U}$ and satisfies the fuzzy differential superordination*

$$F_{h_1(\mathbb{U})} h_1((z)) \leq F_{\psi(\mathbb{C}^2\times\mathbb{U})}(\psi(p_1(z), zp'{}_1(z)); z) \,. \tag{5}$$

(i.e) $h_1(z) \prec_{\mathcal{F}} \psi(p_1(z), zp'{}_1(z); z), z \in \mathbb{U}$, then p_1 is called a fuzzy integral of fuzzy differential superordination. $q_1 \in \mathcal{S}$ is called a fuzzy subordinant of the integrals of the fuzzy differential superordination, if

$$q_1 \prec \mathcal{F} p_1,$$

for all p_1 satisfying (5).

Definition 7. *[42] Denote by Q, the set of all functions f_1 that are analytic and injective on $\overline{\mathbb{U}} - E(f_1)$, where*
$E(f_1) = \{\zeta \in \partial U : \lim_{z\to\infty} f_1(z) = \infty\} \in f_1'(\zeta) \neq 0,\ \zeta \in \partial\mathbb{U} - E(f_1)$.

In this article, two classes of functions have been defined by using Quantum Calculus operators and Differential and Fuzzy Differential Sandwich theorems have been discussed.

2. Subordination results

In this section we have defined two classes $R(q)$ and $\tilde{R}(q)$ and have proved the subordination, superordination and sandwich theorems for the classes defined.

Definition 8. *The function $f_1 \in \mathcal{A}$ is said to be in the class $R(q)$, if*

$$\Re\left[1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\right] > 0,\ z \in \mathbb{U}.$$

Definition 9. *The function $f_1 \in \mathcal{A}$ is said to be in the class $\tilde{R}(q)$, if*

$$\Re\left[1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right] > 0,\ z \in \mathbb{U}.$$

Theorem 2.1. *Let $q_1(z)$ be analytic and univalent with $q_1(0) = 1$ such that $\frac{zq_1'(z)}{q_1(z)}$ is starlike univalent in $\mathbb{U}$ and $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3)$, $(\alpha_3 \neq 0)$. Let $q_1(z)$ satisfy*

$$Re\left[\frac{\alpha_2}{\alpha_3} q_1(z) + 1 + \frac{zq_1''(z)}{q_1'(z)} - \frac{zq_1'(z)}{q_1(z)}\right] > 0. \tag{6}$$

If $f_1 \in \mathcal{A}$ satisfies

$$\Delta^{(\alpha_j)_1^3}(f_1) = \Delta(f_1, \alpha_1, \alpha_2, \alpha_3) \prec \alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}, \tag{7}$$

where

$$\Delta^{(\alpha_j)_1^3}(f_1) = \alpha_1 + \alpha_2\left[1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\right] + \alpha_3\left[\frac{z^2(D_q f_1)''(z)}{(D_q f_1)(z) + z(D_q f_1)'(z)} + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z) + z(D_q f_1)'(z)} - \frac{[z(D_q f)'(z)]^2}{(D_q f_1)(z)[(D_q f_1)(z) + z(D_q f_1)'(z)]}\right], \tag{8}$$

then

$$1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)} \prec q_1(z) \tag{9}$$

and q_1 is the best dominant .

Proof

$$p_1(z) := 1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\ (z \in \mathbb{U}). \tag{10}$$

Here $p_1(0) = 1$ and p_1 is analytic in $\mathbb{U}$. Using Eq. (10)

$$\alpha_1 + \alpha_2\left[1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\right] + \alpha_3\left[\frac{z^2(D_q f_1)''(z)}{(D_q f_1)(z) + z(D_q f_1)'(z)} + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z) + z(D_q f_1)'(z)} - \frac{[z(D_q f)'(z)]^2}{(D_q f_1)(z)[(D_q f_1)(z) + z(D_q f_1)'(z)]}\right] = \alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}. \tag{11}$$

By using Eq. (11) in Eq. (7), we have

$$\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)} \prec \alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}. \tag{12}$$

By setting $\theta(\omega) := \alpha_1 + \alpha_2\omega$ and $\phi(\omega) := \frac{\alpha_3}{\omega}$, we noticed that $\theta(\omega)$ and $\phi(\omega)$ are analytic in $\mathbb{C} - \{0\}$ and $\phi(\omega) \neq 0$.
Let

$$Q_1(z) =: zq_1'(z)\phi(q_1(z)) = \alpha_3 \frac{zq_1'(z)}{q_1(z)}.$$

and

$$h_1(z) := \theta(q_1(z)) + Q_1(z) = \alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}.$$

Since $Q_1(z)$ is starlike univalent in $\mathbb{U}$ with

$$\text{Re}\left\{\frac{zh_1'(z)}{Q_1(z)}\right\} = Re\left[\frac{\alpha_2}{\alpha_3} q_1(z) + 1 + \frac{zq_1''(z)}{q_1'(z)} - \frac{zq_1'(z)}{q_1(z)}\right] > 0\ ,$$

the results follow by the application of [[42],Theorem 3.4h, p.132] and by the assumption of Theorem 2.1.
If $q_1(z) = \frac{1+Az}{1+Bz}; \Big(-1 \leq B < A \leq 1\Big)$ in the previous result, we get the following

Corollary 1. *Let $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3), (\alpha_3 \neq 0)$, q_1 be univalent with $q_1(0) = 1$, and (6) be true. If $f_1 \in \mathcal{A}$ and*

$$\Delta^{(\alpha_j)_1^3}(f_1) \prec \alpha_1 + \alpha_2 \frac{1+Az}{1+Bz} + \alpha_3 \frac{(A-B)z}{(1+Az)(1+Bz)}\ ,$$

then

$$1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)} \prec \frac{1+Az}{1+Bz} \text{ and } \frac{1+Az}{1+Bz} \text{ is the best dominant.}$$

Assigning $\alpha_1 = 1; \alpha_2 = 0$ in the previous result, we obtain the following corollary.

Corollary 2. *Let $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3)$ $(\alpha_3 \neq 0), q_1$ be univalent with $q_1(0) = 1$, and*
$Re\left[1 + \frac{zq_1''(z)}{q_1'(z)} - \frac{zq_1'(z)}{q_1(z)}\right] > 0$. *If $f_1 \in \mathcal{A}$ and*

$$1 + \alpha_3\left[\frac{z^2(D_q f_1)''(z)}{(D_q f_1)(z)+z(D_q f_1)'(z)} + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)+z(D_q f_1)'(z)} - \frac{[z(D_q f)'(z)]^2}{(D_q f_1)(z)[(D_q f_1)(z)+z(D_q f_1)'(z)]}\right] \prec 1 + \alpha_3 \frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)} \prec \frac{1+Az}{1+Bz} \text{ and the best dominant is } \frac{1+Az}{1+Bz}.$$

$q_1(z) = \left(\frac{1+z}{1-z}\right)^\delta (0 < \delta < 1)$ in Theorem 2.1, leads us to the next result.

Corollary 3. *Let q_1 be univalent with $q_1(0) = 1$, $\alpha_j \in \mathbb{C}$ $(j = 1,2,3)$ $(\alpha_3 \neq 0)$ and let (6) hold good. If $f_1 \in \mathcal{A}$ and*

$$\Delta^{(\alpha_j)_1^3}(f_1) \prec \alpha_1 + \alpha_2\left(\frac{1+z}{1-z}\right)^{\delta} + \alpha_3\left(\frac{2\delta z}{1-z^2}\right),$$

then

$$1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta} \text{ and } \left(\frac{1+z}{1-z}\right)^{\delta} \text{ is the best dominant.}$$

2.1. Superordination Results

Theorem 2.2. *Consider $\alpha_j \in \mathbb{C}$ $(j = 1,2,3), (\alpha_3 \neq 0)$ and $q_1 \in \mathcal{S}$ with $q_1(0) = 1 \ni \frac{zq_1'(z)}{q_1(z)}$ is starlike univalent and assume that*

$$Re\left\{\frac{\alpha_2}{\alpha_3} q_1(z)\right\} \geq 0. \tag{13}$$

If $f_1 \in \mathcal{A}$; $\frac{z(D_q f_1)'(z)}{(D_q f_1)(z)} \in \mathcal{H}[1,1] \cap Q, \Delta^{(\alpha_j)_1^3}(f_1)$ is univalent in $\mathbb{U}$ and

$$\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)} \prec \Delta^{(\alpha_j)_1^3}(f_1), \tag{14}$$

where, $\Delta^{(\alpha_j)_1^3}(f_1)$, is given by Eq. (8), then

$$q_1(z) \prec 1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)},$$

and q_1 represents the best subordinant.

Proof: Define

$$p_1(z) := 1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)} \quad (z \in \mathbb{U}). \tag{15}$$

By simple computation, we get

$$\Delta^{(\alpha_j)_1^3}(f_1) = \alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}.$$

Then

$$\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)} \prec \alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}.$$

By setting $v(\omega) = \alpha_1 + \alpha_2\omega$ and $\phi(\omega) = \frac{\alpha_3}{\omega}$, it is easily observed that $v(\omega)$ is analytic in $\mathbb{C}$. Also $\phi(\omega)$ is analytic in $\mathbb{C} - \{0\}$ and $\phi(\omega) \neq 0$.
Using q_1 is convex univalent function in $\mathbb{U}$ and $\alpha_3 \neq 0$, we have,

$$\text{Re}\left\{\frac{v' q_1(z)}{\phi(q_1(z))}\right\} = Re\left\{\frac{\alpha_2}{\alpha_3} q_1(z)\right\} > 0.$$

We get Theorem 2.2 by applying [10] (Theorem 8, p. 822).
The following result is obtained when we replace $q_1(z)$ by $\frac{1+Az}{1+Bz}$, $(-1 \leq B < A \leq 1)$ in Theorem 2.2 .

Corollary 4. *Consider $\alpha_j \in \mathbb{C}$ $(j = 1,2,3), (\alpha_3 \neq 0)$, q_1 a univalent function with $q_1(0) = 1$, and (13) holds true. If $f_1 \in \mathcal{A}$ and*

$$\alpha_1 + \alpha_2 \frac{1+Az}{1+Bz} + \alpha_3 \frac{(A-B)z}{(1+Az)(1+Bz)} \prec \Delta^{(\alpha_j)_1^3}(f_1),$$

then

$$\frac{1+Az}{1+Bz} \prec 1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}$$

and the best subordinant is $\frac{1+Az}{1+Bz}$.

2.2. Sandwich Theorems

Theorem 2.3. *Consider two univalent functions q_2 and q_3 in $\mathbb{U}$ and $\alpha_j \in \mathbb{C}$ $(j = 1,2,3$, $(\alpha_3 \neq 0))$. Let q_2 satisfy (6), q_3 satisfy (13), $f_1 \in \mathcal{A}$, $\left(1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\right) \in \mathcal{H}[1,1] \cap Q$ and $\Delta^{(\alpha_j)_1^3}(f_1)$ by (8) be univalent in $\mathbb{U}$ satisfying*

$$\alpha_1 + \alpha_2 q_2(z) + \alpha_3 \frac{zq_2'(z)}{q_2(z)} \prec \Delta^{(\alpha_j)_1^3}(f_1) \prec \alpha_1 + \alpha_2 q_3(z) + \alpha_3 \frac{zq_3'(z)}{q_3(z)},$$

then

$$q_2(z) \prec 1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)} \prec q_3(z)$$

and q_2 and q_3 seem to be the best subordinant and dominant.

3. Fuzzy Differential Subordination Results

Theorem 3.1. *Let $\alpha_j \in \mathbb{C}(j = 1,2,3), (\alpha_3 \neq 0)$ and $q_1(z)$ be analytic and univalent with $q_1(0) = 1$. Assume that $\frac{zq_1'(z)}{q_1(z)}$ is starlike univalent in $\mathbb{U}$. Consider*

$$Re\left[\frac{\alpha_2}{\alpha_3} q_1(z) + 1 + \frac{zq_1''(z)}{q_1'(z)} - \frac{zq_1'(z)}{q_1(z)}\right] > 0 \quad (z \in \mathbb{U}), \tag{16}$$

and
$\Delta^{(\alpha_j)_1^3}$ is defined by (8). If q_1 satisfies the following fuzzy subordination

$$F_{\Delta^{(\alpha_j)_1^3}} \leq F_{q_1(U)}\left(\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}\right) \tag{17}$$

for $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}, \alpha_3 \neq 0$, then

$$F_{\left(1+\frac{z(D_q f_1)'}{(D_q f_1)}\right)(\mathbb{U})}\left(1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\right) \leq F_{q_{1(\mathbb{U})}}\, q_1(z), \quad z \in \mathbb{U} \tag{18}$$

and q_1 is the fuzzy best dominant .

Proof: Let

$$p_1(z) = 1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}, \quad z \neq 0. \tag{19}$$

After computation, we have

$$\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)} = \Delta^{(\alpha_j)_1^3}(f_1). \tag{20}$$

where $\Delta^{(\alpha_j)_1^3}(f_1)$ is given by (8). By using the above inequality into (17) , we have

$$F_{p_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}\right) \leq F_{q_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}\right).$$

Considering $\theta(\omega) := \alpha_1 + \alpha_2\omega$; $\phi(\omega) := \frac{\alpha_3}{\omega}$, it is noticed that $\theta(\omega)$ and $\phi(\omega)$ in $\mathbb{C} - \{0\}$ are analytic & $\phi(\omega)$ not equal to zero. Consider

$$Q_1(z) =: zq_1'(z)\phi(q_1(z)) = \alpha_3 \frac{zq_1'(z)}{q_1(z)}$$

and

$$h_1(z) := \theta\Big(q_1(z)\Big) + Q_1(z) = \alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}.$$

Since Q_1 is starlike univalent defined in $\mathbb{U}$ with

$$Re\left[\frac{zh_1'(z)}{q_1(z)}\right] = Re\left[\frac{\alpha_2}{\alpha_3}q_1(z) + 1 + \frac{zq_1'(z)}{q_1'(z)} - \frac{zq_1''(z)}{q_1(z)}\right] > 0.(21)$$

By using proposition 2.1 of Ref. [43], we get $F_{p_1(\mathbb{U})}p_1(z) \leq F_{q_1(\mathbb{U})}q_1(z)$ and using (19) we obtain

$$F_{\left(1+\frac{z(D_qf_1)'}{(D_qf_1)}\right)(\mathbb{U})}\left(1 + \frac{z(D_qf_1)'(z)}{(D_qf_1)(z)}\right) \leq F_{q_1(\mathbb{U})}q_1(z),$$

and q_1 indicates the fuzzy best dominant.

Corollary 5. *Consider $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3)$ $(\alpha_3 \neq 0)$. Assume that (16) holds. If $F_{\Delta^{(\alpha_j)_1^3}} \leq F_{q_1(\mathbb{U})}\Big(\alpha_1 + \alpha_2(\frac{1+Az}{1+Bz}) + \alpha_3 \frac{(A-B)z}{(1+Az)(1+Bz)}\Big)$, then*

$$F_{\left(1+\frac{z(D_qf_1)'}{(D_qf_1)}\right)(\mathbb{U})}\left(1 + \frac{z(D_qf_1)'(z)}{(D_qf_1)(z)}\right) \leq F_{q_{1(\mathbb{U})}} \frac{1+Az}{1+Bz}; \quad z \in \mathbb{U}$$

and $\frac{1+Az}{1+Bz}$ represents the best fuzzy dominant.

Proof: In Theorem 3.1, assign $q_1(z) = \frac{1+Az}{1+Bz}; -1 \leq B < A \leq 1$ to obtain the above corollary.

Theorem 3.2. *Let $q_1(z)$ be analytic and univalent in $\mathbb{U}$ such that $q_1(0) \neq 0$ and $\frac{zq_1'(z)}{q_1(z)}$ defines a starlike univalent in $\mathbb{U}$. Assume that*

$$Re\left(\frac{\alpha_2}{\alpha_3}q_1(z)\right) > 0. \tag{22}$$

Let $\Delta^{(\alpha_j)_1^3}(f_1) \in \mathbb{U}$ be univalent and $\left(1 + \frac{z(D_qf_1)'(z)}{(D_qf_1)(z)}\right) \in \mathcal{H}[1, 1] \cap Q$ where $\Delta^{(\alpha_j)_1^3}(f_1)$ has been introduced in (8), then

$$F_{q_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}\right) \leq F_{\left(1+\frac{z(D_qf_1)'}{(D_qf_1)}\right)(\mathbb{U})}\left(1 + \frac{z(D_qf_1)'(z)}{(D_qf_1)(z)}\right)$$

implies

$$F_{q_1(\mathbb{U})}q_1(z) \leq F_{\left(1+\frac{zD_qf_1)'}{(D_qf_1)}\right)(\mathbb{U})}\left(1 + \frac{zD_qf_1)'(z)}{(D_qf_1)(z)}\right)$$

and q_1 represents the best fuzzy subordinant.

Proof: $p_1(z) = 1 + \frac{z(D_qf_1)'(z)}{(D_qf_1)(z)}; z \in \mathbb{U}$. Define $v(\omega) = \alpha_1 + \alpha_2\omega$ and $\phi(\omega) = \frac{\alpha_3}{\omega}$. It is observed that $v(\omega)$ is analytic in $\mathbb{C}$, $\phi(\omega)$ is analytic in $\mathbb{C} - \{0\}$ and $\phi(\omega) \neq 0$.
Under these conditions $\frac{v'(q_1(z)}{\phi(q_1(z))} = \frac{\alpha_2}{\alpha_3}q_1(z) \implies \text{Re}\left(\frac{v'(q_1(z)}{\phi(q_1(z))}\right) = Re\left\{\frac{\alpha_2}{\alpha_2}q_1(z)\right\} > 0$
and we obtain,

$$F_{q_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}\right) \leq F_{p_1(U)}\left(\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}\right).$$

Applying Theorem 2.15 of Ref. [27]], the result is obtained.

Theorem 3.3. *Let $\alpha_j \in \mathbb{C}(j = 1, 2, 3); (\alpha_3 \neq 0), q_2$ and q_3 be univalent in $\mathbb{U}$, q_2 satisfy (16) and q_3 satisfy (22) . Let $f_1 \in \mathcal{A}$, $\left(1 + \frac{z(D_qf_1)'(z)}{(D_qf_1)(z)}\right) \in \mathcal{H}[1, 1] \cap Q$ and $\Delta^{(\alpha_j)_1^3}(f_1)$ given by (8) be univalent in $\mathbb{U}$ satisfying*

$$F_{q_2(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_2(z) + \alpha_3 \frac{zq_2'(z)}{q_2(z)}\right) \leq F_{\Delta^{(\alpha_j)_1^3}(f_1)} \leq F_{q_3(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_3(z) + \alpha_3 \frac{zq_3'(z)}{q_3(z)}\right).$$

Then

$$F_{q_2(\mathbb{U})}\, q_2(z) \leq F_{\left(1+\frac{zD_qf_1)'}{(D_qf_1)}\right)(\mathbb{U})}\left(1 + \frac{zD_qf_1)'(z)}{(D_qf_1)(z)}\right) \leq F_{q_3(\mathbb{U})}\, q_3(z),$$

and q_2 and q_3 are fuzzy best subordiant and best fuzzy dominant.

To get the following result choose $q_2(z) = \frac{1+A_1z}{1+B_1z}$ and $q_3(z) = \frac{1+A_2z}{1+B_2z}; -1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$ in Theorem 3.3.

Corollary 6. *Let $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3)(\alpha_j \neq 0)$. Assume that (16) and (22) hold. If $\left(1 + \frac{z(D_qf_1)'(z)}{(D_qf_1)(z)}\right) \in \mathcal{H}[1, 1] \cap Q$ and $F_{q_1(\mathbb{U})}\Big(\alpha_1 + \alpha_2(\frac{1+A_1z}{1+B_1z}) + \alpha_3 \frac{(A_1-B_1)z}{(1+A_1z)(1+B_1z)}\Big) \leq F_{\Delta^{(\alpha_j)_1^3}} \leq F_{q_2(\mathbb{U})}\Big(\alpha_1 + \alpha_2(\frac{1+A_2z}{1+B_2z}) + \alpha_3 \frac{(A_2-B_2)z}{(1+A_2z)(1+B_2z)}\Big)$, where $\Delta^{(\alpha_j)_1^3}(f_1)$ is introduced in (8), then*

$$F_{q_1(\mathbb{U})} \frac{1+A_1z}{1+B_1z} \leq F_{\left(1+\frac{zD_qf_1)'}{(D_qf_1)}\right)(\mathbb{U})}\left(1 + \frac{zD_qf_1)'(z)}{(D_qf_1)(z)}\right) \leq F_{q_2(\mathbb{U})} \frac{1+A_2z}{1+B_2z},$$

and $\frac{1+A_1z}{1+B_1z}$ and $\frac{1+A_2z}{1+B_2z}$ are the best fuzzy subordinant and best fuzzy dominant respectively.

4. Subordination results for Symmetric q-derivative operator.

Theorem 4.1. *Let $q_1(z)$ be analytic and univalent with $q_1(0) = 1$ such that $\frac{zq_1'(z)}{q_1(z)}$ is starlike univalent in $\mathbb{U}$ and $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3)$ $(\alpha_3 \neq 0)$. Let $q_1(z)$ satisfy*

$$Re\left[\frac{\alpha_2}{\alpha_3}q_1(z) + 1 + \frac{zq_1''(z)}{q_1'(z)} - \frac{zq_1'(z)}{q_1(z)}\right] > 0, \quad (z \in \mathbb{U}).(23)$$

If $f_1 \in \mathcal{A}$ satisfies

$$\Delta^{(\alpha_j)_1^3}(f_1) = \Delta(f_1, \alpha_1, \alpha_2, \alpha_3) \prec \alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}, \tag{24}$$

where $\Delta^{(\alpha_j)_1^3}(f_1)$ is defined by (8) then

$$1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)} \prec q_1(z)$$

and q_1 is the best dominant .

Proof: Let

$$p_1(z) := 1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}, \ (z \in \mathbb{U}). \tag{25}$$

Then the function p_1 is analytic in $\mathbb{U}$ and $p_1(0) = 1$. By using Eq. (25), we get

$$\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)} = \alpha_1 + \alpha_2 \left[1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right] +$$

$$\alpha_3 \Bigg[\frac{z^2(\tilde{D}_q f_1)''(z)}{(\tilde{D}_q f_1)(z) + z(\tilde{D}_q f_1)'(z)} + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z) + z(\tilde{D}_q f_1)'(z)} -$$

$$\frac{[z(\tilde{D}_q f)'(z)]^2}{(\tilde{D}_q f_1)(z)[(\tilde{D}_q f_1)(z) + z(\tilde{D}_q f_1)'(z)]}\Bigg]. \tag{26}$$

By using (26) in (24), we have,

$$\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)} = \alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}. \tag{27}$$

Taking $\theta(\omega) = \alpha_1 + \alpha_2\omega$ and $\phi(\omega) := \frac{\alpha_3}{\omega}$, we note that $\theta(\omega)$ and $\phi(\omega)$ are analytic in $\mathbb{C} - \{0\}$ and $\phi(\omega) \neq 0$. We also notice that

$$Q_1(z) =: zq_1'(z)\phi(q_1(z)) = \alpha_3 \frac{zq_1'(z)}{q_1(z)}.$$

and

$$h_1(z) := \theta(q_1(z)) + Q_1(z) = \alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}.$$

Clearly, $Q_1(z)$ is univalent starlike in $\mathbb{U}$ and

$$\text{Re}\left(\frac{zh_1'(z)}{Q_1(z)}\right) = Re\left[\frac{\alpha_2}{\alpha_3} q_1(z) + 1 + \frac{zq_1''(z)}{q_1'(z)} - \frac{zq_1'(z)}{q_1(z)}\right] > 0\ .$$

By the assertion of the Theorem 4.1, the result follows by the application of [Ref. [42] Theorem 3.4h, p. 132]. By taking $\alpha_1 = 0$ and $\alpha_2 = 1$ in above result , we obtain the following corollary.

Theorem 4.2. *Let $\alpha_j \in \mathbb{C}(j = 1, 2, 3), (\alpha_3 \neq 0)$, q_1 be univalent with $q_1(0) = 1$ such that $\frac{zq_1'(z)}{q_1(z)}$ is starlike univalent and assume that*

$$Re\left(1 + \frac{\alpha_2}{\alpha_3} q_1(z)\right) \geq 0. \tag{28}$$

If $f_1 \in \mathcal{A}; \left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \in \mathcal{H}[1, 1] \cap Q$, $\Delta^{(\alpha_j)_1^3}(f_1)$ be univalent in $\mathbb{U}$ and

$$\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)} \prec \Delta^{(\alpha_j)_1^3}(f_1)$$

where $\Delta^{(\alpha_j)_1^3}(f_1)$ is defined in (8), then

$$q_1(z) \prec \left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right)$$

and q_1 said to be the best subordinant.

Proof: Let

$$p_1(z) = \left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \quad z \in \mathbb{U}; \quad z \neq 0, \tag{29}$$

$v(\omega) = \alpha_1 + \alpha_2\omega$ be analytic in $\mathbb{C}$ and $\phi(\omega) = \frac{\alpha_3}{\omega}$ be analytic in $\mathbb{C} - \{0\}$ and $\phi(\omega) \neq 0$. Under these conditions $\frac{v'(q_1(z)}{\phi(q_1(z))} = \frac{\alpha_2}{\alpha_3} q_1(z)$
$\implies \text{Re}\left\{\frac{v' q_1(z)}{\phi(q_1(z))}\right\} = Re\left\{\frac{\alpha_2}{\alpha_2} q_1(z)\right\} > 0,$
and we obtain,

$$\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z}{q_1(z)}) \leq \alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}.$$

Applying [Ref. [10], Theorem 8, p. 822], we obtain, $q_1(z) \prec p_1(z)$ and q_1 is the best subordinant. Replacing $q_1(z)$ by $= \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 4.2, the following result is obtained:

Corollary 7. *Consider $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3), (\alpha_3 \neq 0)$ and let q_1 be univalent with $q_1(0) = 1$, and (28) be true. If $f_1 \in \mathcal{A}$ and*

$$\alpha_1 + \alpha_2 \frac{1+Az}{1+Bz} + \alpha_3 \frac{(A-B)z}{(1+Az)(1+Bz)} \prec \Delta^{(\alpha_j)_1^3}(f_1),$$

then

$$\frac{1+Az}{1+Bz} \prec 1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}$$

and $\frac{1+Az}{1+Bz}$ represents the best subordinant.

Sandwich theorem is obtained by using Theorems 4.1 and 4.2.

Theorem 4.3. *Let q_2 and q_3 be univalent in $\mathbb{U}$, $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3), (\alpha_j \neq 0), q_2$ satisfy (22) and q_3 satisfy (28). Let $f_1 \in \mathcal{A}$, $\left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \in \mathcal{H}[1, 1] \cap Q$ and $\Delta^{(\alpha_j)_1^3}(f_1)$ defined by (8) be univalent in $\mathbb{U}$ satisfying*

$$\alpha_1 + \alpha_2 q_2(z) + \alpha_3 \frac{zq_2'(z)}{q_2(z)} \prec \Delta^{(\alpha_j)_1^3}(f_1) \prec \alpha_1 + \alpha_2 q_3(z) + \alpha_3 \frac{zq_3'(z)}{q_3(z)},$$

then

$$q_2(z) \prec 1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)} \prec q_3(z),$$

and the best subordinant and best dominant are q_2 and q_3.

5. Fuzzy Differential Subordination for Symmetric q-derivative operator

Theorem 5.1. *Let $q_1(z)$ be an analytic and univalent function with $q_1(z)$ such that $\frac{zq_1'(z)}{q_1(z)} \in \mathbb{U}$ is starlike univalent and $\alpha_j \in \mathbb{C}(j = 1, 2, 3), (\alpha_3 \neq 0)$. Let*

$$Re\left[\frac{\alpha_2}{\alpha_3} q_1(z) + 1 + \frac{zq_1''(z)}{q_1'(z)} - \frac{zq_1'(z)}{q_1(z)}\right] > 0, \quad (z \in \mathbb{U}), \tag{30}$$

and $\Delta^{(\alpha_j)_1^3}(f_1)$ be defined as in Eq. (8). If q_1 satisfies the following fuzzy subordination

$$F_{\Delta^{(\alpha_j)_1^3}(f_1)} \leq F_{q_1(U)}\left(\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}\right), \tag{31}$$

where $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}$ and $(\alpha_3 \neq 0)$, then

$$F_{\left(1+\frac{z(\tilde{D}_q f_1)'}{(\tilde{D}_q f_1)}(U)\right)}\left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \leq F_{q_{1(\mathbb{U})}}\, q_1(z) \quad z \in \mathbb{U}, \tag{32}$$

and q_1 represents the best fuzzy dominant.

Proof: Let

$$p_1(z) = \left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \quad z \in \mathbb{U}; \quad z \neq 0. \tag{33}$$

Here the function p_1 is analytic with $p_1(0) = 1$. Simple calculation leads to

$$\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)} = \Delta^{(\alpha_j)_1^3}(f_1), \tag{34}$$

where $\Delta^{(\alpha_j)_1^3}(f_1)$ is given by (8). By using above inequality in (31) we have

$$F_{p_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}\right) \leq F_{q_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}\right).$$

By considering $\theta(\omega) := \alpha_1 + \alpha_2\omega$ and $\phi(\omega) := \frac{\alpha_3}{\omega}$, it is evident that $\theta(\omega)$, $\phi(\omega) \in \mathbb{C} - \{0\}$ are analytic and $\phi(\omega) \neq 0$.
Consider

$$Q_1(z) =: zq_1'(z)\phi(q_1(z)) = \alpha_3 \frac{zq_1'(z)}{q_1(z)}.$$

We deduce that Q_1 is starlike univalent in $\mathbb{U}$ and

$$Re\left\{\frac{zh_1'(z)}{Q_1(z)}\right\} = Re\left[\frac{\alpha_2}{\alpha_3} q_1(z) + 1 + \frac{zq_1'(z)}{q_1'(z)} - \frac{zq_1''(z)}{q_1(z)}\right] > 0. \tag{35}$$

By [proposition 2.1, Ref. [43]] $F_{p_1(\mathbb{U})}p_1(z) \leq F_{q_1(\mathbb{U})}q_1(z)$ and using (35) we obtain

$$F_{\left(1+\frac{z(\tilde{D}_q f_1)'}{(\tilde{D}_q f_1)}\right)(\mathbb{U})}\left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \leq F_{q_1(\mathbb{U})}q_1(z); \quad z \in \mathbb{U}$$

and q_1 is the best dominant .

Corollary 8. *Let $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3), (\alpha_3 \neq 0)$. Assume that (30) holds if $F_{\Delta^{(\alpha_j)_1^3}(f_1)} \leq F_{q_1(\mathbb{U})}\left(\alpha_1 + \alpha_2(\frac{1+Az}{1+Bz}) + \alpha_3 \frac{(A-B)z}{(1+Az)(1+Bz)}\right)$ then*

$$F_{\left(1+\frac{z(\tilde{D}_q f_1)'}{(\tilde{D}_q f_1)}(\mathbb{U})\right)}\left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \leq F_{q_1(\mathbb{U})} \frac{1+Az}{1+Bz}; \quad z \in \mathbb{U}$$

and $\frac{1+Az}{1+Bz}$ is the fuzzy best dominant.

Theorem 5.2. *Let $\frac{zq_1'(z)}{q_1(z)}$ be starlike univalent in $\mathbb{U}$ and $q_1(z)$ be analytic and univalent in $\mathbb{U}$. Assume that*

$$Re\left(\frac{\alpha_2}{\alpha_3} q_1(z)\right) > 0. \tag{36}$$

Let $\Delta^{(\alpha_j)_1^3}(f_1)$ be univalent in $\mathbb{U}$ and $1 + \frac{z((\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)} \in \mathcal{H}[q(0), 1] \cap Q$ where $\Delta^{(\alpha_j)_1^3}(f_1)$ is given by (8). Then

$$F_{q_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}\right) \leq F_{\Delta^{(\alpha_j)_1^3}(f_1)} \leq F_{\left(1+\frac{z(\tilde{D}_q f_1)'}{(\tilde{D}_q f_1)}\right)(\mathbb{U})}\left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right)$$

$$\implies F_{q_1(\mathbb{U})}q_1(z) \leq F_{\left(1+\frac{z(\tilde{D}_q f_1)'}{(\tilde{D}_q f_1)}(\mathbb{U})\right)}\left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right)$$

and q_1 is said to be the fuzzy best subordinant.

Proof: $p_1(z) = 1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)} \quad (z \in \mathbb{U})$.
Taking $v(\omega) : \alpha_1 + \alpha_2\omega$ and $\phi(\omega) = \frac{\alpha_3}{\omega}$. Also knowing that, $v(\omega)$ is analytic in $\mathbb{C}$, $\phi(\omega)$ is analytic in $\mathbb{C} - \{0\}$, $\phi(q_1(z)) = \frac{\alpha_2}{\alpha_3} q_1(z)$ and $\phi(\omega) \neq 0$, we have $\frac{v'(q_1(z)}{\phi(q_1(z))} = \frac{\alpha_2}{\alpha_3} q_1(z) \implies \text{Re}\left(\frac{v'(q_1(z)}{\phi(q_1(z))}\right) = Re\left\{\frac{\alpha_2}{\alpha_3} q_1(z)\right\} > 0$. We obtain

$$F_{q_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}\right) \leq F_{p_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}\right).$$

Applying [Theorem 2.15 of Ref. [27]], we get $F_{q_1(\mathbb{U})}q_1(z) \leq F_{p_1(\mathbb{U})}p_1(z)$ and q_1 is the fuzzy best subordinant.

Theorem 5.3. *Consider the univalent functions q_2 and q_3 in $\mathbb{U}$, $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3), (\alpha_j \neq 0)$ with q_1 satisfying (30) and q_2 satisfying (36) . Let $f_1 \in \mathcal{A}$, $\left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \in \mathcal{H}[1, 1] \cap Q$ and $\Delta^{(\alpha_j)_1^3}(f_1)$ given by (8) be univalent in $\mathbb{U}$ satisfying*

$$F_{q_2(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_2(z) + \alpha_3 \frac{zq_2'(z)}{q_2(z)}\right) \leq F_{\Delta}^{(\alpha_j)_1^3}(f_1) \leq F_{q_3(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_3(z) + \alpha_3 \frac{zq_3'(z)}{q_3(z)}\right).$$

Then

$$F_{q_2(\mathbb{U})}\, q_2(z) \leq F_{\left(1+\frac{(z\tilde{D}_q f_1)'}{(\tilde{D}_q f_1)}\right)(\mathbb{U})}\left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \leq F_{q_3(\mathbb{U})}\, q_3(z).$$

Here q_2 and q_3 represent fuzzy best subordinant and fuzzy best dominant.

6. Conclusion

In this article, we have defined two classes using quantum calculus operators. We have stated and proved the theorems on differential subordination and superordination. Sandwich theorems have also been stated and proved. These results have been extended to fuzzy set theory as fuzzy differential subordination, superordination, and sandwich theorems.

References

[1] E. L. Lindelöf, "Memoire sur certaines inegalites dans la theorie des fonctions monogenes et sur quelques proprietes nouvelles de ces fonctions dans le voisinage d'un point singulier essentiel", Acta Soc. Sci. Fennicae **35** (1909) 3. http://www.zentralblatt-math.org/zmath/en/advanced/?q=an:40.0439.02.

[2] J. E. Littlewood, "On inequalities in the theory of functions", Proceedings of the London Mathematical Society **2** (1925) 481.

[3] W. Rogosinski, "On subordinate functions", Mathematical Proceedings of the Cambridge Philosophical Society **35** (1939) 1. https://doi.org/10.1017/S0305004100020703.

[4] D. J. Hallenbeck & S. Ruscheweyh, "Subordination by convex functions", Proceedings of the American Mathematical Society **52** (1975) 191. https://doi.org/10.2307/2040127.

[5] W. Ma, *A unified treatment of some special classes of univalent functions*, Proceedings of the Conference on Complex Analysis, International Press Inc., 1992.

[6] G. M. Goluzin, "On the majorization principle in function theory", Dokl. Akad. Nauk. SSSR **42** (1935) 935.

[7] R. M. Robinson," Univalent majorants", Transactions of the American Mathematical Society **61** (1947) 1. https://doi.org/doi:10.2307/1990287

[8] T. J. Suffridge, "Some remarks on convex maps of the unit disk", Duke Mathematical Journal **37** (1970) 775. https://doi.org/10.1215/S0012-7094-70-03792-0

[9] S. S. Miller & P. T. Mocanu, "Differential subordinations and univalent functions", Michigan Mathematical Journal **28** (1981) 157. https://doi.org/10.1307/mmj/1029002507.

[10] S. S Miller & P. T Mocanu, "Subordinants of differential superordinations", Complex variables **48** (2003) 815. https://doi.org/10.1080/02781070310001599322.

[11] T. Bulboacă, "Classes of first-order differential superordinations", Demonstratio Mathematica **35** (2002) 287. https://doi.org/10.1515/dema-2002-0209.

[12] T. Bulboaca, "A class of superordination-preserving integral operators", Indagationes Mathematicae **13** (2002) 301. https://doi.org/10.1016/S0019-3577(02)80013-1.

[13] F. H. Jackson, "on q-functions and a certain difference operator", Earth and Environmental Science Transactions of the Royal Society of Edinburgh **46** (1909) 253. https://doi.org/10.1017/S0080456800002751.

[14] F. H. Jackson, "On q-definite integrals", Quart. J. Pure Appl. Math **41**.(1910) 193.

[15] H. M. Srivastava, *Univalent functions, fractional calculus, and associated generalized hypergeometric functions*. In *Univalent functions, fractional calculus, and their applications*; H. M. Srivastava, S. Owa (Eds.); John Wiley & Sons: New York, USA, 1989.

[16] M. E. H. Ismail, E. Merkes & D. Styer, "A generalization of starlike functions", Complex Variables, Theory and Application: An International Journal **14** (1990) 77. https://doi.org/10.1080/17476939008814407.

[17] S. Abelman, K. A. Selvakumaran, M. M. Rashidi & S. D. Purohit "Subordination conditions for a class of non-Bazilevic type defined by using fractional q-calculus operators", Facta Univ. Ser. Math. Inform **32**(2017) 255. https://doi.org/10.22190/FUMI1702255A.

[18] S. Agrawal & S. K. Sahoo, "A generalization of starlike functions of order alpha", Hokkaido Mathematical Journal **46** (2017) 15. https://doi.org/10.14492/hokmj/1498788094.

[19] M. Govindaraj & S. Sivasubramanian, "On a class of analytic functions related to conic domains involving q-calculus", Analysis Mathematica **43** (2017) 475. https://doi.org/10.1007/s10476-017-0206-5.

[20] H. M. Srivastava, B. Ahmad, N. Khan, M.G. Khan, W. K. Mashwani & B. Khan, "A subclass of multivalent Janowski type q-starlike functions and its consequences", Symmetry **13** (2021) 1275. https://doi.org/10.3390/sym13071275.

[21] Q. Z. Ahmad, N. Khan, M. Raza, M. Tahir & B. Khan, "Certain q-difference operators and their applications to the subclass of meromorphic q-starlike functions", Filomat **33** (2019) 3385. https://doi.org/10.2298/FIL1911385A.

[22] B. Ahmad, W. K. Mashwani, S. Araci, S. Mustafa, M. G. Khan & B.Khan, "A subclass of meromorphic Janowski-type multivalent q-starlike functions involving a q-differential operator", Advances in Continuous and Discrete Models **2022** (2022) 1. https://doi.org/10.1186/s13662-022-03683-y.

[23] L.A. Zadeh, "Fuzzy sets", Information and control **8** (1965) 338. http://dx.doi.org/10.1016/S0019-9958(65)90241-X.

[24] G. I. Oros & G. Oros, "The notion of subordination in fuzzy sets theory", General Mathematics **19** (2011) 97. https://generalmathematics.ro/wp-content/uploads/2020/03/12_Oros.pdf.

[25] G. I. Oros & G. Oros, "Fuzzy differential subordination", Acta Universitatis Apulensis **30** (2012) 55. http://emis.icm.edu.pl/journals/AUA/acta30/Paper6-Acta30-2012.pdf.

[26] G. I. Oros & G. Oros, "Dominants and best dominants in fuzzy differential subordinations", Stud. Univ. Babes-Bolyai Math **57** (2012) 239. https://www.cs.ubbcluj.ro/~studia-m/2012-2/13-oros-final.pdf.

[27] W. G. Atshan & K. O. Hussain, "Fuzzy Differential Superordination", Theory and Applications of Mathematics and Computer Science **7** (2017) 27. https://www.researchgate.net/publication/311518845_Fuzzy_differential_superordination.

[28] A. K. Wanas & A. H. Majeed, "Fuzzy differential subordination properties of analytic functions involving generalized differential operator", Sci. Int. (Lahore) **30** (2018) 297.

[29] S. M. El-Deeb & A. A. Lupas, "A. Fuzzy differential subordinations associated with an integral operator", An. Univ. Oradea Fasc. Mat **27** (2020) 133.

[30] S.M. El-Deeb & G.I Oros, "Fuzzy differential subordinations connected with the linear operator", Mathematica Bohemica **146** (2021) 397. https://doi.org/10.21136/MB.2020.0159-19.

[31] A.Alb Lupas & A. Catas, "Fuzzy differential subordination of the Atangana-Baleanu fractional integral", Symmetry **13** (2021) 1929. https://doi.org/10.3390/sym13101929.

[32] G. I. Oros, "New fuzzy differential subordinations", Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics **70** (2021) 229. https://doi.org/10.31801/cfsuasmas.784080.

[33] A. Alb Lupas, "Applications of the fractional calculus in fuzzy differential subordinations and superordinations", Mathematics **9** (2021) 2601. https://doi.org/10.3390/math9202601.

[34] A. Kareem, "Fuzzy differential subordinations for analytic functions involving Wanas operator and some applications in fractional calculus", Ikonion Journal of Mathematics **2** (2020) 1. https://dergipark.org.tr/en/pub/ikjm/issue/56257/653379.

[35] B. Khan, Z. G. Liu, T.G. Shaba, S .Araci, N. Khan & M.G Khan, "Applications of q-derivative operator to the subclass of bi-univalent functions involving-Chebyshev polynomials", Journal of Mathematics **2022** (2022) 1. https://doi.org/10.1155/2022/8162182.

[36] S. Altinkaya, S. Kanas & S. Yal, "Subclass of k-uniformly starlike functions defined by symmetric q-derivative operator", Ukrains kyi Matematychnyi Zhurnal **70** (2018) 1499. https://doi.org/10.1007/s11253-019-01602-1.

[37] S. Khan, S. Hussain, M. Naeem, M. Darus & A. Rasheed, "A subclass of q-starlike functions defined by using a symmetric q-derivative operator and related with generalized symmetric conic domains", Mathematics **9** (2021) 917.https://doi.org/10.3390/math9090917.

[38] V. O. Atabo & S. O. Adee, "A new special 15-step block method for solving general fourth order ordinary differential equations", Journal of the Nigerian society of Physical Sciences **3** (2021) 308. https://doi.org/10.46481/jnsps.2021.337.

[39] A. Thirumalai, K. Muthunagai & Ritu Agarwal "Pre-functions and Extended pre-functions of Complex Variables", Journal of the Nigerian Society of Physical Sciences **5** (2023) 1427.https://doi.org/10.46481/jnsps.2023.1427.

[40] A.M.C.B. da Cruz & N. Martins, "The q-symmetric variational calculus", Computers and Mathematics with Applications **64** 64 (2012) 2241. https://doi.org/10.1016/j.camwa.2012.01.076.

[41] A. Lavagno, "Basic-deformed quantum mechanics", Reports on Mathematical Physics **64** (2009) 79. https://doi.org/10.1016/S0034-4877(09)90021-0

[42] S. S. Miller & P. T. Mocanu, *Differenatial subordinations theory and applications*, Marcel Dekker Inc., New York, 2000.

[43] E. A. Haydar, "On fuzzy differential subordination", Mathematica Moravica **19** (2015) 123. https://scindeks.ceon.rs/Article.aspx?artid=1450-59321501123H.