Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 6 (2024) 1917

Journal of the Nigerian Society of Physical Sciences

A Chebyshev polynomial based block integrator for the direct numerical solution of fourth order ordinary differential equations

O. M. Ogunlaran[a], M. A. Kehinde[a,*], M. A. Akanbi[b], E. I. Akinola[a]

[a]*Mathematics Programme, College of Agriculture, Engineering and Science, Bowen University, Iwo, Nigeria*
[b]*Department of Mathematics; and Africa Centre of Excellence for Innovative and Transformative STEM Education (ACEITSE), Lagos State University, Ojo, Lagos, Nigeria*

Abstract

This paper introduces an innovative method for numerically integrating fourth-order initial value problems by utilizing Chebyshev polynomials as the fundamental basis function. The block integrator based on Chebyshev polynomial demonstrates significant improvements in accuracy and stability, rendering it a valuable tool across various scientific and engineering fields. By leveraging the characteristics of Chebyshev polynomials, this approach accurately estimates solutions for fourth-order differential equations without reducing it to a system of first order ordinary differential equations while at the same time effectively managing error accumulation within a block integration framework and thereby enhancing its accuracy over extended intervals. Through rigorous numerical experiments, the effectiveness and reliability of the new integrator are demonstrated and compared with existing methods. The new method is consistent, zero stable and convergent. The method also shows an appreciable error constants. The new method performed better in terms of accuracy than the existing methods in the literature in both linear and nonlinear problems.

DOI:10.46481/jnsps.2024.1917

Keywords: Chebyshev polynomials, Continuous scheme, Fourth order, Block method, Initial value problems

Article History :
Received: 12 December 2023
Received in revised form: 18 January 2024
Accepted for publication: 1 February 2024
Published: 26 March 2024

Communicated by: Pankaj Thakur

1. Introduction

This study focuses on addressing fourth-order initial value problems (IVPs) represented by the equation:

$$y^{'v} = f(x, y, y', y'', y'''), y(a) = \alpha, y'(a) = \beta, y''(a) = \gamma, y'''(a) = \delta, \quad (1)$$

where f is continuous within the interval $[a, b]$ of integration.

Numerical methods play a pivotal role in efficiently and accurately solving differential equations, essential in various scientific and engineering applications. While numerous techniques exist for first- and second-order IVPs, addressing higher-order IVPs poses unique challenges due to the intricate nature of the underlying equations. A common strategy involves transforming a fourth-order problem into an equivalent system of first-order initial value problems and then applying a suitable numerical integration method.

However, notable authors such as Awoyemi [1], Familua & Omole [2], Ogunlaran & Kehinde [3] and Lambert [4] have pointed out the drawbacks associated with the reduction of order approach, including increased function evaluations, coding complexity, and greater computational time and storage re-

*Corresponding author: Tel. No.: +234-803-213-8545.
Email address: michaelmaths865@gmail.com (O. M. Ogunlaran)

quirements. Recent interest focuses on innovative approaches to overcome these challenges and develop more efficient numerical integration methods for directly solving higher-order differential equations. Block linear multistep methods (BLMMs) have proven effective for solving IVPs associated with ordinary differential equations, offering stability and computational advantages. Various polynomials have been employed by different authors in BLMMs, such as Power series by Awoyemi [1], Familua & Omole [2], Ramos *et al.* [5], Atabo & Adee [6] and Modebei *et al.* [7], Taylor series by Adoghe & Omole [8], Lucas Polynomials by Adeniran & Longe [9], Legendre Polynomials by Nazreen & Zanariah [10], Chebyshev Polynomials by Olabode & Momoh [11] and Alabi *et al.* [12] and Hermite polynomials by Ogunlaran & Kehinde [3]. The effectiveness of Chebyshev polynomials is widely recognized across scientific and engineering fields, including weather forecasting, surface and interface stress effects in thin films, solving integral equations and two-point boundary value problems, and so on, Khater *et al.* [13].

This paper proposes a novel BLMM to efficiently tackling general fourth-order IVPs by leveraging on orthogonal properties and exceptional approximation capabilities of Chebyshev polynomials which is used as basis function.

2. Derivation of the Method

In this section, we consider the initial value problem of the form (1) defined in a finite interval $a \le x \le b$ and f is a continuous differentiable function.

In order to derive a 4-step multistep method for the solution of (1), we consider

$$y^{iv} = f(x, y(x), y'(x), y''(x), y'''(x)), \quad x_n \le x \le x_{n+4},$$
$$y(x_n) = \alpha, \quad y'(x_n) = \beta, \quad y''(x_n) = \gamma, \quad y'''(x_n) = \delta, \quad (2)$$

and we desire to find the numerical estimation of the analytical solution $y(x)$ using an approximate solution of the form:

$$Y(x) = \sum_{r=0}^{8} a_r T_r\Big(\frac{x - nh - 2h}{2h}\Big), \quad (3)$$

where a_r are coefficients to be determined and $T_r(x)$ are the rth degree Chebyshev polynomial functions of the first kind defined in the interval $[a, b]$ as:

$$T_r(x) = \cos\Big[r\cos^{-1}\Big(\frac{2x-(b+a)}{b-a}\Big)\Big]. \quad (4)$$

Equation (4) satisfies the recurrence relation

$$T_{r+1}(x) = 2\Big(\frac{2x-(b+a)}{b-a}\Big)T_r(x) - T_{r-1}(x), \quad r \ge 1; \quad (5)$$

and with the starting values:

$$T_0(x) = 1 \quad \text{and} \quad T_1(x) = \frac{2x-(b+a)}{b-a}. \quad (6)$$

There are nine equations in nine unknowns to be obtained from (3) and satisfying the following:

$$\sum_{r=0}^{8} a_r T_r\Big(\frac{x - nh - 2h}{2h}\Big) = y_{n+i}, \quad (7)$$

$$\sum_{r=0}^{8} a_r T_r^{iv}\Big(\frac{x - nh - 2h}{2h}\Big) = f_{n+i}. \quad (8)$$

On collocating (8) at $x = x_{n+i}$, $i = 0(1)4$ and interpolating (7) at $x = x_{n+i}$, $i = 0(1)3$ we obtain a system of nine equations with nine unknowns which is solved and gives the values of the undetermined coefficients $a_r (r = 0(1)8)$ as follows:

$$a_0 = \frac{4}{35}h^4 f_{n+2} - \frac{1}{2520}h^4 f_n + \frac{67}{2520}h^4 f_{n+1} + \frac{67}{2520}h^4 f_{n+3} - \frac{1}{2520}h^4 f_{n+4} + y_{n+1} - y_{n+2} + y_{n+3},$$
$$a_1 = -\frac{1}{3789}h^4 f_n + \frac{247}{3780}h^4 f_{n+1} + \frac{79}{360}h^4 f_{n+2} + \frac{187}{3780}h^4 f_{n+3} - \frac{1}{1512}h^4 f_{n+4} + y_{n+1} - 2y_{n+2} + \frac{5}{3}y_{n+3} - \frac{2}{3}y_n,$$
$$a_2 = -\frac{31}{60480}h^4 f_n + \frac{619}{15120}h^4 f_{n+1} + \frac{341}{2016}h^4 f_{n+2} + \frac{619}{15120}h^4 f_{n+3} - \frac{31}{60480}h^4 f_{n+4} + y_{n+1} - 2y_{n+2} + y_{n+3},$$
$$a_3 = y_{n+1} + \frac{79}{720}h^4 f_{n+2} + \frac{31}{1080}h^4 f_{n+1} - y_{n+2} + \frac{31}{1080}h^4 f_{n+3} - \frac{1}{4320}h^4 f_n - \frac{1}{4320}h^4 f_{n+4} - \frac{1}{3}y_n + \frac{1}{3}y_{n+3},$$

$$a_4 = \frac{1}{20}h^4 f_{n+2} + \frac{1}{60}h^4 f_{n+1} + \frac{1}{60}h^4 f_{n+3}, \quad (9)$$

$$a_5 = -\frac{1}{120}h^4 f_{n+1} + \frac{1}{120}h^4 f_{n+3},$$
$$a_6 = \frac{2}{945}h^4 f_{n+1} - \frac{23}{5040}h^4 f_{n+2} + \frac{2}{945}h^4 f_{n+3} + \frac{1}{6048}h^4 f_n + \frac{1}{6048}h^4 f_{n+4},$$
$$a_7 = -\frac{1}{5040}h^4 f_n + \frac{1}{2520}h^4 f_{n+1} - \frac{1}{2520}h^4 f_{n+3} + \frac{1}{5040}h^4 f_{n+4},$$

and,

$$a_8 = \frac{1}{20160}h^4 f_n - \frac{1}{5040}h^4 f_{n+1} + \frac{1}{3360}h^4 f_{n+2} - \frac{1}{5040}h^4 f_{n+3} + \frac{1}{20160}h^4 f_{n+4}.$$

The $a_r (r = 0(1)8)$ in (9) are substituted into (3) and after some algebraic manipulation yields:

$$Y(x) = \sum_{i=0}^{4} \alpha_i(x) y_{n+1} + h^4 \sum_{i=0}^{4} \beta_i(x) f_{n+1}. \quad (10)$$

Evaluating the continuous scheme (10) at $x = x_{n+4}$ leads to the main method:

$$y_{n+4} - 4y_{n+3} + 6y_{n+2} - 4y_{n+1} + y_n = \frac{h^4}{720}\Big(- f_n + 124f_{n+1} + 474f_{n+2} + 124f_{n+3} - f_{n+4}\Big). \quad (11)$$

First, second and third derivatives of the continuous scheme (8) with respect to x yield the following:

$$Y'(x) = \frac{1}{2h}\sum_{i=0}^{4}\alpha_i'(x)y_{n+i} + h^4\sum_{i=0}^{4}\beta_i'(x)f_{n+i}, \quad (12)$$

$$Y''(x) = \frac{1}{4h^2}\sum_{i=0}^{4}\alpha_i''(x)y_{n+i} + h^4\sum_{i=0}^{4}\beta_i''(x)f_{n+i}, \quad (13)$$

and

$$Y'''(x) = \frac{1}{8h^3}\sum_{i=0}^{4}\alpha_i'''(x)y_{n+i} + h^4\sum_{i=0}^{4}\beta_i'''(x)f_{n+i}. \quad (14)$$

Evaluating each of equations (12), (13), and (14) at $x = x_{n+i}, (i = 0(1)4)$ respectively yield the following:

$$- 2y_{n+3} + 9y_{n+2} - 18y_{n+1} + 11y_n + 6hy_n' = \frac{h^4}{10080}\Big(- 579f_n - 10860f_{n+1} - 3762f_{n+2} + 84f_{n+3} - 3f_{n+4}\Big), \quad (15)$$

$$y_{n+3} - 6y_{n+2} + 3y_{n+1} + 2y_n + 6hy_{n+1}' = \frac{h^4}{10080}\Big(- 31f_n + 2908f_{n+1} + 2382f_{n+2} - 260f_{n+3} + 41f_{n+4}\Big), \quad (16)$$

$$- 2y_{n+3} - 3y_{n+2} + 6y_{n+1} - y_n + 6hy_{n+2}' = \frac{h^4}{10080}\Big(55f_n - 1972f_{n+1} - 3318f_{n+2} + 236f_{n+} - 41f_{n+4}\Big), \quad (17)$$

$$- 11y_{n+3} + 18y_{n+2} - 9y_{n+1} + 2y_n + 6hy_{n+3}' = \frac{h^4}{10080}\Big(- 69f_n + 3732f_{n+1} + 10890f_{n+2} + 564f_{n+3} + 3f_{n+4}\Big), \quad (18)$$

$$- 26y_{n+3} + 57y_{n+2} - 42y_{n+1} + 11y_n + 6hy_{n+4}' = \frac{h^4}{10080}\Big(-151f_n+19012f_{n+1}+76758f_{n+2}+29956f_{n+3}+425f_{n+4}\Big), \quad (19)$$

$$4y_{n+3} - 16y_{n+2} + 20y_{n+1} - 8y_n + 4h^2y_n'' = \frac{h^4}{15120}\Big(4463f_n + 42124f_{n+1} + 7962f_{n+2} + 241f_{n+4}\Big), \quad (20)$$

$$- 4y_{n+2} + 8y_{n+1} - 4y_n + 4h^2y_{n+1}'' = \frac{h^4}{15120}\Big(- 157f_n - 4748f_{n+1} - 102f_{n+2} - 44f_{n+3} + 11f_{n+4}\Big), \quad (21)$$

$$- 4y_{n+3} + 8y_{n+2} - 4y_{n+1} + 4h^2y_{n+2}'' = \frac{h^4}{15120}\Big(11f_n - 212f_{n+1} - 4638f_{n+2} - 212f_{n+3} + 11f_{n+4}\Big), \quad (22)$$

$$- 8y_{n+3} + 20y_{n+2} - 16y_{n+1} + 4y_n + 4h^2y_{n+3}'' = \frac{h^4}{15120}\Big(-73f_n+10372f_{n+1}+39714f_{n+2}+5668f_{n+3}-241f_{n+4}\Big), \quad (23)$$

$$- 12y_{n+3} + 32y_{n+2} - 28yn + 1 + 8y_n + 4h^2y_{n+4}'' = \frac{h^4}{15120}\Big(-409f_n+21964f_{n+1}+87594f_{n+2}+62956f_{n+3}+4295f_{n+4}\Big), \quad (24)$$

$$- 8y_{n+3} + 24y_{n+2} - 24y_{n+1} + 8y_n + 8h^3y_n''' = \frac{h^4}{20}\Big(- 53f_n - 184fn + 1 + 10f_{n+2} - 16f_{n+3} + 3f_{n+4}\Big), \quad (25)$$

$$- 8y_{n+3} + 24y_{n+2} - 24y_{n+1} + 8y_n + 8h^3y_{n+1}''' = \frac{h^4}{180}\Big(25f_n - 364f_{n+1} - 438f_{n+2} + 68f_{n+3} - 11f_{n+4}\Big), \quad (26)$$

$$- 8y_{n+3} + 24y_{n+2} - 24y_{n+1} + 8y_n + 8h^3y_n''' = \frac{h^4}{180}\Big(- 13f_n + 328fn + 1 + 474f_{n+2} - 80f_{n+3} + 11f_{n+4}\Big), \quad (27)$$

$$- 8y_{n+3} + 24y_{n+2} - 24y_{n+1} + 8y_n + 8h^3y_{n+3}''' = \frac{h^4}{20}\Big(f_n + 20f_{n+1} + 154f_{n+2} + 68f_{n+3} - 3f_{n+4}\Big), \quad (28)$$

$$- 8y_{n+3} + 24y_{n+2} - 24y_{n+1)} + 8y_n + 8h^3y_{n+4}''' = \frac{h^4}{180}\Big(- 29f_n + 392f_{n+1} + 858f_{n+2} + 1904f_{n+3} + 475f_{n+4}\Big). \quad (29)$$

Equations (11), (15)–(29) combined constitute our block method.

To investigate the properties of the new method, the new block method (11), (15)–(29) can be written in the form of matrix difference equations:

$$AY_m = BY_{m-1}h^4[Cf_n + DF_m], \quad (30)$$

where

$$Y_m = \Big(y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, hy'_{n+1}, hy'_{n+2}, hy'_{n+3}, hy'_{n+4}, h^2y''_{n+1}, h^2y''_{n+2}, h^2y''_{n+3}, h^2y''_{n+4}, h^3y'''_{n+1}, h^3y'''_{n+2}, h^3y'''_{n+3}, h^3y'''_{n+4}\Big)^T,$$

$$Y_{m-1} = \Big(y_{n-3}, y_{n-2}, y_{n-1}, y_n, hy'_{n-3}, hy'_{n-2}, hy'_{n-1}, hy'_n, h^2y''_{n-3}, h^2y''_{n-2}, h^2y''_{n-1}, h^2y''_n, h^3y'''_{n-3}, h^3y'''_{n-2}, h^3y'''_{n-1}, h^3y'''_n\Big)^T,$$

$$F_m = \Big(f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}\Big)^T,$$

$$A = \begin{pmatrix}
-4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-18 & 9 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -6 & 1 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & -3 & -2 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-9 & 18 & 11 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-42 & 57 & -26 & 0 & 0 & 6 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & -16 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4 & 8 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
-16 & 20 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
-28 & 32 & -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
-24 & 24 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-24 & 24 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\
-24 & 24 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\
-24 & 24 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\
-24 & 24 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8
\end{pmatrix}$$

and

$$B = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

3. Analysis of the Method

Basic properties of the block method are considered and analyzed to establish the efficiency and reliability of the method. The following properties are analyzed: Order, error constant, consistency, and zero stability. The convergence of the method is guaranteed by its consistency and zero stability.

3.1. Order and Local Truncation Error

The local truncation error associated with the linear multistep method:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^4 \sum_{i=0}^{k} \beta_i f_{n+i}, \qquad (31)$$

is defined by the difference operator

$$L[y(x) : h] = \sum_{i=0}^{k} [\alpha_i y(x_n + ih) - h^4 \beta_i f(x_n + ih)], \qquad (32)$$

where $y(x)$ is an arbitrary function, continuously differentiable on [a, b], Lambert [4] and Iserles [14]. Expanding (32) in Taylor series about point x leads to the expression:

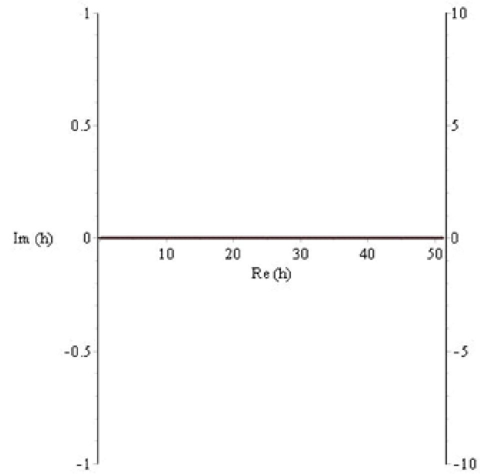

Figure 1. Region of absolute stability of the new block method

$$L[y(x) : h] = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + \cdots + c_p h^p y^{(p)}(x) + \cdots c_{p+4} h^{p+4} y^{(p+4)}(x), \qquad (33)$$

where $c_0, c_1, c_2, \cdots, c_p, \cdots, c_{p+3}$ are obtained as follows:

$$c_0 = \sum_{i=0}^{k} \alpha_i, \qquad c_1 = \sum_{i=1}^{k} i\alpha_i,$$

$$c_2 = \frac{1}{2!} \sum_{i=1}^{k} i^2 \alpha_i,$$

$$\vdots$$

$$c_q = \frac{1}{q!} [\sum_{i=1}^{k} i^q \alpha_i - q(q-1)(q-2) \sum_{i=1}^{k} \beta_i i^{q-3}].$$

Therefore, method (31) is of order p if $c_0 = c_1 = c_2 = \cdots = c_p = c_{p+1} = c_{p+2} = c_{p+3} = 0$ and $c_{p+4} \neq 0$. The constant $c_{p+4} \neq 0$ is called the error constant and $c_{p+4}\ h^{p+4} y^{(p+4)}(x)$ is the principal local truncation error at x_n. Using the above definition, the method (11), (15)–(29) is of order $p = 5$ and error constant

$$c_9 = \Big(\frac{1}{3024}, \frac{559}{8400}, \frac{-11}{5600}, \frac{53}{25200}, \frac{-11}{5600}, \frac{-1}{600}, \frac{23}{1512}, \frac{-11}{15120}, \frac{-11}{15120}, \frac{11}{15120}, \frac{-23}{1512}, \frac{-883}{7560}, \frac{251}{7560}, \frac{-211}{7560}, \frac{251}{7560}, \frac{-883}{7560}\Big)^T.$$

The interval of absolute stability is (0, 51.4286) of the real line.

3.2. Consistency

According to Anake [15], given a continuous implicit 4 - step method with first and second characteristics polynomials defined as: $\rho(r) = \sum_{j=0}^{k} \alpha_j r^j, \qquad \sigma(r) = \sum_{j=0}^{k} \beta_j r^j,$
then the block method is consistency if it satisfies the following conditions:

Table 1. Boundaries of Region of Absolute Stability of the New Block Method

θ	0^0	30^0	60^0	90^0	120^0	150^0	180^0
x	0.0000	0.0752	1.2020	6.0504	18.4615	38.8354	51.4286
$y(10^{-15})$	0.0000	0.0903	-0.0798	0.0000	0.0000	0.9821	0.0000

(i) the order of the method is $p \geq 1$.

(ii) $\sum_{j=0}^{4} \alpha_j = 0$.

(iii) $\rho(1) = \rho'(1)$.

(iv) $\rho^{iv}(1) = 4!\sigma(1)$.

For the main method (11), the first and second characteristic polynomials are given as: $\rho(r) = r^4 - 4r^3 + 6r^2 - 4r + 1$ and $\sigma(r) = \frac{1}{720}(-r^4 + 124r^3 + 474r^2 + 124r - 1)$.
Clearly, the main method (11) satisfies the above four conditions for consistency, therefore the new block method is consistent.

3.3. Zero Stability

To analyze the method for zero stability, we normalize (30) and obtain the normalized matrix difference equation:

$$A^* Y_m = B^* Y_{n-1} + h^4[C^* f_n + D^* F_m], \tag{34}$$

Hence, the zero stability of the method is determined by the expression:

$$\rho(r) = det(rA^* - B^*) = 0 \quad as \quad h \rightarrow 0. \tag{35}$$

Here;

$$A^* = \begin{pmatrix}
1 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$B^* = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Solving equation (35) for r leads to:
$\rho(r) = r^{15}(r-1) = 0 \Rightarrow r = 0, 1$. Therefore, the method is zero stable.

3.4. Convergence

According to Dahlquist [16] and Butcher [17], the necessary and sufficient conditions for a method to be convergent is to be consistent and zero stable. Thus, the block method (11), (15)–(29) is convergent since it is consistent and zero stable.

3.5. Region of Absolute Stability

By boundary locus method, the boundary of absolute stability is given by

$$h(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} = \frac{720(e^{4i\theta} - 4e^{3i\theta} + 6e^{2i\theta} - 4e^{i\theta} + 1)}{(-e^{4i\theta} + 124e^{3i\theta} + 474e^{2i\theta} + 124e^{i\theta} - 1)}, \tag{36}$$

where $\rho(r)$ and $\sigma(r)$ are respectively first and second characteristics polynomials of the main method (11) respectively. Therefore,

$$h(\theta) = \frac{720(\cos 4\theta - 4\cos 3\theta + 6\cos 2\theta - 4\cos\theta + 1)}{(-\cos 4\theta + 124\cos 3\theta + 474\cos 2\theta + 124\cos\theta - 1)} \frac{+720i(\sin 4\theta - 4\sin 3\theta + 6\sin 2\theta - 4\sin\theta)}{+i(-\sin 4\theta + 124\sin 3\theta + 474\sin 2\theta + 124\sin\theta)}.$$

Direct computation of $h(\theta)$ to 4 decimal places for $0^0 \leq \theta \leq 180^0$ with step length of 30^0 gives the following Table 2-5.

4. Numerical Examples

In order to determine the applicability, suitability and accuracy of the method developed in section 2, the following IVPs of fourth order ordinary differential equations were considered:
Example 1 Solve the initial value problem:

$$y^{iv} + x = 0; y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 0. h = 0.1.$$

Analytical Solution is $y(x) = \frac{x^5}{120} + x$ [18].
Example 2
Solve the nonlinear inhomogeneous initial value problem:

$$y^{iv} = (y')^2 - yy''' - 4x^2 + e^x(1 - 4x + x^2); 0 \leq x \leq 1,$$
$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 3, \quad y'''(0) = 1. \quad h = 0.1.$$

Analytical Solution is $y(x) = x^2 + e^x$ [3].
Example 3
Solve the initial value problem:

$$y^{iv} - y = 0; y(0) = 1, y'(0) = 0, y''(0) = -2, y'''(0) = 0. h = \frac{1}{320}$$

Analytical Solution is $y(x) = \frac{-1}{4}e^x - \frac{1}{4}e^{-x} + \frac{3}{2}\cos x$ [18].
Example 4

Table 2. Absolute Errors for Example 1 with $h = 0.1$

x	Approximate Solution	New Method	Kayode *et al.* [20]	Kuboye & Omar [21]	Mohammed [22]	Kuboye *et al.* [18]
0.1	0.100000083	0.00	0.00	1.0E-12	7.00E-10	0.00
0.2	0.200002666	0.00	0.00	0.00	8.00E-10	0.00
0.3	0.300020250	0.00	0.00	0.00	3.00E-09	0.00
0.4	0.400085333	0.00	5.55E-17	0.00	5.10E-09	5.55E-17
0.5	0.500260417	0.00	1.11E-16	1.00E-12	7.80E-09	1.11E-16
0.6	0.600648000	2.00E-20	1.11E-16	2.75E-12	1.18E-08	1.11E-16
0.7	0.701400583	4.00E-20	2.22E-16	3.50E-12	1.24E-08	2.22E-16
0.8	0.802730667	7.00E-20	0.00	3.50E-12	1.41E-08	0.00
0.9	0.904920750	1.90E-19	1.11E-16	4.18E-12	1.88E-08	1.11E-16
1.0	1.008333333	2.10E-19	2.22E-16	4.76E-12	1.01E-08	2.22E-16

Table 3. Absolute Errors for Example 2 with $h = 0.1$

x	Approximate Solution	New Method	Ogunlaran & Kehinde [3]	Alechienu & Oyewola [23]	Alechienu&Oyewola [23] (Adam-Bashforth)
0.1	1.115170918	0.00	3.00E-09	0.00	0.00
0.2	1.261402758	0.00	6.00E-09	0.00	0.00
0.3	1.439858808	0.00	8.00E-09	0.00	0.00
0.4	1.651824698	0.00	1.10E -08	0.00	0.00
0.5	1.898721272	1.00E-09	2.10E-08	1.71E-05	2.53E-01
0.6	2.182118804	4.00E-09	3.30E-08	9.44E-05	7.19E-01
0.7	2.503752715	8.00E-09	4.40E-08	3.11E-04	1.44E-00
0.8	2.865540940	1.20E-08	5.70E-08	7.94E-04	2.33E-00
0.9	3.269603128	1.68E-08	1.12E-07	1.73E-03	3.41E-00
1.0	3.718281870	4.15E-08	1.82E-07	3.38E-03	4.72E-00

Table 4. Absolute Errors for Example 3 with $h = \frac{1}{320}$

x	Approximate Solution	New Method	Kuboye *et al.* [18] EIFBM	Kuboye *et al.* [18] EISBM	Alechienu & Oyewola [23]
0.003125	0.999990234	2.46E-28	2.22E-16	4.44E-16	4.44E-16
0.006250	0.999960937	3.80E-27	0.00	0.00	2.18E-14
0.009375	0.999912110	1.51E-26	2.22E-16	0.00	7.72E-13
0.012500	0.999843751	3.85E-26	4.44E-16	4.44E-16	7.67E-13
0.015625	0.999755862	7.75E-26	0.00	2.22E-16	2.37E-12
0.018750	0.999648443	1.41E-26	4.44E-16	2.22E-16	5.93E-12
0.021875	0.999521494	2.42E-26	2.22E-16	0.00	1.29E-11
0.025000	0.999375016	3.95E-26	2.22E-16	2.22E-16	2.52E-11
0.028125	0.999209010	6.08E-26	4.44E-16	4.44E-16	4.55E-11
0.031250	0.999023477	8.96E-26	0.00	2.22E-16	7.71E-11

Table 5. Absolute Error for Example 4 with $h = \frac{1}{320}$

x	Approximate Solution	New Method	Ukpebor*et al.* [19]	Familua & Omole [2] Method 1	Familua & Omole [2] Method 2
0.003125	0.999999999	3.69E-27	1.90E-19	6.69E-13	5.69E-10
0.006250	0.999999999	5.70E-26	2.30E-19	1.46E-11	1.77E-10
0.009375	0.999999999	2.26E-25	8.60E-19	1.08E-10	5.91E-09
0.012500	0.999999997	5.77E-25	1.38E-18	1.08E-10	5.77E-09
0.015625	0.999999995	1.16E-24	3.53E-18	1.03E-09	1.10E-08
0.018750	0.999999990	2.11E-24	5.31E-18	2.22E-09	6.90E-08
0.021875	0.999999981	3.63E-24	8.88E-18	4.23E-09	4.64E-08
0.025000	0.999999967	5.92E-24	3.92E-17	7.36E-09	5.79E-07
0.028125	0.999999948	9.11E-24	5.85E-17	1.20E-08	2.25E-07
0.031250	0.999999921	1.34E-23	8.48E-17	1.85E-08	2.85E-07

Consider an application problem from ship dynamics below:

$$y'^{v} + 3y'' + y(2 + \epsilon\cos(\Omega x)) = 0; \quad x > 0$$

which is subjected to the following initial conditions

$$y(0) = 1, \quad y'(0) = y''(0) = y'''(0) = 0, \quad h = \frac{1}{320},$$

where $\Omega = 0$ for the theoretical solution $y(x) = 2\cos x - \cos(x\sqrt{2})$ [19].

Tables 2–5 show comparison of absolute errors in the numerical solutions of Examples 1–4 for the new method and other existing methods in the literature.

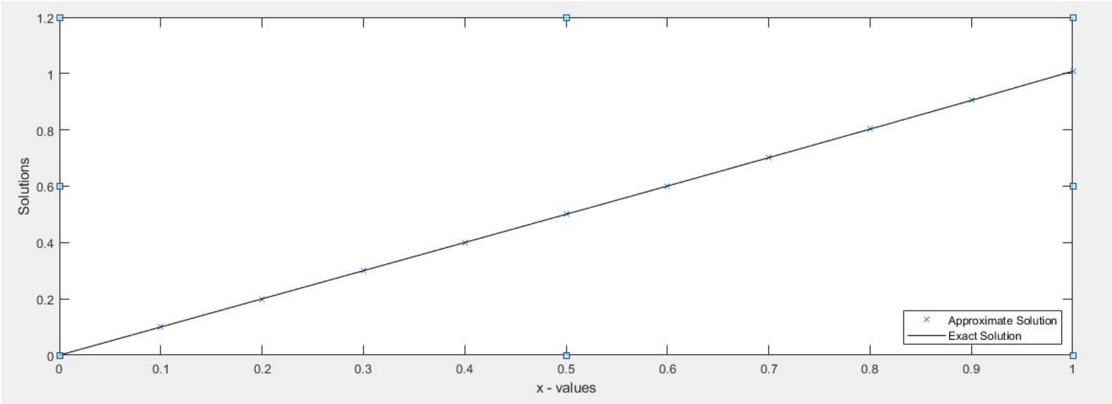

Figure 2. Comparison of Exact and Approximate Solutions for Example 1

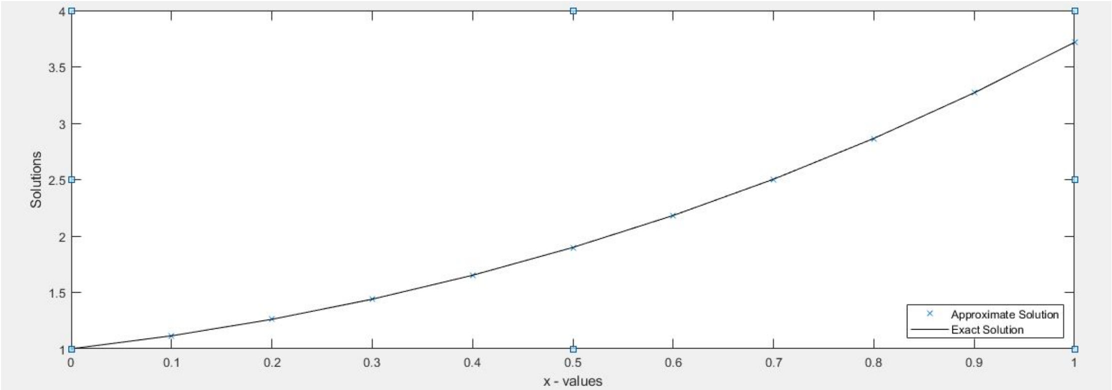

Figure 3. Comparison of Exact and Approximate Solutions for Example 2

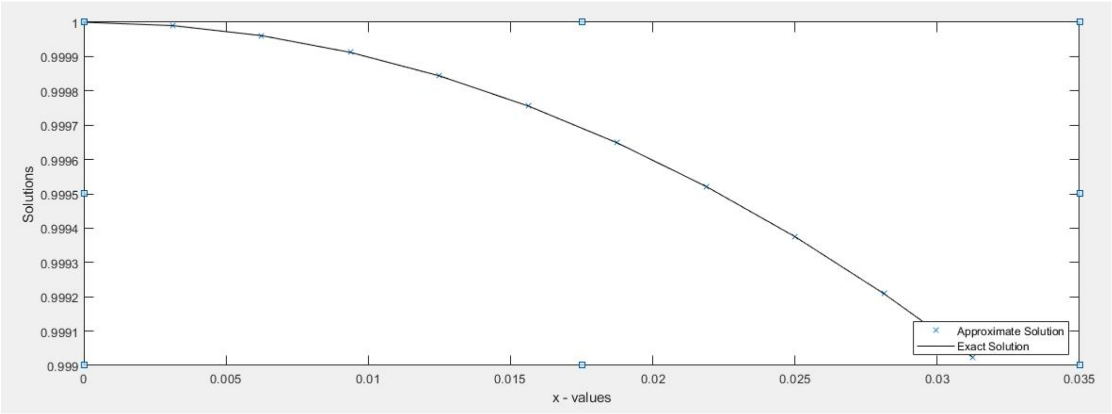

Figure 4. Comparison of Exact and Approximate Solutions for Example 3

5. Discussion of Result

The new method is consistent, zero stable and convergent. The method also shows an appreciable error constants. From Table 1, the interval of absolute stability is given as $(0, 51.4286)$ while the region of absolute stability is presented in Figure 1.

The numerical results of the problems considered using the new block method is compared with that of Kayode *et al.* [20], Kuboye & Omar [21], Mohammed [22] and Kuboye *et al.* [18] in Example 1; Ogunlaran & Kehinde [3], Alechienu & Oyewola [23] and Alechienu & Oyewola [23] using Adams-Bashforth method in Example 2 while Kuboye *et al.* [18] with EIFBM, Kuboye *et al.* [18] with EISBM and Alechienu & Oyewola [23] in Example 3. The new block method is also applied on an application problem from ship dynamics. The results obtained is compared with those of Ukebor *et al.* [19], Familua & Omole [2] (block method) and Familua & Omole [2] (Predictor - Corrector method).

We considered step length $h = 0.1$ for Numerical Examples 1 and 2 while $h = 1/320$ was used for Numerical Exam-

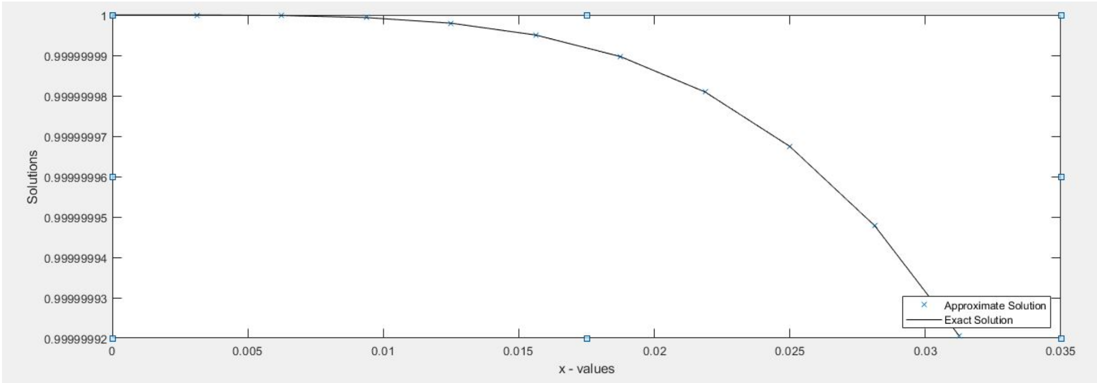

Figure 5. Comparison of Exact and Approximate Solutions for Example 4

ples 3 and 4. Maximum absolute error in Example 1 for the new method as presented in Table 1 is 2.1E-19 while 2.22E-16, 4.76E-12, 1.88E-12, and 2.22E-16 are obtained for Kayode *et al.* [20], Kuboye & Omar [21], Mohammed [22] and Kuboye *et al.* [18] respectively. For Example 2, the maximum absolute error of the new method as presented in Table 2 is 4.15E-08 while 1.82E-07, 3.38E-03 and 4.72E-00 are for Ogunlaran & Kehinde [3], Alechienu & Oyewola [23] and Alechienu & Oyewola [23] using Adams-Bashforth method respectively. The new block method shows a good performance over other methods in the literature for Example 3 as presented in Table 3. The maximum absolute error of the new method is 8.96E-26 while 4.44E-16, 4.44E-16 and 7.71E-11 are for Kuboye *et al.* [18] with EIFBM, Kuboye *et al.* [18] with EISBM and Alechienu & Oyewola [23] respectively. The new block method also performs well in application problem of ship dynamics with maximum absolute error 1.34E-23 which is significantly lower than those of methods in the literature with maximum absolute errors 8.48E-17, 1.85E-08 and 2.85E-07 for Ukpebor *et al.* [19], Familua & Omole [2] (block method) and Familua & Omole [2] (Predictor - Corrector method) respectively as presented in Table 5.

Comparison of exact and approximate solutions for the Examples 1–4 are presented in Figures 2– 5 respectively. Figure 2– 5 show no visible difference in the exact and approximate solutions for the Problems considered. This shows that the new method is applicable and preferable in solving all problems considered. The new method performed better than the existing methods considered in the literature in both linear and nonlinear problems.

Validity of the results can be seen from the analysis of the new block method presented in section 3 and graphical presentation of exact and approximate solutions in section 4.

6. Conclusion

In this study, the Chebyshev-based block integrator is specifically formulated for direct solution of fourth-order IVPs. The method explores the unique properties of Chebyshev polynomials and their efficient use within a block integration framework. Through a series of numerical experiments and comparisons with existing methods, we have demonstrated the remarkable accuracy and stability offered by this innovative technique in solving fourth-order IVPs. The results of the numerical experiments consistently showed that the Chebyshev-based block integrator outperforms existing numerical integration methods. Its ability to control error accumulation and maintain numerical stability even in challenging scenarios is a substantial advancement in the field of numerical analysis. This new method has the potential to impact a wide array of applications across numerous scientific and engineering domains. From the accurate simulation of complex physical systems to the precise modeling of intricate dynamic processes in biology, economics, and other fields, the Chebyshev-based direct block integrator offers an invaluable tool for researchers and practitioners seeking reliable numerical solutions to fourth-order IVPs.

References

[1] D. O. Awoyemi, "A Class of continuous methods for general second order initial value problems in ordinary differential equations", International journal of computer Mathematics **72** (1999) 29. https://doi.org/10.1080/00207169908804832.

[2] A. B. Familua & E. O. Omole, "Five points mono hybrid linear multistep method for solving nth order ordinary differential equations using power series function", Asian Research Journal on Mathematics (Science Domain International) **3** (2017) 1. https://journalarjom.com/index.php/ARJOM/article/view/183.

[3] O. M. Ogunlaran & M. A. Kehinde, "4 Step Hermite based multiderivative block integrator for solving fourth order ordinary differential equations", The Journal of Mathematical Association of Nigeria **49** (2022) 1. https://www.man-nigeria.org.ng/issues/ABA-SCI-2022-14.pdf.

[4] J. D. Lambert, *Computational methods in ordinary differential equation*, John Wiley and Sons, London, U. K, 1974.

[5] H. Ramos, S. N. Jator & M. I. Modebei, "Efficient K-step linear block methods to solve second order initial value problems directly", Mathematics **8** (2020) 1752. https://doi.org/10.3390/math8101752.

[6] V. O. Atebo & S.O. Adee, "A new special 15-step block method for solving general fourth order ordinary differential equations", Journal of Nigerian Society of Physical Sciences **3** (2021) 308. https://doi.org/10.46481/jnsps.2021.337.

[7] M. I. Modebei, O. O. Olaiya & I. P. Ngwangwo, "Computational study of some three-step hybrid integration for solution of third order ordinary differential equations", Journal of Nigerian Society of Physical Sciences **3** (2021) 459. https://doi.org/10.46481/jnsps.2021.323.

[8] L. O. Adoghe & E. O. Omole,"A fifth-fourth continuous block implicit hybdrid method for the solution of third order initial value problems in

ordinary differential equations", Applied and Computational Mathematics **8** (2019). https://www.researchgate.net/publication/335149797.

[9] A. O. Adeniran & I. O. Longe, "Solving directly second order initial value problems with Lucas polynomial", Journal of Advances in Mathematics and Computer Science **32** (2019) 1. https://www.researchgate.net/publication/333853090.

[10] W. Nazreen & A. M. Zanariah, "A 4 – Point block method for solution of higher order ordinary differential equations directly", International Journal of Mathematics and Mathematical Sciences (2016) 1. https://dx.doi.org/10.1155/2016/9823147.

[11] B. T. Olabode & A. L. Momoh, "Chebyshev hybrid multistep method for directly solving second-order initial and boundary value problems", Journal of the Nigerian Mathematical Society **39** (2020) 97. https://www.researchgate.net/publication/341090332.

[12] M. O. Alabi, K. S., Adewoye & O. Z. Babatunde,"Integrator block offgrid points collocation method for direct solution of second order ordinary differential equations using Chebyshev as basis polynimials", International Journal of Mathematics and Statistics Studies **7** (2019) 12. https://www.eajournals.org/wp-content/uploads/Integrator-Block-Off.pdf

[13] A. H. Khater, R. S. Temsah, & M. M. Hassan, "A chebyshev spectral collocation method for solving burgers'-type equations", Journal of Computational and Applied Mathematics **222** (2008) 333. https://doi.org/10.1016/j.cam.2007.11.007.

[14] A. Iserles, "A First Course in the Numerical Analysis of Differential Equations", Cambridge Texts in Applied Mathematics **44** (2009) 252. https://doi.org/10.1017/CBO9780511995569.

[15] T. A Anake, Continuous implicit hybrid 0ne-step methods for the solution of initial value problems of general second order ordinary differential equations, Ph.D Thesis, Department of Mathematics, Covenant University, Ota, Ogun State, 2011. http://eprints.covenantuniversity.edu.ng/id/eprint/7856.

[16] G. Dahlquist, "Convergence and the Dahlquist equivalence theorem" (2010), 54. https://people.maths.ox.ac.uk/trefethen/1all.pdf.

[17] J. C. Butcher, "Numerical Methods for Ordinary Differential Equations", John Wiley & Sons, Chichester, England, 2008. http://dx.doi.org/10.1002/9780470753767.

[18] J. O. Kuboye, O. R. Elusakin & O. F. Quadri, "Numerical algorithm for direct solution of fourth order ordinary differential equations", Journal of the Nigerian Society of Physical Sciences **2** (2020) 218. https://journal.nsps.org.ng/index.php/jnsps/article/view/100.

[19] L. A. Ukebor, E. O. Omole & L. O. Adoghe, "An order four numerical scheme for fourth-order initial value problems using Lucas polynomial with application in ship dynamics", International Journal of Mathematics Research **9** (2020) 28. https://www.researchgate.net/publication/343140406.

[20] S. J. Kayode, M. K. Duromola & B. Bolarinwa, "Direct solution of initial value problems of fourth order ordinary differential equations using modified implicit hybrid block method", Journal of Scientific Research and Reports **3** (2014) 2790. https://www.researchgate.net/subnational/309527154.

[21] J. O. Kuboye & Z. Omar, "New zero- stable block method for direct solution of fourth order ordinary differential equations", Indian Journal of Science and Technology **8** (2015) 1. doi:10.17485/ijst/2015/v8i12/70647.

[22] U. Mohammed, "A six step block method for solution of fourth order ordinary differential equations", The Pacific Journal of Science and Technology **11** (2010) 259. https://staff.futminna.edu.ng/MAT/content/journal/PF1412/2.

[23] B. Alechienu & D. O. Oyewola,"An implicit collocation method for direct solution of fourth order ordinary differential equations", Journal of Applied Sciences and Environmental Management **23** (2019) 2251. DOI:https://dx.doi.org/10.4314/jasem.v23i12.25.