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# Differential and fuzzy differential sandwich theorems involving quantum calculus operators

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## Abstract

The principle of subordination is useful in comparing two holomorphic functions when the range of one holomorphic function is a subset of the other and they comply at a single point. The subordination, when spoken in fuzzy set theory, becomes fuzzy subordination as the comparison between two holomorphic functions is made using the fuzzy membership function. In this article, differential and fuzzy differential Subordination, superordination, and sandwich theorems have been discussed for the classes defined by using q-derivative and symmetric q-derivative operators.


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## 1. Introduction

Let $\mathcal{H}(\mathbb{U})$ consist of all analytic functions in the open unit disc $\mathbb{U} := \{z : |z| < 1\}$ and for $n \in Z^+, a \in \mathbb{C}$ let $\mathcal{H}[a,n] = \{f(z) \ni f(z) = a + \sum_{t=n}^{\infty} a_t z^t\}$ . Clearly $\mathcal{H}[a,n] \subset \mathcal{H}$ and $\mathcal{H}[0,1] = \mathcal{A}$, the class of all normalized analytic functions. According to the principle of subordination, if $f_1, f_2 \in \mathcal{H}$ with $f_1(0) = f_2(0)$, then $f_1$ is subordinate to $f_2$, if $\exists$ a Schwarz function $w_1 \ni w_1(0) = 0, |w_1(z)| < 1$ and $f_1(z) = f_2(w_1(z)), \forall z \in \mathbb{U}$ and we write $f_1 \prec f_2$.

Let $h_1 \in \mathcal{S}$, a subclass of $\mathcal{A}$ that comprises of all univalent functions and let $\phi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$. If $p_1$ and $\phi(p_1(z), zp'_1(z); z) \in \mathcal{S}$ and if $p_1$ satisfies the (first-order) differential superordination

$$h_1(z) \prec \phi(p_1(z), zp'_1(z); z \in \mathbb{U}) , \tag{1}$$

then $p_1$ is called an integral of the differential superordination in (1). An analytic function $q_1$ is called subordinant of the integrals of the differential superordination if $q_1 \prec p_1$ . A subordinant $\tilde{q}_1$ with $q_1 \prec \tilde{q}_1$ for all subordinants of $q_1$ of (1) is said to be the best subordinant.

The principle of subordination, plays a vital part in the theory of univalent functions. Though Lindelof [1] was the first person to initiate this theory, the contributions of Littlewood [2] and Rogosinki [3] were remarkable in this field. Hallenbeck and Ruscheweyh [4] studied the subordinate functions and showed many interesting results for subordination in 1975. Ma and Minda [5] introduced different subclasses of star-like functions using the concept of subordination in 1992.

Differential subordination is the generalisation of differential inequalities with real variables. Goluzin [6] and Robinson [7] gave some important results involving differential implications of order one in 1935 and 1947 respectively. In 1970, Suf-

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fridge [8] extended Goluzin's result. The renowned theory of differential subordination was developed by Miller and Mocanu [9]. They introduced the concept of differential subordination and differential superordination [10].

The results of Miller and Mocanu motivated Bulboaca [11, 12] to study the first-order differential superordination for certain classes and superordination preserving integral operators. Thus, the topics subordination and superordination became the focal point for many researchers. The study of quantum calculus has been captivating the interest of researchers since 1707. Though Leonhard Euler and Carl Gustav Jacobi laid foundation to the study of q-calculus, it was a publication of Albert Einstein that made q- calculus popular because of its applications to quantum Mechanics after 1905. Unlike conventional calculus, q-calculus does not require continuity. In quantum calculus, the notion of limits is not considered.

The $q$-derivatives and $q$-integrals were first introduced by Jackson [13, 14]. The contributions of Srivastava to $q$-calculus for analytic functions and also to $q$-hypergeometric functions in function theory were remarkable. All his contributions were mentioned in his book [15]. Ismail *et al.* [16] have studied about starlike functions using $q$-calculus. Abelman *et al.* [17] have made use of fractional $q$- calculus operators on a class of non-Bazilevic functions to study the subordination conditions. A study on starlike functions in $q$-calculus and an extension of it to $q$-starlikeness for particular subclasses of starlike functions have been carried out by Agrawal and Sahoo [18]. For more details one can refer to Refs.[19–22].

On the other hand, Zadeh [23] introduced the concept of fuzzy in 1965. The notions of fuzzy subordination and fuzzy differential subordination were initially introduced by Oros and Oros [24–26]. This new initiative associates fuzzy set theory with geometric function theory. The duplet notion of fuzzy differential superordination was studied in Ref. [27].

Wanas and Majeed [28] have proved certain results on fuzzy differential subordination of analytic functions using generalized differential operator. Fuzzy differential subordinations involving integral operators can be found in the work of El-Deeb and Lupas [29]. Also El-Deeb and Oros [30] have studied fuzzy differential subordinations using linear operator. Lupas and Catas [31] have obtained the results on fuzzy subordination for analytic functions associated with the Atangana-Baleanu fractional integral of Bessel functions. For further study, see Refs.[32–34].

For results on a subclass of analytic and bi-univalent functions, using the symmetric $q$-derivative operator one can refer to Refs. [35–39]. The theory of symmetric q-calculus has its applications in various fields, specifically in fractional calculus and quantum physics. The importance of symmetric q-calculus in areas like quantum mechanics has been discussed in Ref. [40, 41].

Following are some of the q-analogues in complex (or real) analysis which we need for our work:

**Definition 1.** *[13] For $0 < q < 1$, the Jackson's q-derivative operator of a function is defined as*

$$(D_q f_1)(z) = \begin{cases} \frac{f_1(z)-f_1(qz)}{(1-q)z} & for \quad z \neq 0 \\ f_1'(0) & for \quad z = 0, \end{cases}$$

*where $D_q f_1(0) = f_1'(z)$, if $f_1'(z)$ exists.*

It is to be noted that $(D_q f_1(z)) = 1 + \sum_{t=2}^{\infty}[t]_q a_t z^{t-1}$; where $[t]_q = \frac{1-q^t}{1-q}$.

**Definition 2.** *[41] The symmetric q-derivative operator is defined by $(\tilde{D}_q f_1)(z) = \frac{f_1(qz)-f_1(q^{-1}z)}{(q-q^{-1}z)}$; if $z \neq 0$ and $(\tilde{D}_q f_1)(0) = {f_1}'(0)$ provided ${f_1}'(0)$ exists.*

It is to be observed that $(\tilde{D}_q f_1)(z) = 1 + \sum\limits_{t=2}^{\infty}[t]_q a_t z^{t-1}$, where $[t]_q = \frac{q^t-q^{-t}}{q-q^{-1}}$.

**Definition 3.** *[23] A pair $(K, F_K)$, with $F_K : Y \to [0,1], K = \{x \in Y : 0 < F_K(x) \leq 1\} = supp(K, F_K)$ is named as a fuzzy set of $Y$. $F_K$ is called the membership function of $(K, F_K)$.*

**Definition 4.** *[24] Consider two functions $f_1, g_1 \in \mathcal{H}(\mathcal{D})$, $\mathcal{D} \subseteq \mathbb{C}$ and $z_0 \in \mathcal{D}$ being a fixed point. $f_1$ is said to be fuzzy subordinate to $g_1$ (i.e) $f_1 \prec_F g_1$ otherwise $f_1(z) \prec_F g_1(z)$ if the following conditions are satisfied.*

$$f_1(z_0) := g_1(z_0)\,, \tag{2}$$

$$F_{f_1(D)}(f_1(z)) \leq F_{g_1(D)}(g_1(z)), \quad (z \in \mathcal{D})\,, \tag{3}$$

*where*
$f_1(D) = supp(D, F_{f_1}(D) = \Big\{z \in \mathbb{C} : 0 < F_{f_1}(D)(z) \leq 1\Big\}$,
$g_1(D) = supp(D, F_{g_1}(D) = \Big\{z \in \mathbb{C} : 0 < F_{g_1}(D)(z) \leq 1\Big\}$.

**Definition 5.** *[25] If $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ and $h_1 \in \mathcal{S}$ satisfy $\psi(c, 0; 0) = h_1(0) = c$, then $p_1$ is a fuzzy integral of the fuzzy differential subordination if $p_1$ is an analytic function in $\mathbb{U}$ such that it satifies the (first-order) fuzzy differential subordination*

$$F_{\psi(\mathbb{C}^2\times\mathbb{U})}(\psi(p_1(z), zp_1'(z)) \leq F_{h_1(\mathbb{U})}(h_1 z), \forall\ p_1, \tag{4}$$

*and $p_1(0) = c$. $q_1 \in \mathcal{S}$ is called a fuzzy dominant of the fuzzy integral of the fuzzy differential subordination or simply a fuzzy dominant, if $F_{p_1(\mathbb{U})}p_1(z) \leq F_{q_1(\mathbb{U})}q_1(z)$. A fuzzy dominant $\tilde{q}_1$ which satisfies $F_{\tilde{q}_1(\mathbb{U})}\tilde{q}_1(z) \leq F_{q_1(\mathbb{U})}q_1(z)$ for all fuzzy dominants $q_1$ of (4) is said to be the fuzzy best dominant of (4).*

**Definition 6.** *[25] Consider $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ and $h_1$ an univalent function in $\mathbb{U}$. If $p_1$ is analytic in $\mathbb{U}$ and satisfies the fuzzy differential superordination*

$$F_{h_1(\mathbb{U})}h_1((z)) \leq F_{\psi(\mathbb{C}^2\times\mathbb{U})}(\psi(p_1(z), z{p'}_1(z)); z)\,. \tag{5}$$

*(i.e) $h_1(z) \prec_{\mathcal{F}} \psi(p_1(z), z{p'}_1(z); z), z \in \mathbb{U}$, then $p_1$ is called a fuzzy integral of fuzzy differential superordination. $q_1 \in \mathcal{S}$ is called a fuzzy subordinant of the integrals of the fuzzy differential superordination, if*

$$q_1 \prec \mathcal{F}\, p_1,$$

*for all $p_1$ satisfying (5).*

**Definition 7.** *[42] Denote by $Q$, the set of all functions $f_1$ that are analytic and injective on $\overline{\mathbb{U}} - E(f_1)$, where*
$E(f_1) = \{\zeta \in \partial U : \lim_{z\to\infty} f_1(z) = \infty\} \in f_1'(\zeta) \neq 0,\ \zeta \in \partial\mathbb{U} - E(f_1).$

In this article, two classes of functions have been defined by using Quantum Calculus operators and Differential and Fuzzy Differential Sandwich theorems have been discussed.

## 2. Subordination results

In this section we have defined two classes $R(q)$ and $\tilde{R}(q)$ and have proved the subordination, superordination and sandwich theorems for the classes defined.

**Definition 8.** *The function $f_1 \in \mathcal{A}$ is said to be in the class $R(q)$, if*

$$\Re\left[1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\right] > 0,\ z \in \mathbb{U}.$$

**Definition 9.** *The function $f_1 \in \mathcal{A}$ is said to be in the class $\tilde{R}(q)$, if*

$$\Re\left[1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right] > 0,\ z \in \mathbb{U}.$$

**Theorem 2.1.** *Let $q_1(z)$ be analytic and univalent with $q_1(0) = 1$ such that $\frac{zq_1'(z)}{q_1(z)}$ is starlike univalent in $\mathbb{U}$ and $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3)$, $(\alpha_3 \neq 0)$. Let $q_1(z)$ satisfy*

$$Re\left[\frac{\alpha_2}{\alpha_3} q_1(z) + 1 + \frac{zq_1''(z)}{q_1'(z)} - \frac{zq_1'(z)}{q_1(z)}\right] > 0. \tag{6}$$

*If $f_1 \in \mathcal{A}$ satisfies*

$$\Delta^{(\alpha_j)_1^3}(f_1) = \Delta(f_1, \alpha_1, \alpha_2, \alpha_3) \prec \alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}, \tag{7}$$

*where*

$$\Delta^{(\alpha_j)_1^3}(f_1) = \alpha_1 + \alpha_2\left[1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\right] + \alpha_3\left[\frac{z^2(D_q f_1)''(z)}{(D_q f_1)(z) + z(D_q f_1)'(z)} + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z) + z(D_q f_1)'(z)} - \frac{[z(D_q f)'(z)]^2}{(D_q f_1)(z)[(D_q f_1)(z) + z(D_q f_1)'(z)]}\right], \tag{8}$$

*then*

$$1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)} \prec q_1(z) \tag{9}$$

*and $q_1$ is the best dominant .*

**Proof**

$$p_1(z) := 1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\ (z \in \mathbb{U}). \tag{10}$$

Here $p_1(0) = 1$ and $p_1$ is analytic in $\mathbb{U}$. Using Eq. (10)

$$\alpha_1 + \alpha_2\left[1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\right] + \alpha_3\left[\frac{z^2(D_q f_1)''(z)}{(D_q f_1)(z) + z(D_q f_1)'(z)} + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z) + z(D_q f_1)'(z)} - \frac{[z(D_q f)'(z)]^2}{(D_q f_1)(z)[(D_q f_1)(z) + z(D_q f_1)'(z)]}\right]$$
$$= \alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}. \tag{11}$$

By using Eq. (11) in Eq. (7), we have

$$\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)} \prec \alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}. \tag{12}$$

By setting $\theta(\omega) := \alpha_1 + \alpha_2\omega$ and $\phi(\omega) := \frac{\alpha_3}{\omega}$, we noticed that $\theta(\omega)$ and $\phi(\omega)$ are analytic in $\mathbb{C} - \{0\}$ and $\phi(\omega) \neq 0$.
Let

$$Q_1(z) =: zq_1'(z)\phi(q_1(z)) = \alpha_3 \frac{zq_1'(z)}{q_1(z)}.$$

and

$$h_1(z) := \theta(q_1(z)) + Q_1(z) = \alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}.$$

Since $Q_1(z)$ is starlike univalent in $\mathbb{U}$ with

$$\text{Re}\left\{\frac{zh_1'(z)}{Q_1(z)}\right\} = Re\left[\frac{\alpha_2}{\alpha_3} q_1(z) + 1 + \frac{zq_1''(z)}{q_1'(z)} - \frac{zq_1'(z)}{q_1(z)}\right] > 0\ ,$$

the results follow by the application of [[42],Theorem 3.4h, p.132] and by the assumption of Theorem 2.1.
If $q_1(z) = \frac{1+Az}{1+Bz}; \Big(-1 \le B < A \le 1\Big)$ in the previous result, we get the following

**Corollary 1.** *Let $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3), (\alpha_3 \neq 0)$ , $q_1$ be univalent with $q_1(0) = 1$, and (6) be true. If $f_1 \in \mathcal{A}$ and*

$$\Delta^{(\alpha_j)_1^3}(f_1) \prec \alpha_1 + \alpha_2 \frac{1+Az}{1+Bz} + \alpha_3 \frac{(A-B)z}{(1+Az)(1+Bz)}\ ,$$

*then*

$$1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)} \prec \frac{1+Az}{1+Bz} \text{ and } \frac{1+Az}{1+Bz} \text{ is the best dominant.}$$

Assigning $\alpha_1 = 1; \alpha_2 = 0$ in the previous result, we obtain the following corollary.

**Corollary 2.** *Let $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3)$ $(\alpha_3 \neq 0), q_1$ be univalent with $q_1(0) = 1$, and*
$Re\left[1 + \frac{zq_1''(z)}{q_1'(z)} - \frac{zq_1'(z)}{q_1(z)}\right] > 0$. *If $f_1 \in \mathcal{A}$ and*

$$1 + \alpha_3\left[\frac{z^2(D_q f_1)''(z)}{(D_q f_1)(z)+z(D_q f_1)'(z)} + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)+z(D_q f_1)'(z)} - \frac{[z(D_q f)'(z)]^2}{(D_q f_1)(z)[(D_q f_1)(z)+z(D_q f_1)'(z)]}\right] \prec 1 + \alpha_3 \frac{(A-B)z}{(1+Az)(1+Bz)},$$

*then*

$$1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)} \prec \frac{1+Az}{1+Bz} \text{ and the best dominant is } \frac{1+Az}{1+Bz}.$$

$q_1(z) = \left(\frac{1+z}{1-z}\right)^\delta (0 < \delta < 1)$ in Theorem 2.1, leads us to the next result.

**Corollary 3.** *Let $q_1$ be univalent with $q_1(0) = 1$, $\alpha_j \in \mathbb{C}$ $(j = 1,2,3)$ $(\alpha_3 \neq 0)$ and let (6) hold good. If $f_1 \in \mathcal{A}$ and*

$$\Delta^{(\alpha_j)_1^3}(f_1) \prec \alpha_1 + \alpha_2\left(\frac{1+z}{1-z}\right)^{\delta} + \alpha_3\left(\frac{2\delta z}{1-z^2}\right),$$

*then*

$$1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta} \text{ and } \left(\frac{1+z}{1-z}\right)^{\delta} \text{ is the best dominant.}$$

### 2.1. Superordination Results

**Theorem 2.2.** *Consider $\alpha_j \in \mathbb{C}$ $(j = 1,2,3), (\alpha_3 \neq 0)$ and $q_1 \in \mathcal{S}$ with $q_1(0) = 1 \ni \frac{zq_1'(z)}{q_1(z)}$ is starlike univalent and assume that*

$$Re\left\{\frac{\alpha_2}{\alpha_3} q_1(z)\right\} \geq 0. \tag{13}$$

*If $f_1 \in \mathcal{A}$; $\frac{z(D_q f_1)'(z)}{(D_q f_1)(z)} \in \mathcal{H}[1,1] \cap Q, \Delta^{(\alpha_j)_1^3}(f_1)$ is univalent in $\mathbb{U}$ and*

$$\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)} \prec \Delta^{(\alpha_j)_1^3}(f_1), \tag{14}$$

*where, $\Delta^{(\alpha_j)_1^3}(f_1)$, is given by Eq. (8), then*

$$q_1(z) \prec 1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)},$$

*and $q_1$ represents the best subordinant.*

**Proof**: Define

$$p_1(z) := 1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)} \quad (z \in \mathbb{U}). \tag{15}$$

By simple computation, we get

$$\Delta^{(\alpha_j)_1^3}(f_1) = \alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}.$$

Then

$$\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)} \prec \alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}.$$

By setting $v(\omega) = \alpha_1 + \alpha_2\omega$ and $\phi(\omega) = \frac{\alpha_3}{\omega}$, it is easily observed that $v(\omega)$ is analytic in $\mathbb{C}$. Also $\phi(\omega)$ is analytic in $\mathbb{C} - \{0\}$ and $\phi(\omega) \neq 0$.
Using $q_1$ is convex univalent function in $\mathbb{U}$ and $\alpha_3 \neq 0$, we have,

$$\mathrm{Re}\left\{\frac{v' q_1(z)}{\phi(q_1(z))}\right\} = Re\left\{\frac{\alpha_2}{\alpha_3} q_1(z)\right\} > 0.$$

We get Theorem 2.2 by applying [10] (Theorem 8, p. 822).
The following result is obtained when we replace $q_1(z)$ by $\frac{1+Az}{1+Bz}$, $(-1 \leq B < A \leq 1)$ in Theorem 2.2 .

**Corollary 4.** *Consider $\alpha_j \in \mathbb{C}$ $(j = 1,2,3), (\alpha_3 \neq 0)$ , $q_1$ a univalent function with $q_1(0) = 1$, and (13) holds true. If $f_1 \in \mathcal{A}$ and*

$$\alpha_1 + \alpha_2 \frac{1+Az}{1+Bz} + \alpha_3 \frac{(A-B)z}{(1+Az)(1+Bz)} \prec \Delta^{(\alpha_j)_1^3}(f_1),$$

*then*

$$\frac{1+Az}{1+Bz} \prec 1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}$$

*and the best subordinant is $\frac{1+Az}{1+Bz}$.*

### 2.2. Sandwich Theorems

**Theorem 2.3.** *Consider two univalent functions $q_2$ and $q_3$ in $\mathbb{U}$ and $\alpha_j \in \mathbb{C}$ $(j = 1,2,3$ , $(\alpha_3 \neq 0))$. Let $q_2$ satisfy (6), $q_3$ satisfy (13), $f_1 \in \mathcal{A}$, $\left(1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\right) \in \mathcal{H}[1,1] \cap Q$ and $\Delta^{(\alpha_j)_1^3}(f_1)$ by (8) be univalent in $\mathbb{U}$ satisfying*

$$\alpha_1 + \alpha_2 q_2(z) + \alpha_3 \frac{zq_2'(z)}{q_2(z)} \prec \Delta^{(\alpha_j)_1^3}(f_1) \prec \alpha_1 + \alpha_2 q_3(z) + \alpha_3 \frac{zq_3'(z)}{q_3(z)},$$

*then*

$$q_2(z) \prec 1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)} \prec q_3(z)$$

*and $q_2$ and $q_3$ seem to be the best subordinant and dominant.*

## 3. Fuzzy Differential Subordination Results

**Theorem 3.1.** *Let $\alpha_j \in \mathbb{C}(j = 1,2,3), (\alpha_3 \neq 0)$ and $q_1(z)$ be analytic and univalent with $q_1(0) = 1$. Assume that $\frac{zq_1'(z)}{q_1(z)}$ is starlike univalent in $\mathbb{U}$. Consider*

$$Re\left[\frac{\alpha_2}{\alpha_3} q_1(z) + 1 + \frac{zq_1''(z)}{q_1'(z)} - \frac{zq_1'(z)}{q_1(z)}\right] > 0 \quad (z \in \mathbb{U}), \tag{16}$$

*and*

*$\Delta^{(\alpha_j)_1^3}$ is defined by (8). If $q_1$ satisfies the following fuzzy subordination*

$$F_{\Delta^{(\alpha_j)_1^3}} \leq F_{q_1(U)}\left(\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}\right) \tag{17}$$

*for $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}, \alpha_3 \neq 0$, then*

$$F_{\left(1+\frac{z(D_q f_1)'}{(D_q f_1)}\right)(\mathbb{U})}\left(1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\right) \leq F_{q_{1(\mathbb{U})}}\, q_1(z), \quad z \in \mathbb{U} \tag{18}$$

*and $q_1$ is the fuzzy best dominant .*

**Proof**: Let

$$p_1(z) = 1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}, \quad z \neq 0. \tag{19}$$

After computation, we have

$$\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)} = \Delta^{(\alpha_j)_1^3}(f_1). \tag{20}$$

where $\Delta^{(\alpha_j)_1^3}(f_1)$ is given by (8). By using the above inequality into (17) , we have

$$F_{p_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}\right) \leq F_{q_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}\right).$$

Considering $\theta(\omega) := \alpha_1 + \alpha_2\omega$ ; $\phi(\omega) := \frac{\alpha_3}{\omega}$, it is noticed that $\theta(\omega)$ and $\phi(\omega)$ in $\mathbb{C} - \{0\}$ are analytic & $\phi(\omega)$ not equal to zero. Consider

$$Q_1(z) =: zq_1'(z)\phi(q_1(z)) = \alpha_3 \frac{zq_1'(z)}{q_1(z)}$$

and

$$h_1(z) := \theta\Big(q_1(z)\Big) + Q_1(z) = \alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}.$$

Since $Q_1$ is starlike univalent defined in $\mathbb{U}$ with

$$Re\left[\frac{zh_1'(z)}{q_1(z)}\right] = Re\left[\frac{\alpha_2}{\alpha_3}q_1(z) + 1 + \frac{zq_1'(z)}{q_1'(z)} - \frac{zq_1''(z)}{q_1(z)}\right] > 0.(21)$$

By using proposition 2.1 of Ref. [43], we get $F_{p_1(\mathbb{U})}p_1(z) \leq F_{q_1(\mathbb{U})}q_1(z)$ and using (19) we obtain

$$F_{\left(1+\frac{z(D_q f_1)'}{(D_q f_1)}\right)(\mathbb{U})}\left(1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\right) \leq F_{q_1(\mathbb{U})}q_1(z),$$

and $q_1$ indicates the fuzzy best dominant.

**Corollary 5.** *Consider* $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3)$ $\quad(\alpha_3 \neq 0)$*. Assume that (16) holds. If* $F_{\Delta^{(\alpha_j)_1^3}} \leq F_{q_1(\mathbb{U})}\left(\alpha_1 + \alpha_2(\frac{1+Az}{1+Bz}) + \alpha_3\frac{(A-B)z}{(1+Az)(1+Bz)}\right)$*, then*

$$F_{\left(1+\frac{z(D_q f_1)'}{(D_q f_1)}\right)(\mathbb{U})}\left(1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\right) \leq F_{q_{1(\mathbb{U})}} \frac{1+Az}{1+Bz}; \quad z \in \mathbb{U}$$

*and* $\frac{1+Az}{1+Bz}$ *represents the best fuzzy dominant.*

**Proof**: In Theorem 3.1, assign $q_1(z) = \frac{1+Az}{1+Bz}; -1 \leq B < A \leq 1$ to obtain the above corollary.

**Theorem 3.2.** *Let* $q_1(z)$ *be analytic and univalent in* $\mathbb{U}$ *such that* $q_1(0) \neq 0$ *and* $\frac{zq_1'(z)}{q_1(z)}$ *defines a starlike univalent in* $\mathbb{U}$*. Assume that*

$$Re\left(\frac{\alpha_2}{\alpha_3}q_1(z)\right) > 0. \tag{22}$$

*Let* $\Delta^{(\alpha_j)_1^3}(f_1) \in \mathbb{U}$ *be univalent and* $\left(1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\right) \in \mathcal{H}[1, 1] \cap Q$ *where* $\Delta^{(\alpha_j)_1^3}(f_1)$ *has been introduced in (8), then*

$$F_{q_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}\right) \leq F_{\left(1+\frac{z(D_q f_1)'}{(D_q f_1)}\right)(\mathbb{U})}\left(1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\right)$$

*implies*

$$F_{q_1(\mathbb{U})}q_1(z) \leq F_{\left(1+\frac{zD_q f_1)'}{(D_q f_1)}\right)(\mathbb{U})}\left(1 + \frac{zD_q f_1)'(z)}{(D_q f_1)(z)}\right)$$

*and* $q_1$ *represents the best fuzzy subordinant.*

**Proof**: $p_1(z) = 1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}; z \in \mathbb{U}$. Define $v(\omega) = \alpha_1 + \alpha_2\omega$ and $\phi(\omega) = \frac{\alpha_3}{\omega}$. It is observed that $v(\omega)$ is analytic in $\mathbb{C}$, $\phi(\omega)$ is analytic in $\mathbb{C} - \{0\}$ and $\phi(\omega) \neq 0$.
Under these conditions $\frac{v'(q_1(z))}{\phi(q_1(z))} = \frac{\alpha_2}{\alpha_3}q_1(z) \implies \text{Re}\left(\frac{v'(q_1(z))}{\phi(q_1(z))}\right) = Re\left\{\frac{\alpha_2}{\alpha_2}q_1(z)\right\} > 0$
and we obtain,

$$F_{q_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}\right) \leq F_{p_1(U)}\left(\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}\right).$$

Applying Theorem 2.15 of Ref. [27]], the result is obtained.

**Theorem 3.3.** *Let* $\alpha_j \in \mathbb{C}(j = 1, 2, 3); (\alpha_3 \neq 0), q_2$ *and* $q_3$ *be univalent in* $\mathbb{U}$*,* $q_2$ *satisfy (16) and* $q_3$ *satisfy (22) . Let* $f_1 \in \mathcal{A}$*,* $\left(1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\right) \in \mathcal{H}[1, 1] \cap Q$ *and* $\Delta^{(\alpha_j)_1^3}(f_1)$ *given by (8) be univalent in* $\mathbb{U}$ *satisfying*

$$F_{q_2(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_2(z) + \alpha_3 \frac{zq_2'(z)}{q_2(z)}\right) \leq F_{\Delta^{(\alpha_j)_1^3}(f_1)} \leq F_{q_3(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_3(z) + \alpha_3 \frac{zq_3'(z)}{q_3(z)}\right).$$

*Then*

$$F_{q_2(\mathbb{U})}\, q_2(z) \leq F_{\left(1+\frac{zD_q f_1)'}{(D_q f_1)}\right)(\mathbb{U})}\left(1 + \frac{zD_q f_1)'(z)}{(D_q f_1)(z)}\right) \leq F_{q_3(\mathbb{U})}\, q_3(z),$$

*and* $q_2$ *and* $q_3$ *are fuzzy best subordiant and best fuzzy dominant.*

To get the following result choose $q_2(z) = \frac{1+A_1 z}{1+B_1 z}$ and $q_3(z) = \frac{1+A_2 z}{1+B_2 z}; -1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$ in Theorem 3.3.

**Corollary 6.** *Let* $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3)(\alpha_j \neq 0)$*. Assume that (16) and (22) hold. If* $\left(1 + \frac{z(D_q f_1)'(z)}{(D_q f_1)(z)}\right) \in \mathcal{H}[1, 1] \cap Q$ *and* $F_{q_1(\mathbb{U})}\left(\alpha_1 + \alpha_2(\frac{1+A_1 z}{1+B_1 z}) + \alpha_3\frac{(A_1-B_1)z}{(1+A_1 z)(1+B_1 z)}\right) \leq F_{\Delta^{(\alpha_j)_1^3}} \leq F_{q_2(\mathbb{U})}\left(\alpha_1 + \alpha_2(\frac{1+A_2 z}{1+B_2 z}) + \alpha_3\frac{(A_2-B_2)z}{(1+A_2 z)(1+B_2 z)}\right)$*, where* $\Delta^{(\alpha_j)_1^3}(f_1)$ *is introduced in (8), then*

$$F_{q_1(\mathbb{U})}\frac{1+A_1 z}{1+B_1 z} \leq F_{\left(1+\frac{zD_q f_1)'}{(D_q f_1)}\right)(\mathbb{U})}\left(1 + \frac{zD_q f_1)'(z)}{(D_q f_1)(z)}\right) \leq F_{q_2(\mathbb{U})}\frac{1+A_2 z}{1+B_2 z},$$

*and* $\frac{1+A_1 z}{1+B_1 z}$ *and* $\frac{1+A_2 z}{1+B_2 z}$ *are the best fuzzy subordinant and best fuzzy dominant respectively.*

## 4. Subordination results for Symmetric q-derivative operator.

**Theorem 4.1.** *Let* $q_1(z)$ *be analytic and univalent with* $q_1(0) = 1$ *such that* $\frac{zq_1'(z)}{q_1(z)}$ *is starlike univalent in* $\mathbb{U}$ *and* $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3)$ $(\alpha_3 \neq 0)$ *. Let* $q_1(z)$ *satisfy*

$$Re\left[\frac{\alpha_2}{\alpha_3}q_1(z) + 1 + \frac{zq_1''(z)}{q_1'(z)} - \frac{zq_1'(z)}{q_1(z)}\right] > 0, \quad (z \in \mathbb{U}).(23)$$

*If* $f_1 \in \mathcal{A}$ *satisfies*

$$\Delta^{(\alpha_j)_1^3}(f_1) = \Delta(f_1, \alpha_1, \alpha_2, \alpha_3) \prec \alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}, \tag{24}$$

*where* $\Delta^{(\alpha_j)_1^3}(f_1)$ *is defined by (8) then*

$$1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)} \prec q_1(z)$$

*and $q_1$ is the best dominant .*

**Proof**: Let

$$p_1(z) := 1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}, \ (z \in \mathbb{U}). \tag{25}$$

Then the function $p_1$ is analytic in $\mathbb{U}$ and $p_1(0) = 1$. By using Eq. (25), we get

$$\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)} = \alpha_1 + \alpha_2 \left[1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right] +$$

$$\alpha_3 \left[\frac{z^2(\tilde{D}_q f_1)''(z)}{(\tilde{D}_q f_1)(z) + z(\tilde{D}_q f_1)'(z)} + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z) + z(\tilde{D}_q f_1)'(z)} - \right.$$

$$\left. \frac{[z(\tilde{D}_q f)'(z)]^2}{(\tilde{D}_q f_1)(z)[(\tilde{D}_q f_1)(z) + z(\tilde{D}_q f_1)'(z)]}\right]. \tag{26}$$

By using (26) in (24), we have,

$$\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)} = \alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}. \tag{27}$$

Taking $\theta(\omega) = \alpha_1 + \alpha_2\omega$ and $\phi(\omega) := \frac{\alpha_3}{\omega}$, we note that $\theta(\omega)$ and $\phi(\omega)$ are analytic in $\mathbb{C} - \{0\}$ and $\phi(\omega) \neq 0$. We also notice that

$$Q_1(z) =: zq_1'(z)\phi(q_1(z)) = \alpha_3 \frac{zq_1'(z)}{q_1(z)}.$$

and

$$h_1(z) := \theta(q_1(z)) + Q_1(z) = \alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}.$$

Clearly, $Q_1(z)$ is univalent starlike in $\mathbb{U}$ and

$$\text{Re}\left(\frac{zh_1'(z)}{Q_1(z)}\right) = Re\left[\frac{\alpha_2}{\alpha_3}q_1(z) + 1 + \frac{zq_1''(z)}{q_1'(z)} - \frac{zq_1'(z)}{q_1(z)}\right] > 0 \ .$$

By the assertion of the Theorem 4.1, the result follows by the application of [Ref. [42] Theorem 3.4h, p. 132]. By taking $\alpha_1 = 0$ and $\alpha_2 = 1$ in above result , we obtain the following corollary.

**Theorem 4.2.** *Let $\alpha_j \in \mathbb{C}(j = 1, 2, 3), (\alpha_3 \neq 0)$ , $q_1$ be univalent with $q_1(0) = 1$ such that $\frac{zq_1'(z)}{q_1(z)}$ is starlike univalent and assume that*

$$Re\left(1 + \frac{\alpha_2}{\alpha_3}q_1(z)\right) \geq 0. \tag{28}$$

*If $f_1 \in \mathcal{A}$; $\left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \in \mathcal{H}[1, 1] \cap Q$, $\Delta^{(\alpha_j)_1^3}(f_1)$ be univalent in $\mathbb{U}$ and*

$$\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)} \prec \Delta^{(\alpha_j)_1^3}(f_1)$$

*where $\Delta^{(\alpha_j)_1^3}(f_1)$ is defined in (8), then*

$$q_1(z) \prec \left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right)$$

*and $q_1$ said to be the best subordinant.*

**Proof**: Let

$$p_1(z) = \left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \quad z \in \mathbb{U}; \quad z \neq 0, \tag{29}$$

$v(\omega) = \alpha_1 + \alpha_2\omega$ be analytic in $\mathbb{C}$ and $\phi(\omega) = \frac{\alpha_3}{\omega}$ be analytic in $\mathbb{C} - \{0\}$ and $\phi(\omega) \neq 0$. Under these conditions $\frac{v'(q_1(z)}{\phi(q_1(z))} = \frac{\alpha_2}{\alpha_3}q_1(z)$
$\implies \text{Re}\left\{\frac{v'q_1(z)}{\phi(q_1(z))}\right\} = Re\left\{\frac{\alpha_2}{\alpha_2}q_1(z)\right\} > 0,$
and we obtain,

$$\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z}{q_1(z)}) \leq \alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}.$$

Applying [Ref. [10], Theorem 8, p. 822], we obtain, $q_1(z) \prec p_1(z)$ and $q_1$ is the best subordinant. Replacing $q_1(z)$ by $= \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 4.2, the following result is obtained:

**Corollary 7.** *Consider $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3), (\alpha_3 \neq 0)$ and let $q_1$ be univalent with $q_1(0) = 1$, and (28) be true. If $f_1 \in \mathcal{A}$ and*

$$\alpha_1 + \alpha_2 \frac{1+Az}{1+Bz} + \alpha_3 \frac{(A-B)z}{(1+Az)(1+Bz)} \prec \Delta^{(\alpha_j)_1^3}(f_1),$$

*then*

$$\frac{1+Az}{1+Bz} \prec 1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}$$

*and $\frac{1+Az}{1+Bz}$ represents the best subordinant.*

Sandwich theorem is obtained by using Theorems 4.1 and 4.2.

**Theorem 4.3.** *Let $q_2$ and $q_3$ be univalent in $\mathbb{U}$, $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3), (\alpha_j \neq 0), q_2$ satisfy (22) and $q_3$ satisfy (28). Let $f_1 \in \mathcal{A}$, $\left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \in \mathcal{H}[1, 1] \cap Q$ and $\Delta^{(\alpha_j)_1^3}(f_1)$ defined by (8) be univalent in $\mathbb{U}$ satisfying*

$$\alpha_1 + \alpha_2 q_2(z) + \alpha_3 \frac{zq_2'(z)}{q_2(z)} \prec \Delta^{(\alpha_j)_1^3}(f_1) \prec \alpha_1 + \alpha_2 q_3(z) + \alpha_3 \frac{zq_3'(z)}{q_3(z)},$$

*then*

$$q_2(z) \prec 1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)} \prec q_3(z),$$

*and the best subordinant and best dominant are $q_2$ and $q_3$.*

## 5. Fuzzy Differential Subordination for Symmetric q-derivative operator

**Theorem 5.1.** *Let $q_1(z)$ be an analytic and univalent function with $q_1(z)$ such that $\frac{zq_1'(z)}{q_1(z)} \in \mathbb{U}$ is starlike univalent and $\alpha_j \in \mathbb{C}(j = 1, 2, 3), (\alpha_3 \neq 0)$. Let*

$$Re\left[\frac{\alpha_2}{\alpha_3}q_1(z) + 1 + \frac{zq_1''(z)}{q_1'(z)} - \frac{zq_1'(z)}{q_1(z)}\right] > 0, \quad (z \in \mathbb{U}), \tag{30}$$

*and $\Delta^{(\alpha_j)^3_1}(f_1)$ be defined as in Eq. (8). If $q_1$ satisfies the following fuzzy subordination*

$$F_{\Delta^{(\alpha_j)^3_1}(f_1)} \leq F_{q_1(U)}\left(\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}\right), \tag{31}$$

*where $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}$ and $(\alpha_3 \neq 0)$, then*

$$F_{\left(1+\frac{z(\tilde{D}_q f_1)'}{(\tilde{D}_q f_1)}(U)\right)}\left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \leq F_{q_{1(\mathbb{U})}}\, q_1(z) \quad z \in \mathbb{U}, \tag{32}$$

*and $q_1$ represents the best fuzzy dominant.*

**Proof**: Let

$$p_1(z) = \left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \quad z \in \mathbb{U}; \quad z \neq 0. \tag{33}$$

Here the function $p_1$ is analytic with $p_1(0) = 1$. Simple calculation leads to

$$\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)} = \Delta^{(\alpha_j)^3_1}(f_1), \tag{34}$$

where $\Delta^{(\alpha_j)^3_1}(f_1)$ is given by (8). By using above inequality in (31) we have

$$F_{p_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}\right) \leq F_{q_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}\right).$$

By considering $\theta(\omega) := \alpha_1 + \alpha_2\omega$ and $\phi(\omega) := \frac{\alpha_3}{\omega}$, it is evident that $\theta(\omega)$ , $\phi(\omega) \in \mathbb{C} - \{0\}$ are analytic and $\phi(\omega) \neq 0$.
Consider

$$Q_1(z) =: zq_1'(z)\phi(q_1(z)) = \alpha_3 \frac{zq_1'(z)}{q_1(z)}.$$

We deduce that $Q_1$ is starlike univalent in $\mathbb{U}$ and

$$Re\left\{\frac{zh_1'(z)}{Q_1(z)}\right\} = Re\left[\frac{\alpha_2}{\alpha_3} q_1(z) + 1 + \frac{zq_1'(z)}{q_1'(z)} - \frac{zq_1''(z)}{q_1(z)}\right] > 0. \tag{35}$$

By [proposition 2.1, Ref. [43]] $F_{p_1(\mathbb{U})} p_1(z) \leq F_{q_1(\mathbb{U})} q_1(z)$ and using (35) we obtain

$$F_{\left(1+\frac{z(\tilde{D}_q f_1)'}{(\tilde{D}_q f_1)}\right)(\mathbb{U})}\left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \leq F_{q_1(\mathbb{U})} q_1(z); \quad z \in \mathbb{U}$$

and $q_1$ is the best dominant .

**Corollary 8.** *Let $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3), (\alpha_3 \neq 0)$. Assume that (30) holds if $F_{\Delta^{(\alpha_j)^3_1}(f_1)} \leq F_{q_1(\mathbb{U})}\left(\alpha_1 + \alpha_2(\frac{1+Az}{1+Bz}) + \alpha_3 \frac{(A-B)z}{(1+Az)(1+Bz)}\right)$ then*

$$F_{\left(1+\frac{z(\tilde{D}_q f_1)'}{(\tilde{D}_q f_1)}(\mathbb{U})\right)}\left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \leq F_{q_1(\mathbb{U})} \frac{1+Az}{1+Bz}; \quad z \in \mathbb{U}$$

*and $\frac{1+Az}{1+Bz}$ is the fuzzy best dominant.*

**Theorem 5.2.** *Let $\frac{zq_1'(z)}{q_1(z)}$ be starlike univalent in $\mathbb{U}$ and $q_1(z)$ be analytic and univalent in $\mathbb{U}$. Assume that*

$$Re\left(\frac{\alpha_2}{\alpha_3} q_1(z)\right) > 0. \tag{36}$$

*Let $\Delta^{(\alpha_j)^3_1}(f_1)$ be univalent in $\mathbb{U}$ and $1 + \frac{z((\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)} \in \mathcal{H}[q(0), 1] \cap Q$ where $\Delta^{(\alpha_j)^3_1}(f_1)$ is given by (8). Then*

$$F_{q_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}\right) \leq F_{\Delta^{(\alpha_j)^3_1}(f_1)} \leq F_{\left(1+\frac{z(\tilde{D}_q f_1)'}{(\tilde{D}_q f_1)}\right)(\mathbb{U})}\left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right)$$

$$\implies F_{q_1(\mathbb{U})} q_1(z) \leq F_{\left(1+\frac{z(\tilde{D}_q f_1)'}{(\tilde{D}_q f_1)}(\mathbb{U})\right)}\left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right)$$

*and $q_1$ is said to be the fuzzy best subordinant.*

**Proof**: $p_1(z) = 1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)} \quad (z \in \mathbb{U})$.
Taking $\nu(\omega) : \alpha_1 + \alpha_2\omega$ and $\phi(\omega) = \frac{\alpha_3}{\omega}$ . Also knowing that, $\nu(\omega)$ is analytic in $\mathbb{C}$, $\phi(\omega)$ is analytic in $\mathbb{C} - \{0\}$, $\phi(q_1(z)) = \frac{\alpha_2}{\alpha_3} q_1(z)$ and $\phi(\omega) \neq 0$, we have $\frac{\nu'(q_1(z)}{\phi(q_1(z))} = \frac{\alpha_2}{\alpha_3} q_1(z) \implies \text{Re}\left(\frac{\nu'(q_1(z)}{\phi(q_1(z))}\right) = Re\left\{\frac{\alpha_2}{\alpha_3} q_1(z)\right\} > 0$. We obtain

$$F_{q_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_1(z) + \alpha_3 \frac{zq_1'(z)}{q_1(z)}\right) \leq F_{p_1(\mathbb{U})}\left(\alpha_1 + \alpha_2 p_1(z) + \alpha_3 \frac{zp_1'(z)}{p_1(z)}\right).$$

Applying [Theorem 2.15 of Ref. [27]], we get $F_{q_1(\mathbb{U})} q_1(z) \leq F_{p_1(\mathbb{U})} p_1(z)$ and $q_1$ is the fuzzy best subordinant.

**Theorem 5.3.** *Consider the univalent functions $q_2$ and $q_3$ in $\mathbb{U}$, $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3), (\alpha_j \neq 0)$ with $q_1$ satisfying (30) and $q_2$ satisfying (36) . Let $f_1 \in \mathcal{A}$, $\left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \in \mathcal{H}[1, 1] \cap Q$ and $\Delta^{(\alpha_j)^3_1}(f_1)$ given by (8) be univalent in $\mathbb{U}$ satisfying*

$$F_{q_2(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_2(z) + \alpha_3 \frac{zq_2'(z)}{q_2(z)}\right) \leq F_{\Delta}^{(\alpha_j)^3_1}(f_1) \leq F_{q_3(\mathbb{U})}\left(\alpha_1 + \alpha_2 q_3(z) + \alpha_3 \frac{zq_3'(z)}{q_3(z)}\right).$$

*Then*

$$F_{q_2(\mathbb{U})}\, q_2(z) \leq F_{\left(1+\frac{(z\tilde{D}_q f_1)'}{(\tilde{D}_q f_1)}\right)(\mathbb{U})}\left(1 + \frac{z(\tilde{D}_q f_1)'(z)}{(\tilde{D}_q f_1)(z)}\right) \leq F_{q_3(\mathbb{U})}\, q_3(z).$$

*Here $q_2$ and $q_3$ represent fuzzy best subordinant and fuzzy best dominant.*

## 6. Conclusion

In this article, we have defined two classes using quantum calculus operators. We have stated and proved the theorems on differential subordination and superordination. Sandwich theorems have also been stated and proved. These results have been extended to fuzzy set theory as fuzzy differential subordination, superordination, and sandwich theorems.

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