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Original Research

Common Fixed Point Theorems for Multivalued Generalized F-Suzuki-Contraction Mappings in Complete Strong $b-$Metric Spaces

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Abstract

This paper introduces a new version of multivalued generalized F-Suzuki-Contraction mapping and then establish some new common fixed point theorems for these new multivalued generalized F-Suzuki-Contraction Mappings in complete strong $b-$Metric Spaces.

Keywords: Common Fixed Point Problem, Multivalued Generalized F-Suzuki-Contraction Mapping, Complete Strong $b-$metric Space.

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1. Introduction

Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \rightarrow \mathbb{R}^*$ is said to be a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space with constant s.
A strong $b-$metric is a semimetric space (X, d) if there exists $s \geq 1$ for which d satisfies the following triangular inequality.

$$d(x, y) \leq d(x, z) + sd(z, y), \ for\ each\ x, y, z \in X. \tag{1}$$

In 1922, a mathematician Banach [1] proved a very important result regarding a contraction mapping, known as the Banach contraction principle, which states that every self-mapping T defined on a complete metric space (X, d) satisfying

$$\forall x, y \in X, d(Tx, Ty) \leq \lambda d(x, y), \ where\ \lambda \in (0, 1)$$

has a unique fixed point and for every $x_0 \in X$ a sequence $\{T_n x_0\}_{n=1}^{\infty}$ converges to the fixed point. Subsequently, in 1962, Edelstein [2] proved the following version of the Banach contraction principle. Let (X, d) be a compact metric space and let $T : X \rightarrow X$ be a self-mapping. Assume that for all $x, y \in X$ with $x \neq y$,

$$d(x, Tx) < d(x, y) \Longrightarrow d(Tx, Ty) < d(x, y).$$

Then T has a unique fixed point in X. In 2012, Wardowski [3] introduced a new type of contractions called F-contraction and proved a new fixed point theorem concerning F-contractions.

Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to be an F-contraction if there exists $\tau > 0$ such that

$$\forall x, y \in X, d(Tx, Ty) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

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where $F : R^+ \to R$ is a mapping satisfying the following conditions:

F1 F is strictly increasing, i.e. for all $x, y \in R^+$ such that $x < y$, $F(x) < F(y)$;
F2 For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
F3 There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We denote by ζ, the set of all functions satisfying the conditions $(F1) - (F3)$. Wardowski [3] then stated a modified version of the Banach contraction principle as follows. Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T_n x\}_{n=1}^{\infty}$ converges to x^*. In 2014, Hossein, P. and Poom, K. [15] defined the F-Suzuki contraction as follows and gave another version of theorem. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-Suzuki-contraction if there exists $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$

$$d(x, Tx) < d(x, y) \Longrightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $F : R^+ \to R$ is a mapping satisfying the following conditions:

F1 F is strictly increasing, i.e. for all $x, y \in R^+$ such that $x < y$, $F(x) < F(y)$;
F2 For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
F3 F is continuous on $(0, \infty)$

We denote by ζ, the set of all functions satisfying the conditions $(F1) - (F3)$.

Let T be a self-mapping of a complete metric space X into itself. Suppose $F \in \zeta$ and there exists $\tau > 0$ such that

$$\forall x, y \in X, d(Tx, Ty) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T_n x_0\}_{n=1}^{\infty}$ converges to x^*.

Following this direction of research (see examples, [4, 5, 6, 7, 8, 9, 10, 16, 17]), in this paper, fixed point results of Piri and Kumam [11], Ahmad *et al.* [9], Suzuki [18] and Suzuki [19] are extended by introducing common fixed point problem for multivalued generalized F-Suzuki-contraction mappings in strong b-metric spaces.

Definition 1.1. *(Hardy and Rogers [14])*

(1) There exist non-negative constants a, satisfying $\sum_{i=1}^{5} a_i < 1$ *such that, for each* $x, y \in X$, $d(f(x), f(y)) < a_1 d(x, y) + a_2 d(x, f(x)) + a_3 d(y, f(y)) + a_4 d(x, f(y)) + a_5 d(y, f(x))$.

(2) There exist monotonically decreasing functions $a_i(t) : (0, \infty) \to [0, 1)$ *satisfying* $\sum_{i=1}^{5} a_i(t) < 1$ *such that, for each* $x, y \in X$, $x \neq y$, $d(f(x), f(y)) < a_1(d(x, y))d(x, f(x)) + a_2(d(x, y))d(y, f(y)) + a_3(d(x, y))d(x, f(y)) + a_4(d(x, y))d(y, f(x)) + a_5(d(x, y))d(x, y)$.

(3) For each $x, y \in X$, $x \neq y$, $d(f(x), f(y)) < max\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}$.

Lemma 1.1. *[13] From definition 1.1,* (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

Denote by $CB(X)$, the collection of all nonempty closed and bounded subsets of X and let H be the Hausdorff metric with respect to the metric d; that is,

$$H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}$$

for all $A, B \in CB(X)$, where $d(a, B) = \inf_{b\in B} d(a, b)$ is the distance from the point a to the subset B.

2. Main Results

Definition 2.1. *Let* $\mho$ *be the family of all functions* $F : R^+ \to R$ *such that:*

(F1) F is strictly increasing, i.e. for all $x, y \in R^+$ *such that* $x < y$, $F(x) < F(y)$;
(F2) for each sequence $\{\alpha_n\}_{n=1}^{\infty}$ *of positive numbers,* $\lim_{n\to\infty} \alpha_n = 0$ *if and only if* $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) F is continuous on $(0, \infty)$.

Definition 2.2. *Let* Ψ *be the family of all functions* $\psi : [0, \infty) \to [0, \infty)$ *such that* ψ *is continuous and* $\psi(t) = 0$ *iff* $t = 0$.

Definition 2.3. *Let* (X, d) *be a strong b−metric space. Mappings* $T, S : X \to CB(X)$ *are said to be multivalued generalized F-Suzuki-Contraction on* (X, d) *if there exists* $F \in \mho$ *and* $\psi \in \Psi$ *such that,* $\forall x, y \in X$, $x \neq y$,

$$\frac{1}{1+s} d(x, Tx) < d(x, y) \text{ and } \frac{1}{1+s} d(y, Sy) < d(y, STx)$$

$\Rightarrow \psi(N_\phi(x, y)) + F(s^4 H(Tx, Sy)) \leq F(N_\phi(x, y))$ *in which*

$$\begin{aligned} N_\phi(x, y) &= \phi_1(d(x, y))(d(x, y)) + \phi_2(d(x, y))(d(y, STx)) \\ &+ \phi_3(d(x, y))\left(\frac{(d(y, Tx)) + d(x, Sy)}{2s}\right) \\ &+ \phi_4(d(x, y))\left(\frac{(d(x, STx)) + H(STx, Sy)}{2s}\right) \\ &+ \phi_5(d(x, y))(H(STx, Sy) + H(STx, Tx)) \\ &+ \phi_6(d(x, y))(H(STx, Sy) + d(Tx, x)) \\ &+ \phi_7(d(x, y))(d(Tx, y)) + d(y, Sy)) \end{aligned} \tag{2}$$

for which $\phi : R^+ \to [0, 1)$, *with* $\sum_{i=1}^{7} \phi_i(d(x, y)) < 1$, *is monotonically decreasing function.*

Comsidering the definition $STx := \{Sy \subseteq CB(X) : \forall y \in Tx\}$, we have the following result.

Theorem 2.1. *Let* (X, d) *be a complete strong b−metric space and let* $T, S : X \to CB(X)$ *be multivalued generalized F-Suzuki-Contraction mappings. Then T and S has a common* *fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_n^\infty$ and $\{S^n x\}_n^\infty$ converge to x^*.*

Proof Let $x_0 = x \in X$. Let $x_{n+1} \in Tx_n$ and $x_{n+2} \in Sx_{n+1}$ $\forall n \in N$. If there exists $n \in N$ such that $d(x_n, Tx_n) = d(x_{n+1}, Sx_{n+1}) = 0$ then $x_{n+1} = x_n = x$ becomes a fixed point of T and S, respectively, therefore the proof is complete. Now, suppose that $d(x_n, Tx_n) > 0$ *and* $d(x_{n+1}, Sx_{n+1}) > 0$ $\forall n \in N$ then the proof will be divided in to two steps.

Step one. We show that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence.

Let

$$d(x_n, Tx_n) > 0 \text{ and } d(x_{n+1}, Sx_{n+1}) > 0 \ \forall n \in N. \tag{3}$$

therefore, we have that

$$\begin{aligned}&\frac{1}{s+1} d(x_n, Tx_n) < d(x_n, Tx_n) \text{ and}\\ &\frac{1}{s+1} d(x_{n+1}, Sx_{n+1}) < d(x_{n+1}, Sx_{n+1}) \ \forall n \in N.\end{aligned} \tag{4}$$

By Definition 2.3, we get

$$F(H(Tx_n, Sx_{n+1})) \le F(N_\phi(x_n, x_{n+1})) - \psi(N_\phi(x_n, x_{n+1})).$$

Since that

$$\begin{aligned}
&N_\phi(x_n, x_{n+1})\\
&= \phi_1(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_2(d(x_n, x_{n+1}))(d(x_{n+1}, x_{n+2}))\\
&+ \phi_3(d(x_n, x_{n+1}))\left(\frac{d(x_n, x_{n+2})}{2s}\right) + \phi_4(d(x_n, x_{n+1}))\left(\frac{(d(x_n, x_{n+2}))}{2s}\right)\\
&+ \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_6(d(x_n, x_{n+1}))(d(x_n, x_{n+1}))\\
&+ \phi_7(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})\\
&\le \phi_1(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_2(d(x_n, x_{n+1}))(d(x_{n+1}, x_{n+2}))\\
&+ \phi_3(d(x_n, x_{n+1}))\left(\frac{d(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2})}{2s}\right)\\
&+ \phi_4(d(x_n, x_{n+1}))\left(\frac{d(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2})}{2s}\right)\\
&+ \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_6(d(x_n, x_{n+1}))(d(x_n, x_{n+1}))\\
&+ \phi_7(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})\\
&\le \phi_1(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_2(d(x_n, x_{n+1}))(d(x_{n+1}, x_{n+2}))\\
&+ \phi_3(d(x_n, x_{n+1}))\left(\frac{s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]}{2s}\right)\\
&+ \phi_4(d(x_n, x_{n+1}))\left(\frac{s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]}{2s}\right)\\
&+ \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_6(d(x_n, x_{n+1}))(d(x_n, x_{n+1}))\\
&+ \phi_7(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})\\
&\le \phi_1(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_2(d(x_n, x_{n+1}))(d(x_{n+1}, x_{n+2}))\\
&+ \phi_3(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_3(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})\\
&+ \phi_4(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_4(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})\\
&+ \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_6(d(x_n, x_{n+1}))(d(x_n, x_{n+1}))\\
&+ \phi_7(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})\\
&= [\phi_1(d(x_n, x_{n+1})) + \phi_3(d(x_n, x_{n+1})) + \phi_4(d(x_n, x_{n+1}))\\
&+ \phi_6(d(x_n, x_{n+1}))](d(x_n, x_{n+1}))\\
&+ [\phi_2(d(x_n, x_{n+1})) + \phi_3(d(x_n, x_{n+1})) + \phi_4(d(x_n, x_{n+1}))\\
&+ \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_7(d(x_n, x_{n+1}))](d(x_{n+2}, x_{n+1})\\
&= \phi'(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi''(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1}))
\end{aligned} \tag{5}$$

then by (5) and definition 2.3, we get

$$\begin{aligned}
&F(d(x_{n+1}, x_{n+2}))\\
&\quad \le F(\phi'(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi''(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})))\\
&\quad - \psi(\phi'(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi''(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1}))).
\end{aligned} \tag{6}$$

On contrary, if $d(x_{n+1}, x_{n+2}) > d(x_n, x_{n+1})$, then

$$\begin{aligned}&\phi'(d(x_n, x_{n+1}))(d(x_n, x_{n+1}))\\ &+\phi''(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) < d(x_{n+1}, x_{n+2})\end{aligned}$$

and therefore (6) becomes

$$F(d(x_{n+1}, x_{n+2})) \le F(d(x_{n+1}, x_{n+2})) - \psi(d(x_{n+1}, x_{n+2})).$$

But, from (3) and the fact that $\psi(d(x_{n+1}, x_{n+2})) > 0$, this is a contradiction. Thus, we conclude that

$$\begin{aligned}F(d(x_{n+1}, x_{n+2})) &\le F(d(x_n, x_{n+1})) - \psi(d(x_n, x_{n+1}))\\ &< F(d(x_n, x_{n+1})).\end{aligned} \tag{7}$$

By (7) and Definition 2.1($F1$), we have that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \ \forall n \in N. \tag{8}$$

Therefore $\{d(x_n, x_{n+1})\}$ is a nonnegative decreasing sequence of real numbers. Thus there exists $\gamma \ge 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = \gamma$. From (7) as $n \to \infty$, we have that

$$F(\gamma) \le F(\gamma) - \psi(\gamma).$$

This implies that $\psi(\gamma) = 0$ and thus $\gamma = 0$. Consequently we arrive at

$$\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{9}$$

Now, we claim that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. On contrary, we assume that there exists $\epsilon > 0$ and $n, m \in N$ such that, for all $n \ge n_\epsilon$ and $n_\epsilon < n < m$,

$$d(x_n, x_m) \ge \epsilon \text{ and } d(x_{n-1}, x_m) < \epsilon. \tag{10}$$

It implies that

$$\begin{aligned}\epsilon \le d(x_n, x_m) &\le d(x_n, x_{n-1}) + sd(x_{n-1}, x_m)\\ &< d(x_n, x_{n-1}) + s\epsilon.\end{aligned} \tag{11}$$

By (11) and (9), we have that

$$\epsilon \le \limsup_{n\to\infty} d(x_n, x_m) < s\epsilon. \tag{12}$$

By triangle inequality, we have that

$$\begin{aligned}\epsilon \le d(x_n, x_m) &\le d(x_n, x_{m+1}) + sd(x_{m+1}, x_m)\\ &\le d(x_n, x_m) + 2sd(x_{m+1}, x_m).\end{aligned} \tag{13}$$

By (9),(10), (12) and (13), we have that

$$\epsilon \le \limsup_{n\to\infty} d(x_n, x_{m+1}) < s\epsilon. \tag{14}$$

Similarly, we have that

$$\begin{aligned}\epsilon \le d(x_n, x_m) &\le d(x_n, x_{n+1}) + sd(x_{n+1}, x_m)\\ &\le sd(x_n, x_m) + (s^2+1)d(x_n, x_{n+1}).\end{aligned} \tag{15}$$

By (9),(10), (12) and (15), we have that

$$\epsilon \le \limsup_{n\to\infty} d(x_n, x_{n+1}) < s\epsilon. \tag{16}$$

Observe that

$$\begin{aligned}d(x_n, x_{m+1}) &\le d(x_n, x_{n+1}) + sd(x_{n+1}, x_{m+1})\\ &\le d(x_n, x_{n+1}) + s[d(x_{n+1}, x_m) + sd(x_{m+1}, x_m)]\\ &\le d(x_n, x_{n+1}) + s[d(x_n, x_{n+1}) + sd(x_n, x_m)\\ &+ sd(x_{m+1}, x_m)].\end{aligned} \tag{17}$$

By (17), we have that

$$\frac{\epsilon}{s} \le \limsup_{n\to\infty} d(x_{n+1}, x_{m+1}) < s^2\epsilon. \tag{18}$$

By (9)and (10), we select $n_\epsilon > 0 \in N$ such that

$$\begin{aligned}\frac{1}{s+1}d(x_n, Tx_n) &< \frac{1}{s+1}\epsilon < \epsilon \le d(x_n, x_m)\ \forall n \ge n(\epsilon)\\ &\Leftrightarrow \frac{1}{s+1}d(x_n, Tx_n) < \frac{1}{s+1}\epsilon < d(x_n, x_m)\\ &\forall n \ge n(\epsilon)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{s+1}d(x_{n+1}, Sx_{n+1}) &< \frac{1}{s+1}\epsilon < \frac{\epsilon}{s} \le d(x_{n+1}, x_{m+1})\ \forall n \ge n_\epsilon\\ &\Leftrightarrow \frac{1}{s+1}d(x_{n+1}, Sx_{n+1}) < \frac{1}{s+1}\epsilon\\ &< d(x_{n+1}, x_{m+1})\ \forall n \ge n_\epsilon\end{aligned}$$

It follows that from Definition 2.3, we have, for every $n \ge n_\epsilon$

$$F(H(x_{n+1}, x_{m+1})) \le F(N_\phi(x_n, x_m)) - \psi(N_\phi(x_n, x_m)). \tag{19}$$

Since that

$$\begin{aligned}&d(x_n, x_m) \le N_\phi(x_n, x_m)\\ &= \phi_1(d(x_n, x_m))(d(x_n, x_m)) + \phi_2(d(x_n, x_m))(d(x_{n+2}, x_m))\\ &+ \phi_3(d(x_n, x_m))\left(\frac{d(x_{n+1}, x_m) + d(x_n, x_{m+1})}{2s}\right)\\ &+ \phi_4(d(x_n, x_m))\left(\frac{(d(x_{n+2}, x_n) + d(x_{n+2}, x_{m+1}))}{2s}\right)\\ &+ \phi_5(d(x_n, x_m))(d(x_{n+2}, x_{m+1}) + d(x_{n+2}, x_{n+1}))\\ &+ \phi_6(d(x_n, x_m))(d(x_{n+2}, x_{m+1}) + d(x_n, x_{n+1}))\\ &+ \phi_7(d(x_n, x_m))(d(x_m, x_{n+1} + d(x_m, x_{m+1})))\\ &\le \phi_1(d(x_n, x_m))(d(x_n, x_m)) + \phi_2(d(x_n, x_m))(d(x_{n+2}, x_{n+1}) + sd(x_{n+1}, x_m))\\ &+ \phi_3(d(x_n, x_m))\left(\frac{d(x_{n+1}, x_m) + d(x_n, x_{m+1})}{2s}\right)\\ &+ \phi_4(d(x_n, x_m))\left(\frac{(d(x_{n+2}, x_{n+1}) + sd(x_{n+1}, x_n) + d(x_{n+2}, x_{n+1})) + sd(x_{n+1}, x_{m+1}))}{2s}\right)\\ &+ \phi_5(d(x_n, x_m))(d(x_{n+2}, x_{n+1}) + sd(x_{n+1}, x_{m+1}) + d(x_{n+2}, x_{n+1}))\\ &+ \phi_6(d(x_n, x_m))(d(x_{n+2}, x_{n+1}) + sd(x_{n+1}, x_{m+1}) + d(x_n, x_{n+1}))\\ &+ \phi_7(d(x_n, x_m))(d(x_m, x_{n+1}) + d(x_m, x_{m+1}))).\end{aligned} \tag{20}$$

By (12), (14), (16), (18) and (20), we have that

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, x_m) &\le \limsup_{n\to\infty} N_\phi(x_n, x_m) < \phi_1(\epsilon)(s\epsilon) + \phi_2(\epsilon)(s^2\epsilon)\\ &+ \phi_3(\epsilon)(\epsilon) + \phi_4(\epsilon)(\frac{s^2\epsilon}{2}) + \phi_5(\epsilon)(s^3\epsilon) + \phi_6(\epsilon)(s^3\epsilon) + \phi_7(\epsilon)(s\epsilon)\\ &\le max\{s\epsilon, s^2\epsilon, \epsilon, \frac{s\epsilon}{2}, s^3\epsilon, s\epsilon\}\\ &= s^3\epsilon\end{aligned}$$

and therefore

$$\epsilon \le \limsup_{n\to\infty} N_\phi(x_n, x_m) < s^3\epsilon. \tag{21}$$

Similarly

$$\epsilon \le \liminf_{n\to\infty} N_\phi(x_n, x_m) < s^3\epsilon. \tag{22}$$

By (19), (21) and (22), we have that

$$\begin{aligned}F(s^3\epsilon) = F(s^4\frac{\epsilon}{s}) &\le F(s^4 \limsup_{n\to\infty} d(x_{n+1}, x_{m+1}))\\ &\le F(\limsup_{n\to\infty} N_\phi(x_n, x_m)) - \psi(\limsup_{n\to\infty} N_\phi(x_n, x_m))\\ &\le F(s^3\epsilon) - \psi(\epsilon).\end{aligned} \tag{23}$$

By (23) and the fact that $\epsilon > 0$, this is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d), $\{x_n\}_{n=1}^\infty$ and $\{x_{n+1}\}_{n=1}^\infty$ converge to some point $x^* \in X$, that is,

$$\lim_{n\to\infty} d(x_n, x^*) = 0 \text{ and } \lim_{n\to\infty} d(x_{n+1}, x^*) = 0. \tag{24}$$

There exists increasing sequences $\{n_k\}, \{n+1_k\} \subset N$ such that $x_{n_k} \in Tx^*$ and $x_{n+1_k} \in Sx^*$ for all $k \in N$. Since Tx^* and Sx^* are closed and

$$\lim_{n\to\infty} d(x_{n_k}, x^*) = 0 \text{ and } \lim_{n\to\infty} d(x_{n+1_k}, x^*) = 0,$$

we get $x^* \in Tx^*$ and $x^* \in Sx^*$.

Step two. We show that x^* is a common fixed point of T and S. It suffices to show that

$$\frac{1}{1+s}d(x_n, Tx_n) < d(x_n, x^*) \text{ and } \frac{1}{1+s}d(x_{n+1}, Sx_{n+1}) < d(x_{n+1}, x^*),\quad for\ every\ n \in N, \tag{25}$$

implies

$$F(d(Tx^*, x^*)) \leq F(N_\phi(x^*, Tx^*)) - \psi(N_\phi(x^*, Tx^*))$$

and

$$F(d(Sx^*, x^*)) \leq F(N_\phi(Sx^*, x^*)) - \psi(N_\phi(Sx^*, x^*)),$$

respectively.
On contrary, suppose there exists $m \in N$ such that

$$\frac{1}{1+s}d(x_m, Tx_m) \geq d(x_m, x^*) \text{ or } \frac{1}{1+s}d(x_{m+1}, Sx_{m+1}) \geq d(x_{m+1}, x^*). \quad (26)$$

By (26), we have that

$$(s+1)d(x_m, x^*) \leq d(x_m, Tx_m) \leq d(x_m, x^*) + sd(Tx_m, x^*)$$

or

$$(s+1)d(x_{m+1}, x^*) \leq d(x_{m+1}, Sx_{m+1}) \leq d(x_{m+1}, x^*) + sd(Sx_{m+1}, x^*),$$

and therefore

$$\begin{aligned} &d(x_m, x^*) \leq d(Tx_m, x^*) = d(x_{m+1}, x^*) \text{ and} \\ &d(x_{m+1}, x^*) \leq d(Sx_{m+1}, x^*) = d(x_{m+2}, x^*). \end{aligned} \quad (27)$$

By (8), (26) and (27), this is a contradiction. Hence, (25) holds, and therefore

$$\begin{aligned} F(d(x_{n+1}, x^*)) &= F(H(Tx_n, Sx^*)) \\ &\leq F(N_\phi(x_n, x^*)) - \psi(N_\phi(x_n, x^*)), \end{aligned} \quad (28)$$

and

$$\begin{aligned} F(d(x_{n+2}, x^*)) &= F(H(Sx_{n+1}, Tx^*)) \\ &\leq F(N_\phi(x_{n+1}, x^*)) - \psi(N_\phi(x_{n+1}, x^*)). \end{aligned} \quad (29)$$

Since that

$$\begin{aligned} d(x^*, Tx^*) &\leq N_\phi(x_n, x^*) \\ &= \phi_1(d(x_n, x^*))(d(x_n, x^*)) + \phi_2(d(x_n, x^*))(d(x_{n+2}, x^*)) \\ &+ \phi_3(d(x_n, x^*))\left(\frac{d(x_{n+1}, x^*) + d(x_n, Sx^*)}{2s}\right) \\ &+ \phi_4(d(x_n, x^*))\left(\frac{d(x_n, Sx^*) + d(Sx^*, x_{n+2})}{2s}\right) \\ &+ \phi_5(d(x_n, x^*))(d(Sx^*, x_{n+2}) + d(x_{n+1}, Sx^*)) \\ &+ \phi_6(d(x_n, x^*))(d(x_n, x_{n+1}) + d(x_{n+2}, Tx^*)) \\ &+ \phi_7(d(x_n, x^*))(d(Tx^*, x^*) + d(x^*, x_{n+1})) \\ &\leq max\{(d(x_n, x^*), d(x_{n+2}, x^*), \\ &\frac{d(x_{n+1}, x^*) + d(x_n, Sx^*)}{2s}, \\ &\frac{d(x_n, Sx^*) + sd(Sx^*, x_{n+2}) + d(Sx^*, x_{n+2})}{2s}, \\ &d(Sx^*, x_{n+2}) + d(x_{n+1}, Sx^*), d(x_n, x_{n+1}) + d(x_{n+2}, Tx^*), \\ &d(Tx^*, x^*) + d(x^*, x_{n+1})\} \end{aligned} \quad (30)$$

and

$$\begin{aligned} d(x^*, Sx^*) &\leq N_\phi(x_{n+1}, x^*) \\ &= \phi_1(d(x_{n+1}, x^*))(d(x_{n+1}, x^*)) + \phi_2(d(x_{n+1}, x^*))(d(x^*, x_{n+3})) \\ &+ \phi_3(x_{n+1}, x^*))\left(\frac{d(x_{n+2}, x^*) + d(x_{n+1}, x^*)}{2s}\right) \\ &+ \phi_4(d(x_{n+1}, x^*))\left(\frac{d(x_{n+1}, Sx^*) + d(Sx^*, x_{n+3})}{2s}\right) \\ &+ \phi_5(d(x_{n+1}, x^*))(d(x_{n+3}, Sx^*) + d(x_{n+2}, Sx^*)) \\ &+ \phi_6(d(x_{n+1}, x^*))(d(x_{n+1}, x_{n+2}) + d(x_{n+3}, Sx^*)) \\ &+ \phi_7(d(x_{n+1}, x^*))(d(Sx^*, x^*) + d(x^*, x_{n+2})) \\ &\leq max\{d(x_{n+1}, x^*), d(x^*, x_{n+3}), \\ &\frac{d(x_{n+2}, x^*) + d(x_{n+1}, x^*)}{2s}, \\ &\frac{d(x_{n+1}, x_{n+2}) + sd(x_{n+2}, Sx^*) + d(Sx^*, x_{n+3})}{2s}, \\ &d(x_{n+3}, Sx^*) + d(x_{n+2}, Sx^*), d(x_{n+1}, x_{n+2}) + d(x_{n+3}, Sx^*), \\ &d(Sx^*, x^*) + d(x^*, x_{n+2})\}. \end{aligned} \quad (31)$$

By (24) and (30), we have that

$$\lim_{n\to\infty} N_\phi(x_n, x^*) = d(Tx^*, x^*).$$

By (24) and (31), we have that

$$\lim_{n\to\infty} N_\phi(x_{n+1}, x^*) = d(Sx^*, x^*).$$

By (28)and (29) and by the continuity of F and ψ, we have that

$$F(d(x^*, Tx^*)) \leq F(N_\phi(x^*, Tx^*)) - \psi(N_\phi(x^*, Tx^*)),$$

and

$$F(d(x^*, Sx^*)) \leq F(N_\phi(x^*, Sx^*)) - \psi(N_\phi(x^*, Sx^*)).$$

Hence, since Tx^* and Sx^* are closed then we have $x^* \in Tx^*$ and $x^* \in Sx^*$, that is, x^* is a fixed point of T and S.

In Theorem 2.1, when $T = S = U$, then we have the following result.

Corollary 2.1.1. *Let (X, d) be a complete strong $b-$metric space and let $U : X \to CB(X)$ be a multivalued generalized F-Suzuki-Contraction mapping. Then U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^{\infty}$ converges to x^*.*

In Corollary 2.1.1, when U is a single-valued then we have another new result as follows.

Corollary 2.1.2. *Let (X, d) be a complete strong $b-$metric space and let $U : X \to X$ be a single-valued generalized F-Suzuki-Contraction mapping. Then U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^{\infty}$ converges to x^*.*

In Theorem 2.1, when T and S are two single-valued then the following result holds.

Corollary 2.1.3. *Let (X, d) be a complete strong $b-$metric space and let $T, S : X \to X$ be two single-valued generalized F-Suzuki-Contraction mappings. Then T and S have a common fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ and $\{S^n x\}_{n=1}^{\infty}$ converge to x^*.*

In Theorem 2.1, when (X, d) is a complete $b-$metric space then the following new result holds.

Corollary 2.1.4. *Let (X, d) be a complete $b-$metric space and let $T, S : X \to X$ be two single-valued generalized F-Suzuki-Contraction mappings. Then T and S have a common fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ and $\{S^n x\}_{n=1}^{\infty}$ converge to x^*.*

In corollary 2.1.4, when $T = S = U$, then we have the following result.

Corollary 2.1.5. *Let (X, d) be a complete $b-$metric space and let $U : X \to CB(X)$ be a multivalued generalized F-Suzuki-Contraction mapping. Then U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^{\infty}$ converges to x^*.*

Corollary 2.1.6. *Let (X, d) be a complete strong $b-$metric space and let $U : X \to CB(X)$ be a multivalued generalized F-Suzuki-Contraction mapping such that there exists $F \in \mho$ and $\psi \in \Psi$, $\forall x, y \in X$, $x \neq y$, $\frac{1}{s+1} d(x, Ux) < d(x, y) \Rightarrow \psi(N(x, y)) + F(s^4 d(Ux, Uy)) \leq F(N(x, y))$ in which*

$$N(x,y) = max\{d(x,y), d(y,U^2x), \frac{(d(y,Ux)) + d(x,Uy)}{2s}, \frac{(d(x,Uy)) + d(U^2x,Uy)}{2s}, d(U^2x,Uy) + d(Uy,Ux), d(U^2x,Uy) + d(Ux,x), d(Ux,y)) + d(y,Uy)\}. \quad (32)$$

Then U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^{\infty}$ converges to x^*.

Proof from Lemma 1.1, since (2) $\Rightarrow$ (32) then by the corollary 2.1.1 the result follows immediately.

Corollary 2.1.7. *Let (X, d) be a complete strong $b-$metric space and let $U : X \to X$ be a single-valued generalized F-Suzuki-Contraction mapping such that there exists $F \in \mho$ and $\psi \in \Psi$, $\forall x, y \in X$, $x \neq y$, $\frac{1}{s+1} d(x, Ux) < d(x, y) \Rightarrow \psi(N(x, y)) + F(s^4 d(Ux, Uy)) \leq F(N(x, y))$ in which*

$$N(x,y) = max\{d(x,y), d(y,U^2x), \frac{(d(y,Ux)) + d(x,Uy)}{2s}, \frac{(d(x,Uy)) + d(U^2x,Uy)}{2s}, d(U^2x,Uy) + d(Uy,Ux), d(U^2x,Uy) + d(Ux,x), d(Ux,y)) + d(y,Uy)\}. \quad (33)$$

Then U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^{\infty}$ converges to x^*.

Proof from Lemma 1.1, since (2) $\Rightarrow$ (33) then by the corollary 2.1.2 the result holds.

Corollary 2.1.8. *Let (X, d) be a complete strong $b-$metric space and let $T, S : X \to X$ be two single-valued generalized F-Suzuki-Contraction mappings such that there exists $F \in \mho$ and $\psi \in \Psi$, $\forall x, y \in X$, $x \neq y$, $\frac{1}{s+1} d(x, Tx) < d(x, y)$ and $\frac{1}{s+1} d(y, Sx) < d(y, STx) \Rightarrow \psi(N(x, y)) + F(s^4 H(Tx, Sy)) \leq F(N(x, y))$ in which*

$$N(x,y) = max\{d(x,y), d(y,STx), \frac{(d(y,Tx)) + d(x,Sy)}{2s}, \frac{(d(x,Sy)) + d(STx,Sy)}{2s}, d(STx,Sy) + d(Sy,Tx), d(STx,Sy) + d(Tx,x), d(Tx,y)) + d(y,Sy)\}. \quad (34)$$

Then T and S have a common fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ and $\{S^n x\}_{n=1}^{\infty}$ converge to x^*.

Proof from Lemma 1.1, since (2) $\Rightarrow$ (34) then by the corollary 2.1.4 the result holds.

Corollary 2.1.9. *Let (X, d) be a complete $b-$metric space and let $U : X \to CB(X)$ be a multivalued generalized F-Suzuki-Contraction mapping such that there exists $F \in \mho$ and $\psi \in \Psi$, $\forall x, y \in X$, $x \neq y$, $\frac{1}{2s} d(x, Ux) < d(x, y) \Rightarrow \psi(N(x, y)) + F(s^6 d(Ux, Uy)) \leq F(N(x, y))$ in which*

$$N(x,y) = max\{d(x,y), d(y,U^2x), \frac{(d(y,Ux)) + d(x,Uy)}{2s}, \frac{(d(x,Uy)) + d(U^2x,Uy)}{2s}, d(U^2x,Uy) + d(Uy,Ux), d(U^2x,Uy) + d(Ux,x), d(Ux,y)) + d(y,Uy)\}. \quad (35)$$

Then U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^{\infty}$ converges to x^*.

Proof from Lemma 1.1, since (2) $\Rightarrow$ (35) then by the corollary 2.1.5 the result holds.

3. Example

Let $X = [0, 1]$. $T, S : [0, 1] \to CB([0, 1])$ be defined by $Tx = [0, \frac{x}{2}]$ and $Sy = [0, \frac{y}{2}]$ such that $STx = [0, \frac{x}{8}]$ for all $x \in [0, 1]$. Let d be the usual metric on X. Taking $F(t) = \frac{t}{10}$ and let $x < y$, then $\forall x, y \in [0, 1]$ $d(x, y) > 0$ and $d(y, STx) = |y - \frac{x}{8}| > |y - \frac{y}{8}| = \frac{7}{8}y > \frac{y}{4}$. Now, for $s = 1$, we have that $\frac{1}{2} d(x, Tx) = 0 < d(x, y)$ and $\frac{1}{2} d(y, Sy) = \frac{y}{4} < d(y, STx)$. Without lose of generality, let $\phi_1(d(x, y)) = \phi_2(d(x, y)) = \phi_3(d(x, y)) = \frac{1}{5}$; and $\phi_4(d(x, y)) = \phi_5(d(x, y)) = \phi_6(d(x, y)) = \phi_7(d(x, y)) = \frac{1}{10^2}$. Therefore, we have that

$$\begin{aligned} F(H(Tx,Sy)) &= ln\,(H(Tx,Sy)) + H(Tx,Sy) \\ &= \frac{1}{10}\left|\frac{y}{2} - \frac{x}{4}\right| = \frac{1}{10}\left|y - \frac{y}{2} - \frac{x}{4}\right| \\ &\leq \frac{1}{10}\left(\left|y - \frac{x}{4}\right| + \left|x - \frac{y}{2}\right|\right) \\ &= \frac{1}{10}\left(\frac{\left|y - \frac{x}{4}\right| + \left|x - \frac{y}{2}\right|}{2}\right) + \frac{1}{10}\left(\frac{\left|y - \frac{x}{4}\right| + \left|x - \frac{y}{2}\right|}{2}\right) \end{aligned}$$

$$\leq \frac{1}{10}\left(\frac{\left|y-\frac{x}{4}\right|+\left|x-\frac{y}{2}\right|}{2}\right)+\frac{1}{10}\left(\frac{\left|y-\frac{x}{4}\right|+\left|x-\frac{x}{8}\right|+\left|\frac{x}{8}-\frac{y}{2}\right|}{2}\right)$$
$$=\frac{1}{10}\left(\frac{\left|y-\frac{x}{4}\right|+\left|x-\frac{y}{2}\right|}{2}\right)+\frac{1}{10}\left(\frac{\left|\frac{x}{8}-\frac{y}{2}\right|+\left|x-\frac{x}{8}\right|}{2}\right)$$
$$+\frac{1}{10}\left(\left|\frac{y}{2}-\frac{x}{8}\right|\right)\leq\frac{1}{10}\left(\frac{\left|y-\frac{x}{4}\right|+\left|x-\frac{y}{2}\right|}{2}\right)$$
$$+\frac{1}{10}\left(\frac{\left|\frac{x}{8}-\frac{y}{2}\right|+\left|x-\frac{x}{8}\right|}{2}\right)+\frac{1}{10}\left(\left|\frac{y}{2}-\frac{x}{8}\right|+\left|\frac{x}{8}-\frac{x}{4}\right|\right)$$
$$=\frac{1}{5}\left(\frac{\left|y-\frac{x}{4}\right|+\left|x-\frac{y}{2}\right|}{2}\right)+\frac{1}{5}\left(\frac{\left|\frac{x}{8}-\frac{y}{2}\right|+\left|x-\frac{x}{8}\right|}{2}\right)$$
$$+\frac{1}{10}\left(\left|\frac{y}{2}-\frac{x}{8}\right|+\left|\frac{x}{8}-\frac{x}{4}\right|\right)+\frac{1}{10^2}\left(|x-y|\right)+\frac{1}{10^2}\left(\left|y-\frac{x}{8}\right|\right)$$
$$+\frac{1}{10^2}\left(\left|\frac{x}{8}-\frac{y}{2}\right|+\left|\frac{x}{4}-x\right|\right)+\frac{1}{10^2}\left(\left|\frac{x}{4}-y\right|+\left|y-\frac{y}{2}\right|\right)$$
$$-\frac{1}{10^2}\left[\left(|x-y|\right)+\left(\left|y-\frac{x}{8}\right|\right)+\left(\left|\frac{x}{8}-\frac{y}{2}\right|+\left|\frac{x}{4}-x\right|\right)\right]$$
$$+\left(\left|\frac{x}{4}-y\right|+\left|y-\frac{y}{2}\right|\right)-\frac{1}{10}\left(\left|\frac{y}{2}-\frac{x}{8}\right|+\left|\frac{x}{8}-\frac{x}{4}\right|\right)$$
$$-\frac{1}{10}\left(\frac{\left|y-\frac{x}{4}\right|+\left|x-\frac{y}{2}\right|}{2}\right)-\frac{1}{10}\left(\frac{\left|\frac{x}{8}-\frac{y}{2}\right|+\left|x-\frac{x}{8}\right|}{2}\right).$$
$$=\phi_1(d(x,y))(d(x,y))+\phi_2(d(x,y))(d(y,STx))$$
$$+\phi_3(d(x,y))\left(\frac{(d(y,Tx))+d(x,Sy)}{2s}\right)+\phi_4(d(x,y))$$
$$\left(\frac{(d(x,STx))+d(STx,Sy)}{2s}\right)$$
$$+\phi_5(d(x,y))(d(STx,Sy)+d(STx,Tx))$$
$$+\phi_6(d(x,y))(d(STx,Sy)+d(Tx,x))$$
$$+\phi_7(d(x,y))(d(Tx,y))+d(y,Sy))-\psi(N_\phi(x,y)).$$

4. Conclusion

Fixed point results of Piri and Kumam [11], Ahmad *et al.* [9], Suzuki [18] and Suzuki [19] are extended by introducing common fixed point problem for multivalued generalized F-Suzuki-contraction mappings in strong b-metric spaces. In specific, Corollary 2.1.1 and corollary 2.1.2 generalize and extend the work of Ahmad *et al.* [9] and Kumam and Hossein [5], respectively.

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