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Iterative method for the numerical solution of optimal control model for mosquito and insecticide

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Abstract

A linear multistep method is transformed into an iterative method based on the Patade and Bhalekar technique for the numerical solution of an optimal control problem modeled for mosquito and insecticide management using forward-backward sweep methods via Pontryagin's principle. Stability and convergence analysis of the iterative method are carried out and it is found to be stable, convergent, and of order four. Results obtained by the method clearly show that the population of mosquitoes can be minimized to a large extent using the new iterative method while reducing the harmful effects of the insecticide, which subsequently reduces the spread of malaria.

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1. Introduction

Researchers widely use mathematical models to explain various phenomena arising in science, technology, engineering, economics and social sciences which are based on ordinary and partial differential equations [1–4]. In our environment today, many physical problems give rise to first order ordinary differential equations of Initial Value Problems (IVPs) [4–6].

Malaria, which is a mosquito - borne disease is among the leading causes of human deaths especially in Africa, needs to be controlled by eradicating the mosquitoes [7, 8]. Vector control is the most common global strategy for management of mosquito-associated diseases, and the application of insecticide [7, 8].

Insecticides are the easiest way to get rid of mosquitoes around the yard, but, they are only temporary measures; as soon as the insecticide drifts away or dries out, the mosquitoes are back [7]. Insecticides are formulated specifically to kill flying and crawling insects. It has a long lasting residual protection against flying and crawling insects, but has adverse effect on human being if it comes in contact with eye or skin. Excessive inhalation of insecticides also has a devastating effect on the health. Hence, there is a need to drastically reduce the population of mosquitoes around the yard, while minimizing the harmful effect of the application of insecticide on human beings.

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This study aims at developing an iterative method for the minimization of both the mosquito population and the effect of insecticide used for the eradication of the mosquitoes. This paper considers the numerical solution to optimal control problems of the form

$$\min J\left[x(t), u(t)\right] = \int_{t_0}^{T_f} f(t, x(t), u(t))dt, \tag{1}$$

subject to the dynamic system

$$x' = g(t,\ x(t),\ u(t)), \tag{2}$$

$$x(t_0) = x_0, \tag{3}$$

where $x(t)$ and $u(t)$ are real valued functions, f and g are smooth functions.

Fatmawati *et al.* [9] studied some mathematical models of malaria transmission with and without seasonal factor and numerical simulation of the model show that, providing controls in the form of insecticide, prevention, and treatment simultaneously are effective in reducing the number of the exposed and infectious human population, and also the infectious mosquito population. Araujo, *et al.* [8] studied the theoretical and numerical optimal control problems concerning the extinction of mosquito populations by using moving devices whose main role is to spread insecticide. The optimal trajectory are computed using optimality conditions, and performed a rigorous analysis of an optimal control problem.

Forward Backward Sweep (FBS) is an iterative method based on how the algorithm solves the problem state and control [10, 11]. To approximate the control function, FBS solves the state 'forward' in time first, before solving the adjoint 'backward' in time. Finding the optimal result by solving the necessary conditions is commonly known as the indirect approach. Pontryagin's Maximum (or Minimum) Principle (PMP) is a powerful method for the computation of optimal controls, which has the crucial advantage that it does not require prior evaluation of the infimal cost function [10–13].

Forward-backward sweep method is extensively discussed by Araujo *et al.* [8] for the solution of optimal control problem, using classical order Runge-Kutta via Pontryagin's maximum principle. Orakwelua *et al*. [5] used the optimal control strategies to minimize the population of mosquitoes in the ponds and swamps by applying forward-backward sweep method using classical Runge-Kutta method. Garret [10] also adopted FBS to solve Optimal Control Problem (OCP) using Classical Runge-Kutta Method (CRKM). Hence CRKM became the most popular choice method for solving optimal control problems using FBS.

Runge-Kutta (R-K) formulas are among the oldest schemes in numerical analysis [14] and are an important family of iterative methods for the approximation of solutions of ODEs [15, 16]. Runge-Kutta methods are typically single-step methods, however, with multiple stages per step which were first studied by Carle Runge and Martin Kutta [2, 13, 15]. The Runge-Kutta methods are fairly simple to program and easy to implement.

Patade and Bhalekar [1] transformed a single step trapezoidal method to 3 stage Runge-Kutta type method using constant step length for the solution of ordinary differential equations. The techniques proved to be efficient for the solution of ODEs, but to the best of our knowledge, it has never been used for the solution of optimal control problems. The proposed technique is highly promising for the numerical solution of optimal control model, as evidenced by the fact that, hybrid points, especially of lower step size, enhance stability and approximation properties in one-step approaches [17]. According to Davaeifar and Rashidinia [18], the validity and dependability of the findings produced are the main benefits of adopting FBPs in the construction of the collocation method. As a result, it stands to reason that using FBPs in conjunction with the hybrid point will result in an approach that is more accurate in approximation.

In this paper, new four-stage iterative method is developed based on Patade and Bhalekar [1] approach for the numerical solution of optimal control model for mosquito population growth and insecticide.

2. Methodology

2.1. Development of linear multistep methods

Consider the polynomial approximate solution of the form

$$x(t) = \sum_{n=0}^{3} \alpha_n t^n, \tag{4}$$

where $t \in [a, b]$, $a_n \in \mathbb{R}$ are unknown parameters to be determined.

Interpolating and collocating Eq. (4) using the points

$$x\left(t_{n+j}\right) = x_{n+j}, \quad j = 0;$$

$$x'\left(t_{n+j}\right) = f_{n+j}, \quad j = 0, \frac{49}{100}, 1$$

gives a system of equations

$$XA = U, \tag{5}$$

where

$$X = \begin{bmatrix} 3 & 2t_n & t_n^2 & t_n^3 \\ 0 & 2 & 2t_n & 3t_n^2 \\ 0 & 2 & 2t_{n+\frac{49}{100}} & 3t^2_{n+\frac{49}{100}} \\ 0 & 2 & 2t_{n+1} & 3t^2_{n+1} \end{bmatrix},$$

$$A = \left[a_0, a_1, a_2, a_3\right]^T,$$

$$U = \left[x_n,\ f_n,\ f_{n+\frac{49}{100}},\ f_{n+1}\right]^T.$$

Solving the system Eq. (5) for the unknown parameters and substitute it into the approximate solution Eq. (4) to get the continuous scheme

$$x_{n+t} = \alpha_0(t)x_n + \beta_0(t)hf_n + \beta_{\frac{49}{100}}(t)hf_{n+\frac{49}{100}} + \beta_1(t)hf_{n+1}, \tag{6}$$

where the coefficients $\alpha_0(t), \beta_0(t), \beta_v(t), \beta_1(t)$ are:

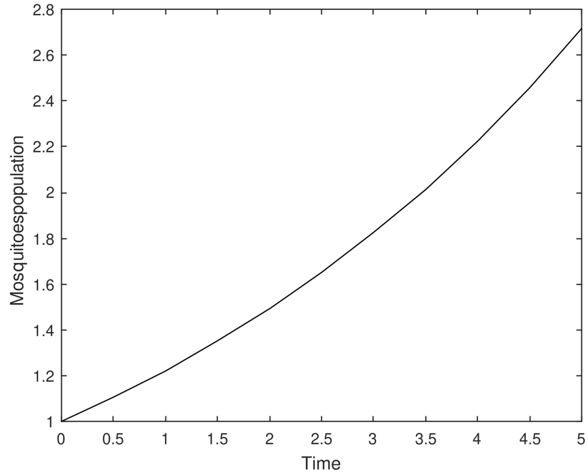

Figure 1. The Mosquito population without control case 1.

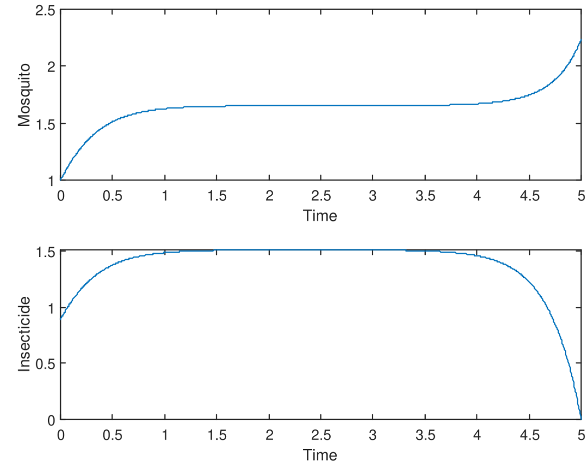

Figure 2. The optimal Mosquito population and Insecticide for case 2.

$\alpha_0(t) = 1,$
$\beta_0(t) = \frac{100}{147}t^3 - \frac{149}{98}t^2 + t,$
$\beta_v(t) = \frac{5000}{2499}t^2 - \frac{10\,000}{7497}t^3,$
$\beta_1(t) = \frac{100}{153}t^3 - \frac{49}{102}t^2.$

Evaluating Equation (6) at point x_{n+1} to get the discrete scheme

$$x_{n+1} = x_n + \frac{h}{14\,994}\left(2397 f_n + 10\,000 f_{n+\frac{49}{100}} + 2597 f_{n+1}\right). \quad (7)$$

2.2. Development of the Iterative Method

Equation (7) can be written as

$$x_{n+1} = x_n + \frac{47}{294}hf_n + \frac{5000}{7497}hf_{n+\frac{49}{100}} + \frac{53}{306}hf_{n+1}. \quad (8)$$

Let

$$x = x_{n+1}, \quad (9)$$

$$x_0 = f = x_n + \frac{47h}{294}f(t_n, x_n) + \frac{5000h}{7497}f\left(t_{n+\frac{49}{100}}, x_{n+\frac{49}{100}}\right), \quad (10)$$

$$N(x_{n+1}) = N(x) = \frac{53h}{306}f\left(t_{n+1}, x_{n+1}\right). \quad (11)$$

Using 3-terms solution in Daftardar-Gejji and Jafari Method (DJM) series [1], then

$$x = x_0 + x_1 + x_2, \quad (12)$$

where $x_1 = N(x_0)$ and $x_2 = N(x_0 + x_1) - N(x_0)$. Equation (12) becomes

$$x = x_0 + N(x_0) + N(x_0 + x_1) - N(x_0)$$

and simplifying it to get

$$x = x_0 + N\left(x_0 + N\left(x_0\right)\right). \quad (13)$$

Substitutes Eqs. (9) and (10) into Eq. (13) to get

$$x_{n+1} = x_n + \frac{47}{294}hf(t_n, x_n) + \frac{5000}{7497}hf\left(t_{n+\frac{49}{100}}, x_{n+\frac{49}{100}}\right) \quad (14)$$

$$+ N\left(\begin{array}{c} x_n + \frac{47}{294}hf(t_n, x_n) + \frac{5000}{7497}hf\left(t_{n+\frac{49}{100}}, x_{n+\frac{49}{100}}\right) \\ +N\left(x_n + \frac{47}{294}hf(t_n, x_n) + \frac{5000}{7497}hf\left(t_{n+\frac{49}{100}}, x_{n+\frac{49}{100}}\right)\right) \end{array}\right).$$

This can be simplified, by considering Equation (11) as

$$x_{n+1} = x_n + \frac{47}{294}hf(t_n, x_n) + \frac{5000}{7497}hf\left(t_{n+\frac{49}{100}}, x_{n+\frac{49}{100}}\right) + \frac{53h}{306}f \quad (15)$$

$$\left(\begin{array}{c} t_{n+1}, x_n + \frac{47}{294}hf(t_n, x_n) + \frac{5000}{7497}hf\left(t_{n+\frac{49}{100}}, x_{n+\frac{49}{100}}\right) \\ +\frac{53h}{306}f\left(t_{n+1}, x_n + \frac{47}{294}hf(t_n, x_n) + \frac{5000}{7497}hf\left(t_{n+\frac{49}{100}}, x_{n+\frac{49}{100}}\right)\right) \end{array}\right).$$

Equation (15) reduces to

$$x_{n+1} = x_n + \frac{h}{14994}\left(2397k_1 + 10000k_2 + 2597k_4\right), \quad (16)$$

where

$$k_1 = f(t_n, x_n), \quad k_2 = f\left(t_{n+\frac{49}{100}}, x_{n+\frac{49}{100}}\right),$$

$$k_3 = f\left(t_{n+1}, x_n + \frac{47h}{294}k_1 + \frac{5000h}{7497}k_2\right),$$

$$k_4 = f\left(t_{n+1}, x_n + \frac{47h}{294}k_1 + \frac{5000h}{7497}k_2 + \frac{53h}{306}k_3\right).$$

Proposition. The established iterative method in Eq. (16) is Non-Runge-Kutta Method.

Proof. To show that, the iterative method Eq. (16) is not a Runge-Kutta method, Runge-Kutta properties are tested. The iterative method is written as:

$$x_{n+1} = x_n + h\left(b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4\right),$$

while k_i is written as

$$k_i = f\left(\begin{array}{c} t_j + c_1h, x_j + h(a_{i1}k_1 \\ +a_{i2}k_2 + a_{i3}k_3 + a_{i4}k_4) \end{array}\right), i = 1, 2, 3, 4.$$

The following parameters are extracted from the iterative method

$$b_1 = \frac{47}{294}, \; b_2 = \frac{5000}{7497}, \; b_3 = 0, \; b_4 = \frac{53}{306};$$

$$c_1 = 0,\ c_2 = \tfrac{49}{100}, c_3 = c_4 = 1;$$

$$a_{11} = a_{12} = a_{13} = a_{14} = 0,\ a_{21} = a_{22} = a_{23} = a_{24} = 0,$$

$$a_{31} = \tfrac{47}{294},\ a_{32} = \tfrac{5000}{7497},\ a_{33} = a_{34} = 0,$$

$$a_{41} = \tfrac{47}{294}, a_{42} = \tfrac{5000}{7497},\ a_{43} = \tfrac{53}{306}, a_{44} = 0,$$

Thus, the Butcher table for iterative method is

$$\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
\frac{49}{100} & 0 & 0 & 0 & 0 \\
1 & \frac{47}{294} & \frac{5000}{7497} & 0 & 0 \\
1 & \frac{47}{294} & \frac{5000}{7497} & \frac{53}{306} & 0 \\
\hline
 & \frac{47}{294} & \frac{5000}{7497} & 0 & \frac{53}{306}
\end{array}$$

For Runge-Kutta method, it is necessary that $\sum_{j=1}^{4} a_{ij} = c_i$ [19].

From the parameters, $a_{31} + a_{32} + a_{33} + a_{34} = \frac{253}{306} \neq c_3$. This shows that iterative method is not Runge-Kutta method.

2.3. Analysis of the Iterative Method

2.3.1. Order of the method

Definition 2.1. The order of a single step method is the largest integer p such that

$$\left\| h^{-1} T_j \right\| = O\left(h^p\right). \tag{17}$$

Source: [1]

Theorem 2.1. The order of the new iterative method in Eq. (16) is four.

Proof. Having

$$k_1 = f_n. \tag{18}$$

Considering the second order Taylor's series, k_2, k_3 and k_4 can be presented as

$$k_2 = f_n + \frac{49h}{100} f_{n,t} + \frac{49h}{200} f_{n,x} + \frac{49h^2}{200} f_{n,tt} + \frac{49h^2}{200} f_{n,tx} + \frac{49h^2}{400} f_{n,xx} + \ldots \tag{19}$$

$$k_3 = f_n + h f_{n,t} + \frac{h^2}{2} f_{n,tt} + \frac{47h^2}{294} f_n f_{n,tx} + \frac{47h}{294} f_n f_{n,x} + \frac{47h^2}{21168} f_n^2 f_{n,xx} + \ldots \tag{20}$$

$$k_4 = f_n + h f_{n,t} + \frac{h^2}{2} f_{n,tt} + W_2 \left[h f_{n,tx} + f_{n,x} + \frac{h}{4} f_n f_{n,xx} \right] + \ldots, \tag{21}$$

where

$$W_2 = \frac{47h}{294} f_n + \frac{5000h}{7497} \left(\begin{array}{c} \frac{49h}{100} f_{n,t} + \frac{49h}{200} f_{n,x} + \frac{49h^2}{200} f_{n,tt} \\ + \frac{49h^2}{200} f_{n,tx} + \frac{49h^2}{400} f_{n,xx} \end{array} \right) + \frac{53h}{306} \left(\begin{array}{c} f_n + h f_{n,t} + \frac{h^2}{2} f_{n,tt} + \frac{47h^2}{294} f_n f_{n,tx} \\ + \frac{47h}{294} f_n f_{n,x} + \frac{47h^2}{21168} f_n^2 f_{n,xx} \end{array} \right),$$

$$f_n = f(t_n, x_n),\ f_{n,t} = \left(\frac{\partial f(t,x)}{\partial t} \right)_{(t_n,x_n)}, f_{n,x} = \left(\frac{\partial f(t,x)}{\partial x} \right)_{(t_n,x_n)},$$

$$f_{n,tt} = \left(\frac{\partial^2 f(t,x)}{\partial t^2} \right)_{(t_n,x_n)},\ f_{n,xx} = \left(\frac{\partial^2 f(t,x)}{\partial x^2} \right)_{(t_n,x_n)}.$$

Using Eqs. (18) – (21) in Eq. (16), we get

$$x_{n+1} = x_n + \frac{h}{14994} \left(\begin{array}{c} 2397 f_n + 10000 \\ \left(f_n + \frac{49h}{100} f_{n,t} + \frac{49h}{200} f_{n,x} + \frac{49h^2}{200} f_{n,tt} + \frac{49h^2}{200} f_{n,tx} + \frac{49h^2}{400} f_{n,xx} \right) \\ +2597 \left(f_n + h f_{n,t} + \frac{h^2}{2} f_{n,tt} + W_2 \left[h f_{n,tx} + f_{n,x} + \frac{h}{4} f_n f_{n,xx} \right] \right) \end{array} \right)$$

$$\begin{aligned} x_{n+1} = {} & x_n + h f_n + h^2 \left(\tfrac{132\,341}{2294\,082} f_n f_{n,x} + \tfrac{1}{2} f_{n,t} + \tfrac{25}{153} f_{n,x} \right) \\ & + h^3 \left(\begin{array}{c} \frac{132\,341}{9176\,328} f_{n,x^2} f_n^2 + \frac{132\,023}{27\,528\,984} f_n f_{n,x}^2 + \frac{132\,341}{2294\,082} f_{n,tx} f_n \\ + \frac{1325}{46\,818} f_{n,x}^2 + \frac{53}{612} f_{n,t} f_{n,x} + \frac{25}{153} f_{n,tx} + \frac{1}{4} f_{n,t^2} + \frac{25}{306} f_{n,x^2} \end{array} \right) \\ & + h^4 \left(\begin{array}{c} \frac{53}{612} f_{n,tx} f_{n,t} + \frac{1325}{23\,409} f_{n,tx} f_{n,x} + \frac{53}{1224} f_{n,t^2} f_{n,x} \\ + \frac{1325}{93\,636} f_{n,x^2} f_{n,x} \\ + \frac{132\,023}{13\,764\,492} f_n f_{n,tx} f_{n,x} + \frac{53}{2448} f_n f_{n,x^2} f_{n,t} + \frac{1325}{187\,272} f_n f_{n,x^2} f_{n,x} \\ + \frac{2508\,437}{1982\,086\,848} f_n^2 f_{n,x^2} f_{n,x} \end{array} \right) \\ & + h^5 \left(\begin{array}{c} \frac{132\,023}{7928\,347\,392} f_n^3 f_{n,x^2}^2 + \frac{2508\,437}{1982\,086\,848} f_n^2 f_{n,tx} f_{n,x^2} \\ + \frac{132\,023}{27\,528\,984} f_n f_{n,tx}^2 \\ + \frac{1325}{187\,272} f_n f_{n,tx} f_{n,x^2} \\ + \frac{1325}{374\,544} f_n f_{n,x^2}^2 + \frac{53}{4896} f_{n,t^2} f_n f_{n,x^2} + \frac{1325}{46\,818} f_{n,tx}^2 \\ + \frac{1325}{93\,636} f_{n,tx} f_{n,x^2} + \frac{53}{1224} f_{n,t^2} f_{n,tx} \end{array} \right) \ldots \end{aligned}$$

Expanding the of exact value of $x(t)$ about t_n in Taylor's series

$$\begin{aligned} x(t_{n+1}) = {} & x_n + h f_n + h^2 \left(\frac{132\,341}{2294\,082} f_n f_{n,x} + \frac{1}{2} f_{n,t} + \frac{25}{153} f_{n,x} \right) + h^3 \\ & \left(\begin{array}{c} \frac{132\,341}{9176\,328} f_{n,x^2} f_n^2 + \frac{132\,023}{27\,528\,984} f_n f_{n,x}^2 + \frac{132\,341}{2294\,082} f_{n,tx} f_n \\ + \frac{1325}{46\,818} f_{n,x}^2 + \frac{53}{612} f_{n,t} f_{n,x} + \frac{25}{153} f_{n,tx} + \frac{1}{4} f_{n,t^2} + \frac{25}{306} f_{n,x^2} \end{array} \right) + O(h^4). \end{aligned} \tag{22}$$

The truncation error gives

$$T_{n+1} = x_{i+1} - x(t_{i+1})$$

$$\begin{aligned} T_{n+1} = {} & x_n + h f_n + h^2 \left(\frac{132\,341}{2294\,082} f_n f_{n,x} + \frac{1}{2} f_{n,t} + \frac{25}{153} f_{n,x} \right) \\ & + h^3 \left(\begin{array}{c} \frac{132\,341}{9176\,328} f_{n,x^2} f_n^2 + \frac{132\,023}{27\,528\,984} f_n f_{n,x}^2 + \frac{132\,341}{2294\,082} f_{n,tx} f_n \\ + \frac{1325}{46\,818} f_{n,x}^2 + \frac{53}{612} f_{n,t} f_{n,x} + \frac{25}{153} f_{n,tx} + \frac{1}{4} f_{n,t^2} + \frac{25}{306} f_{n,x^2} \end{array} \right) \\ & + h^4 \left(\begin{array}{c} \frac{53}{612} f_{n,tx} f_{n,t} + \frac{1325}{23\,409} f_{n,tx} f_{n,x} + \frac{53}{1224} f_{n,t^2} f_{n,x} \\ + \frac{1325}{93\,636} f_{n,x^2} f_{n,x} \\ + \frac{132\,023}{13\,764\,492} f_n f_{n,tx} f_{n,x} + \frac{53}{2448} f_n f_{n,x^2} f_{n,t} + \frac{1325}{187\,272} f_n f_{n,x^2} f_{n,x} \\ + \frac{2508\,437}{1982\,086\,848} f_n^2 f_{n,x^2} f_{n,x} \end{array} \right) \\ & + h^5 \left(\begin{array}{c} \frac{132\,023}{7928\,347\,392} f_n^3 f_{n,x^2}^2 + \frac{2508\,437}{1982\,086\,848} f_n^2 f_{n,tx} f_{n,x^2} \\ + \frac{132\,023}{27\,528\,984} f_n f_{n,tx}^2 \\ + \frac{1325}{187\,272} f_n f_{n,tx} f_{n,x^2} \\ + \frac{1325}{374\,544} f_n f_{n,x^2}^2 + \frac{53}{4896} f_{n,t^2} f_n f_{n,x^2} + \frac{1325}{46\,818} f_{n,tx}^2 \\ + \frac{1325}{93\,636} f_{n,tx} f_{n,x^2} + \frac{53}{1224} f_{n,t^2} f_{n,tx} \end{array} \right) \end{aligned} \tag{23}$$

$$T_{n+1} = h^5 \left(\begin{array}{c} \frac{132\,023}{7928\,347\,392} f_n^3 f_{n,x^2}^2 + \frac{2508\,437}{1982\,086\,848} f_n^2 f_{n,tx} f_{n,x^2} \\ + \frac{132\,023}{27\,528\,984} f_n f_{n,tx}^2 + \frac{1325}{187\,272} f_n f_{n,tx} f_{n,x^2} \\ + \frac{1325}{374\,544} f_n f_{n,x^2}^2 + \frac{53}{4896} f_{n,t^2} f_n f_{n,x^2} + \frac{1325}{46\,818} f_{n,tx}^2 \\ + \frac{1325}{93\,636} f_{n,tx} f_{n,x^2} + \frac{53}{1224} f_{n,t^2} f_{n,tx} \end{array} \right). \quad (24)$$

Hence, the new iterative method is of fourth order.

2.3.2. *Convergence of the iterative method*

Definition 2.2. A single step method is said to be consistent if

$$\phi(x, y, 0) = f(x, y) \ [1]. \quad (25)$$

Definition 2.3. A single step method is called regular if the function $\phi(x, y, h)$ is defined and continuous in the domain $a \le x \le b, -\infty < y^j < \infty, i = 1, 2, .., n, 0 \le h \le h_0$ and if there exist a constant L such that

$$\|\phi(x, y, h) - \phi(x, y^*, h)\| \le L \|y - y^*\|, \quad (26)$$

for every $x \in [x_0, b], y, y^* \in (-\infty, \infty), h \in (0, h_0)$ [1].

Theorem 2.2. Suppose the single step method is regular, then the relation Eq. (25) is a necessary and sufficient condition for the convergence of the method [1].

Lemma 2.3. An iterative method Eq. (16) is said to be convergent by Theorem 2.2 if the following results hold:

- Consistency holds
- Regularity holds

Proof (Consistency). Consider the increment function in Eq. (16),

$$\phi(x_n, u_n, h) = h^{-1} \left(\frac{47h}{294} k_1 + \frac{5000h}{7497} k_2 + \frac{53h}{306} k_4 \right)$$

$$\phi(x_n, u_n, h) = \frac{1}{14994} (2397k_1 + 10000k_2 + 2597k_4). \quad (27)$$

If $h = 0$, then

$$\phi(x_n, u_n, 0) = f(t, x). \quad (28)$$

Therefore, by Definition 2.2, the method Eq. (16) is consistent.

Proof (Regularity). Let

$$k_1^* = f(t_n, x_n^*),$$

$$k_2^* = f\left(t_{n+\frac{49}{100}}, x^*_{n+\frac{49}{100}} \right),$$

$$k_3^* = f\left(t_{n+1}, x_n^* + \frac{47h}{294} k_1^* + \frac{5000}{7497} k_2^* \right) \text{ and}$$

$$k_4^* = f\left(t_{n+1}, x_n^* + \frac{47h}{294} k_1^* + \frac{5000h}{7497} k_2^* + \frac{53h}{306} k_3^* \right)$$

for every $(t, x), (t, x^*) \in S$ and k_n $(n = 1, 2, 3, 4)$ are defined in Eq. (16).

Considering $f(t, x)$ to be Lipschitz, then

$$\|k_1 - k_1^*\| = \|f(t_n, x_n) - f(t_n, x_n^*)\|,$$

$$\le L \|x_n - x_n^*\|,$$

$$\|k_2 - k_2^*\| = \left\| f\left(t_n + \frac{49h}{100}, x_n + \frac{49h}{100} \right) - f\left(t_n + \frac{49h}{100}, x_n^* + \frac{49h}{100} \right) \right\|,$$

$$\le L \|x_n - x_n^*\|,$$

$$\|k_3 - k_3^*\| =$$
$$\left\| f\left(t_{n+1}, x_n + \tfrac{47h}{294} k_1 + \tfrac{5000h}{7497} k_2 \right) - f\left(t_{n+1}, x_n^* + \tfrac{47h}{294} k_1^* + \tfrac{5000h}{7497} k_2^* \right) \right\|,$$

$$\le L \left(1 + \frac{253h}{306} L \right) \|x_n - x_n^*\|,$$

$$\|k_4 - k_4^*\| = \left\| \begin{array}{c} f\left(t_n + h, x_n + \frac{47h}{294} k_1 + \frac{5000h}{7497} k_2 + \frac{53h}{306} k_3 \right) \\ -f\left(t_n + h, x_n^* + \frac{47h}{294} k_1^* + \frac{5000h}{7497} k_2^* + \frac{53h}{306} k_3^* \right) \end{array} \right\|,$$

$$\le L \|x_n - x_n^*\| \left(1 + \left(\frac{253h}{306} L + \frac{53h}{306} L \left(\frac{253h}{306} L + 1 \right) \right) \right),$$

Now using Eq. (27),

$$\phi(x_n, u_n, h) - \phi(x_n, u_n^*, h) = \frac{1}{14994} \left(\begin{array}{c} 2397k_1 + 10000k_2 + 2597k_4 \\ -2397k_1^* - 10000k_2^* - 2597k \end{array} \right.$$

$$= L \left(\frac{710\,677}{28\,652\,616} L^2 h^2 + \frac{53}{306} Lh + 1 \right) \|x_n - x_n^*\|$$

$$= L' \|x_n - x_n^*\|,$$

where $L' = L\left(\frac{710\,677}{28\,652\,616} L^2 h^2 + \frac{53}{306} Lh + 1 \right)$.

Therefore, the increment ϕ satisfies the Lipschitz conditions in x, hence, by Definition 2.3, method Eq. (16) is said to be regular.

Lemma 2.4. Given the function $f(t, x)$ defined and continuous in the strip

Lemma 2.5. $S\ (|t - t_0| \le \alpha,\ \|x\| < \infty, \alpha > 0)$ satisfying the Lipschitz condition

$$\|f(t, x) - f(t, x^*)\| \le L \|x - x^*\|. \quad (29)$$

$\forall\ (t, x), (t, x^*) \in S$, here Lipschitz constant is denoted by L; then, the method Eq. (16) is said to be convergent.

Proof. Since consistency and regularity holds for method Eq. (16), by Theorem 2.2, it is convergent.

2.3.3. *Stability of the iterative method*

Definition 2.4. For a method given by the tableau $\begin{array}{c|c} c & A \\ \hline & b \end{array}$, the stability for a $y' = qy$ is the set of points in the complex plane satisfying $|R(z)| \le 1$ [19].

Theorem 2.3. For the iterative method Eq. (16) to be stable, we must have

$$\left|\frac{\left(1+\frac{97h\lambda}{147}+\frac{5141h^2\lambda^2}{44\,982}+\frac{20\,497\,273h^3\lambda^3}{1403\,978\,184}\right)}{\left(1-\frac{50h\lambda}{153}-\frac{1325h^2\lambda^2}{23\,409}-\frac{70\,225h^3\lambda^3}{7163\,154}\right)}\right|\leq 1.$$

Proof. Using Definition 2.4. Applying the iterative method Eq. (16) to the test equation $x'=\lambda x$, we obtain

$$k_1=\lambda x_n,$$

$$k_2=\lambda x_{n+\frac{49}{100}},$$

$$k_3=\lambda\left[\left(1+\frac{47h\lambda}{294}\right)x_n+\frac{5000h\lambda}{7497}x_{n+\frac{49}{100}}\right],$$

$$k_4=\lambda\left[\left(1+\frac{2497}{7497}h\lambda+\frac{2491}{89\,964}h^2\lambda^2\right)x_n+\left(\frac{5000h\lambda}{7497}+\frac{132\,500h^2\lambda^2}{1147\,041}\right)x_{n+\frac{49}{100}}\right].$$

From Eq. (16), we can write

$$x_{n+1}=x_n+\frac{47h\lambda}{294}x_n+\frac{5000h\lambda}{7497}x_{n+\frac{49}{100}}+\frac{53h\lambda}{306}\left[\begin{array}{c}\left(1+\frac{2497}{7497}h\lambda+\frac{2491}{89\,964}h^2\lambda^2\right)x_n\\+\left(\frac{5000h\lambda}{7497}+\frac{132\,500h^2\lambda^2}{1147\,041}\right)x_{n+\frac{49}{100}}\end{array}\right].$$

$$x_{n+1}=\left(1+\frac{2497h\lambda}{7497}+\frac{132\,341h^2\lambda^2}{2294\,082}+\frac{132\,023h^3\lambda^3}{27\,528\,984}\right)x_n+\left(\frac{5000h\lambda}{7497}+\frac{132\,500h^2\lambda^2}{1147\,041}+\frac{3511\,250h^3\lambda^3}{175\,497\,273}\right)x_{n+\frac{49}{100}}.$$

Writing $x_{n+\frac{49}{100}}$ as $\frac{49}{100}\left(x_{n+1}+x_n\right)$, then

$$x_{n+1}=\left(1+\frac{2497h\lambda}{7497}+\frac{132\,341h^2\lambda^2}{2294\,082}+\frac{132\,023h^3\lambda^3}{27\,528\,984}\right)x_n+\frac{49}{100}\left(\frac{5000h\lambda}{7497}+\frac{132\,500h^2\lambda^2}{1147\,041}+\frac{3511\,250h^3\lambda^3}{175\,497\,273}\right)\left(x_{n+1}+x_n\right).$$

$$x_{n+1}=\frac{\left(1+\frac{97h\lambda}{147}+\frac{5141h^2\lambda^2}{44\,982}+\frac{20\,497\,273h^3\lambda^3}{1403\,978\,184}\right)}{\left(1-\frac{50h\lambda}{153}-\frac{1325h^2\lambda^2}{23\,409}-\frac{70\,225h^3\lambda^3}{7163\,154}\right)}x_n.$$

Now, for stability of iterative method and by Definition 2.6, we must have

$$\left|\frac{\left(1+\frac{97h\lambda}{147}+\frac{5141h^2\lambda^2}{44\,982}+\frac{20\,497\,273h^3\lambda^3}{1403\,978\,184}\right)}{\left(1-\frac{50h\lambda}{153}-\frac{1325h^2\lambda^2}{23\,409}-\frac{70\,225h^3\lambda^3}{7163\,154}\right)}\right|\leq 1.$$

2.4. Algorithm for Forward Backward Sweep Methods

2.4.1. Algorithm for FBS implementation for the iterative method

Applying Eq. (16), the algorithms are given as
Forward algorithm
for $n=1:N$
$k_1=f(t_n,x_n,u_n)$
$k_2=f\left(t_n+\frac{49}{100}h,\ \frac{49}{100}(x_n+x_{n+1}),\ \frac{49}{100}\left(u_n+u_{n+1}\right)\right)$
$k_3=f\left(t_n+h,\ x_n+\frac{47h}{294}k_1+\frac{5000h}{7497}k_2,\ u_{n+1}\right)$
$k_4=f\left(t_n+h,\ x_n+\frac{47h}{294}k_1+\frac{5000h}{7497}k_2+\frac{53h}{306}k_3,\ u_{n+1}\right)$
$x_{n+1}=x_n+\frac{h}{14994}\left(2397k_1+10000k_2+2597k_4\right)$
Backward algorithm
for $j=1:N$

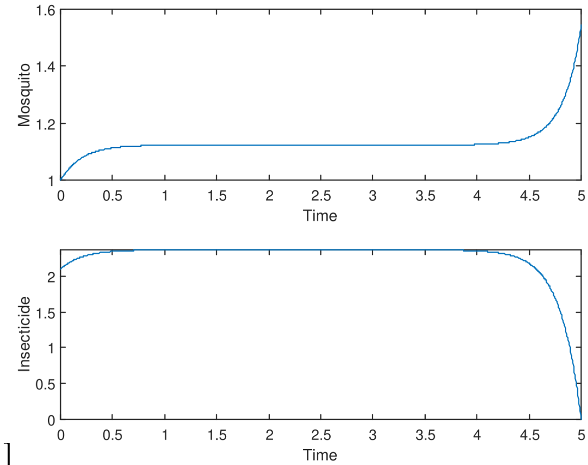

Figure 3. The optimal Mosquito population and Insecticide for case 3.

$n=N+2-j$
$k_1=f\left(t_n,x_n,\lambda_n,u_n\right)$
$k_2=f\left(t_n-\frac{49}{100}h,\ \frac{49}{100}(x_n+x_{n-1}),\ \frac{49}{100}(\lambda_n+\lambda_{n-1}),\ \frac{49}{100}(u_n+u_{n-1})\right)$
$k_3=f\left(t_n-h,\ x_{n-1},\lambda_n-\frac{47h}{294}k_1-\frac{5000h}{7497}k_2,u_{n-1}\right)$
$k_4=f\left(t_n-h,x_{n-1},\lambda_n-\frac{47h}{294}k_1-\frac{5000h}{7497}k_2-\frac{53h}{306}k_3,u_{n-1}\right)$
$\lambda_{n-1}=\lambda_n-\frac{h}{14994}\left(2397k_1+10000k_2+2597k_4\right)$

2.5. Numerical Experiment

In this section, optimal control model for mosquito and insecticide is solved. All computations in this section are done with the aid of a written code in MATLAB 2018a, which is run on a Window 8.1 computer.

Problem. Supposed the population concentration of mosquitoes at time t is given by $x(t)$, and the population over a fixed period of time wished to be reduced. Assuming x has a carrying capacity M and growth rate r. An application of the substance which is known to decrease the rate of change of $x(t)$, by decreasing the proportional rate to the amount of $u(t)$ and $x(t)$. Assumed the amount of the substance to be added at time t is $u(t)$. Then the concentration of mosquitoes is taken to be $x(t)$ and the insecticide known to kill it taken to be $u(t)$. The mosquito is represented in differential equation

$$x'(t)=r(M-x(t))-u(t)x(t),\ x(0)=x_0,$$

where the population size at the initial point is given as $x_0>0$. Here, term $u(t)x(t)$ pulls down the growth rate of the mosquito. Both mosquito and insecticide have negative effects on individuals around them, so both need to be minimized. Little amount is acceptable for both, there is then need to penalize for amounts too large, so quadratic terms for both will be analyzed. Hence, the optimal control problem for 5 day regimen is

$$\min_u J(x,u)=\int_0^5 Ax(t)+u(t)^2dt,$$

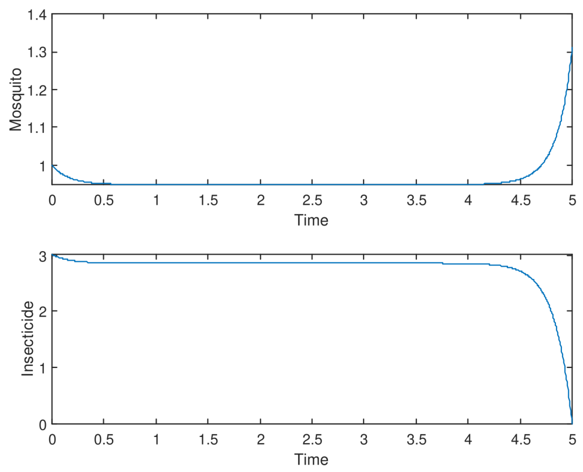

Figure 4. The optimal Mosquito population and Insecticide for case 4.

subject to

$$x'(t) = r(M - x(t)) - u(t)x(t), \ x(0) = x_0.$$

The coefficient A is the weight parameter, balancing the relative importance of the two terms in the objective functional [7, 12].

Solution. The optimality system of the problem is developed by first constructing the Hamiltonian

$$H(t, x, u, \lambda) = f(t, x, u) + \lambda g(t, x, u),$$

$$H(t, x, u, \lambda) = Ax + u^2 + \lambda r(M - x) - \lambda xu.$$

The optimality condition is

$$0 = \frac{\partial H}{\partial u} = 2u - \lambda x \Rightarrow u^* = \frac{\lambda x}{2}.$$

The adjoint equation is

$$\lambda'(t) = -\frac{\partial H}{\partial x} = -A + \lambda r + \lambda u$$

$$= -A + \lambda r + \frac{1}{5}\lambda^2 x$$

$$x'(t) = Mr - x(r + u), \ x(0) = x_0.$$

$$\lambda'(t) = -A + \lambda r + \frac{1}{5}\lambda^2 x, \lambda(T) = 0.$$

Using the optimality system, the numerical code is generated, written in MATLAB R2018a. This problem is solved with $N = 1000$: The results are shown in Figures and Tables.

First considering parameters $M = 10$, $r = 0.3$, $x_0 = 1$, $A = 1$.

Case 1: No control.

In Figure 1, the mosquito population continue to increase steadily without any interuption at any point. Now considering the following parameters and with control.

Case 2: M=10, r=0.3, x_0=1, A=1.

The aim is to lower the population of mosquito as well as minimizing the effect of insecticide on individuals around. In Figure 2,when the carrying capacity $M = 10$, and the weight parameter $A = 1$, the population of the mosquito is increasing but when the insecticide is introduce and maintained at constant level between time 1.2 to 3.5, the population of mosquito level up and become constant between the time 1 to 4.5; here the insecticide and mosquito population are paarallel. As the insecticide begin to decrease at day 5, the population of the mosquito begin to increase again, with heavy growth at the beginning and end.

Case 3: Varying the weight parameter A to A = 5.

In Figure 3, when carrying capacity is maintained at $M = 10$, and the weight parameter is varied to $A = 5$, the level of insecticide applied is slightly higher. It is seen that, state and control are at equilibrium for a longer period. The control starts at its highest point, then decreasing slightly before becoming constant for a moment, then drop to zero. The state growth rate drop and become constant up to day 4. However, at the end of the interval, when the effect of the insecticide is no longer harmful to the mosquitoes, the population of the mosquito rapidly increases.

Case 4: Varying the parameter A to A = 10.

In Figure 4, when the carrying capacity is maintained at $M = 10$, and the weight parameter is varied again to $A = 10$, a much higher level of insecticide is used. It is observed that, the mosquito population and insecticide are at equilibrium for a long period. The insecticide is at its highest point at the beginning before decreasing slightly and then become constant, and later decreasing to zero. The state growth rate also drop and become constant up to day 4.5. However, at the end of the interval, when the effect of the insecticide is no longer harmful to the mosquitoes, the population of the mosquito rapidly increases.

Therefore, studying the results, it can be seen that, when an insecticide that last for five days is applied, it reduced the population of the mosquito at minimal level but as soon as the effect of the insecticide approach 0, the mosquito population begin to increase again. Also, the higher the insecticide applied, the lower the population of the mosquitoes. The best way to maintain a 5 days regimen, is to start a second regimen on about day 4.5.

3. Conclusion

An order-four iterative method is developed based on Patade and Bhalekar's approach for the numerical solution of the optimal control problem for mosquito population growth and insecticide using the forward-backward sweep method. The stability properties of the iterative method are tested and found to be stable and convergent. The new iterative method is used to solve the optimal control problem, and the results obtained clearly show that mosquitoes cannot be completely eradicated, but through optimal control, their population can be substantially minimized, thereby reducing the spread of malaria disease.

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