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A new Maxwell-Log logistic distribution and its applications for mortality rate data

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Abstract

In this research, we extended the Log-Logistic distribution by incorporating it into the Maxwell generalized class, resulting in the Maxwell-Log Logistic (Max-LL) distribution. The probability density function and cumulative distribution function of the proposed distribution have been defined. The proposed distribution's density shapes can be left or right-skewed and symmetric. The failure function of this distribution might be increasing, decreasing, or inverted bathtub forms. We discussed some essential properties of the Max-LL distribution, including moments, moment generating function, probability weighted moments, stress-strength, and order statistics. The efficiency of the model parameters has been evaluated through a simulation study utilizing a quantile function. To assess the proposed distribution's adaptability, we applied it to two lifetime datasets: global COVID-19 mortality rates (for nations with more than 100,000 cases) and Canadian COVID-19 mortality rates. The Maxwell-Log Logistic distribution outperformed other distributions on both datasets, as evidenced by several accuracy measures. This shows that the proposed distribution is the best fit for COVID-19 mortality rate data in Canada and around the world.

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1. Introduction

Numerous statistical frameworks that provide accurate and reliable forecasts of underlying processes are being developed by researchers to address the increased complexity and diversity of data sets [1]. In recent years, modeling lifetime distributions has garnered significant interest, with its relevance expanding due to the fundamental importance of accurately modeling phenomena. Researchers in distribution theory often enhance the flexibility of data modeling by introducing new parameters, thereby making the distribution more adaptable across a wide range of practical disciplines [2]. These fields include biology, engineering, econometrics, survival analysis, finance, environmental studies, medical research, survey sampling, and the biological sciences [3]. The flexibility of these models makes

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them valuable for accurately simulating and evaluating various real-world phenomena, which in turn supports well-informed decision-making in both academic and practical applications [4].

The Log-Logistic (LL) distribution, which was originally developed by Verhulst [5] to simulate population expansion has gained popularity as the Fisk distribution because of its suitability for use with income data, as reported by Fisk [6]. This distribution was studied by Bennett [7] as the statistical distribution suitable for modeling lifetime phenomena in survival analysis as its failure rate described the non-monotonic function [8]. The LL distribution can be used as a substitute for the Weibull distribution and has some similarities with log-normal and normal distributions; however, it is more useful in application to reliability, survival analysis, and life-testing experiments. When dealing with censored observations, the LL distribution may be preferred over log-normal and normal distributions. This distribution is useful for modeling rainfall and stream flow in hydrology and studying income and wealth data in economics, among other scientific domains. Some properties of LL distribution are explained in Refs. [9–13], and its parameters were determined by Refs. [14, 15], among others. The LL distribution's cumulative distribution function (cdf) is presented as:

$$W(x;c,d) = 1 - \left(\frac{1}{1+\left(\frac{x}{d}\right)^{c}}\right) = \frac{\left(\frac{x}{d}\right)^{c}}{\left\{1+\left(\frac{x}{d}\right)^{c}\right\}}, \quad x>0. \tag{1}$$

The probability density function (pdf) that corresponds to equation (1) is

$$w(x;c,d) = \frac{c\left(\frac{x}{d}\right)^{c-1}}{d\left\{1+\left(\frac{x}{d}\right)^{c}\right\}^{2}}, \tag{2}$$

where $c > 0$ and $d > 0$ are the parameters representing the shape and scale, respectively.

Many researchers generalized LL distribution to increase its flexibility. For instance, De Santana *et al.* [16] generalized LL distribution within the logit of the Kumaraswamy class defined by Ref. [17] to study the Kumaraswamy-LL distribution. This model can have a bathtub, decreasing, upside-down bathtub, or increasing failure rates. The pdf of the proposed distribution was presented as a mixture of representations by Ref. [18], where the moments, mean deviations, quantile function, Bonferroni, and Lorenz curves were discussed. The regression model was introduced based on the Kumaraswamy-LL distribution. Two data sets relating to acquired immunodeficiency syndrome (AIDS) and cancer were employed to assess the flexibility of the suggested distribution against the existing ones. They found that the proposed Kumaraswamy-LL distribution outperformed the comparative distributions. The Marshall-Olkin LL distribution was studied by Ref. [19], and its density as well as hazard shapes were provided. Some important properties, such as stochastic ordering and representations, moments, distribution of order statistics, and quantile function were studied. Some of the features of the Marshall-Olkin-LL distribution and its minification procedure were introduced in Ref. [19]. Another study was conducted by Ref. [20] to extend the LL model from the McDonald family to propose the McDonald-LL distribution. The Kumaraswamy-LL, Dagum, Beta-LL, Singh-Maddala, LL, and standard LL distributions were considered special cases of McDonald-LL distribution. Some of its properties were discussed, including moments, mean residual, entropies, and quantile function. The characterization of the M-LL distribution and the parameters of its estimates were studied. The data set relating to breast cancer was used, and it was discovered that the proposed distribution could be chosen by having a smaller value of accuracy measures. The LL distribution was generalized by Ref. [21] to explore the exponentiated alpha-power log-logistic distribution, some features, and parameter estimation of this distribution are provided in Ref. [21]. An extension of LL distribution was introduced in Ref. [22] by considering the quadratic transmutation technique as the class of distribution obtained by Ref. [23] to study transmuted LL distribution. The model structural properties including moments, reliability analysis, quantile, random number generation, and order statistics, were determined. The simulation study, as well as the parameters of its estimates, were established by utilizing the maximum likelihood (ML) method. The adaptability of the suggested distribution was illustrated using a real-world data set, and ultimately it was found that the transmuted-LL distribution could be prepared efficiently.

This study, which builds on the work of several scholars, attempts to make the LL distribution more versatile and robust, allowing it to better simulate and capture complicated real-world occurrences. Among them are the unit-LL distribution by Ref. [24], the Odd-logistic generalized exponential distribution by Ref. [25], the LL distribution's survival analysis and hazard using type I censored data by Ref. [26], the Bayesian study of the unit LL distribution with non-informative priors by Ref. [27], the odd LL generalized Lindley distribution by Ref. [28], the Odd LL-Burr XII distribution by Ref. [29], the LL distribution's robust explicit estimation by Ref. [30], among others.

Over the last few years, there has been an increasing focus on extending conventional models to better represent real-world phenomena by integrating a broader family of distributions. Among others are the study and application of the exponentiated Log-Logistic Weibull distribution to censored data by Fagbamigbe *et al.* [31], the Zero-truncated Poisson-power function by Okorie *et al.* [32], A new extended gumbel by Fayomi *et al.* [33], the new flexible exponentiated Weibull by Arif *et al.* [34], the odd F-Weibull by Ishaq *et al.* [35], the generalized odd beta prime by Suleiman *et al.* [36], the new extended Topp-Leone exponential by Muhammad *et al.* [37], the exponentiated odd Lomax exponential by Ref. [38], the odd beta prime-logistic by Ref. [39], the generalized Odd Maxwell-Kumaraswamy by Ref. [40], the Multivariate Birnbaum-Saunders by Ref. [41], the modified Lomax model and its bivariate extension by Ref. [42], the Marshall-Olkin extended Gumbel type-II by Ref. [43], the inverse Weibull model to evaluate wind speed data by Ref. [44], the McDonald generalized power Weibull by Ref. [45], the odd beta prime-Burr X by Ref. [46], the new updated Weibull by Ref. [47], the log-Topp-Leone by Ref. [48], and many others.

The Maxwell generalized (M-G) distribution class, described by Ref. [49], evolved from the Maxwell-Boltzmann model, initially utilized to elucidate data related to molecular behavior in physics, chemistry, and biology, it was primarily suited for right-skewed datasets. This family was developed by applying the odd ratio approach developed by Ref. [50]. The cdf and pdf for the M-G family are respectively provided as:

$$F(x;b,\Phi) = \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\ \frac{1}{2b^2}\left(\frac{W(x;\Phi)}{1-W(x;\Phi)}\right)^2\right),$$
$$x \in \Re;\ \ \Phi \in \Re,\ \ b > 0, \tag{3}$$

and

$$f(x;b,\Phi) = \frac{2w(x;\Phi)}{b^3\sqrt{2\pi}\{1-W(x;\Phi)\}^2}\left(\frac{W(x;\Phi)}{1-W(x;\Phi)}\right)^2$$
$$\exp\left\{-\frac{1}{2b^2}\left(\frac{W(x;\Phi)}{1-W(x;\Phi)}\right)^2\right\},$$
$$x \in \Re;\ \ \Phi \in \Re,\ \ b > 0. \tag{4}$$

A scaling parameter is denoted by b, $w(x;\Phi)$ and $W(x;\Phi)$ are respectively the baseline cdf and pdf of the LL model with the parameter Φ. Some baseline distributions have been extended using the M-G family, developing novel compound distributions with various features and uses. Consider distributions such as Maxwell-exponential [51], Maxwell-Mukherjee Islam [52], Maxwell-Dagum [53], Maxwell-Burr X [54], and so on.

This study presents the Maxwell-LL distribution, an extension of the LL distribution. Notably, the LL distribution finds relevance in conjunction with COVID-19 mortality rate data, a contemporary and critical field of this study. The distribution's adaptability is assessed through the analysis of two data sets representing COVID-19 mortality rates in the global and Canada. This research contributes to the ongoing efforts to model and understand the patterns of COVID-19 impact using statistical distributions with robust applications.

A new virus called COVID-19 has been connected to the severe acute respiratory syndrome coronavirus, or SARS-CoV-2, which has caused a worldwide outbreak. Various statistical models have been put forth in research to comprehend and illustrate the progression of this pandemic [55, 56], and several authors who studied the SARS-CoV-2 pandemic can be found in Refs. [57–59]. It is crucial to emphasize that the features of pandemic data can vary, rendering the fitting of classical probability distributions challenging in certain instances [60]. Consequently, We employed the Maxwell-Log Logistic distribution to accurately simulate the mortality rate linked with this disease. The rationale and basis for introducing the Maxwell-LL distribution are to:

1. enhance the overall efficacy of the traditional LL distribution, which handles skewed, and symmetric data sets more effectively than other competing models;
2. produce a distribution with various structural, including symmetrical, right-, and left-skewed;
3. present a novel approach with many hazard functions capable of capturing forms that are decreasing, upside-down bathtub, and increasing; and
4. reliably provide a better fit when compared to existing developed distributions for the identical traditional distribution.

This paper has been laid out as follows: The development of the Maxwell-LL distribution is demonstrated in section 2. Section 3 delves into some of its statistical features. Section 4 describes the methods used to estimate parameters. Section 5 introduces a simulation study, while section 6 presents the newly developed model's numerical applications. Finally, section 6 offers concluding remarks.

2. Maxwell-Log logistic distribution

This section utilizes the M-G family proposed by Ref. [49] to introduce the Maxwell-LL (Max-LL) distribution. Inserting equation (1) into equation (3) provided the cdf of the Max-LL distribution given as

$$F(x;b,c,d) = \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\ \frac{1}{2b^2}\left(\frac{x}{d}\right)^{2c}\right), \qquad x > 0;\ \ b,c,d > 0. \tag{5}$$

The associated pdf is obtained by substituting equations (1) and (2) into equation (4) to obtain:

$$f(x;b,c,d) = \frac{2cx^{3c-1}}{b^3 d^{3c}\sqrt{2\pi}}\exp\left(-\frac{1}{2b^2}\left(\frac{x}{d}\right)^{2c}\right). \tag{6}$$

In this case, d and b are the scale parameters and c is the shape parameter. A random variable X with pdf in equation (6) is denoted as $X \sim MaxLL(b,c,d)$. The parameters on the cdf and pdf presented in equations (5) and (6) can be eliminated for simplicity by writing $F(x;b,c,d) = F(x)$ and $f(x;b,c,d) = f(x)$. Figure 1 depicts plots for the pdf of the suggested Max-LL distribution with various parameter values. The observed patterns include (a) left-skewed, (b) symmetrical, and (c) right-skewed. In general, modeling lifetime phenomena like COVID-19 data sets reaps the advantages of these attributes.

2.1. Mixture representations of the Max-LL distribution for the cdf and pdf

According to Gradshteyn and Ryzhik [61], the series expansion for incomplete gamma function is:

$$\gamma(a,y) = \sum_{j=0}^{\infty}\frac{y^{a+j}(-1)^j}{(a+j)\,j!}, \qquad y > 0;\ \ a > 0. \tag{7}$$

In this regard, $a = \frac{3}{2}$, and $y = \frac{1}{2b^2}\left(\frac{x}{d}\right)^{2c}$. Employing equation (7) into equation (5) gives:

$$F(x;b,c,d) = \frac{4}{\sqrt{\pi}}\sum_{j=0}^{\infty}\frac{(-1)^j\left(\frac{x}{d}\right)^{c(3+2j)}}{b^{3+2j}2^{\frac{3+2j}{2}}\,j!\,(3+2j)}$$
$$= \sum_{j=0}^{\infty}\varphi_j x^{c(3+2j)}, \tag{8}$$

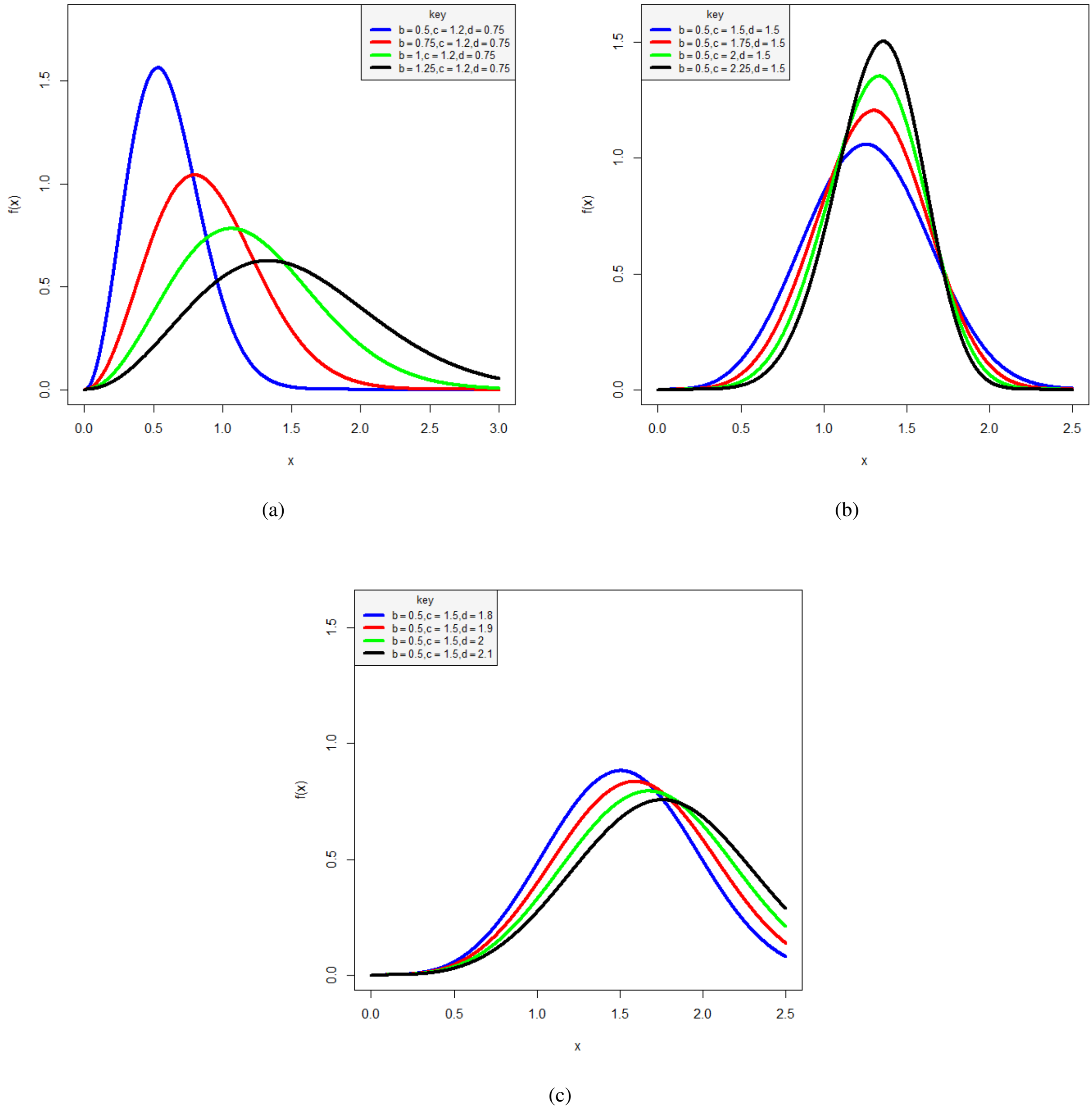


Figure 1: The pdf plots showing different parameter values for the Max-LL distribution.

which is the linear representation for the cdf of Max-LL distribution, where

$$\varphi_j = \frac{\sqrt{2}\,(-1)^j}{\sqrt{\pi}2^j\, j!\,(3+2j)\, b^{3+2j} d^{c(3+2j)}}.$$

For the expansion of pdf, let us consider the series expansion for the exponential function as:

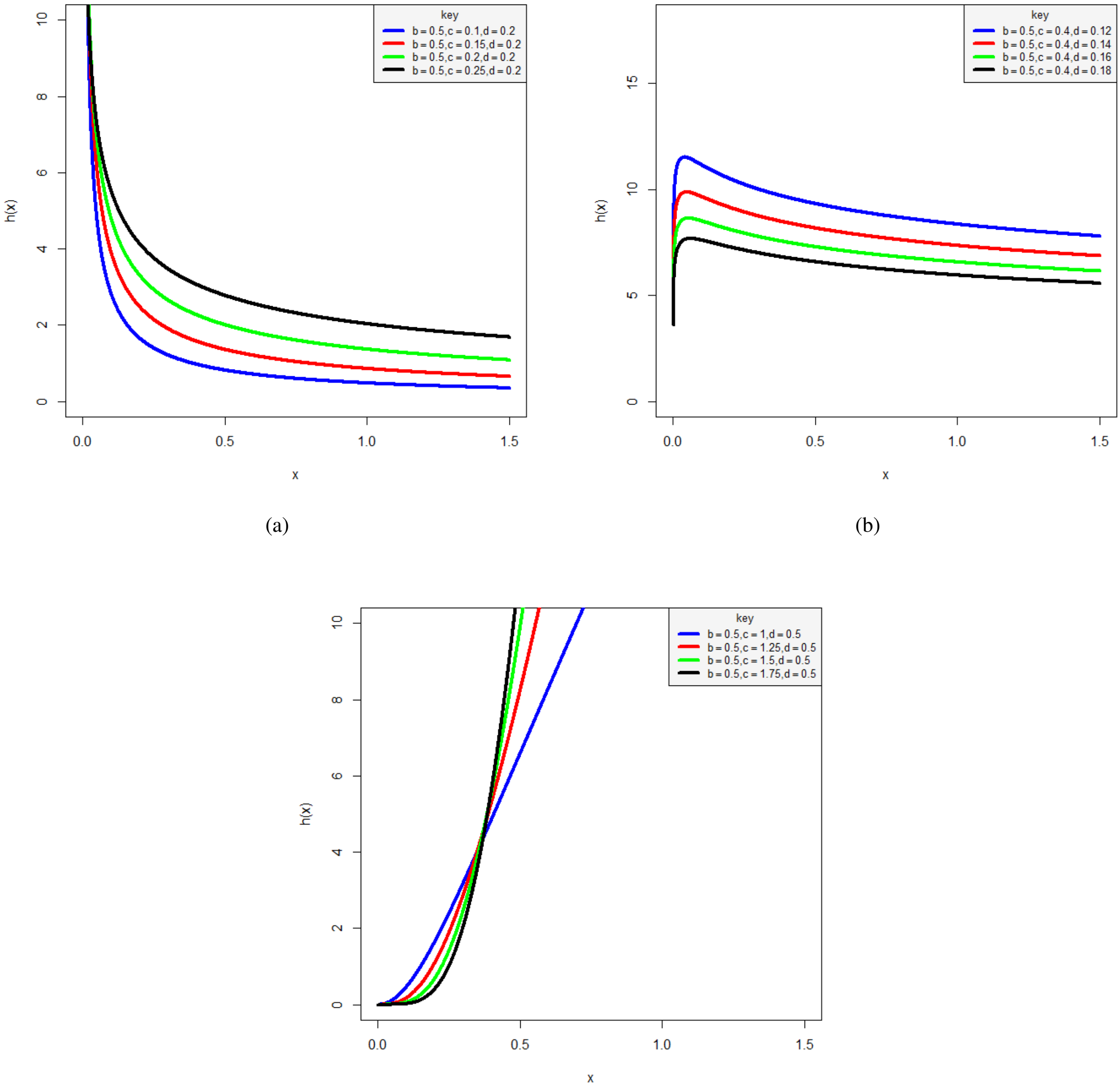
$$e^{-y} = \sum_{l=0}^{\infty} \frac{y^l\,(-1)^l}{l!}. \tag{9}$$

Applying equation (9) to equation (6), we have:

$$f\,(x; b, c, d) = \sum_{l=0}^{\infty} \Omega_l x^{3c+2cl-1}, \tag{10}$$

which is the linear representation for the pdf of Max-LL distribution, where

$$\Omega_l = \frac{2c\,(-1)^l}{b^3 d^{3c}\,\sqrt{2\pi} l!\,(2b^2)^l\, d^{2cl}}.$$



(c)

Figure 2: Failure plots showing different parameter values for the Max-LL distribution.

2.2. Survival, Failure, and Quantile Functions for the Maxwell-Log Logistic Distribution

This section presents the Max-LL distribution's quantile, failure, and survival functions.

2.2.1. Survival Function (SF)

Taking into account the CDF provided in equation (5), the Max-LL distribution's SF is obtained from equation (5) as:

$$S\left(x\right) = 1 - \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\ \frac{1}{2b^2}\left(\frac{x}{d}\right)^{2c}\right), \qquad x \in [0, \infty)\,;\ b, c, d > 0. \tag{11}$$

2.2.2. Failure Function (FF)

The failure function (FF) of the Max-LL distribution can be derived by considering equations (6) and (11) as:

$$h\left(x\right) = \frac{2cx^{3c-1}\exp\left(-\frac{1}{2b^2}\left(\frac{x}{d}\right)^{2c}\right)}{b^3 d^{3c}\sqrt{2\pi}\left\{1 - \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\ \frac{1}{2b^2}\left(\frac{x}{d}\right)^{2c}\right)\right\}}. \tag{12}$$

Figure 2 depicts plots of the Max-LL distribution's FF. As shown in Figure 2, the FF for the Max-LL distribution could have (a) decreasing, (b) upside-down bathtub, or (c) increasing.

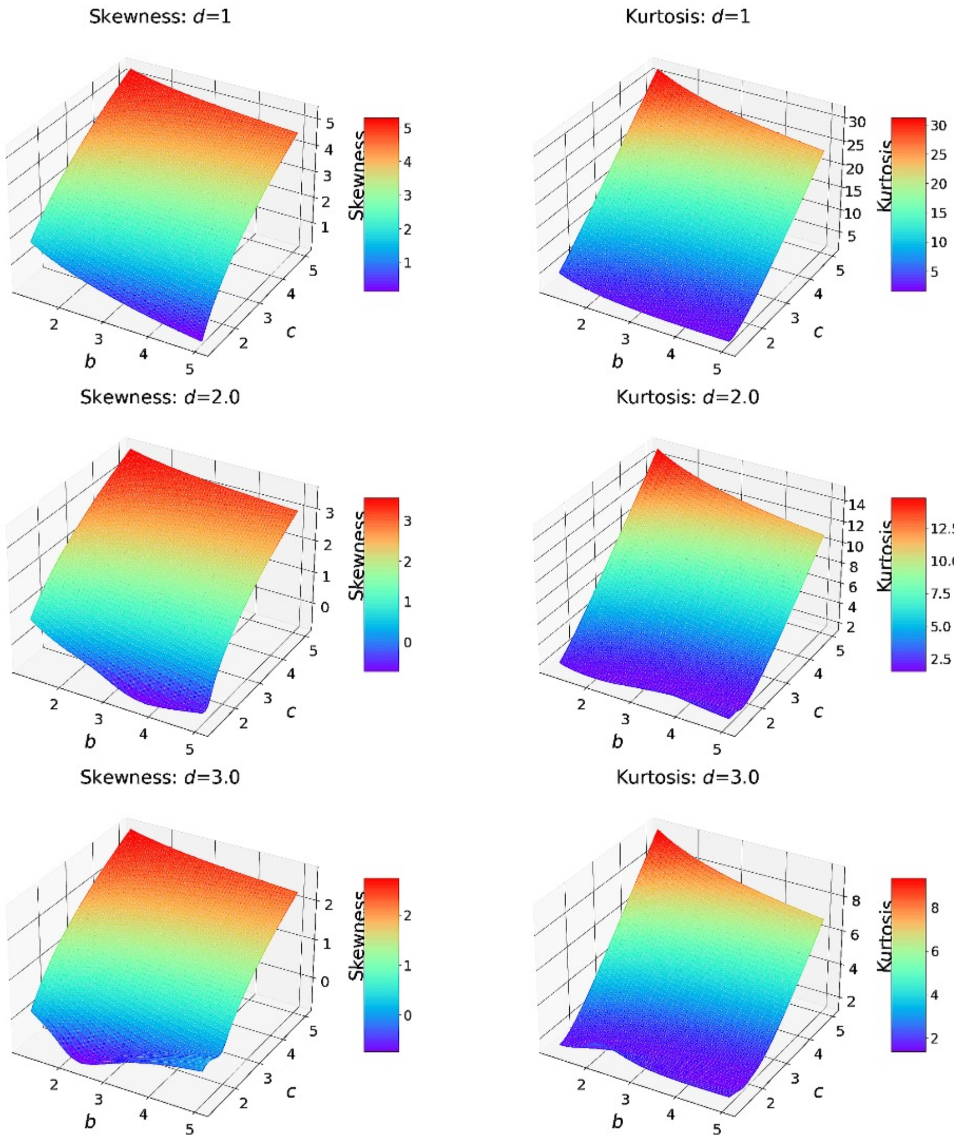


Figure 3: 3D plots of skewness and kurtosis for the Max-LL distribution across different combinations of b, c and d .

2.2.3. Quantile Function (QF)

Considering equation (5), the cdf is represented by:

$$F(x) = \frac{\gamma(a,\ z)}{\Gamma(a)}, \tag{13}$$

where $a = \frac{3}{2}$ and $z = \frac{1}{2b^2}\left(\frac{x}{d}\right)^{2c}$. Inverting equation (13) yields the QF of the Max-LL model as:

$$\left(\frac{x}{d}\right)^{2c} = 2b^2\gamma^{-1}\left(\frac{3}{2},\ u\Gamma\left(\frac{3}{2}\right)\right), \tag{14}$$

which on simplification becomes:

$$x = d\left\{2b^2\gamma^{-1}\left(\frac{3}{2},\ u\Gamma\left(\frac{3}{2}\right)\right)\right\}^{\frac{1}{2c}}, \tag{15}$$

where u represents a uniform random variable with values ranging from 0 to 1.

3. Properties of the Maxwell-Log logistic distribution

Moments, moment generating function, probability weighted moments, stress-strength as well as order statistics are among the Max-LL distribution properties explored.

3.1. Moments

The Max-LL distribution's moments is examined as:

$$E(X^r) = \frac{2c}{b^3 d^{3c}\sqrt{2\pi}}\int_0^\infty x^{r+3c-1}\exp\left(-\frac{1}{2b^2}\left(\frac{x}{d}\right)^{2c}\right)dx. \tag{16}$$

Let

$$A = \frac{1}{2b^2}\left(\frac{x}{d}\right)^{2c}, \quad \Rightarrow \quad x = \left(2b^2 d^{2c} A\right)^{\frac{1}{2c}}, \quad dx = \frac{b^2 d^{2c}}{c x^{2c-1}} dA. \tag{17}$$

Table 1: Simulation findings for the Maxwell-LL distribution when $b = 1$, $c = 1$ and $d = 2$.

n	MLE			MPS		
	Mean	Bias	MSE	Mean	Bias	MSE
5	1.2733	0.2733	1.1167	0.9959	0.0041	0.1695
	1.4159	0.4159	0.8121	0.9569	0.0431	0.2621
	2.2928	0.2928	1.2156	2.1470	0.1470	0.0781
10	1.1392	0.1392	0.2432	0.9511	0.0489	0.0725
	1.1551	0.1551	0.1396	0.9180	0.0820	0.0791
	2.1105	0.1105	0.6089	2.1154	0.1154	0.0496
15	1.1163	0.1163	0.1394	0.9515	0.0485	0.0431
	1.0980	0.0980	0.0673	0.9257	0.0743	0.0457
	2.0401	0.0401	0.3909	2.0894	0.0894	0.0349
20	1.0913	0.0913	0.1020	0.9561	0.0439	0.0351
	1.0793	0.0793	0.0467	0.9405	0.0595	0.0340
	2.0396	0.0396	0.3033	2.0843	0.0843	0.0298
50	1.0471	0.0471	0.0514	0.9620	0.0380	0.0137
	1.0281	0.0281	0.0144	0.9595	0.0405	0.0134
	2.0161	0.0161	0.1517	2.0559	0.0559	0.0229
100	1.0281	0.0281	0.0297	0.9684	0.0316	0.0066
	1.0155	0.0155	0.0063	0.9750	0.0250	0.0062
	2.0115	0.0115	0.0891	2.0493	0.0493	0.0131
250	1.0174	0.0174	0.0184	0.9803	0.0197	0.0025
	1.0057	0.0057	0.0026	0.9861	0.0139	0.0027
	2.0045	0.0045	0.0564	2.0296	0.0296	0.0068
500	1.0111	0.0111	0.0119	0.9861	0.0139	0.0011
	1.0030	0.0030	0.0012	0.9917	0.0083	0.0013
	2.0040	0.0040	0.0373	2.0218	0.0218	0.0037

Substituting equation (17) into equation (16) we get:

$$E\left(X^r\right) = \frac{2^{1+\frac{r}{2c}} d^r b^{\frac{r}{c}}}{\sqrt{\pi}} \Gamma\left(\frac{r}{2c} + \frac{3}{2}\right). \tag{18}$$

Equation (18) yields the mean as well as the variance of the Max-LL model by setting $r = 1$ and $r = 2$ so that:

$$\text{Mean} = E(X) = \frac{2^{1+\frac{1}{2c}} d b^{\frac{1}{c}}}{\sqrt{\pi}} \Gamma\left(\frac{1}{2c} + \frac{3}{2}\right), \tag{19}$$

and

$$\begin{aligned} \text{Varianve} &= E(X^2) - (E(X))^2 \\ &= \frac{2^{1+\frac{1}{c}} d^2 b^{\frac{2}{c}}}{\sqrt{\pi}} \Gamma\left(\frac{1}{c} + \frac{3}{2}\right) - \left[\frac{2^{1+\frac{1}{2c}} d b^{\frac{1}{c}}}{\sqrt{\pi}} \Gamma\left(\frac{1}{2c} + \frac{3}{2}\right)\right]^2 \qquad (20) \\ &= \frac{b^{\frac{2}{c}} d^2 2^{1+\frac{1}{c}}}{\sqrt{\pi}} \left\{\Gamma\left(\frac{1}{c} + \frac{3}{2}\right) - \frac{2}{\sqrt{\pi}} \Gamma^2\left(\frac{1}{2c} + \frac{3}{2}\right)\right\}. \qquad (21) \end{aligned}$$

Figure 3 provides 3D plots for the skewness and kurtosis across varying combinations of the parameters b, c and d. The plots illustrate how these parameters influence the shape and behavior of the Max-LL distribution.

3.2. *Moment Generating Function (MGF)*

For the random variable X, the MGF of the Max-LL distribution is:

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \sum_{r=0}^{\infty} \frac{(t)^r}{r!} \int_{-\infty}^{\infty} x^r f(x) dx. \tag{22}$$

The integral part of equation (22) has been defined in equation (18). Therefore, the Max-LL distribution's MGF is:

$$M_X(t) = \sum_{r=0}^{\infty} \frac{(t)^r}{r!} \left\{\frac{2^{1+\frac{r}{2c}} d^r b^{\frac{r}{c}}}{\sqrt{\pi}} \Gamma\left(\frac{r}{2c} + \frac{3}{2}\right)\right\}. \tag{23}$$

3.3. *Probability Weighted Moments (PWM)*

According to Greenwood *et al.* [62], the PWM of the random variable X for real numbers r, s and $p = 0$ is:

$$P_{r,s,0} = \int_{-\infty}^{\infty} x^r f(x) F(x)^s dx. \tag{24}$$

Therefore, the term $f(x)F(x)^s$ in equation (24) is simplified as:

$$f(x)F(x)^s = f(x)\left[1 - \{1 - \gamma_1(a,\ z)\}\right]^s, \tag{25}$$

Table 2: Simulation findings for the Maxwell-LL distribution when $b = 1$, $c = 1$ and $d = 2$.

n	MLE			MPS		
	Mean	Bias	MSE	Mean	Bias	MSE
5	2.1997	0.6997	2.2529	1.4338	0.0662	0.4968
	1.4160	0.4160	0.8123	0.9571	0.0429	0.2701
	2.2472	0.2472	1.4810	2.1695	0.1695	0.2836
10	1.8558	0.3558	0.7353	1.3967	0.1033	0.1824
	1.1551	0.1551	0.1396	0.9154	0.0846	0.0756
	2.0740	0.0740	0.7209	2.0938	0.0938	0.1312
15	1.7686	0.2686	0.4224	1.4024	0.0976	0.1204
	1.0980	0.0980	0.0673	0.9257	0.0743	0.0457
	2.0155	0.0155	0.4402	2.0714	0.0714	0.1007
20	1.7210	0.2210	0.3307	1.4171	0.0829	0.1067
	1.0793	0.0793	0.0467	0.9405	0.0597	0.0338
	2.0175	0.0175	0.3511	2.0698	0.0698	0.0829
50	1.6129	0.1129	0.1433	1.4391	0.0609	0.0448
	1.0280	0.0281	0.0144	0.9595	0.0404	0.0135
	1.9906	0.0094	0.1550	2.0348	0.0348	0.0451
100	1.5699	0.0699	0.0783	1.4528	0.0472	0.0179
	1.0154	0.0154	0.0063	0.9751	0.0249	0.0062
	1.9897	0.0103	0.0880	2.0285	0.0285	0.0239
250	1.5438	0.0438	0.0454	1.4708	0.0292	0.0063
	1.0057	0.0057	0.0026	0.9861	0.0139	0.0027
	1.9855	0.0145	0.0533	2.0159	0.0159	0.0105
500	1.5256	0.0256	0.0293	1.4806	0.0194	0.0030
	1.0029	0.0029	0.0012	0.9918	0.0082	0.0012
	1.9949	0.0051	0.0364	2.0125	0.0125	0.0050

where a and z are as defined in equation (13), $\gamma_1(a,\ z)$ is the ratio of the incomplete gamma function. For $p > 0$ and $|x| < 1$, the binomial expansion of this becomes:

$$(1-x)^p = \sum_{i=0}^{\infty}\binom{p}{i}(-1)^i x^i. \tag{26}$$

Applying equation (26) to equation (25) gives:

$$\begin{aligned} f(x)F(x)^s &= \sum_{i=0}^{\infty}\binom{s}{i}(-1)^i f(x)\{1-\gamma_1(a,\ z)\}^i \\ &= \sum_{k=0}^{i}\sum_{i=0}^{\infty}(-1)^{k+i}\binom{s}{i}\binom{i}{k}f(x)\sum_{k=0}^{i}\gamma_1^k(a,\ z). \end{aligned} \tag{27}$$

Replace $\sum_{i=0}^{\infty}\sum_{k=0}^{i}$ with $\sum_{k=0}^{\infty}\sum_{i=k}^{\infty}$ in equation (27), we get:

$$f(x)F(x)^s = f(x)\sum_{k=0}^{\infty}\Lambda_i(s)\gamma_1^k(a,\ z), \tag{28}$$

where $\Lambda_i(s) = \sum_{i=k}^{\infty}(-1)^{i+k}\binom{s}{i}\binom{i}{k}$. Take into consideration the series expansion provided in [61] as:

$$\gamma_1^k(a,\ z) = \frac{z^{ak}}{\{\Gamma(a)\}^k}\sum_{m=0}^{\infty}\Pi_{k,m}z^m, \tag{29}$$

where $\Pi_{k,m} = (me_0)^{-1}\sum_{p=1}^{m}(kp-m+p)e_p\Pi_{k,m-p}$ with $e_p = (-1)^p/p!(a+p)$. Applying equation (29) into equation (28), it becomes:

$$f(x)F(x)^s = \sum_{k,m=0}^{\infty}V_{k,m}x^{3c+c(2m+3k)-1}\exp\left(-\frac{1}{2b^2}\left(\frac{x}{d}\right)^{2c}\right), \tag{30}$$

where $V_{k,m} = \frac{c\Lambda_i(s)\Pi_{k,m}}{\{\Gamma(\frac{3}{2})\}^k 2^{\frac{2m+3k}{2}}b^{3+3k+2m}d^{c(2m+3k)+3c}}\sqrt{\frac{2}{\pi}}$. Substituting equation (30) into equation (24), we get:

$$P_{r,s,0} = \sum_{k,m=0}^{\infty}V_{k,m}\int_0^{\infty}x^{3c+c(2m+3k)+r-1}\exp\left(-\frac{1}{2b^2}\left(\frac{x}{d}\right)^{2c}\right)dx. \tag{31}$$

Equation (17) can be applied to equation (31) to obtain:

$$\begin{aligned} P_{r,s,0} &= \sum_{k,m=0}^{\infty}V_{k,m}\frac{b^{2+2m+3k+1+\frac{r}{c}}d^{2c+c(2m+3k+1)+r}2^{\frac{2m+3k+1}{2}+\frac{r}{2c}}}{c} \\ &\quad\times\int_0^{\infty}A^{\frac{2m+3k+1}{2}+\frac{r}{2c}}e^{-A}dA \\ &= \sum_{k,m=0}^{\infty}\Omega_{k,m}\Gamma\left(\frac{2m+3k+1}{2}+\frac{r}{2c}+1\right), \end{aligned} \tag{32}$$

which is the Max-LL distribution's probability weighted moments, where $\Omega_{k,m} = \frac{2^{1+\frac{r}{2c}}d^r b^{\frac{r}{c}}\Lambda_i(s)\Pi_{k,m}}{\sqrt{\pi}\{\Gamma(\frac{3}{2})\}^k}$.

Table 3: Simulation findings for the Maxwell-LL distribution when $b = 1$, $c = 2$ and $d = 2$.

n	MLE			MPS		
	Mean	Bias	MSE	Mean	Bias	MSE
5	1.2173	0.2173	1.0575	1.0929	0.0929	0.1628
	2.8320	0.8320	3.2492	1.8824	0.1176	0.7499
	2.2117	0.2117	0.4561	1.8969	0.1031	0.3079
10	1.1063	0.1063	0.2922	1.0882	0.0882	0.1112
	2.3101	0.3103	0.5582	1.8313	0.1687	0.3007
	2.1039	0.1039	0.2600	1.8917	0.1083	0.1736
15	1.0755	0.0755	0.1991	1.0738	0.0738	0.0709
	2.1962	0.1963	0.2692	1.8509	0.1491	0.1817
	2.0700	0.0700	0.1827	1.9017	0.0983	0.1051
20	1.0622	0.0622	0.1688	1.0512	0.0512	0.0512
	2.1587	0.1587	0.1866	1.8803	0.1197	0.1337
	2.0653	0.0653	0.1529	1.9305	0.0695	0.0701
50	1.0264	0.0264	0.0733	1.0139	1.0139	0.0149
	2.0562	0.0562	0.0575	1.9189	0.0811	0.0536
	2.0334	0.0334	0.0745	1.9699	0.0301	0.0197
100	1.0206	0.0206	0.0424	1.0160	0.0160	0.0062
	2.0310	0.0310	0.0253	1.9502	0.0498	0.0248
	2.0158	0.0158	0.0415	1.9736	0.0264	0.0071
250	1.0177	0.0177	0.0244	1.0142	0.0142	0.0031
	2.0115	0.0115	0.0105	1.9723	0.0277	0.0108
	2.0026	0.0026	0.0217	1.9796	0.0204	0.0035
500	1.0104	0.0104	0.0153	1.0136	0.0136	0.0022
	2.0060	0.0060	0.0049	1.9835	0.0165	0.0050
	2.0030	0.0029	0.0136	1.9834	0.0166	0.0022

3.4. Order statistics

Suppose that $X_i, i = 1, \ldots, n$ represents the random sample of size n from Max-LL distribution and let $X_{(i)}$ indicate the sample's order statistics. The pdf of the i^{th} order statistics can be defined as:

$$f_{i,n}(x) = \frac{n!f(x)F(x)^{i-1}}{(n-i)!\,(i-1)!}\left[1 - F(x)\right]^{n-i}$$
$$= \frac{n!f(x)}{(n-i)!\,(i-1)!}\sum_{q=0}^{\infty}\binom{n-i}{q}(-1)^q F(x)^{i+q-1}. \tag{33}$$

Substituting equation (30) into equation (33) for $s = i + q - 1$ gives:

$$f_{i,n}(x) = \frac{n!}{(n-i)!\,(i-1)!}\sum_{q=0}^{\infty}\binom{n-i}{q}(-1)^q\sum_{k,m=0}^{\infty}V_{k,m}x^{3c+c(2m+3k)-1}$$
$$\times \quad \exp\left(-\frac{1}{2b^2}\left(\frac{x}{d}\right)^{2c}\right), \tag{34}$$

that is the proposed distribution's order statistics. The Max-LL distribution's moments of the i^{th} order statistics are obtained as:

$$E(X_{i,n}^r) = \int_0^{\infty} x^r f_{i,n}(x)dx$$
$$= \frac{n!\sum_{q=0}^{\infty}\binom{n-i}{q}(-1)^k}{(i-1)!\,(n-i)!}\sum_{k,m=0}^{\infty}V_{k,m}\int_0^{\infty}x^r \times x^{3c+c(2m+3k)-1}$$
$$\times \quad \exp\left(-\frac{1}{2b^2}\left(\frac{x}{d}\right)^{2c}\right)dx$$
$$= \frac{n!\sum_{q=0}^{\infty}\binom{n-i}{q}(-1)^k}{(i-1)!\,(n-i)!}\sum_{k,m=0}^{\infty}V_{k,m}\int_0^{\infty}x^{r+3c+c(2m+3k)-1}$$
$$\times \quad \exp\left(-\frac{1}{2b^2}\left(\frac{x}{d}\right)^{2c}\right)dx, \tag{35}$$

where $f_{i,n}(x)$ is the pdf of the i^{th} order statistics presented in equation (34). Applying equation (17), equation (35) becomes:

$$E\left(X_{i,n}^r\right) = \frac{2^{1+\frac{r}{2c}}b^{\frac{r}{c}}d^r n!}{(i-1)!\,(n-i)!\sqrt{\pi}}\sum_{k,m,q=0}^{\infty}(-1)^k\binom{n-i}{q}\frac{\Lambda_t(s)\,\Pi_{k,m}}{\left\{\Gamma\left(\frac{3}{2}\right)\right\}^k}$$
$$\times \quad \int_0^{\infty}A^{\frac{1+(2m+3k)}{2}+\frac{r}{2c}}\mathrm{e}^{-A}dA$$
$$= \sum_{k,m=0}^{\infty}\Psi_{k,m}\Gamma\left(\frac{2m+3k+3}{2}+\frac{r}{2c}\right), \tag{36}$$

where $\Psi_{k,m} = \sum_{q=0}^{\infty}\frac{(-1)^k n! 2^{1+\frac{r}{2c}}b^{\frac{r}{c}}d^r\Lambda_t(s)\Pi_{k,m}}{\sqrt{\pi}\left\{\Gamma\left(\frac{3}{2}\right)\right\}^k(i-1)!(n-i)!}\binom{n-i}{q}$.

3.5. Stress-strength model

Suppose X_1 and X_2 follows Max-LL model with probability density functions $f(x_1)$ and $f(x_2)$. Then the stress-strength model denoted by is R

$$R = \int_0^\infty f(x_1, b, c_1, d) F(x_1, b, c_2, d)\, dx_1. \tag{37}$$

The cdf and pdf defined in equation (37) are provide in equations (5) and (6). Thus, subsituting equations (28) and (29) into equation (37) gives:

$$\begin{aligned} R &= \int_0^\infty f(x_1, b, c_1, d) \sum_{k=0}^{\infty} \Lambda_i(s)\, \gamma_1^k(a,\ z) dx_1 \\ &= \frac{c_1\sqrt{2}}{\sqrt{\pi}} \sum_{k,m=0}^{\infty} \frac{\Lambda_i(s)\,\Pi_{k,m}}{2^{\frac{2m+3k}{2}} b^{3+2m+3k} d^{3c_1+c_2(2m+3k)} \left\{\Gamma\left(\frac{3}{2}\right)\right\}^k} \\ &\times \int_0^\infty x_1^{3c_1+c_2(2m+3k)-1} \exp\left(-\frac{1}{2b^2}\left(\frac{x_1}{d}\right)^{2c_1}\right) dx_1. \end{aligned} \tag{38}$$

Using equation (17), equation (38) becomes:

$$\begin{aligned} R &= \sum_{k,m=0}^{\infty} \frac{\Lambda_i(s)\,\Pi_{k,m} 2^{1+\frac{c_2(2m+3k)}{2c_1}} \{b\}^{\frac{c_2(2m+3k)}{c_1}}}{\sqrt{\pi} 2^{\frac{2m+3k}{2}} b^{2m+3k} \left\{\Gamma\left(\frac{3}{2}\right)\right\}^k} \int_0^\infty A^{\frac{c_2(2m+3k)}{2c_1}+\frac{1}{2}} e^{-A} dA \\ &= \sum_{k,m=0}^{\infty} \Psi_{k,m} \Gamma\left(\frac{c_2(2m+3k)}{2c_1} + \frac{3}{2}\right), \end{aligned} \tag{39}$$

where $\Psi_{k,m} = \frac{2\Lambda_i(s)\Pi_{k,m}\{2b^2\}^{\frac{c_2(2m+3k)}{2c_1}}}{\sqrt{\pi}\{\Gamma(\frac{3}{2})\}^k\{2b^2\}^{\frac{2m+3k}{2}}}$.

4. Parameter estimation

The literature describes a variety of approaches for estimating distribution parameters, with the maximum likelihood approach being one of the most popular. The following is a review of recent literature on several estimating methods, Refs. [63–67]. This section presents the maximum likelihood approach for estimating the parameters of the Max-LL distribution.

Suppose X_i for $i = 1, \ldots, n$ indicates the random sample of size n with observed values x_i from the Max-LL Model. Let $\Theta = (b, c, d)^T$ be the $p \times 1$ parameter vector. The likelihood function of the parameter Θ is derived from equation (6) as:

$$\ell(x_i/\Theta) = \left(\frac{2c}{b^3 d^{3c}\sqrt{2\pi}}\right)^n \prod_{i=1}^{n} x_i^{3c-1} \exp\left(-\frac{1}{2b^2}\left(\frac{x_i}{d}\right)^{2c}\right). \tag{40}$$

The log-likelihood function of equation (40) is:

$$\begin{aligned} \ell &= n\log(2) + n\log(c) - 3n\log(b) - 3nc\log(d) - \frac{n}{2}\log(2\pi) \\ &+ (3c-1)\sum_{i=1}^{n} \log(x_i) - \frac{1}{2b^2}\sum_{i=1}^{n}\left(\frac{x_i}{d}\right)^{2c}. \end{aligned} \tag{41}$$

As a result, the estimates of the Max-LL distribution parameters are determined by partially differentiating equation (41) about parameters b, c and d, and then setting the results to zero.

$$\frac{\partial \ell}{\partial b} = \frac{-3n}{b} + \frac{1}{b^3}\sum_{i=1}^{n}\left(\frac{x_i}{d}\right)^{2c} = 0, \tag{42}$$

$$\begin{aligned} \frac{\partial \ell}{\partial c} &= \frac{n}{c} - 3nlog(d) + 3\sum_{i=1}^{n}(log(x_i)) - \frac{1}{b^2}\sum_{i=1}^{n}\left(\frac{x_i}{d}\right)^{2c}\log\left(\frac{x_i}{d}\right) \\ &= 0, \end{aligned} \tag{43}$$

and

$$\frac{\partial \ell}{\partial d} = \frac{-3nc}{d} + \frac{c}{db^2}\sum_{i=1}^{n}\left(\frac{x_i}{d}\right)^{2c} = 0. \tag{44}$$

Using equation (42), the estimate of the parameter b is:

$$\hat{b} = \sqrt{\frac{1}{3n}\sum_{i=1}^{n}\left(\frac{x_i}{d}\right)^{2c}}. \tag{45}$$

As a result, the parameter estimations c and d are obtained from equations (43) and (44) by substituting equation (45). Moreover, the second derivatives of equations (42), (43) and (44) are given as:

$$\frac{\partial^2 \ell}{\partial b^2} = \frac{3n}{b^2} - \frac{3}{b^4}\sum_{i=1}^{n}\left(\frac{x_i}{d}\right)^{2c}, \tag{46}$$

$$\frac{\partial^2 \ell}{\partial c^2} = \frac{-n}{c^2} - \frac{2}{b^2}\sum_{i=1}^{n}\left(\frac{x_i}{d}\right)^{2c}\left\{\log\left(\frac{x_i}{d}\right)\right\}^2, \tag{47}$$

$$\frac{\partial^2 \ell}{\partial d^2} = \frac{3nc}{d^2} - \frac{2(1+2c)}{b^2 d^2}\sum_{i=1}^{n}\left(\frac{x_i}{d}\right)^{2c}, \tag{48}$$

$$\frac{\partial^2 \ell}{\partial b \partial c} = \frac{2}{b^3}\sum_{i=1}^{n}\left(\frac{x_i}{d}\right)^{2c} log\left(\frac{x_i}{d}\right), \tag{49}$$

$$\frac{\partial^2 \ell}{\partial b \partial d} = \frac{-2c}{b^3 d}\sum_{i=1}^{n}\left(\frac{x_i}{d}\right)^{2c}, \tag{50}$$

and

$$\frac{\partial^2 \ell}{\partial c \partial d} = \frac{-3n}{d} + \frac{1}{b^2 d}\sum_{i=1}^{n}\left(\frac{x_i}{d}\right)^{2c}\left\{1 + 2c\log\left(\frac{x_i}{d}\right)\right\}. \tag{51}$$

In this case, the second derivatives of the parameters of the Max-LL distribution might be used to obtain the inferences about the proposed distribution.

5. Simulation study

A simulation analysis was performed to investigate the accuracy of the Max-LL distribution, employing the quantile function defined in equation (15) for a variety of sample sizes (n) of 5, 10, 15, 20, 50, 100, 250, and 500, as well as parameter values b, c and d. The simulation study employed the MLE and Maximum Product of Spacing (MPS) methodologies. The mean of the estimations (mean), absolute bias (bias), and mean square error (MSE) were determined by conducting the simulation 1000 times. Tables 1, 2, and 3 provide the simulation results, accordingly.

As seen in Table 1, when the sample size increases, the mean of each parameter using MLE decreases and approaches

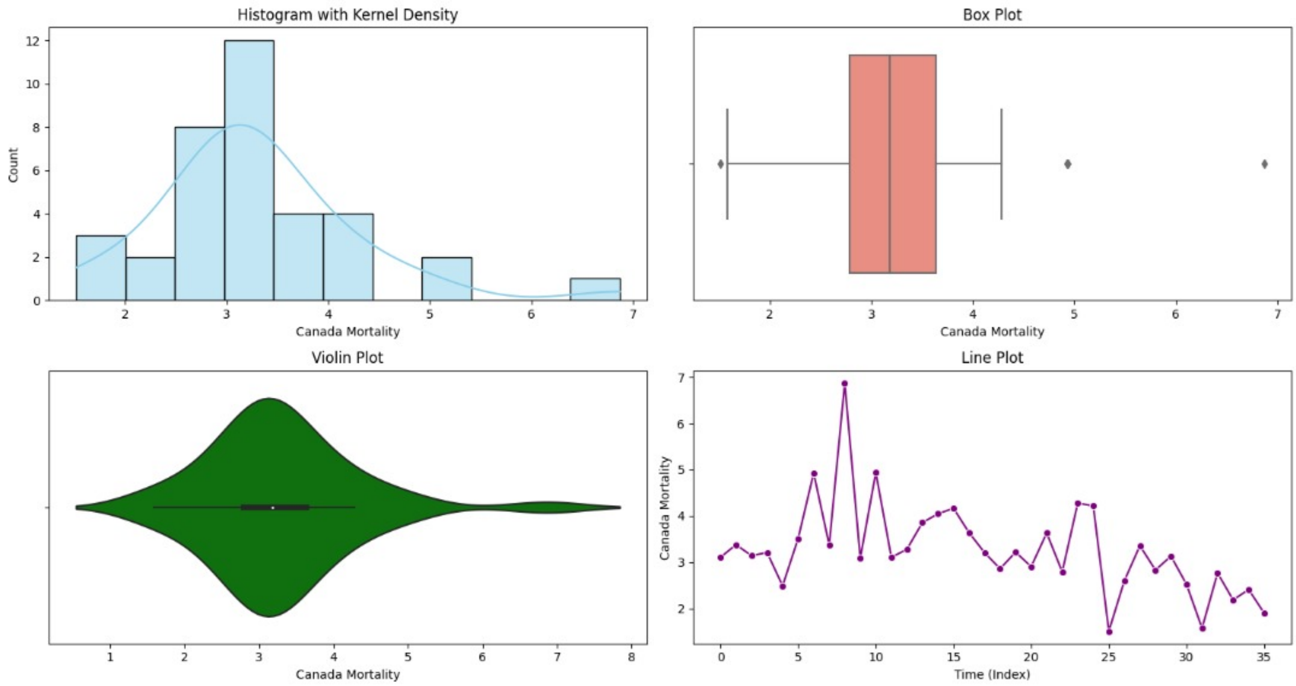


Figure 4: Histogram, box, violin, and line plots of the first data set.

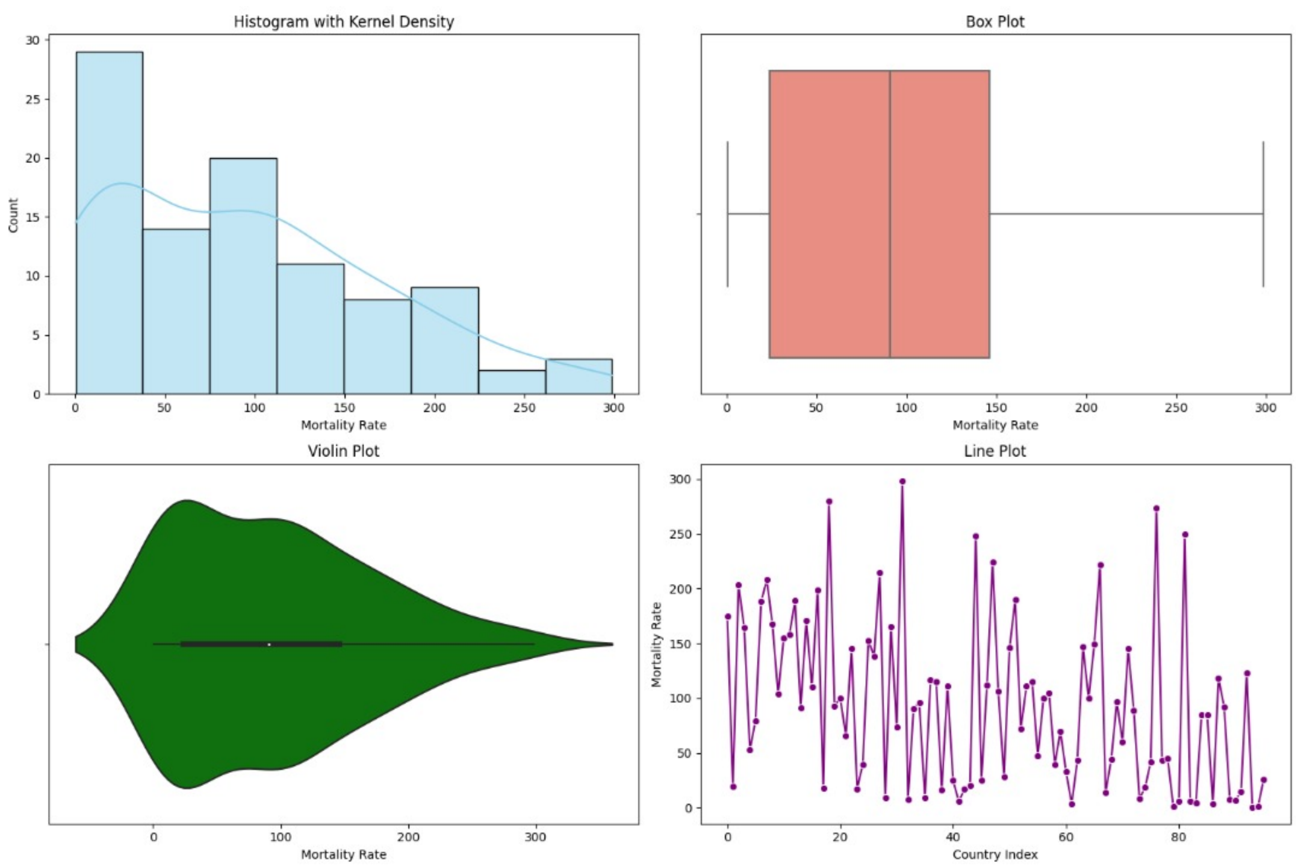


Figure 5: Histogram, box, violin, and line plots of the second data set.

the true parameter values of $b = 1$, $c = 1$ and $d = 2$. The mean and MSE of each estimate are decreases and approaching zero. In terms of MPS, the means approach true parameters, whereas the bias and MSE both drop and approach zero.

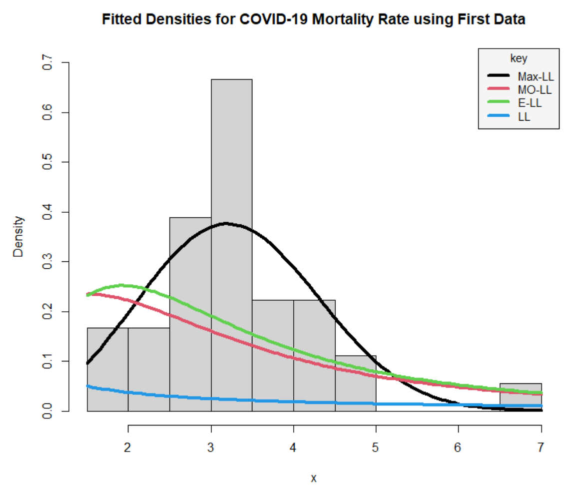


Figure 6: Density plots utilizing first data, showing the Max-LL distribution as well as various competing distributions.

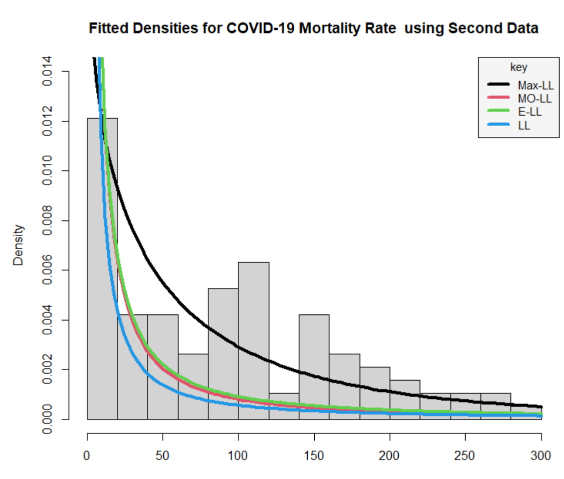


Figure 7: Density plots utilizing second data, showing the Max-LL distribution as well as various competing distributions.

The mean of parameters b, c and d using MLE decreased and approached the true parameter values of $b = 1.5$, $c = 1$ and $d = 2$, as shown in Table 2. Additionally, as the sample size increases, the mean and MSE of each estimate decrease and approach zero. When employing the MPS strategy, the means of each estimate approach's real parameter values, bias, and MSE all dropped as sample size increased.

The mean of parameters b, c and d using MLE decreases and approaches the true parameter values of $b = 1$, $c = 2$ and $d = 2$, as shown in Table 3. Furthermore, as sample sizes increase, the mean square error of each estimate approaches zero. Using the MPS, the means of each estimate approach their true parameter values, but bias and MSE decrease as sample size increases. In conclusion, Tables 1-3 show that the MSEs of the estimates decrease as the sample sizes increase, and both MPSs and MLEs work consistently. Finally, we observed that the MPS is more suitable for small sample sizes ($n \leq 20$), whereas the MLE is more suitable for large sample sizes ($n \geq 100$). This observation is supported by the results presented in Tables 1-3, which suggest that the MPS performs better for smaller sample sizes, while the MLE performs better for larger sample sizes.

6. Application to data sets

We offer two COVID-19 datasets to evaluate the Max-LL distribution's performance. The first dataset comprises the COVID-19 mortality rate for 36 days pertaining to Canada from April 10, 2020 to May 15, 2020, as provided in Ref. [68]. The data set is provided as follows:

3.1091, 3.3825, 3.1444, 3.2135, 2.4946, 3.5146, 4.9274, 3.3769, 6.8686, 3.0914, 4.9378, 3.1091, 3.2823, 3.8594, 4.0480, 4.1685, 3.6426, 3.2110, 2.8636, 3.2218, 2.9078, 3.6346, 2.7957, 4.2781, 4.2202, 1.5157, 2.6029, 3.3592, 2.8349, 3.1348, 2.5261, 1.5806, 2.7704, 2.1901, 2.4141, 1.9048.

The second data set covers the COVID-19 mortality rate for 95 countries with at least 100,000 confirmed COVID-19 cases as of May 17, 2021. See the website [https://covid19.who.int/] for further information. The data set is provided as follows:

175.12, 19.59, 203.53, 164.17, 52.81, 79.4, 188.07, 208.01, 167.5, 103.52, 154.56, 157.72, 188.8, 91.24, 170.75, 109.93, 198.98, 17.54, 279.61, 93.04, 100.16, 65.89, 145.08, 17.39, 39.6, 152.54, 138.22, 214.24, 8.81, 165.17, 73.72, 298.63, 7.36, 90.59, 96.21, 9.06, 116.95, 114.67, 16.47, 111.13, 24.65, 5.77, 16.67, 20.53, 248.15, 25.35, 111.61, 223.97, 106.03, 28.48, 145.82, 189.94, 72.35, 111.35, 115.23, 47.03, 99.66, 104.9, 39.5, 69.63, 32.9, 3.46, 42.97, 146.85, 99.53, 149.13, 221.53, 13.94, 43.79, 96.99, 59.94, 145 89, 8.32, 18.26, 42.06, 273.74, 43.31, 44.9, 1 5.58, 249.45, 5.9, 4.39, 84.97, 84.4, 3.71, 118.31, 91.95, 7.68, 7.03, 14.42, 123.24, 0.33, 0.84.

Figures 4 and 5 display the histogram with kernel density, box, violin, and line plots for the first and second datasets, respectively. The histogram plots affirm the right-tailed nature of the data, with the presence of extreme values evident in the box plot. These support the pdf's shape presented in Figure 1 for the proposed distribution and are well-suited for modeling both the first and second datasets.

Furthermore, we will use the two data sets to compare the performance of the Max-LL distribution to existing models to gain insight into its adaptability. The present research looked at three competing models: the LL distribution by Ref. [5], the Marshall-Olkin-LL (MO-LL) distribution by Ref. [19], and the Exponentiated-LL (E-LL) distribution by Ref. [69].

Figures 6 and 7 show density plots of the first and second data sets, respectively, for the Max-LL and competing distributions. The figures show that the Max-LL distribution appears to be appropriate for modeling the first and second datasets, respectively.

Some of the information criteria considered in this study is the Akaike Information Criterion (AIC). Other criteria include the Hannan Quinn Information Criterion (HQIC), Bayesian Information Criterion (BIC), and Corrected Akaike Information Criterion (CAIC). The model that best fits these types of data sets should be labeled as having the lowest value for these criteria.

Table 4 displays the estimated parameters (with standard errors) and log-likelihood values for the Max-LL and competing models using the first data. Table 5 displays the AIC, BIC, HQIC, and CAIC for both the proposed and competing models. Table 5 shows that the Max-LL distribution provided the lowest values for the AIC, BIC, HQIC, and CAIC compared to other competing models, followed by the E-LL, MO-LL, and LL models.

Table 4: The estimates, standard errors, and ℓ for the first data set

Models	Estimates	Std Errors	ℓ
Max-LL	$b = 14.4951$	8.0626	-528.6200
	$c = 0.3939$	0.0330	
	$d = 0.0242$	0.0386	
E-LL	$h = 0.2852$	0.0297	-537.6770
	$c = 2.4246$	0.0740	
	$d = 109.7914$	0.0250	
MO-LL	$a = 16.4969$	3.2419	-539.9730
	$c = 1.3071$	0.1315	
	$d = 7.3389$	2.6793	
LL	$c = 1.3070$	0.1119	-539.9730
	$d = 62.6640$	2.9679	

Table 5: The information criteria values for the first data set.

Models	AIC	CAIC	BIC	HQIC
Max-LL	1063.2400	1063.5040	1070.9020	1066.3360
E-LL	1081.3540	1081.6180	1089.0160	1084.4500
MO-LL	1085.9460	1086.2100	1093.6080	1089.0420
LL	1083.9460	1084.0760	1089.0540	1086.0100

Table 6 compares the estimates (with the standard errors) and log-likelihood values for the proposed distribution to competing ones using the second dataset. Table 7 shows the information criteria findings for the Max-LL and other competing distributions using the second data set. The proposed distribution offered the lowest values for those criteria, followed by the E-LL, MO-LL, and LL distributions.

Table 6: The estimates, standard errors, and ℓ for the second data set

Models	Estimates	Std Errors	ℓ
Max-LL	$b = 3.9139$	11.3225	-45.3062
	$c = 1.3662$	0.1571	
	$d = 0.8722$	1.8517	
E-LL	$h = 0.7489$	0.3599	-46.9145
	$c = 7.2007$	1.7980	
	$d = 3.3671$	0.3528	
MO-LL	$a = 9.1709$	8.0635	-47.0914
	$c = 6.4111$	0.9175	
	$d = 2.2410$	0.3466	
LL	$c = 6.4111$	0.9181	-47.0914
	$d = 3.1664$	0.1401	

Table 7: The information criteria values for the second data set.

Models	AIC	CAIC	BIC	HQIC
Max-LL	96.6124	97.3624	101.3630	98.2705
E-LL	99.8289	100.5789	104.5795	101.4870
MO-LL	100.1829	100.9329	104.9334	101.8409
LL	98.1829	98.5465	101.3499	99.2882

7. Conclusion

The present study introduced the Max-LL distribution, which is based on the Maxwell generalized distributions. The Maxwell-Logistic distribution's density forms might be left, right, or symmetric. The failure function of the Max-LL distribution can be increasing, decreasing, or upside-down bathtub forms. The parameters for the proposed distribution were determined using the maximum likelihood technique. The quantile function was used in the simulation study, and the findings revealed that as sample size increased, the means approached true parameter values while the variance and mean squared errors of each estimate decreased. The Max-LL distribution fitted both data sets because it produced higher log-likelihood values (as seen in Tables 4 and 6) and lower information criteria values than competing distributions. This demonstrated that the Max-LL distribution is best suitable for modeling the COVID-19 mortality data utilized in this study.

Data Availability

The link to the dataset used in this study is provided below: https://covid19.who.int/.

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