



Common Fixed Point Theorems for Multivalued Generalized F -Suzuki-Contraction Mappings in Complete Strong b -Metric Spaces

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Abstract

This paper introduces a new version of multivalued generalized F -Suzuki-Contraction mapping and then establish some new common fixed point theorems for these new multivalued generalized F -Suzuki-Contraction Mappings in complete strong b -Metric Spaces.

Keywords: Common Fixed Point Problem, Multivalued Generalized F -Suzuki-Contraction Mapping, Complete Strong b -metric Space.

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1. Introduction

Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \rightarrow \mathbb{R}^*$ is said to be a b -metric if for all $x, y, z \in X$ the following conditions are satisfied:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b -metric space with constant s .

A strong b -metric is a semimetric space (X, d) if there exists $s \geq 1$ for which d satisfies the following triangular inequality.

$$d(x, y) \leq d(x, z) + sd(z, y), \text{ for each } x, y, z \in X. \quad (1)$$

In 1922, a mathematician Banach [1] proved a very important result regarding a contraction mapping, known as the Banach contraction principle, which states that every self-mapping T defined on a complete metric space (X, d) satisfying

$$\forall x, y \in X, d(Tx, Ty) \leq \lambda d(x, y), \text{ where } \lambda \in (0, 1)$$

has a unique fixed point and for every $x_0 \in X$ a sequence $\{T_n x_0\}_{n=1}^{\infty}$ converges to the fixed point. Subsequently, in 1962, Edelstein [2] proved the following version of the Banach contraction principle. Let (X, d) be a compact metric space and let $T : X \rightarrow X$ be a self-mapping. Assume that for all $x, y \in X$ with $x \neq y$,

$$d(x, Tx) < d(x, y) \implies d(Tx, Ty) < d(x, y).$$

Then T has a unique fixed point in X . In 2012, Wardowski [3] introduced a new type of contractions called F -contraction and proved a new fixed point theorem concerning F -contractions.

Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to be an F -contraction if there exists $\tau > 0$ such that

$$\forall x, y \in X, d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

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where $F : R^+ \rightarrow R$ is a mapping satisfying the following conditions:

- F1 F is strictly increasing, i.e. for all $x, y \in R^+$ such that $x < y, F(x) < F(y)$;
- F2 For each sequence $\{\alpha_n\}_{n=1}^\infty$ of positive numbers, $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$;
- F3 There exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

We denote by ζ , the set of all functions satisfying the conditions (F1) – (F3). Wardowski [3] then stated a modified version of the Banach contraction principle as follows. Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T_n x\}_{n=1}^\infty$ converges to x^* . In 2014, Hossein, P. and Poom, K. [15] defined the F-Suzuki contraction as follows and gave another version of theorem. Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to be an F-Suzuki-contraction if there exists $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$

$$d(x, Tx) < d(x, y) \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $F : R^+ \rightarrow R$ is a mapping satisfying the following conditions:

- F1 F is strictly increasing, i.e. for all $x, y \in R^+$ such that $x < y, F(x) < F(y)$;
- F2 For each sequence $\{\alpha_n\}_{n=1}^\infty$ of positive numbers, $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$;
- F3 F is continuous on $(0, \infty)$

We denote by ζ , the set of all functions satisfying the conditions (F1) – (F3).

Let T be a self-mapping of a complete metric space X into itself. Suppose $F \in \zeta$ and there exists $\tau > 0$ such that

$$\forall x, y \in X, d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T_n x_0\}_{n=1}^\infty$ converges to x^* .

Following this direction of research (see examples, [4, 5, 6, 7, 8, 9, 10, 16, 17]), in this paper, fixed point results of Piri and Kumam [11], Ahmad *et al.* [9], Suzuki [18] and Suzuki [19] are extended by introducing common fixed point problem for multivalued generalized F-Suzuki-contraction mappings in strong b-metric spaces.

Definition 1.1. (Hardy and Rogers [14])

- (1) There exist non-negative constants a_i , satisfying $\sum_{i=1}^5 a_i < 1$ such that, for each $x, y \in X, d(f(x), f(y)) < a_1 d(x, y) + a_2 d(x, f(x)) + a_3 d(y, f(y)) + a_4 d(x, f(y)) + a_5 d(y, f(x))$.
- (2) There exist monotonically decreasing functions $a_i(t) : (0, \infty) \rightarrow [0, 1)$ satisfying $\sum_{i=1}^5 a_i(t) < 1$ such that, for each $x, y \in X, x \neq y, d(f(x), f(y)) < a_1(d(x, y))d(x, f(x)) + a_2(d(x, y))d(y, f(y)) + a_3(d(x, y))d(x, f(y)) + a_4(d(x, y))d(y, f(x)) + a_5(d(x, y))d(x, y)$.

- (3) For each $x, y \in X, x \neq y,$
 $d(f(x), f(y)) < \max\{d(x, y), d(x, f(x)),$
 $d(y, f(y)), d(x, f(y)), d(y, f(x))\}$.

Lemma 1.1. [13] From definition 1.1, (1) \implies (2) \implies (3).

Denote by $CB(X)$, the collection of all nonempty closed and bounded subsets of X and let H be the Hausdorff metric with respect to the metric d ; that is,

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$$

for all $A, B \in CB(X)$, where $d(a, B) = \inf_{b \in B} d(a, b)$ is the distance from the point a to the subset B .

2. Main Results

Definition 2.1. Let \mathcal{U} be the family of all functions $F : R^+ \rightarrow R$ such that:

- (F1) F is strictly increasing, i.e. for all $x, y \in R^+$ such that $x < y, F(x) < F(y)$;
- (F2) for each sequence $\{\alpha_n\}_{n=1}^\infty$ of positive numbers, $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$;
- (F3) F is continuous on $(0, \infty)$.

Definition 2.2. Let Ψ be the family of all functions $\psi : [0, \infty) \rightarrow [0, \infty)$ such that ψ is continuous and $\psi(t) = 0$ iff $t = 0$.

Definition 2.3. Let (X, d) be a strong b–metric space. Mappings $T, S : X \rightarrow CB(X)$ are said to be multivalued generalized F-Suzuki-Contraction on (X, d) if there exists $F \in \mathcal{U}$ and $\psi \in \Psi$ such that, $\forall x, y \in X, x \neq y,$

$$\frac{1}{1+s} d(x, Tx) < d(x, y) \text{ and } \frac{1}{1+s} d(y, Sy) < d(y, STx)$$

$\implies \psi(N_\phi(x, y)) + F(s^4 H(Tx, Sy)) \leq F(N_\phi(x, y))$ in which

$$\begin{aligned} N_\phi(x, y) &= \phi_1(d(x, y))(d(x, y)) + \phi_2(d(x, y))(d(y, STx)) \\ &+ \phi_3(d(x, y)) \left(\frac{(d(y, Tx) + d(x, Sy))}{2s} \right) \\ &+ \phi_4(d(x, y)) \left(\frac{(d(x, STx) + H(STx, Sy))}{2s} \right) \\ &+ \phi_5(d(x, y))(H(STx, Sy) + H(STx, Tx)) \\ &+ \phi_6(d(x, y))(H(STx, Sy) + d(Tx, x)) \\ &+ \phi_7(d(x, y))(d(Tx, y)) + d(y, Sy) \end{aligned} \tag{2}$$

for which $\phi : R^+ \rightarrow [0, 1)$, with $\sum_{i=1}^7 \phi_i(d(x, y)) < 1$, is monotonically decreasing function.

Considering the definition $STx := \{Sy \subseteq CB(X) : \forall y \in Tx\}$, we have the following result.

Theorem 2.1. Let (X, d) be a complete strong b–metric space and let $T, S : X \rightarrow CB(X)$ be multivalued generalized F-Suzuki-Contraction mappings. Then T and S has a common

fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_n^\infty$ and $\{S^n x\}_n^\infty$ converge to x^* .

Proof Let $x_0 = x \in X$. Let $x_{n+1} \in Tx_n$ and $x_{n+2} \in Sx_{n+1} \forall n \in N$. If there exists $n \in N$ such that $d(x_n, Tx_n) = d(x_{n+1}, Sx_{n+1}) = 0$ then $x_{n+1} = x_n = x$ becomes a fixed point of T and S , respectively, therefore the proof is complete. Now, suppose that $d(x_n, Tx_n) > 0$ and $d(x_{n+1}, Sx_{n+1}) > 0 \forall n \in N$ then the proof will be divided in to two steps.

Step one. We show that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence.

Let

$$d(x_n, Tx_n) > 0 \text{ and } d(x_{n+1}, Sx_{n+1}) > 0 \forall n \in N. \quad (3)$$

therefore, we have that

$$\begin{aligned} \frac{1}{s+1}d(x_n, Tx_n) &< d(x_n, Tx_n) \text{ and} \\ \frac{1}{s+1}d(x_{n+1}, Sx_{n+1}) &< d(x_{n+1}, Sx_{n+1}) \forall n \in N. \end{aligned} \quad (4)$$

By Definition 2.3, we get

$$F(H(Tx_n, Sx_{n+1})) \leq F(N_\phi(x_n, x_{n+1})) - \psi(N_\phi(x_n, x_{n+1})).$$

Since that

$$\begin{aligned} N_\phi(x_n, x_{n+1}) &= \phi_1(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_2(d(x_n, x_{n+1}))(d(x_{n+1}, x_{n+2})) \\ &+ \phi_3(d(x_n, x_{n+1})) \left(\frac{d(x_n, x_{n+2})}{2s} \right) + \phi_4(d(x_n, x_{n+1})) \left(\frac{d(x_n, x_{n+2})}{2s} \right) \\ &+ \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_6(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) \\ &+ \phi_7(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) \\ &\leq \phi_1(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_2(d(x_n, x_{n+1}))(d(x_{n+1}, x_{n+2})) \\ &+ \phi_3(d(x_n, x_{n+1})) \left(\frac{d(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2})}{2s} \right) \\ &+ \phi_4(d(x_n, x_{n+1})) \left(\frac{d(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2})}{2s} \right) \\ &+ \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_6(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) \\ &+ \phi_7(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) \\ &\leq \phi_1(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_2(d(x_n, x_{n+1}))(d(x_{n+1}, x_{n+2})) \\ &+ \phi_3(d(x_n, x_{n+1})) \left(\frac{s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]}{2s} \right) \\ &+ \phi_4(d(x_n, x_{n+1})) \left(\frac{s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]}{2s} \right) \\ &+ \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_6(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) \\ &+ \phi_7(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) \\ &\leq \phi_1(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_2(d(x_n, x_{n+1}))(d(x_{n+1}, x_{n+2})) \\ &+ \phi_3(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_3(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) \\ &+ \phi_4(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_4(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) \\ &+ \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_6(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) \\ &+ \phi_7(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) \\ &= [\phi_1(d(x_n, x_{n+1})) + \phi_3(d(x_n, x_{n+1})) + \phi_4(d(x_n, x_{n+1})) \end{aligned}$$

$$\begin{aligned} &+ \phi_6(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) \\ &+ [\phi_2(d(x_n, x_{n+1})) + \phi_3(d(x_n, x_{n+1})) + \phi_4(d(x_n, x_{n+1})) \\ &+ \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_7(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) \\ &= \phi'(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi''(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) \end{aligned} \quad (5)$$

then by (5) and definition 2.3, we get

$$\begin{aligned} F(d(x_{n+1}, x_{n+2})) &\leq F(\phi'(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi''(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1}))) \\ &- \psi(\phi'(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi''(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1}))). \end{aligned} \quad (6)$$

On contrary, if $d(x_{n+1}, x_{n+2}) > d(x_n, x_{n+1})$, then

$$\begin{aligned} \phi'(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) \\ + \phi''(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) &< d(x_{n+1}, x_{n+2}) \end{aligned}$$

and therefore (6) becomes

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_{n+1}, x_{n+2})) - \psi(d(x_{n+1}, x_{n+2})).$$

But, from (3) and the fact that $\psi(d(x_{n+1}, x_{n+2})) > 0$, this is a contradiction. Thus, we conclude that

$$\begin{aligned} F(d(x_{n+1}, x_{n+2})) &\leq F(d(x_n, x_{n+1})) - \psi(d(x_n, x_{n+1})) \\ &< F(d(x_n, x_{n+1})). \end{aligned} \quad (7)$$

By (7) and Definition 2.1(F1), we have that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \forall n \in N. \quad (8)$$

Therefore $\{d(x_n, x_{n+1})\}$ is a nonnegative decreasing sequence of real numbers. Thus there exists $\gamma \geq 0$ such that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = \gamma$. From (7) as $n \rightarrow \infty$, we have that

$$F(\gamma) \leq F(\gamma) - \psi(\gamma).$$

This implies that $\psi(\gamma) = 0$ and thus $\gamma = 0$. Consequently we arrive at

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (9)$$

Now, we claim that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. On contrary, we assume that there exists $\epsilon > 0$ and $n, m \in N$ such that, for all $n \geq n_\epsilon$ and $n_\epsilon < n < m$,

$$d(x_n, x_m) \geq \epsilon \text{ and } d(x_{n-1}, x_m) < \epsilon. \quad (10)$$

It implies that

$$\begin{aligned} \epsilon &\leq d(x_n, x_m) \leq d(x_n, x_{n-1}) + sd(x_{n-1}, x_m) \\ &< d(x_n, x_{n-1}) + s\epsilon. \end{aligned} \quad (11)$$

By (11) and (9), we have that

$$\epsilon \leq \limsup_{n \rightarrow \infty} d(x_n, x_m) < s\epsilon. \quad (12)$$

By triangle inequality, we have that

$$\begin{aligned} \epsilon &\leq d(x_n, x_m) \leq d(x_n, x_{m+1}) + sd(x_{m+1}, x_m) \\ &\leq d(x_n, x_m) + 2sd(x_{m+1}, x_m). \end{aligned} \tag{13}$$

By (9),(10), (12) and (13), we have that

$$\epsilon \leq \limsup_{n \rightarrow \infty} d(x_n, x_{m+1}) < s\epsilon. \tag{14}$$

Similarly, we have that

$$\begin{aligned} \epsilon &\leq d(x_n, x_m) \leq d(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ &\leq sd(x_n, x_m) + (s^2 + 1)d(x_n, x_{n+1}). \end{aligned} \tag{15}$$

By (9),(10), (12) and (15), we have that

$$\epsilon \leq \limsup_{n \rightarrow \infty} d(x_n, x_{n+1}) < s\epsilon. \tag{16}$$

Observe that

$$\begin{aligned} d(x_n, x_{m+1}) &\leq d(x_n, x_{n+1}) + sd(x_{n+1}, x_{m+1}) \\ &\leq d(x_n, x_{n+1}) + s[d(x_{n+1}, x_m) + sd(x_{m+1}, x_m)] \\ &\leq d(x_n, x_{n+1}) + s[d(x_n, x_{n+1}) + sd(x_n, x_m) \\ &\quad + sd(x_{m+1}, x_m)]. \end{aligned} \tag{17}$$

By (17), we have that

$$\frac{\epsilon}{s} \leq \limsup_{n \rightarrow \infty} d(x_{n+1}, x_{m+1}) < s^2\epsilon. \tag{18}$$

By (9)and (10), we select $n_\epsilon > 0 \in N$ such that

$$\begin{aligned} \frac{1}{s+1}d(x_n, Tx_n) &< \frac{1}{s+1}\epsilon < \epsilon \leq d(x_n, x_m) \quad \forall n \geq n(\epsilon) \\ \Leftrightarrow \frac{1}{s+1}d(x_n, Tx_n) &< \frac{1}{s+1}\epsilon < d(x_n, x_m) \\ \forall n \geq n(\epsilon) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{s+1}d(x_{n+1}, Sx_{n+1}) &< \frac{1}{s+1}\epsilon < \frac{\epsilon}{s} \leq d(x_{n+1}, x_{m+1}) \quad \forall n \geq n_\epsilon \\ \Leftrightarrow \frac{1}{s+1}d(x_{n+1}, Sx_{n+1}) &< \frac{1}{s+1}\epsilon \\ &< d(x_{n+1}, x_{m+1}) \quad \forall n \geq n_\epsilon \end{aligned}$$

It follows that from Definition 2.3, we have, for every $n \geq n_\epsilon$

$$F(H(x_{n+1}, x_{m+1})) \leq F(N_\phi(x_n, x_m)) - \psi(N_\phi(x_n, x_m)). \tag{19}$$

Since that

$$\begin{aligned} d(x_n, x_m) &\leq N_\phi(x_n, x_m) \\ &= \phi_1(d(x_n, x_m))(d(x_n, x_m)) + \phi_2(d(x_n, x_m))(d(x_{n+2}, x_m)) \\ &\quad + \phi_3(d(x_n, x_m))\left(\frac{d(x_{n+1}, x_m) + d(x_n, x_{m+1})}{2s}\right) \\ &\quad + \phi_4(d(x_n, x_m))\left(\frac{d(x_{n+2}, x_n) + d(x_{n+2}, x_{m+1})}{2s}\right) \\ &\quad + \phi_5(d(x_n, x_m))(d(x_{n+2}, x_{m+1}) + d(x_{n+2}, x_{n+1})) \end{aligned}$$

$$\begin{aligned} &+ \phi_6(d(x_n, x_m))(d(x_{n+2}, x_{m+1}) + d(x_n, x_{n+1})) \\ &+ \phi_7(d(x_n, x_m))(d(x_m, x_{n+1}) + d(x_m, x_{m+1})) \\ &\leq \phi_1(d(x_n, x_m))(d(x_n, x_m)) + \phi_2(d(x_n, x_m))(d(x_{n+2}, x_{n+1}) + sd(x_{n+1}, x_m)) \\ &\quad + \phi_3(d(x_n, x_m))\left(\frac{d(x_{n+1}, x_m) + d(x_n, x_{m+1})}{2s}\right) \\ &\quad + \phi_4(d(x_n, x_m))\left(\frac{(d(x_{n+2}, x_{n+1}) + sd(x_{n+1}, x_n) + d(x_{n+2}, x_{n+1})) + sd(x_{n+1}, x_{m+1}))}{2s}\right) \\ &\quad + \phi_5(d(x_n, x_m))(d(x_{n+2}, x_{n+1}) + sd(x_{n+1}, x_{m+1}) + d(x_{n+2}, x_{n+1})) \\ &\quad + \phi_6(d(x_n, x_m))(d(x_{n+2}, x_{n+1}) + sd(x_{n+1}, x_{m+1}) + d(x_n, x_{n+1})) \\ &\quad + \phi_7(d(x_n, x_m))(d(x_m, x_{n+1}) + d(x_m, x_{m+1})). \end{aligned} \tag{20}$$

By (12), (14), (16), (18) and (20), we have that

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(x_n, x_m) &\leq \limsup_{n \rightarrow \infty} N_\phi(x_n, x_m) < \phi_1(\epsilon)(s\epsilon) + \phi_2(\epsilon)(s^2\epsilon) \\ &\quad + \phi_3(\epsilon)(\epsilon) + \phi_4(\epsilon)\left(\frac{s^2\epsilon}{2}\right) + \phi_5(\epsilon)(s^3\epsilon) + \phi_6(\epsilon)(s^3\epsilon) + \phi_7(\epsilon)(s\epsilon) \\ &\leq \max\{s\epsilon, s^2\epsilon, \epsilon, \frac{s\epsilon}{2}, s^3\epsilon, s\epsilon\} \\ &= s^3\epsilon \end{aligned}$$

and therefore

$$\epsilon \leq \limsup_{n \rightarrow \infty} N_\phi(x_n, x_m) < s^3\epsilon. \tag{21}$$

Similarly

$$\epsilon \leq \liminf_{n \rightarrow \infty} N_\phi(x_n, x_m) < s^3\epsilon. \tag{22}$$

By (19), (21) and (22), we have that

$$\begin{aligned} F(s^3\epsilon) &= F(s^4\frac{\epsilon}{s}) \leq F(s^4\limsup_{n \rightarrow \infty} d(x_{n+1}, x_{m+1})) \\ &\leq F(\limsup_{n \rightarrow \infty} N_\phi(x_n, x_m)) - \psi(\limsup_{n \rightarrow \infty} N_\phi(x_n, x_m)) \\ &\leq F(s^3\epsilon) - \psi(\epsilon). \end{aligned} \tag{23}$$

By (23) and the fact that $\epsilon > 0$, this is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in X . By completeness of (X, d) , $\{x_n\}_{n=1}^\infty$ and $\{x_{n+1}\}_{n=1}^\infty$ converge to some point $x^* \in X$, that is,

$$\lim_{n \rightarrow \infty} d(x_n, x^*) = 0 \text{ and } \lim_{n \rightarrow \infty} d(x_{n+1}, x^*) = 0. \tag{24}$$

There exists increasing sequences $\{n_k\}, \{n+1_k\} \subset N$ such that $x_{n_k} \in Tx^*$ and $x_{n+1_k} \in Sx^*$ for all $k \in N$. Since Tx^* and Sx^* are closed and

$$\lim_{n \rightarrow \infty} d(x_{n_k}, x^*) = 0 \text{ and } \lim_{n \rightarrow \infty} d(x_{n+1_k}, x^*) = 0,$$

we get $x^* \in Tx^*$ and $x^* \in Sx^*$.

Step two. We show that x^* is a common fixed point of T and S . It suffices to show that

$$\begin{aligned} \frac{1}{1+s}d(x_n, Tx_n) &< d(x_n, x^*) \text{ and } \frac{1}{1+s}d(x_{n+1}, Sx_{n+1}) < d(x_{n+1}, x^*), \\ &\text{for every } n \in N, \end{aligned} \tag{25}$$

implies

$$F(d(Tx^*, x^*)) \leq F(N_\phi(x^*, Tx^*)) - \psi(N_\phi(x^*, Tx^*))$$

and

$$F(d(Sx^*, x^*)) \leq F(N_\phi(Sx^*, x^*)) - \psi(N_\phi(Sx^*, x^*)),$$

respectively.

On contrary, suppose there exists $m \in N$ such that

$$\frac{1}{1+s}d(x_m, Tx_m) \geq d(x_m, x^*) \text{ or } \frac{1}{1+s}d(x_{m+1}, Sx_{m+1}) \geq d(x_{m+1}, x^*). \tag{26}$$

By (26), we have that

$$(s+1)d(x_m, x^*) \leq d(x_m, Tx_m) \leq d(x_m, x^*) + sd(Tx_m, x^*)$$

or

$$(s+1)d(x_{m+1}, x^*) \leq d(x_{m+1}, Sx_{m+1}) \leq d(x_{m+1}, x^*) + sd(Sx_{m+1}, x^*),$$

and therefore

$$\begin{aligned} d(x_m, x^*) &\leq d(Tx_m, x^*) = d(x_{m+1}, x^*) \text{ and} \\ d(x_{m+1}, x^*) &\leq d(Sx_{m+1}, x^*) = d(x_{m+2}, x^*). \end{aligned} \tag{27}$$

By (8), (26) and (27), this is a contradiction. Hence, (25) holds, and therefore

$$\begin{aligned} F(d(x_{n+1}, x^*)) &= F(H(Tx_n, Sx^*)) \\ &\leq F(N_\phi(x_n, x^*)) - \psi(N_\phi(x_n, x^*)), \end{aligned} \tag{28}$$

and

$$\begin{aligned} F(d(x_{n+2}, x^*)) &= F(H(Sx_{n+1}, Tx^*)) \\ &\leq F(N_\phi(x_{n+1}, x^*)) - \psi(N_\phi(x_{n+1}, x^*)). \end{aligned} \tag{29}$$

Since that

$$\begin{aligned} d(x^*, Tx^*) &\leq N_\phi(x_n, x^*) \\ &= \phi_1(d(x_n, x^*))(d(x_n, x^*)) + \phi_2(d(x_n, x^*))(d(x_{n+2}, x^*)) \\ &+ \phi_3(d(x_n, x^*)) \left(\frac{d(x_{n+1}, x^*) + d(x_n, Sx^*)}{2s} \right) \\ &+ \phi_4(d(x_n, x^*)) \left(\frac{d(x_n, Sx^*) + d(Sx^*, x_{n+2})}{2s} \right) \\ &+ \phi_5(d(x_n, x^*))(d(Sx^*, x_{n+2}) + d(x_{n+1}, Sx^*)) \\ &+ \phi_6(d(x_n, x^*))(d(x_n, x_{n+1}) + d(x_{n+2}, Tx^*)) \\ &+ \phi_7(d(x_n, x^*))(d(Tx^*, x^*) + d(x^*, x_{n+1})) \\ &\leq \max\{d(x_n, x^*), d(x_{n+2}, x^*), \\ &\frac{d(x_{n+1}, x^*) + d(x_n, Sx^*)}{2s}, \\ &\frac{d(x_n, Sx^*) + sd(Sx^*, x_{n+2}) + d(Sx^*, x_{n+2})}{2s}, \\ &d(Sx^*, x_{n+2}) + d(x_{n+1}, Sx^*), d(x_n, x_{n+1}) + d(x_{n+2}, Tx^*), \end{aligned}$$

$$d(Tx^*, x^*) + d(x^*, x_{n+1})\} \tag{30}$$

and

$$\begin{aligned} d(x^*, Sx^*) &\leq N_\phi(x_{n+1}, x^*) \\ &= \phi_1(d(x_{n+1}, x^*))(d(x_{n+1}, x^*)) + \phi_2(d(x_{n+1}, x^*))(d(x^*, x_{n+3})) \\ &+ \phi_3(x_{n+1}, x^*) \left(\frac{d(x_{n+2}, x^*) + d(x_{n+1}, x^*)}{2s} \right) \\ &+ \phi_4(d(x_{n+1}, x^*)) \left(\frac{d(x_{n+1}, Sx^*) + d(Sx^*, x_{n+3})}{2s} \right) \\ &+ \phi_5(d(x_{n+1}, x^*))(d(x_{n+3}, Sx^*) + d(x_{n+2}, Sx^*)) \\ &+ \phi_6(d(x_{n+1}, x^*))(d(x_{n+1}, x_{n+2}) + d(x_{n+3}, Sx^*)) \\ &+ \phi_7(d(x_{n+1}, x^*))(d(Sx^*, x^*) + d(x^*, x_{n+2})) \\ &\leq \max\{d(x_{n+1}, x^*), d(x^*, x_{n+3}), \\ &\frac{d(x_{n+2}, x^*) + d(x_{n+1}, x^*)}{2s}, \\ &\frac{d(x_{n+1}, x_{n+2}) + sd(x_{n+2}, Sx^*) + d(Sx^*, x_{n+3})}{2s}, \\ &d(x_{n+3}, Sx^*) + d(x_{n+2}, Sx^*), d(x_{n+1}, x_{n+2}) + d(x_{n+3}, Sx^*), \\ &d(Sx^*, x^*) + d(x^*, x_{n+2})\}. \end{aligned} \tag{31}$$

By (24) and (30), we have that

$$\lim_{n \rightarrow \infty} N_\phi(x_n, x^*) = d(Tx^*, x^*).$$

By (24) and (31), we have that

$$\lim_{n \rightarrow \infty} N_\phi(x_{n+1}, x^*) = d(Sx^*, x^*).$$

By (28) and (29) and by the continuity of F and ψ , we have that

$$F(d(x^*, Tx^*)) \leq F(N_\phi(x^*, Tx^*)) - \psi(N_\phi(x^*, Tx^*)),$$

and

$$F(d(x^*, Sx^*)) \leq F(N_\phi(x^*, Sx^*)) - \psi(N_\phi(x^*, Sx^*)).$$

Hence, since Tx^* and Sx^* are closed then we have $x^* \in Tx^*$ and $x^* \in Sx^*$, that is, x^* is a fixed point of T and S .

In Theorem 2.1, when $T = S = U$, then we have the following result.

Corollary 2.1.1. *Let (X, d) be a complete strong b -metric space and let $U : X \rightarrow CB(X)$ be a multivalued generalized F -Suzuki-Contraction mapping. Then U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^\infty$ converges to x^* .*

In Corollary 2.1.1, when U is a single-valued then we have another new result as follows.

Corollary 2.1.2. *Let (X, d) be a complete strong b -metric space and let $U : X \rightarrow X$ be a single-valued generalized F -Suzuki-Contraction mapping. Then U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^\infty$ converges to x^* .*

In Theorem 2.1, when T and S are two single-valued then the

following result holds.

Corollary 2.1.3. *Let (X, d) be a complete strong b -metric space and let $T, S : X \rightarrow X$ be two single-valued generalized F -Suzuki-Contraction mappings. Then T and S have a common fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^\infty$ and $\{S^n x\}_{n=1}^\infty$ converge to x^* .*

In Theorem 2.1, when (X, d) is a complete b -metric space then the following new result holds.

Corollary 2.1.4. *Let (X, d) be a complete b -metric space and let $T, S : X \rightarrow X$ be two single-valued generalized F -Suzuki-Contraction mappings. Then T and S have a common fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^\infty$ and $\{S^n x\}_{n=1}^\infty$ converge to x^* .*

In corollary 2.1.4, when $T = S = U$, then we have the following result.

Corollary 2.1.5. *Let (X, d) be a complete b -metric space and let $U : X \rightarrow CB(X)$ be a multivalued generalized F -Suzuki-Contraction mapping. Then U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^\infty$ converges to x^* .*

Corollary 2.1.6. *Let (X, d) be a complete strong b -metric space and let $U : X \rightarrow CB(X)$ be a multivalued generalized F -Suzuki-Contraction mapping such that there exists $F \in \mathcal{U}$ and $\psi \in \Psi, \forall x, y \in X, x \neq y, \frac{1}{s+1}d(x, Ux) < d(x, y) \Rightarrow \psi(N(x, y)) + F(s^4 d(Ux, Uy)) \leq F(N(x, y))$ in which*

$$N(x, y) = \max\{d(x, y), d(y, U^2x), \frac{(d(y, Ux) + d(x, Uy))}{2s}, \frac{(d(x, Uy)) + d(U^2x, Uy)}{2s}, d(U^2x, Uy) + d(Uy, Ux), d(U^2x, Uy) + d(Ux, x), d(Ux, y)) + d(y, Uy)\}. \tag{32}$$

Then U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^\infty$ converges to x^* .

Proof from Lemma 1.1, since (2) \Rightarrow (32) then by the corollary 2.1.1 the result follows immediately.

Corollary 2.1.7. *Let (X, d) be a complete strong b -metric space and let $U : X \rightarrow X$ be a single-valued generalized F -Suzuki-Contraction mapping such that there exists $F \in \mathcal{U}$ and $\psi \in \Psi, \forall x, y \in X, x \neq y, \frac{1}{s+1}d(x, Ux) < d(x, y) \Rightarrow \psi(N(x, y)) + F(s^4 d(Ux, Uy)) \leq F(N(x, y))$ in which*

$$N(x, y) = \max\{d(x, y), d(y, U^2x), \frac{(d(y, Ux) + d(x, Uy))}{2s}, \frac{(d(x, Uy)) + d(U^2x, Uy)}{2s}, d(U^2x, Uy) + d(Uy, Ux), d(U^2x, Uy) + d(Ux, x), d(Ux, y)) + d(y, Uy)\}. \tag{33}$$

Then U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^\infty$ converges to x^* .

Proof from Lemma 1.1, since (2) \Rightarrow (33) then by the corollary 2.1.2 the result holds.

Corollary 2.1.8. *Let (X, d) be a complete strong b -metric space and let $T, S : X \rightarrow X$ be two single-valued generalized F -Suzuki-Contraction mappings such that there exists $F \in \mathcal{U}$ and $\psi \in \Psi, \forall x, y \in X, x \neq y, \frac{1}{s+1}d(x, Tx) < d(x, y)$ and $\frac{1}{s+1}d(y, Sx) < d(y, STx) \Rightarrow \psi(N(x, y)) + F(s^4 H(Tx, Sy)) \leq F(N(x, y))$ in which*

$$N(x, y) = \max\{d(x, y), d(y, STx), \frac{(d(y, Tx) + d(x, Sy))}{2s}, \frac{(d(x, Sy)) + d(STx, Sy)}{2s}, d(STx, Sy) + d(Sy, Tx), d(STx, Sy) + d(Tx, x), d(Tx, y)) + d(y, Sy)\}. \tag{34}$$

Then T and S have a common fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^\infty$ and $\{S^n x\}_{n=1}^\infty$ converge to x^* .

Proof from Lemma 1.1, since (2) \Rightarrow (34) then by the corollary 2.1.4 the result holds.

Corollary 2.1.9. *Let (X, d) be a complete b -metric space and let $U : X \rightarrow CB(X)$ be a multivalued generalized F -Suzuki-Contraction mapping such that there exists $F \in \mathcal{U}$ and $\psi \in \Psi, \forall x, y \in X, x \neq y, \frac{1}{2s}d(x, Ux) < d(x, y) \Rightarrow \psi(N(x, y)) + F(s^6 d(Ux, Uy)) \leq F(N(x, y))$ in which*

$$N(x, y) = \max\{d(x, y), d(y, U^2x), \frac{(d(y, Ux) + d(x, Uy))}{2s}, \frac{(d(x, Uy)) + d(U^2x, Uy)}{2s}, d(U^2x, Uy) + d(Uy, Ux), d(U^2x, Uy) + d(Ux, x), d(Ux, y)) + d(y, Uy)\}. \tag{35}$$

Then U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^\infty$ converges to x^* .

Proof from Lemma 1.1, since (2) \Rightarrow (35) then by the corollary 2.1.5 the result holds.

3. Example

Let $X = [0, 1]$. $T, S : [0, 1] \rightarrow CB([0, 1])$ be defined by $Tx = [0, \frac{x}{2}]$ and $Sy = [0, \frac{y}{2}]$ such that $STx = [0, \frac{x}{8}]$ for all $x \in [0, 1]$. Let d be the usual metric on X . Taking $F(t) = \frac{t}{10}$ and let $x < y$, then $\forall x, y \in [0, 1] d(x, y) > 0$ and $d(y, STx) = |y - \frac{x}{8}| > |y - \frac{y}{8}| = \frac{7}{8}y > \frac{y}{4}$. Now, for $s = 1$, we have that $\frac{1}{2}d(x, Tx) = 0 < d(x, y)$ and $\frac{1}{2}d(y, Sy) = \frac{y}{4} < d(y, STx)$. Without lose of generality, let $\phi_1(d(x, y)) = \phi_2(d(x, y)) = \phi_3(d(x, y)) = \frac{1}{5}$; and $\phi_4(d(x, y)) = \phi_5(d(x, y)) = \phi_6(d(x, y)) = \phi_7(d(x, y)) = \frac{1}{10^2}$. Therefore, we have that

$$\begin{aligned} F(H(Tx, Sy)) &= \ln(H(Tx, Sy)) + H(Tx, Sy) \\ &= \frac{1}{10} \left| \frac{y}{2} - \frac{x}{4} \right| = \frac{1}{10} \left| y - \frac{y}{2} - \frac{x}{4} \right| \\ &\leq \frac{1}{10} \left(\left| y - \frac{x}{4} \right| + \left| x - \frac{y}{2} \right| \right) \\ &= \frac{1}{10} \left(\frac{|y - \frac{x}{4}| + |x - \frac{y}{2}|}{2} \right) + \frac{1}{10} \left(\frac{|y - \frac{x}{4}| + |x - \frac{y}{2}|}{2} \right) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{10} \left(\frac{|y - \frac{x}{4}| + |x - \frac{y}{2}|}{2} \right) + \frac{1}{10} \left(\frac{|y - \frac{x}{4}| + |x - \frac{x}{8}| + |\frac{x}{8} - \frac{y}{2}|}{2} \right) \\
&= \frac{1}{10} \left(\frac{|y - \frac{x}{4}| + |x - \frac{y}{2}|}{2} \right) + \frac{1}{10} \left(\frac{|\frac{x}{8} - \frac{y}{2}| + |x - \frac{x}{8}|}{2} \right) \\
&+ \frac{1}{10} \left(\frac{|y}{2} - \frac{x}{8} \right) \leq \frac{1}{10} \left(\frac{|y - \frac{x}{4}| + |x - \frac{y}{2}|}{2} \right) \\
&+ \frac{1}{10} \left(\frac{|\frac{x}{8} - \frac{y}{2}| + |x - \frac{x}{8}|}{2} \right) + \frac{1}{10} \left(\left| \frac{y}{2} - \frac{x}{8} \right| + \left| \frac{x}{8} - \frac{x}{4} \right| \right) \\
&= \frac{1}{5} \left(\frac{|y - \frac{x}{4}| + |x - \frac{y}{2}|}{2} \right) + \frac{1}{5} \left(\frac{|\frac{x}{8} - \frac{y}{2}| + |x - \frac{x}{8}|}{2} \right) \\
&+ \frac{1}{10} \left(\left| \frac{y}{2} - \frac{x}{8} \right| + \left| \frac{x}{8} - \frac{x}{4} \right| \right) + \frac{1}{10^2} (|x - y|) + \frac{1}{10^2} \left(\left| y - \frac{x}{8} \right| \right) \\
&+ \frac{1}{10^2} \left(\left| \frac{x}{8} - \frac{y}{2} \right| + \left| \frac{x}{4} - x \right| \right) + \frac{1}{10^2} \left(\left| \frac{x}{4} - y \right| + \left| y - \frac{y}{2} \right| \right) \\
&- \frac{1}{10^2} \left[(|x - y|) + \left(\left| y - \frac{x}{8} \right| \right) + \left(\left| \frac{x}{8} - \frac{y}{2} \right| + \left| \frac{x}{4} - x \right| \right) \right] \\
&+ \left(\left| \frac{x}{4} - y \right| + \left| y - \frac{y}{2} \right| \right) - \frac{1}{10} \left(\left| \frac{y}{2} - \frac{x}{8} \right| + \left| \frac{x}{8} - \frac{x}{4} \right| \right) \\
&- \frac{1}{10} \left(\frac{|y - \frac{x}{4}| + |x - \frac{y}{2}|}{2} \right) - \frac{1}{10} \left(\frac{|\frac{x}{8} - \frac{y}{2}| + |x - \frac{x}{8}|}{2} \right). \\
&= \phi_1(d(x, y))(d(x, y)) + \phi_2(d(x, y))(d(y, STx)) \\
&+ \phi_3(d(x, y)) \left(\frac{(d(y, Tx)) + d(x, Sy)}{2s} \right) + \phi_4(d(x, y)) \\
&\left(\frac{(d(x, STx)) + d(STx, Sy)}{2s} \right) \\
&+ \phi_5(d(x, y))(d(STx, Sy) + d(STx, Tx)) \\
&+ \phi_6(d(x, y))(d(STx, Sy) + d(Tx, x)) \\
&+ \phi_7(d(x, y))(d(Tx, y)) + d(y, Sy) - \psi(N_\phi(x, y)).
\end{aligned}$$

4. Conclusion

Fixed point results of Piri and Kumam [11], Ahmad *et al.* [9], Suzuki [18] and Suzuki [19] are extended by introducing common fixed point problem for multivalued generalized F-Suzuki-contraction mappings in strong b-metric spaces. In specific, Corollary 2.1.1 and corollary 2.1.2 generalize and extend the work of Ahmad *et al.* [9] and Kumam and Hossein [5], respectively.

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