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The dynamics of hybrid-immune and immunodeficient susceptible individuals and the three stages of COVID-19 vaccination

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Abstract

The World Health Organization has disclosed that the hybrid-immune and immunodeficient individuals are two distinct classes of individuals susceptible to the COVID-19 virus. To model this unique characteristics of two distinct categories of susceptible individuals and the dynamics of the three phases of a vaccination program implemented by the World Health Organization in Malaysia, which have not been accounted for in previous studies, a twelve compartmental *SSVEIHQR-D* epidemiological model was developed. This model aimed to accurately capture the spread of the COVID-19 virus by fitting real-life data to the model and obtaining updated estimates of the reproduction number. The study also focused on assessing current control measures and exploring strategies to eradicate the virus and mitigate future outbreaks. Mathematical analyses of the model included investigations into stability, equilibrium points, the basic reproduction number R_0, optimal control strategies, and sensitivity analyses. Estimation and fitting of the model parameters were conducted using daily situation reports from the Ministry of Health of Malaysia. The obtained value of the basic reproduction number, based on fitted parameter values, indicated stability and reflected the current pandemic situation more realistically. Additionally, the herd immunity threshold was calculated and interpreted in the context of the study findings. Finally, practical insights and recommendations derived from the model's results were provided to inform government agencies, public health organizations, and intervention bodies, aimed at controlling the spread of the COVID-19 virus.

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1. Introduction

The novel coronavirus (COVID-19) which was first confirmed in Malaysia on the 25th of January, 2020 [1] emerged from the city of Wuhan, China, in December, 2019 [2, 3]. The COVID-19 virus is said to belong to a family of coronaviruses that causes illness such as common cold and severe acute respiratory syndrome (SARS). The World Health Organization (WHO) on the 30th of January 2020 declared the spread of the virus a public health emergency of international concern (PHEIC), and called of the PHEIC declaration on 5th

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May 2023. As of September 4, 2023 the COVID-19 Pandemic records show that 770,084,949 confirmed cases, and 6,956,160 confirmed death cases [4]; while the situation data reports from the Ministry of Health of Malaysia (MOH) as of October 16 2023 showed a total 5,128,668 confirmed cases and 37,179 deaths. These figures according to Wikipedia contributors [4] and Joshua[5] makes the COVID-19 pandemic the fifth deadliest epidemic and pandemic in history.

The COVID-19 pandemic has caused a lot mayhem in almost all spheres of life. It brought about social and economic setback, disruption of livelihood, such as travel restrictions, mandatory wearing of masks, observation of social distancing, and unprecedented deaths of loved ones. The WHO has made it known that the transmission of the COVID-19 virus occurs through either direct contact with infected persons/objects or through inhalation from the air. This transmission processes is supported by a number of research studies which includes but not limited to the works of Anand and Mayya [6], Anfinrud *et al.* [7], Jayaweera *et al.* [8], Kwon *et al.* [9] and Lai *et al.* [10]. These transmissions occur when a COVID-19 symptomatic patient coughs or sneezes at a close range to susceptible person, or when a susceptible individual uses or make a contact with items contaminated with droplets (from cough or sneezing) of symptomatic person. WHO has stated that prevention from inhaling droplets in the air could be achieved through the frequent wearing of face-masks, while prevention from direct contact with contaminated objects could be achieved through the frequent washing of hands and the use of hand sanitizers.

The need to capture in a model the dynamics of the spread of COVID-19 virus in order to appropriately predict and curb its future anticipated variants (a variant is where the virus contains at least one new change to the original virus and sometimes variants of the virus may develop [11]) or second waves has led to the proposal of variety of models, both new and modification of existing ones. Ibidoja and Fowobaje [12] extracted COVID-19 data from the Nigerian Center for Disease and Control (NCDC) repository to forecast and fit the trend of the virus. When it comes to modeling the spread of infectious diseases, the compartmental models are the general modeling techniques adopted by researchers [13]. The compartmental model approach originated from the 20th century works of Ross [14], Ross and Hudson [15], Kermack and McKendrick[16], Kendall [17]. This modeling approach has been applied to other malicious objects such as computer worm and virus [18], scam rumor [13], and many other diseases such as Ebola [19], malaria [20] and many more.

A number of works have been done in modelling the dynamics of the spread of COVID-19 which are purely based on deterministic approach. Among such works are the works of Ngonghala *et al.* [21], Adewole *et al.* [22], Omede *et al.* [23]; and a some modifications/improvements have been made to capture the dynamics of the spread of COVID-19. For example, in order to improve the accuracy for prediction of COVID-19 spread, Zhu *et al.* [24] proposed a deterministic model by introducing the re-infection rate and social distancing factor into the traditional SEIRD to account for the effects of re-infection and social distancing on COVID-19 spread. The deterministic SEIRD(R)D-SD model was further transformed into the stochastic form to account for fluctuations due environmental distortions. Danane *et al.* [25] in their work investigated the dynamics of a COVID-19 stochastic model with isolation strategy. Adewole *et al.* [22] and Ahmed *et al.* [26] divided the infected compartment into two groups: the symptomatic and asymptomatic infected persons. In July, 2023, Manaqib *et al.* [27] developed an SVEIHQR model for the spread of COVID-19 where the vaccination group is divided into three: the first dose, the second dose and the booster dose.

But the WHO has hinted that there are two distinct classes of persons susceptible to the COVID-19 virus, whose susceptibility differs from each other: the strong immune (hybrid immune) individuals and the weak immune (immunodeficient/immunocompromised) individuals. Over the past few years, series of studies have shown that some people have a powerful immune response against the COVID-19 [11]. According to Dr. Paul Brieniasz, "One could reasonably predict that these people will be quite well protected against most and perhaps all of the SARV-COV-2 variants that we are likely to see in the foreseeable future" [28]. While people with weak/immunocompromised system are group of persons with low/weak immunity against the COVID-19 virus due to some reasons such as medical conditions like, acute treatment for solid tumor, hematologic malignancies, advanced or untreated HIV, severe combined immunodeficiency (SCID), flu infection, diabetes, liver or kidney disease. Immune system can also be weakened/compromised through smoking, alcohol, and poor nutrition [11]. This significant gap has not been taken into consideration in any model.

Having distinct classes of susceptible persons is believed to have played a significant role in the dynamics of the spread of COVID-19 by varying the infection force of the spread of COVID-19 in the population, but scholars have not taken that into consideration. In order to take into consideration this significant gap, following the step of Manaqib *et al.* [27], we developed a new SSVEIHQR-D model for the spread of COVID-19, where the susceptible compartment is divided into two groups: the strong/hybrid immune individuals and the weak/compromised immune individuals. Three stages of vaccination are incorporated into the model,which resulted in having three classes/groups of vaccinated individuals: those who have received the first dose, those who have received the second dose, and those who have received the booster dose. In this model, two classes of infected/infectious persons are considered: the symptomatic and the asymptomatic individuals. Furthermore, the model captured the hospitalized and dead patients in order to account for treatment and deaths record as a result of the COVID-19 virus.

The objectives of this research are: to assess the dynamics of heterogeneous susceptibility, since it is medical fact that we have varying levels of susceptibility to the COVID-19 virus; to assess possible means of tracking vulnerable population that are at higher risk of being infected with COVID-19 due to weaker immune system; to highlight the need for booster shot vaccination in population with weaker immune system; to support the findings of other researchers and highlight the importance of vaccination, control strategies, preventive measures, and risk awareness for individuals with weaker immune system; finally, to explore the long-term dynamics of COVID-19 in Malaysia, considering the impact of vaccination campaigns, natural immunity, and potential future variants. In order to achieve these objectives, we conduct a theoretical analysis of the SSVEIHQR-D model to understand its mathematical properties, stability, and equilibrium points; fitting of daily data as reported by the National Center for Disease and Control in order to estimate a more accurate and updated basic reproduction number and other parameters using current data; to perform a sensitivity analysis in order to identify the key parameters that significantly influence the basic reproduction number (R_0) of the model, which is crucial for assessing the disease's potential for spreading; to estimate the threshold head immunity and assess control strategies for more effective interventions in optimal setting.

2. Model formulation

In order to describe the dynamics of COVID-19 that is peculiar to the Malaysia populace, we modified the work of Manaqib *et al.* [27] by dividing the susceptible population into two subgroups: the weak and strong immune, since it is a proven medical fact that some people have strong immune systems that provide better protection, while others have weaker immune systems and are more vulnerable to COVID-19 virus [29–32]. To have a more realistic model it becomes necessary to subdivide the susceptible class into these subgroups to account for the differences in susceptibility. Thus, a deterministic epidemic *SSVEIHQR-D* model is proposed, where the population is divided into twelve compartments, the susceptible group of people with high/strong immune system (S_h), the susceptible group of people with low/weak immune system (S_w), the group of persons who have received the first dose of vaccination (V_1), the group of persons who have received the second dose of vaccination (V_2), the group of persons who have received the booster dose of vaccination (V_b), the group exposed/latent individuals (E), the group of symptomatic infected persons (I_s), the of asymptomatic infected persons (I_a) the group of hospitalized persons (H), the quarantined group/class (Q), the group of persons that have recovered or have gained immunity to COVID-19 (R), and total deaths as a result of COVID-19 (D) (but here, the death compartment is an artificial or "auxiliary" compartment which is introduced to account for deaths as a result of COVID-19 disease, so it it not part of the sum of N; where N is the sum of the population at a given time t, so D is not part of it. So death would not be part of the analysis, but shall be use for simulation purpose in order to obtain a more accurate result. The overview of the description of the *SSVEIHQR-D* model state variable is presented in Table 1. The total population at a given time t denoted by $N(t)$ is equal to the sum of the compartments with the exception of D (see reason above), that is,

$$N = S_h + S_w + V_1 + V_2 + V_b + E + I_s + I_a + H + Q + R$$

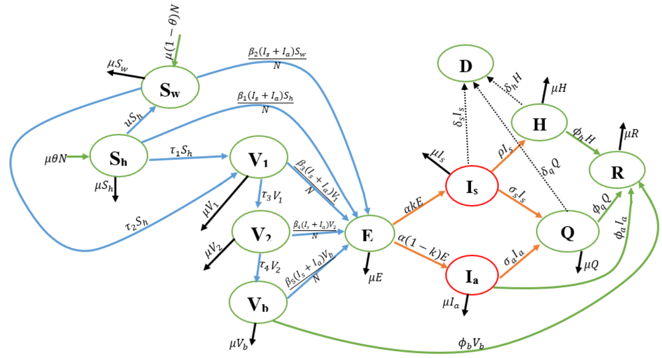

Figure 1: Epidemiological interaction dynamics of the COVID-19 SSVEIHQR model.

Table 1: Description of the State Variables of the Model.

Variables	Description
S_h	Susceptible group of people with high/ strong immune system at a given time (t)
S_w	Susceptible group of people with low/ weak immune system at a given time (t)
V_1	Group of persons who have received the first dose of COVID-19 vaccination
V_2	Group of persons who have received the second dose of COVID-19 vaccination
V_b	Group of persons who have received the booster dose of COVID-19 vaccine
E	Group of exposed/latent individuals at a given time (t)
I_s	Group of COVID-19 symptomatic infected persons at a given time (t)
I_a	Group of COVID-19 asymptomatic infected persons at a given time (t)
H	Group of persons hospitalized as a result of COVID-19 infection at time (t)
Q	The set of COVID-19 quarantined individuals at a given time (t)
R	Recovered group of person (or those who have gained immunity over COVID-19)

2.1. Assumptions of the Model

Following the tips of the spread of COVID-19 outlined by the WHO, and the peculiarities of the spread of COVID-19 virus in the Malaysian populace, the following assumptions are made for the model:

- The population is assumed to be homogeneous, where each individual has equal chance of being contaminated with the virus.
- In order to have a constant population, we assume equal natural birth rate and death rate (μ).
- Individuals with strong immune system could enter the group of persons with weak immune system at a rate (u).

Table 2: Parameter Description.

Parameter	Parameter Description	values	Reference
μ	Natural birth rate and death rate	$\frac{1}{73\times365}$	[27]
θ	Natural birth rate and death rate ratio	0.7	Assumed
u	The rate at which strong immune becomes weak immune	0.01	Assumed
β_1	Rate of infection among strong immune group	0.73138	Data fitting
β_2	Rate of infection among weak immune group	0.25688	Data fitting
β_3	Rate of infection among those who have taken first dose	0.99657	Data fitting
β_4	Rate of infection among those who have taken second dose	0.42067	Data fitting
β_5	Rate of infection among those who have taken the booster dose	0.84937	Data fitting
τ_1	Rate at which strong immune receives first dose	0.3910×10^{-7}	Data fitting
τ_2	Rate at which weak immune receives first dose	0.0001×10^{-7}	Data fitting
τ_3	Rate at which individuals receives second dose	0.99657	Data fitting
τ_4	Rate at which individuals receives booster dose	0.58838	Data fitting
α	The incubation period	1/8	[27]
k	The proportion of symptomatic individuals		
σ_s	The rate at which the symptomatic are quarantined	0.25675	Data fitting
σ_a	The rate at which the asymptomatic are quarantined	0.67314	Data fitting
ρ	The rate at which the symptomatic are hospitalized	0.12590	[27]
ϕ_a	The rate at which the asymptomatic recover from the infection	0.03671	Data fitting
ϕ_h	The rate at which the hospitalized recover from COVID-19	0.995192	Data fitting
ϕ_b	The rate at which those who have taken booster dose recover	0.14286	[27]
ϕ_q	The rate at which the quarantined recover from COVID-19	0.027115	Data fitting
δ_s	Symptomatic individuals death rate due to COVID-19.	1.64×10^{-5}	[22]
δ_h	Hospitalized individuals death rate due to COVID-19.	1.64×10^{-3}	[22]
δ_q	Quarantined individuals death rate due to COVID-19.	1.64×10^{-6}	[22]

Table 3: Summary of Parameter and Sensitivity indices.

Parameter	Sensitivity indices	Parameter	Sensitivity indices
μ	0.0184	α	−0.0142
θ	−0.0003	k	0.2236
u	−0.2521	σ_s	0.9215
β_1	0.0269	σ_a	0.8398
β_2	0.9620	ρ	0.8245
β_3	0.0053	ϕ_a	−0.0061
β_4	0.0165	ϕ_h	−0.0163
β_5	0.0016	ϕ_b	0.0022
τ_1	−0.0197	ϕ_q	0.9220
τ_2	−0.0089	δ_s	0.0178
τ_3	0.0258	δ_h	0.0023
τ_4	0.0110	δ_q	0.0085

- We assume that new births are either individuals with strong immune system or weak immune system.
- All compartments/classes have a natural death rate of (μ).
- The model assumes three groups of vaccinated persons: those who have taken the first dose (i.e. partially vaccinated), those who have taken the second dose (complete dose), and those who have takes the booster dose.
- A fraction of those who have taken the booster dose are assumed to have obtained immunity from the COVID-19 virus.
- An exposed/latent individuals cannot spread/infect another person
- All vaccinated persons are likely to be exposed with the COVID-19 virus (according to WHO [11]).
- After a given period of time (incubation period), exposed individuals could either become symptomatic or asymptomatic patients of the disease.
- Symptomatic individuals are either quarantined, hospitalized or they die as a result of the virus.
- Symptomatic, hospitalized and quarantined group of individuals do not have equal death rate as a result of the virus.
- Asymptomatic infected individuals are either quarantined or they recovered from the infection.
- The quarantined either recover or die as result of the infection.
- Recovered persons are assumed to obtain temporal immunity from the virus.
- Death rate as a result of the disease are not equal for all compartments.

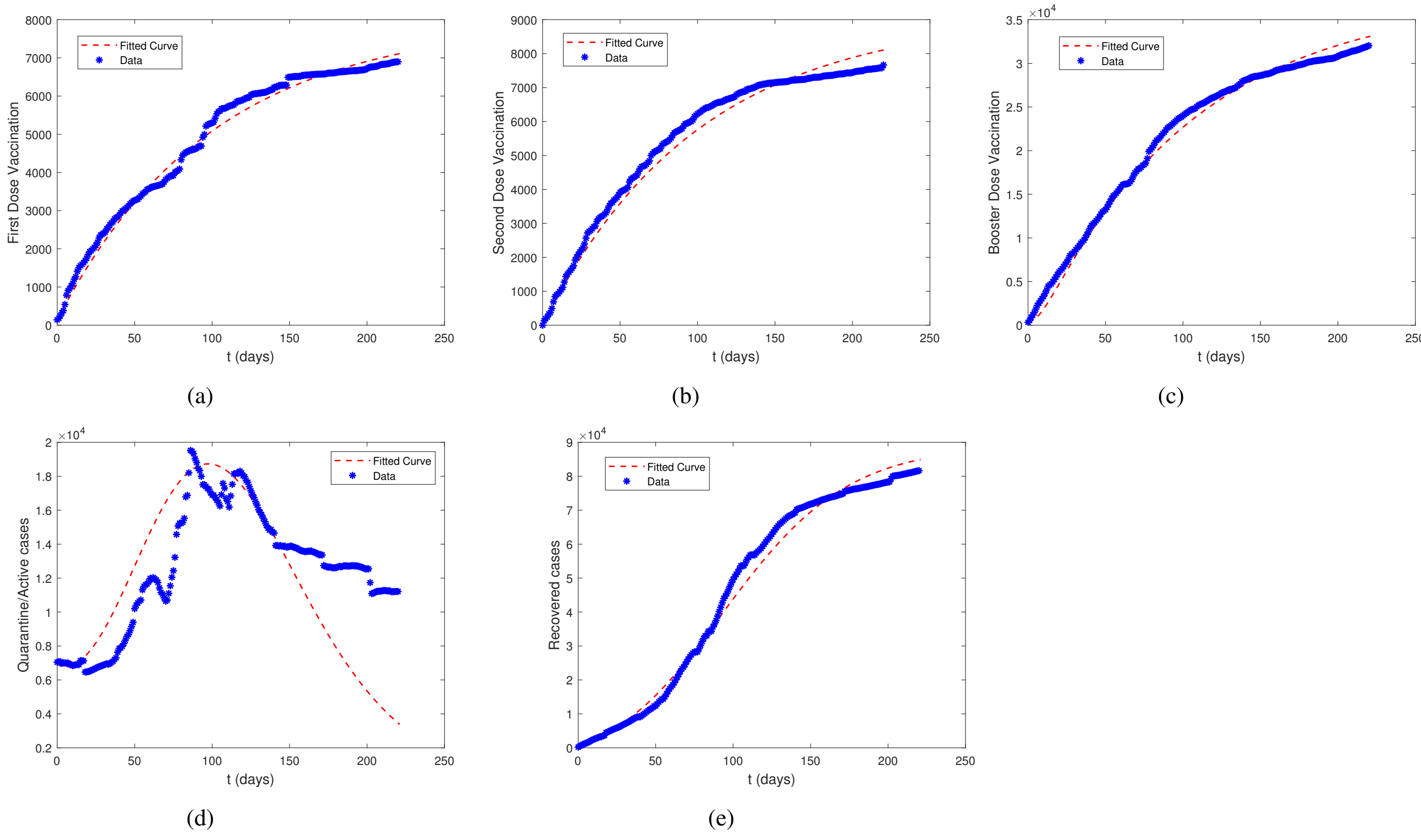

Figure 2: Data fitting simulation graph of COVID-19 data with the SSVEIHQR-D model from February 2023 to September 2023.

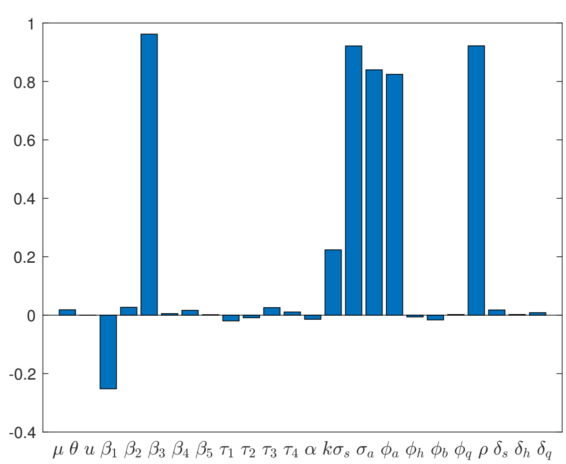

Figure 3: The PRCC of the influence of each parameter on the basic reproduction number (R_0).

Based on our assumptions and the flow of transmission of COVID-19 within groups of people as depicted in Figure 1, we have the following system of differential equations:

$$
\begin{aligned}
\frac{dS_h}{dt} &= \mu\theta N - uS_h - \frac{\beta_1(I_s+I_a)S_h}{N} - \tau_1 S_h - \mu S_h\\
\frac{dS_w}{dt} &= \mu(1-\theta)N + uS_h - \frac{\beta_2(I_s+I_a)S_w}{N} - \tau_2 S_w - \mu S_w\\
\frac{dV_1}{dt} &= \tau_1 S_h + \tau_2 S_w - \frac{\beta_3(I_s+I_a)V_1}{N} - \tau_3 V_1 - \mu V_1\\
\frac{dV_2}{dt} &= \tau_3 V_1 - \frac{\beta_4(I_s+I_a)V_2}{N} - \tau_4 V_2 - \mu V_2\\
\frac{dV_b}{dt} &= \tau_4 V_2 - \frac{\beta_5(I_s+I_a)V_b}{N} - \phi_b V_b - \mu V_b\\
\frac{dE}{dt} &= \frac{\beta_1(I_s+I_a)S_h}{N} + \frac{\beta_2(I_s+I_a)S_w}{N} + \frac{\beta_3(I_s+I_a)V_1}{N}\\
&\quad + \frac{\beta_4(I_s+I_a)V_2}{N} + \frac{\beta_5(I_s+I_a)V_b}{N} - \alpha E - \mu E \qquad (1)\\
\frac{dI_s}{dt} &= \alpha kE - \sigma_s I_s - \rho I_s - \delta_s I_s - \mu I_s\\
\frac{dI_a}{dt} &= \alpha(1-k)E - \sigma_a I_a - \phi_a I_a - \mu I_a\\
\frac{dH}{dt} &= \rho I_s - \phi_h H - \delta_h H - \mu H\\
\frac{dQ}{dt} &= \sigma_s I_s + \sigma_a I_a - \phi_q Q - \delta_q Q - \mu Q\\
\frac{dR}{dt} &= \phi_b V_b + \phi_a I_a + \phi_q Q + \phi_h H - \mu R\\
\frac{dD}{dt} &= \delta_s I_s + \delta_h H + \delta_q Q.
\end{aligned}
$$

With $N = S_h+S_w+V_1+V_2+V_b+E+I_s+I_a+H+Q+R$. Observe that $\frac{dN}{dt} = -(\delta_s I_s + \delta_h H + \delta_q Q) \leq 0 \implies N(t) \leq c$, where c is a positive constant (of integration) integer; showing that $N(t)$ is bounded at all time t. Now, since $N(t)$ is a constant at any given time t, we can transform the system of equation (Equation (1)) into a non-dimensional system. Following the step of Manaqib *et al.* [27] let the proportion of individuals in each compartment at a given time be expressed as in Equation (2)

$$\begin{aligned} S_h &= \frac{1}{N}S_h, \quad S_w = \frac{1}{N}S_w, \quad V_1 = \frac{1}{N}V_1, \quad V_2 = \frac{1}{N}V_2, \\ V_b &= \frac{1}{N}V_b, \quad E = \frac{1}{N}E, \quad I_s = \frac{1}{N}I_s, \quad I_a = \frac{1}{N}I_a, \\ H &= \frac{1}{N}H, \quad Q = \frac{1}{N}Q, \quad R = \frac{1}{N}R. \end{aligned} \tag{2}$$

Therefore, from Equation (1) and (2) we now form the dimensionless equations as given in Equation (3).

$$\begin{aligned} \frac{dS_h}{dt} &= \mu\theta - \beta_1(I_s + I_a)S_h - \xi_1 S_h \\ \frac{dS_w}{dt} &= \mu(1-\theta) + uS_h - \beta_2(I_s + I_a)S_w - \xi_2 S_w \\ \frac{dV_1}{dt} &= \tau_1 S_h + \tau_2 S_w - \beta_3(I_s + I_a)V_1 - \xi_3 V_1 \\ \frac{dV_2}{dt} &= \tau_3 V_1 - \beta_4(I_s + I_a)V_2 - \xi_4 V_2 \\ \frac{dV_b}{dt} &= \tau_4 V_2 - \beta_5(I_s + I_a)V_b - \xi_5 V_b \\ \frac{dE}{dt} &= \beta_1(I_s + I_a)S_h + \beta_2(I_s + I_a)S_w + \beta_3(I_s + I_a)V_1 \\ &\quad + \beta_4(I_s + I_a)V_2 + \beta_5(I_s + I_a)V_b - \xi_6 E \\ \frac{dI_s}{dt} &= \alpha k E - \xi_7 I_s \\ \frac{dI_a}{dt} &= \alpha(1-k)E - \xi_8 I_a \\ \frac{dH}{dt} &= \rho I_s - \xi_9 H \\ \frac{dQ}{dt} &= \sigma_s I_s + \sigma_a I_a - \xi_{10} Q \\ \frac{dR}{dt} &= \phi_b V_b + \phi_a I_a + \phi_q Q + \phi_h H - \mu R \\ \frac{dD}{dt} &= \delta_s I_s + \delta_h H + \delta_q Q, \end{aligned} \tag{3}$$

where $\xi_1 = u+\tau_1+\mu$, $\xi_2 = u+\tau_2+\mu$, $\xi_3 = \tau_3+\mu$, $\xi_4 = \tau_4+\mu$, $\xi_5 = \phi_b+\mu$, $\xi_6 = \alpha+\mu$, $\xi_7 = \sigma_s+\rho+\delta_s+\mu$, $\xi_8 = \sigma_a + \phi_a + \mu$, $\xi_9 = \phi_h + \delta_h + \mu$, $\xi_{10} = \phi_q + \delta_q + \mu$.

3. Model analysis

3.1. The invariant region

Since the model is that of human interaction, we can assume that all initial conditions are non-negative and it suffices to show that the solutions of the system are non-negative and bounded (recall as stated earlier, death compartment shall not be part of the analysis). The following theorem ensures that:

Theorem 3.1. Equation (3) have non-negative solutions and is bounded, i.e $S_h(0) = S_{h_0} \geq 0,\ S_w(0) = S_{w_0} \geq 0,\ V_1(0) = V_{1_0} \geq 0,\ V_2(0) = V_{2_0} \geq 0,\ V_b(0) = V_{b_0} \geq 0,\ E(0) = E0 \geq 0,\ I_s(0) = I_{s_0} \geq 0,\ I_a(0) = I_{a_0} \geq 0,\ H(0) = H_0 \geq 0,\ Q(0) = Q_0 \geq 0,\ R(0) = R_0 \geq 0,\ D(0) = D_0 \geq 0$.

Proof. From the system (Equation (3), we have $\frac{dS_h}{dt} = \mu k - \beta_1(I_s+I_a)S_h - \xi_1 S_h \implies \frac{dS_h}{dt} \geq -\beta_1(I_s+I_a)S_h - \xi_1 S_h, \implies \int \frac{dS_h}{S_h} \geq -\int[\beta_1(I_s+I_a)+\xi_1]dt \implies S_h \geq Ae^{-(\xi_1)t-\beta_1\int(I_s+I_a)dt} \geq 0$, where $A = e^C$; $C\ is\ the\ constant\ of\ integration$. Showing that $S_h(t) \geq 0\ \forall t \geq 0$. Following the same steps we can show that $S_w(t), V_1(t), V_2(t), V_b(t), E(t), I_s(t), I_a(t), H(t), Q(t), R(t), D(t)$ are non-negative. Next, we show that the system (Equation (3)) is bounded. Observe that by summing up the system (Equation (3)) (with the exception of $D(t)$) we have $\frac{dN}{dt} \leq 0 \implies N(t) \leq c$, where c is a positive constant. Thus, $S_w(t) + S_w(t) + V_1(t) + V_2(t) + V_b(t) + E(t) + I_s(t) + I_a(t) + H(t) + Q(t) + R(t) \leq 1$, but since we can't have say a half human, it is okay to assume the constant to be 1, this imply that we can define a positive invariant set of the system (Equation (3)) as given in Equation (4)

$$\begin{aligned} \Omega = \{&(S_h, S_w, V_1, V_2, V_b, E, I_s, I_a, H, Q, R) \in \mathbb{R}^{11}_+ | S_h + S_w \\ &+ S_w + V_1 + V_2 + V_b + E + I_s + I_a + H + Q + R = 1\}. \end{aligned} \tag{4}$$

3.2. Equilibrium points

The equilibrium points are obtained by setting all the compartments in the system (Equation (3)) to zero [33], i.e

$$\begin{aligned} \frac{dS_h}{dt} = \frac{dS_w}{dt} = \frac{dV_1}{dt} = \frac{dV_2}{dt} = \frac{dV_b}{dt} = \frac{dE}{dt} = \frac{dI_s}{dt} \\ = \frac{dI_a}{dt} = \frac{dH}{dt} = \frac{dQ}{dt} = \frac{dR}{dt} = 0. \end{aligned} \tag{5}$$

Now, there are two forms of equilibrium points: the COVID-19-free equilibrium point/ Disease-free equilibrium point (DFE) and the endemic equilibrium point (EEP).

3.2.1. The Disease-free equilibrium point (DFE) of the system

The COVID-19-free equilibrium point (DFE) of the (Equation (3)) is:

$$\begin{aligned} &\varepsilon^0 = (S_h^0, S_w^0, V_1^0, V_2^0, V_b^0, E^0, I_s^0, I_a^0, H^0, Q^0, R^0) = \\ &\Bigg(\frac{\mu\theta}{\xi_1}, \frac{\mu(\theta u - \theta\xi_1 + \xi_1)}{\xi_1\xi_2}, \frac{\mu(\theta u\tau_2 + \theta\tau_1\xi_2 - \theta\tau_2\xi_1 + \tau_2\xi_1)}{\xi_1\xi_2\xi_3}, \\ &\frac{\tau_3\mu(\theta u\tau_2 + \theta\tau_1\xi_2 - \theta\tau_2\xi_1 + \tau_2\xi_1)}{\xi_1\xi_2\xi_3\xi_4}, \\ &\frac{\tau_4\tau_3\mu(\theta u\tau_2 + \theta\tau_1\xi_2 - \theta\tau_2\xi_1 + \tau_2\xi_1)}{\xi_1\xi_2\xi_3\xi_4\xi_5}, \\ &0, 0, 0, 0, 0, \frac{\tau_4\tau_3(\theta u\tau_2 + \theta\tau_1\xi_2 - \theta\tau_2\xi_1 + \tau_2\xi_1)\phi_b}{\xi_1\xi_2\xi_3\xi_4\xi_5}\Bigg). \end{aligned} \tag{6}$$

3.2.2. The Endemic Equilibrium Point (EED) of the system

The endemic equilibrium point (EED) of the system (Equation (3)) is:

$$\varepsilon^* = (S_h^*, S_w^*, V_1^*, V_2^*, V_b^*, E^*, I_s^*, I_a^*, H^*, Q^*, R^*), \tag{7}$$

where

$$S_h^* = \frac{\mu\theta\xi_7\xi_8}{\beta_1(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_1\xi_7\xi_8},$$

$$S_w^* = \frac{\mu(1-\theta)\xi_7\xi_8}{\beta_2(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_2\xi_7\xi_8} + \frac{u\xi_7\xi_8}{\beta_2(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_2\xi_7\xi_8} \Bigg(\frac{\mu\theta\xi_7\xi_8}{\beta_1(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_1\xi_7\xi_8}\Bigg),$$

$$V_1^* = \frac{\tau_1\xi_7\xi_8}{\beta_3(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_3\xi_7\xi_8} \Bigg(\frac{\mu\theta\xi_7\xi_8}{\beta_1(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_1\xi_7\xi_8}\Bigg) + \frac{\tau_2\xi_7\xi_8}{\beta_3(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_3\xi_7\xi_8} \Bigg(\frac{\mu(1-\theta)\xi_7\xi_8}{\beta_2(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_2\xi_7\xi_8} + \frac{u\xi_7\xi_8}{\beta_2(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_2\xi_7\xi_8} \Bigg(\frac{\mu\theta\xi_7\xi_8}{\beta_1(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_1\xi_7\xi_8}\Bigg)\Bigg)\Bigg),$$

$$V_2^* = \frac{\tau_3\xi_7\xi_8}{\beta_4(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_4\xi_7\xi_8} \Bigg(\frac{\tau_1\xi_7\xi_8}{\beta_3(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_3\xi_7\xi_8} \Bigg(\frac{\mu\theta\xi_7\xi_8}{\beta_1(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_1\xi_7\xi_8}\Bigg) + \frac{\tau_2\xi_7\xi_8}{\beta_3(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_3\xi_7\xi_8} \Bigg(\frac{\mu(1-\theta)\xi_7\xi_8}{\beta_2(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_2\xi_7\xi_8} + \frac{u\xi_7\xi_8}{\beta_2(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_2\xi_7\xi_8} \Bigg(\frac{\mu\theta\xi_7\xi_8}{\beta_1(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_1\xi_7\xi_8}\Bigg)\Bigg)\Bigg)\Bigg),$$

$$V_b^* = \frac{\tau_4\xi_7\xi_8}{\beta_5(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_5\xi_7\xi_8} \Bigg(\frac{\tau_3\xi_7\xi_8}{\beta_4(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_4\xi_7\xi_8} \Bigg(\frac{\tau_1\xi_7\xi_8}{\beta_3(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_3\xi_7\xi_8} \Bigg(\frac{\mu\theta\xi_7\xi_8}{\beta_1(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_1\xi_7\xi_8}\Bigg) + \frac{\tau_2\xi_7\xi_8}{\beta_3(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_3\xi_7\xi_8} \Bigg(\frac{\mu(1-\theta)\xi_7\xi_8}{\beta_2(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_2\xi_7\xi_8} + \frac{u\xi_7\xi_8}{\beta_2(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_2\xi_7\xi_8} \Bigg(\frac{\mu\theta\xi_7\xi_8}{\beta_1(\alpha k\xi_8 + \alpha(1-k)\xi_7)E^* + \xi_1\xi_7\xi_8}\Bigg)\Bigg)\Bigg)\Bigg)\Bigg),$$

$$I_s = \frac{\alpha k}{\xi_7}E^*,$$

$$I_a = \frac{\alpha(1-k)}{\xi_8}E^*,$$

$$H = \frac{\alpha\rho k}{\xi_7\xi_9}E^*,$$

$$Q = \frac{\alpha(\sigma_s k\xi_8 + \sigma_a(1-k)\xi_7)\rho k}{\xi_7\xi_8\xi_9}E^*,$$

$$R = \frac{\phi_b V_b^* + \phi_a I_a^* + \phi_h H^* + \phi_q Q^*}{\mu}E^*.$$

E^* is the solution of a third degree polynomial $a_0z^3 + a_1z^2 + a_2z^1 + a_3$, see Appendix A for the expression of a_0, a_1, a_2 and a_3.

Observe from Table 2, that all denominators are well defined since they're all positive parameters.

3.2.3. The basic reproduction number

The basic reproduction number R_0 of the proposed $SSVEIHQR-D$ COVID-19 model is given by:

$$R_0 = \frac{A\alpha\,(k\xi_8 + \xi_7 - k\xi_7)}{\xi_6\xi_7\xi_8}, \tag{8}$$

where $A = \beta_1 S_h^0 + \beta_2 S_w^0 + \beta_3 V_1^0 + \beta_4 V_2^0 + \beta_5 V_b^0$; $S_h^0 = \frac{\mu\theta}{\xi_1}$; $S_w^0 = \frac{\mu(\theta u - \theta\xi_1 + \xi_1)}{\xi_1\xi_2}$; $V_1^0 = \frac{\mu(\theta u\tau_2 + \theta\tau_1\xi_2 - \theta\tau_2\xi_1 + \tau_2\xi_1)}{\xi_1\xi_2\xi_3}$; $V_2^0 = \frac{\tau_3\mu(\theta u\tau_2 + \theta\tau_1\xi_2 - \theta\tau_2\xi_1 + \tau_2\xi_1)}{\xi_1\xi_2\xi_3\xi_4}$; $V_b^0 = \frac{\tau_4\tau_3\mu(\theta u\tau_2 + \theta\tau_1\xi_2 - \theta\tau_2\xi_1 + \tau_2\xi_1)}{\xi_1\xi_2\xi_3\xi_4\xi_5}$

We use the *Next Generation Matrix* as seen in Van den Driessche and Watmough [34] to obtain our reproduction number using the system of equations (Equation (3)) which comprise of two parts: F and V^{-1}, where, we define the matrix $\mathscr{F}$ as the new infections, while the component of matrix $\mathscr{V}$ are transfers of infections from one compartment to another. ε^0 is the COVID-19-free equilibrium state. R_0 is the dominant eigenvalue of the matrix $G = FV^{-1}$.

Now,

$$\mathscr{F} = \begin{pmatrix} (I_s + I_a)(\beta_1 S_w + \beta_2 S_h + \beta_3 V_1 + \beta_4 V_2 + \beta_5 V_b) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{9}$$

$$\mathscr{V} = \begin{pmatrix} \xi_6 E \\ -\alpha_1 k E + \xi_7 I_s \\ -\alpha(1-k)E + \xi_8 I_a \\ -\rho I_s + \xi_9 H \\ -\sigma_s I_s - \sigma_a I_a + \xi_{10} Q \end{pmatrix}. \tag{10}$$

We now obtain the F and V by computing the Jacobian $\mathscr{F}$ and $\mathscr{V}$. Thus,

$$F = \begin{pmatrix} 0 & A & A & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{11}$$

where $S_h^0, S_w^0, V_1^0, V_2^0, V_b^0$,

$$V = \begin{pmatrix} \xi_6 & 0 & 0 & 0 & 0 \\ -\alpha k & \xi_7 & 0 & 0 & 0 \\ -\alpha(1-k) & 0 & \xi_8 & 0 & 0 \\ 0 & -\rho & 0 & \xi_9 & 0 \\ 0 & -\sigma_s & -\sigma_a & 0 & \xi_{10} \end{pmatrix}, \tag{12}$$

$$V^{-1} = \begin{pmatrix} \frac{1}{\xi_6} & 0 & 0 & 0 & 0 \\ \frac{\alpha k}{\xi_6\xi_7} & \frac{1}{\xi_7} & 0 & 0 & 0 \\ -\frac{\alpha(-1+k)}{\xi_6\xi_8} & 0 & \frac{1}{\xi_8} & 0 & 0 \\ \frac{\rho\alpha k}{\xi_7\xi_6\xi_9} & \frac{\rho}{\xi_7\xi_9} & 0 & \frac{1}{\xi_9} & 0 \\ -\frac{\alpha(\sigma_a\xi_7 k - \sigma_s k\xi_8 - \sigma_a\xi_7)}{\xi_7\xi_6\xi_8\xi_{10}} & \frac{\sigma_s}{\xi_7\xi_{10}} & \frac{\sigma_a}{\xi_8\xi_{10}} & 0 & \frac{1}{\xi_{10}} \end{pmatrix}. \tag{13}$$

Now,

$$FV^{-1} = \begin{pmatrix} \frac{A\alpha k}{\xi_6\xi_7} - \frac{A\alpha(-1+k)}{\xi_6\xi_8} & \frac{A}{\xi_7} & \frac{A}{\xi_8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

Hence, the basic reproduction number R_0 which is the spectra radius of FV^{-1} is

$$R_0 = \rho(FV^{-1}) = \frac{\alpha A\,(k\xi_8 + \xi_7 - k\xi_7)}{\xi_7\xi_6\xi_8}.$$

4. Stability analysis of the DFE and EEP

4.1. Local stability of the COVID-19-Free equilibrium

Theorem 4.1. The COVID-19-free equilibrium point of model (Equation (3)) is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

proof. We begin by linearizing the system of differential equations (Equation (3)) by computing it Jacobian matrix, $J_{(\xi)}$ at the COVID-19-free equilibrium point.

$$J_{(DFE)} = \begin{bmatrix} -\xi_1 & 0 & 0 & 0 & 0 & 0 & -b_1 & -b_1 & 0 & 0 & 0 \\ u & -\xi_2 & 0 & 0 & 0 & 0 & -b_2 & -b_2 & 0 & 0 & 0 \\ \tau_1 & \tau_2 & -\xi_3 & 0 & 0 & 0 & -b_3 & -b_3 & 0 & 0 & 0 \\ 0 & 0 & \tau_3 & -\xi_4 & 0 & 0 & -b_4 & -b_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_4 & -\xi_5 & 0 & -b_5 & -b_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\xi_6 & A & A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha k & -\xi_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha\,(k-1) & 0 & -\xi_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho & 0 & -\xi_9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_s & \sigma_a & 0 & -\xi_{10} & 0 \\ 0 & 0 & 0 & 0 & \phi_b & 0 & 0 & \phi_a & \phi_h & \phi_q & -\mu \end{bmatrix} \tag{15}$$

Using the relation $|J(\xi_0) - \lambda I| = 0$, where I is an identity matrix; $b_1 = \beta_1 S_h^0$, $b_2 = \beta_2 S_w^0$, $b_3 = \beta_3 V_1^0$, $b_4 = \beta_4 V_2^0$, $b_5 = \beta_5 V_b^0$, and $A = \beta_1 S_h^0 + \beta_2 S_w^0 + \beta_3 V_1^0 + \beta_4 V_2^0 + \beta_5 V_b^0$ the characteristic polynomial of the matrix (Equation (13)) is

$$\begin{aligned} |J(\xi_0) - \lambda I| = (\lambda^3 - (-\xi_8 - \xi_6 - \xi_7)\lambda^2 - (\alpha A - \xi_6\xi_7 \\ - \xi_8\xi_6 - \xi_8\xi_7)\lambda + A\alpha k\xi_7 - A\alpha k\xi_8 - A\alpha\xi_7 \\ + \xi_6\xi_7\xi_8)(\lambda + \xi_1)(\lambda + \xi_2)(\lambda + \xi_3)(\lambda + \xi_4) \\ (\lambda + \xi_5)(\lambda + \xi_9)(\lambda + \xi_{10})(\lambda + \mu) = 0. \end{aligned} \tag{16}$$

Showing that eight of the roots are less that zero. To show that the system is stable, it suffices to show that the remaining roots are less than zero. Now, let $f(\lambda) = \lambda^3 - (-\xi_8 - \xi_6 - \xi_7)\lambda^2 - (\alpha A - \xi_6\xi_7 - \xi_8\xi_6 - \xi_8\xi_7)\lambda + A\alpha k\xi_7 - A\alpha k\xi_8 - A\alpha\xi_7 + \xi_6\xi_7\xi_8 = \lambda^3 + (\xi_8 + \xi_6 + \xi_7)\lambda^2 + (\xi_6\xi_7 + \xi_8\xi_6 + \xi_8\xi_7 - \alpha A)(R_0 - 1)$. By the Routh-Hurwitz criteria on local stability [35], the COVID-19-free equilibrium is locally asyptotically stable if $(\xi_6\xi_7 + \xi_8\xi_6 + \xi_8\xi_7 - \alpha A) > 0$ and $R_0 < 1$. This completes the proof.

4.1.1. Local stability of the Endemic Equilibrium Point (EEP)

Theorem 4.2. The COVID-19 endemic equilibrium $\varepsilon^* = (S_h^*, S_w^*, V_1^*, V_2^*, V_b^*, E^*, I_s^*, I_a^*, H^*, Q^*, R^*)$ (Equation (7)) is locally asymptotically stable if $R_0 > 1$.

Proof. The COVID-19 endemic point ε^* the Jacobian matrix is given as

$$J_{(EEP)} = \begin{bmatrix} -l_1 & 0 & 0 & 0 & 0 & 0 & -\beta_1 S_h & -\beta_1 S_h & 0 & 0 & 0 \\ u & -l_2 & 0 & 0 & 0 & 0 & -\beta_2 S_w & -\beta_2 S_w & 0 & 0 & 0 \\ \tau_1 & \tau_2 & -l_3 & 0 & 0 & 0 & -\beta_3 V_1 & -\beta_3 V_1 & 0 & 0 & 0 \\ 0 & 0 & \tau_3 & -l_4 & 0 & 0 & -\beta_4 V_2 & -\beta_4 V_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_4 & -l_5 & 0 & -\beta_5 V_b & -\beta_5 V_b & 0 & 0 & 0 \\ -I\beta_1 & -I\beta_2 & -I\beta_3 & -I\beta_4 & -I\beta_5 & -l_6 & A & A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_1 & -l_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_2 & 0 & -l_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho & 0 & -l_9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_1 & \sigma_2 & 0 & -l_{10} & 0 \\ 0 & 0 & 0 & 0 & \phi_b & 0 & 0 & \phi_a & \phi_h & \phi_q & -\mu \end{bmatrix} \quad (17)$$

where $l_1 = \beta_1(I_s + I_a) + \xi_1$; $l_2 = \beta_1(I_s + I_a) + \xi_2$; $l_3 = \beta_1(I_s + I_a) + \xi_3$; $l_4 = \beta_1(I_s + I_a) + \xi_4$; $l_5 = \beta_1(I_s + I_a) + \xi_5$; $l_6 = \xi_6$; $l_7 = \xi_7$; $l_8 = \xi_8$; $l_9 = \xi_9$; $l_{10} = \xi_{10}$; $I = I_s + I_a$. Now, using the relation $Det(J_{(EEP)} - \lambda I) = 0$, where I is an identity matrix, we obtain the characteristic polynomial: $Det(J_{(EEP)} - \lambda I)(f(\lambda)(\lambda + \mu)(\lambda + l_{10})(\lambda + l_9)) = 0$; $f(\lambda) = b_0\lambda^8 + b_1\lambda^7 + b_2\lambda^6 + b_3\lambda^5 + b_4\lambda^b + b_5\lambda^3 + b_6\lambda^2 + b_7\lambda^1 + b_8$, see Appendix B for the expression of b_i for $i = 1, 2, \cdots 8$.

The Lienard-Chipart (LC) criteria [36] for a system having a characteristic polynomial of degree $n = 8$ to be locally asymptotically stable are: the coefficient a_1, a_3, a_5, a_7 and a_8 must be positive, and the Hurwitz determinant $|H_3|, |H_5|, |H_7|$, must be positive. After using an algebraic package to obtain the characteristic polynomial, we were able to obtain the expression of the coefficients. We can observe from the coefficients obtained that only a_1 is positive, while the rest are not. We therefore conclude that the endemic point of the system is not locally stable since all parameters are positive.

4.2. Global stability analysis of DFE and EEP

4.2.1. Global stability of the DFE

We shall show the condition for the global stability of the system when there are no COVID-19 infectious persons in the population.

Theorem 4.3. The COVID-19 free equilibrium point of model (Equation (1)) is globally asymptotically stable for $R_0 < min\left(\frac{\xi_6\xi_7}{\xi_8}, \frac{\xi_6\xi_8}{\xi_7}\right)$

Proof. Consider the following Lyapunov function

$$U(S_h, S_w, V_1, V_2, V_b, E, I_s, I_a, H, Q, R) = w_1 E + w_2 I_s + w_3 I_a. \quad (18)$$

Defined in $\Omega = \{(S_h, S_w, V_1, V_2, V_b, E, I_s, I_a, H, Q, R) \in \mathbb{R}_+^{11}) | S_h^0 > 0, S_w^0 > 0, V_1^0 > 0, V_2^0 > 0, V_b^0 > 0, E^0 > 0, I_s^0 > 0, I_a^0 > 0, H^0 > 0, Q^0 > 0, R^0 > 0\}$, such that $R_0 < min\left(\frac{\xi_6\xi_7}{\xi_8}, \frac{\xi_6\xi_8}{\xi_7}\right)$, where w_1, w_2 and w_3 are positive constants to be determined. Then differentiating $U(S_h, S_w, V_1, V_2, V_b, E, I_s, I_a, H, Q, R)$ along the trajectory of the solution of the model Equation (3) yields

$$\begin{aligned} U' &= w_1(\beta_1(I_s + I_a)S_h + \beta_2(I_s + I_a)S_w + \beta_3(I_s + I_a)V_1 \\ &+ \beta_4(I_s + I_a)V_2 + \beta_5(I_s + I_a)V_b - \xi_6 E) + w_2(\alpha k E - \xi_7 I_s) \\ &+ w_3(\alpha(1-k)E - \xi_8 I_a) \\ &= w_1(I_s + I_a)(\beta_1 S_h + \beta_2 S_w + \beta_3 V_1 + \beta_4 V_2 + \beta_5 V_b) \\ &+ (w_3\alpha - w_1\xi_6 + w_2\alpha k - w_3\alpha k)E - w_2\xi_7 I_s - w_3\xi_8 I_a. \end{aligned} \quad (19)$$

Now, taking $w_1 = \frac{\alpha(k\xi_8 + (1-k)\xi_7)}{\xi_6\xi_7\xi_8}$, $w_2 = \frac{\xi_6}{\xi_8}$ and $w_3 = \frac{\xi_6}{\xi_7}$ implies

$$\begin{aligned} U' &= \frac{\alpha(k\xi_8 + (1-k)\xi_7)(\beta_1 S_h + \beta_2 S_w + \beta_3 V_1 + \beta_4 V_2 + \beta_5 V_b)}{\xi_6\xi_7\xi_8} I_s \\ &- \frac{\xi_6\xi_7}{\xi_8} I_s \\ &+ \frac{\alpha(k\xi_8 + (1-k)\xi_7)(\beta_1 S_h + \beta_2 S_w + \beta_3 V_1 + \beta_4 V_2 + \beta_5 V_b)}{\xi_6\xi_7\xi_8} I_a \\ &- \frac{\xi_6\xi_8}{\xi_7} I_a = \left(R_0 - \frac{\xi_6\xi_7}{\xi_8}\right) I_s + \left(R_0 - \frac{\xi_6\xi_8}{\xi_7}\right) I_a \leq 0 \\ &\quad for\ R_0 < min\left(\frac{\xi_6\xi_7}{\xi_8}, \frac{\xi_6\xi_8}{\xi_7}\right). \end{aligned} \quad (20)$$

Showing that this is Lyapunov function for $R_0 < min\left(\frac{\xi_6\xi_7}{\xi_8}, \frac{\xi_6\xi_8}{\xi_7}\right)$. Also, $U' = 0$ if only $I_s = I_a = 0$. Implying that the trajectory of the solution for which $U' = 0$ is the point ε^0. Thus, the largest compact invariant set $\{(S_h, S_w, V_1, V_2, V_b, E, I_s, I_a, H, Q, R) \in \Omega : U' = 0\}$ is the singleton set $\{\varepsilon^0\}$. Hence, by the (cite Lassle) invariant principle, the COVID-19 free equilibrium point (DFE) is globally asymptotically stable in Ω if $R_0 < min\left(\frac{\xi_6\xi_7}{\xi_8}, \frac{\xi_6\xi_8}{\xi_7}\right)$. This completes the proof.

4.2.2. Global stability of the EEP

Theorem 4.4. The COVID-19 endemic point given by $\varepsilon^* = (S_h^*, S_w^*, V_1^*, V_2^*, V_b^*, E^*, I_s^*, I_a^*, H^*, Q^*, R^*)$ is asymptotically stable in Ω if $R_0 < 0$.

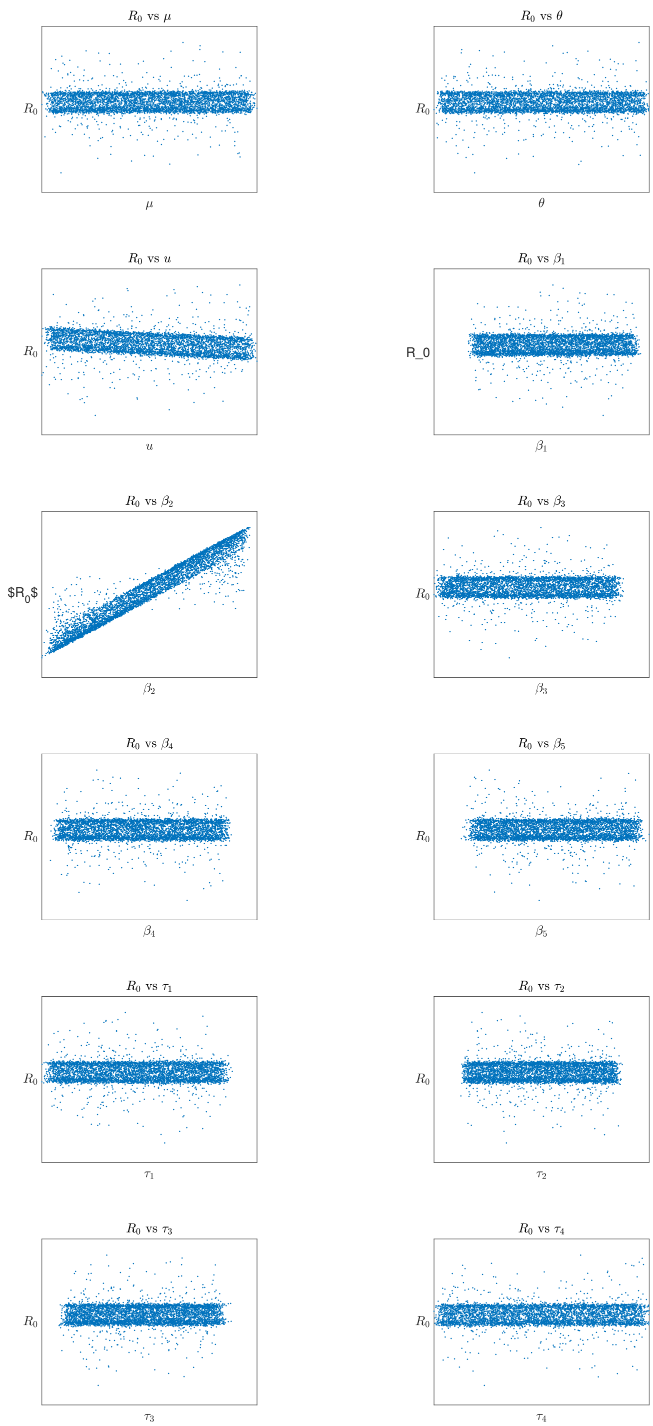

Figure 4: Scatter plots showing the relationship of the model parameters on the basic reproduction number R_0

Proof. Consider a Lyapunov function whose domain Ω, define by:

$$\begin{aligned}
&V(S_h, S_w, V_1, V_2, V_b, E, I_s, I_a, H, Q, R)\\
&= \left(S_h - S_h^* - S_h^* log\frac{S_h}{S_h^*}\right) + \left(S_w - S_w^* - S_w^* log\frac{S_w}{S_w^*}\right)\\
&+ \left(V_1 - V_1^* - V_1^* log\frac{V_1}{V_1^*}\right) + \left(V_2 - V_2^* - V_2^* log\frac{V_2}{V_2^*}\right)\\
&+ \left(V_b - V_b^* - V_b^* log\frac{V_b}{V_b^*}\right) + \left(E - E^* - E^* log\frac{E}{E^*}\right)\\
&+ \left(I_s - I_s^* - I_s^* log\frac{I_s}{I_s^*}\right) + \left(I_a - I_a^* - I_a^* log\frac{I_a}{I_a^*}\right)\\
&+ \left(H - H^* - H^* log\frac{H}{H^*}\right) + \left(Q - Q^* - Q^* log\frac{Q}{Q^*}\right)\\
&+ \left(R - R^* - R^* log\frac{R}{R^*}\right).
\end{aligned} \tag{21}$$

Differentiating $V(S_h, S_w, V_1, V_2, V_b, E, I_s, I_a, H, Q, R)$ along the trajectory of the solution of the model (Equation (3)), we obtain:

$$\begin{aligned}
\frac{dV}{dt} &= \left(\frac{S_h - S_h^*}{S_h}\right)\frac{dS_h}{dt} + \left(\frac{S_w - S_w^*}{S_w}\right)\frac{dS_w}{dt} + \left(\frac{V_1 - V_1^*}{V_1}\right)\frac{dV_1}{dt}\\
&+ \left(\frac{V_2 - V_2^*}{V_2}\right)\frac{dV_2}{dt} + \left(\frac{V_b - V_b^*}{V_b}\right)\frac{dV_b}{dt}\\
&+ \left(\frac{E - E^*}{E}\right)\frac{dE}{dt} + \left(\frac{I_s - I_s^*}{I_s}\right)\frac{dI_s}{dt}\\
&+ \left(\frac{I_a - I_a^*}{I_a}\right)\frac{dI_a}{dt} + \left(\frac{H - H^*}{H}\right)\frac{dH}{dt}\\
&+ \left(\frac{Q - Q^*}{Q}\right)\frac{dQ}{dt} + \left(\frac{R - R^*}{R}\right)\frac{dR}{dt}\\
&= \left(\frac{S_h - S_h^*}{S_h}\right)(\mu\theta - \beta_1(I_s + I_a)S_h - \xi_1 S_h)\\
&+ \left(\frac{S_w - S_w^*}{S_w}\right)(\mu(1-\theta) + uS_h - \beta_2(I_s + I_a)S_w - \xi_2 S_w)\\
&+ \left(\frac{V_1 - V_1^*}{V_1}\right)(\tau_1 S_h + \tau_2 S_w - \beta_3(I_s + I_a)V_1 - \xi_3 V_1)\\
&+ \left(\frac{V_2 - V_2^*}{V_2}\right)(\tau_3 V_1 - \beta_4(I_s + I_a)V_2 - \xi_4 V_2)\\
&+ \left(\frac{V_b - V_b^*}{V_b}\right)(\tau_4 V_2 - \beta_5(I_s + I_a)V_b - \xi_5 V_b)\\
&+ \left(\frac{E - E^*}{E}\right)(\beta_1(I_s + I_a)S_h + \beta_2(I_s + I_a)S_w\\
&+ \beta_3(I_s + I_a)V_1 + \beta_4(I_s + I_a)V_2 + \beta_5(I_s + I_a)V_b - \xi_6 E)\\
&+ \left(\frac{I_s - I_s^*}{I_s}\right)(\alpha k E - \xi_7 I_s)\\
&+ \left(\frac{I_a - I_a^*}{I_a}\right)(\alpha(1-k)E - \xi_8 I_a)\\
&+ \left(\frac{H - H^*}{H}\right)(\rho I_s - \xi_9 H)
\end{aligned}$$

$$+\left(\frac{Q-Q^*}{Q}\right)(\sigma_s I_s+\sigma_a I_a-\xi_{10}Q)$$
$$+\left(\frac{R-R^*}{R}\right)\Big(\phi_b V_b+\phi_a I_a+\phi_h H+\phi_q Q-\mu R\Big). \tag{22}$$

$$\frac{dV}{dt}=\left(\frac{S_h-S_h^*}{S_h}\right)\Big(\mu\theta-\beta_1(I_s+I_a)(S_h-S_h^*)-(S_h-S_h^*)\xi_1$$
$$-(\beta_1(I_s+I_a)S_h^*+\xi_1S_h^*)\Big)+\left(\frac{S_w-S_w^*}{S_w}\right)(\mu(1-\theta)+uS_h$$
$$-\beta_2(I_s+I_a)(S_w-S_w^*)-(S_w-S_w^*)\xi_2$$
$$-(\beta_2(I_s+I_a)S_w^*+\xi_2S_w^*))+\left(\frac{V_1-V_1^*}{V_1}\right)$$
$$(\tau_1S_h+\tau_2S_w-\beta_3(I_s+I_a)(V_1-V_1^*)-(V_1-V_1^*)\xi_3$$
$$-(\beta_3(I_s+I_a)V_1^*+\xi_3V_1^*))+\left(\frac{V_2-V_2^*}{V_2}\right)$$
$$(\tau_3V_1-\beta_4(I_s+I_a)(V_2-V_2^*)-(V_2-V_2^*)\xi_4$$
$$-(\beta_4(I_s+I_a)V_2^*+\xi_4V_2^*))+\left(\frac{V_b-V_b^*}{V_b}\right)$$
$$\Big(\tau_4V_2-\beta_5(I_s+I_a)(V_b-V_b^*)-(V_b-V_b^*)\xi_5$$
$$-(\beta_5(I_s+I_a)V_b^*+\xi_5V_b^*)\Big)+\left(\frac{E-E^*}{E}\right)$$
$$(\beta_1(I_s+I_a)S_h+\beta_2(I_s+I_a)S_w+\beta_3(I_s+I_a)V_1+\beta_4(I_s+I_a)V_2$$
$$+\beta_5(I_s+I_a)V_b-(E-E*)\xi_6-\xi_6E)+\left(\frac{I_s-I_s^*}{I_s}\right)$$
$$(\alpha kE-(I_s-I_s^*)\xi_7-\xi_7I_s^*)+\left(\frac{I_a-I_a^*}{I_a}\right)$$
$$(\alpha(1-k)E-(I_a-I_a^*)\xi_8-\xi_8I_a^*)+\left(\frac{H-H^*}{H}\right)$$
$$(\rho I_s-(H-H^*)\xi_9-\xi_9H^*)+\left(\frac{Q-Q^*}{Q}\right)$$
$$(\sigma_sI_s+\sigma_aI_a-(Q-Q^*)\xi_{10}-\xi_{10}Q^*)$$
$$+\left(\frac{R-R^*}{R}\right)\Big(\phi_bV_b+\phi_aI_a+\phi_hH+\phi_qQ(R-R^*)-\mu R^*\Big). \tag{23}$$

Observe that we can write $\frac{dV}{dt}$ as $\ell_1-\ell_2$, where

$$\ell_1=\mu\theta+\beta_1(I_s+I_a)\frac{S_h^{*2}}{S_h}+\xi_1\frac{S_h^{*2}}{S_h}\mu(1-\theta)$$
$$+uS_h+\beta_2(I_s+I_a)\frac{S_w^{*2}}{S_w}+\xi_2\frac{S_h^{*2}}{S_h}\tau_1S_h+\tau_2S_w+\beta_3(I_s+I_a)\frac{V_1^{*2}}{V_1}$$
$$+\xi_3\frac{V_1^{*2}}{V_1}\tau_3V_1+\beta_4(I_s+I_a)\frac{V_2^{*2}}{V_2}+\xi_4\frac{V_2^{*2}}{V_2}\tau_4V_2$$
$$+\beta_5(I_s+I_a)\frac{V_b^{*2}}{V_v}+\xi_5\frac{V_b^{*2}}{V_b}\beta_1(I_s+I_a)S_h+\beta_2(I_s+I_a)S_w$$
$$+\beta_3(I_s+I_a)V_1+\beta_4(I_s+I_a)V_2+\beta_5(I_s+I_a)V_b+\xi_6E^*$$
$$+\alpha kE+\xi_7\frac{I_s^{*2}}{I_s}\alpha(1-k)E+\xi_8\frac{I_a^{*2}}{I_a}\rho I_s+\xi_9\frac{H^{*2}}{H}\sigma_sI_s$$
$$+\sigma_aI_a+\xi_{10}\frac{Q^{*2}}{Q}\phi_bV_b+\phi_aI_a+\phi_qQ+\phi_hH+\mu\frac{R^{*2}}{R}, \tag{24}$$

$$\ell_2=\frac{(S_h-S_h^*)^2}{S_h}(\beta_1(I_s+I_a)+\xi_1)+\left(\frac{S_h^*}{S_h}\mu\theta+\beta_1(I_s+I_a)S_h^*+\xi_1S_h^*\right)$$
$$+\frac{(S_w-S_w^*)^2}{S_w}(\beta_2(I_s+I_a)+\xi_2)$$
$$+\left(\frac{S_w^*}{S_w}(\mu(1-\theta)+\mu S_h)+\beta_2(I_s+I_a)S_w^*+\xi_2S_w^*\right)$$
$$+\frac{(V_1-V_1^*)^2}{V_1}(\beta_3(I_s+I_a)+\xi_3)$$
$$+\left(\frac{V_1^*}{V_1}\mu(\tau_1S_h+\tau_2S_w)+\beta_3(I_s+I_a)V_1^*+\xi_3V_1^*\right)$$
$$+\frac{(V_2-V_2^*)^2}{V_2}(\beta_4(I_s+I_a)+\xi_4)$$
$$+\left(\frac{V_2^*}{V_2}\tau_3V_1+\beta_4(I_s+I_a)V_2^*+\xi_4V_2^*\right)$$
$$+\frac{(V_b-V_b^*)^2}{V_b}(\beta_5(I_s+I_a)+\xi_5)$$
$$+\left(\frac{V_b^*}{V_b}\tau_4V_2+\beta_5(I_s+I_a)V_b^*+\xi_5V_b^*\right)$$
$$+\frac{(E-E^*)^2}{E}\xi_6+(\beta_1(I_s+I_a)S_h+\beta_2(I_s+I_a)S_w+\beta_3(I_s+I_a)V_1$$
$$+\beta_4(I_s+I_a)V_2+\beta_5(I_s+I_a)V_b)\frac{E^*}{E}+\xi_6\frac{E^{*2}}{E}$$
$$+\frac{(I_s-I_s)^2}{I_s}\xi_7+\alpha kE\frac{I_s^*}{I_s}+\xi_7I_s^*$$
$$\frac{(I_a-I_a)^2}{I_a}\xi_8+\alpha(1-k)E\frac{I_a^*}{I_a}+\xi_8I_a^*$$
$$+\frac{(H-H)^2}{H}\xi_9+\rho I_s\frac{H^*}{H}+\xi_9H^*$$
$$+\frac{(Q-Q)^2}{Q}\xi_{10}+(\sigma_sI_s+\sigma_aI_a)\frac{Q^*}{Q}+\xi_{10}Q^*$$
$$+\frac{(R-R)^2}{R}\mu+(\phi_bV_b+\phi_aI_a+\phi_qQ+\phi_hH)\frac{R^*}{R}+\mu R^*. \tag{25}$$

Showing that $\frac{dV}{dt}\le 0$ for $\ell_1\le\ell_2$ and $\frac{dV}{dt}=0$ if and only if $\ell_1=\ell_2$ (this is true since all parameters in the model are non-negative). Observe immediately that $\frac{dV}{dt}=0$ if and only if $S_h=S_h^*$, $S_w=S_w^*$, $V_1=V_1^*$, $V_2=V_2^*$, $V_b=V_b^*$, $E=E^*$, $I_s=I_s^*$, $I_a=I_a^*$, $H=H^*$, $Q=Q^*$, $R=R^*$. Hence, by the LaSalle's invariant principle the result follows.

5. Parameter estimation

Parameter estimation is one of the objectives of this research. Parameter estimation is the process of computing or determination of the best values of a model's parameter using sample data through numerical simulations. From the onset of the COVID-19 outbreak till date efforts have been made by various researchers to estimate the value of parameters that influ-

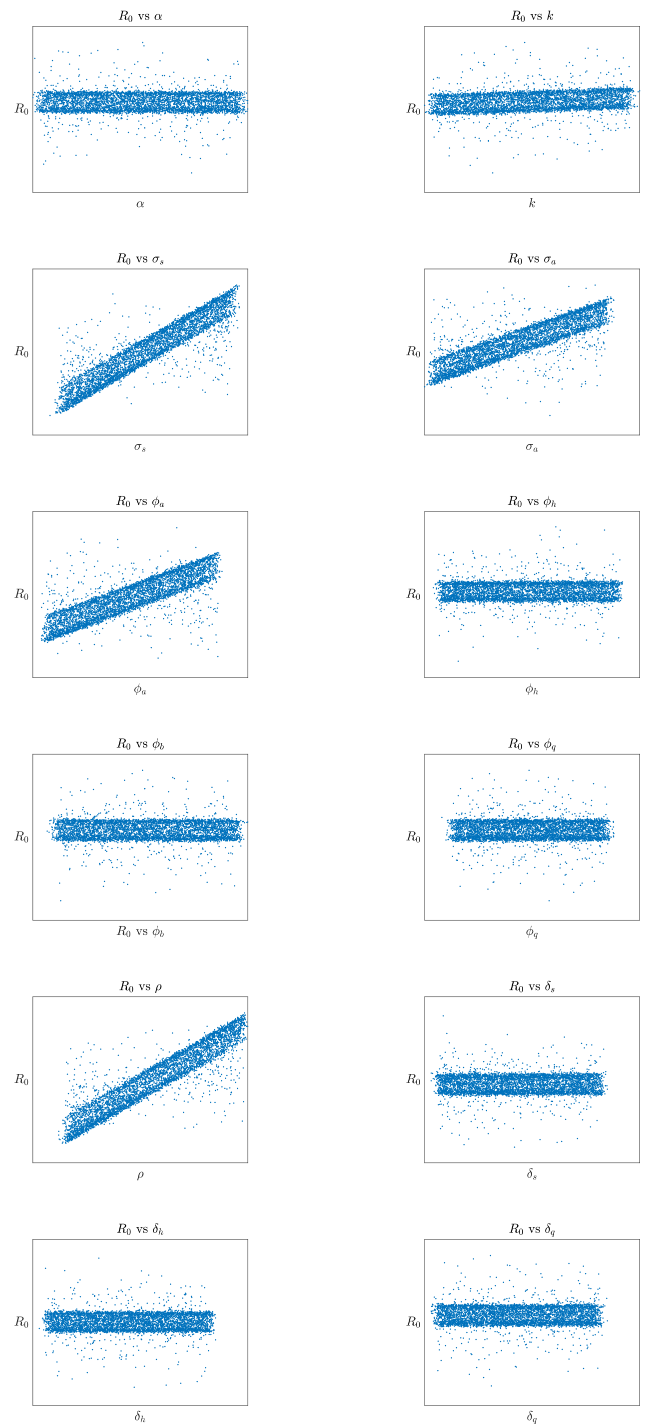

Figure 5: Scatter plots showing the relationship of the model parameters on the basic reproduction number R_0

ence the spread of the virus over time, and the basic reproduction number. The estimation of such parameters are very important because they give information on the dynamics of the spread of the disease. At the early stage of COVID-19 outbreak, in February 2020, when there was nothing like COVID vaccine, using the daily reported cases of COVID-19 in Malaysia, Gill *et al.* [37] estimated the basic reproduction to be $R_0 = 1.68$. In 2021, Abidemi *et al.* [38] by using the daily reported cases of COVID-19 in Malaysia and applying the least square curve fitting techniques estimated the basic reproduction number to be $R_0 = 2.287$ and contact rate to be 2.6576×10^{-5}. The work further shows that head immunity with respect to the basic reproduction number is $\hat{p} = 1 - \frac{1}{R_0} = 0.56$, while in 2021, Ganasegeran *et al.* [39] estimated a time-dependent reproduction number (T_t) at three different intervention periods after the Movement Control Order: fourteen days prior to MCO the R_0 was 3.91 (2.69, 5.35), at the first phase of the MCO the value decreased to 2.52 (2.34, 2.70), subsquently it decreased to 1.12 (1.07, 1.17) and then to 0.97. Based on the fact that parameters are not constant at all time, Ganasegeran *et al.* [39] estimated the value of R_0 at five different intervention stages in Malaysia. In Nigeria, Ibrahim and Oladipo[40], Okuonghae and Omame [41] and Adewole *et al.* [22] estimated the R_0 using the National Center for Disease Control and Prevention (NCDC) weekly situation data report to be around 2.2, 2.10 and 0.6 at three different periods. Negi *et al.* [42] studied the pandemic dynamics of four different countries: Brazil, India, Italy and USA, and obtained the estimates of their reproduction number to 1.4607, 0.7507, 3.5301, and 1.34268 respectively.

Among the importance of parameter estimation are future predictions of new cases of the disease under study and the provision of insight towards understanding the impacts of various employed preventive mechanisms. But estimated parameters within the same period of time and future predictions did not correspond to real life outcomes, as shown in Ganasegeran *et al.* [39] and Negi *et al.* [42]. The resulting discrepancies of estimated parameters for predictions of new cases with the real life out come over time could be the result of using only one compartment [40, 41] or two compartments [22] for estimation and data fitting. The exemption of other compartments could be the source of errors in the estimated values. Furthermore, factors such as incorporation of preventive mechanisms and the introduction of vaccination programmes have contributed in reducing the basic reproduction number. Hence, the need for a more accurate estimation of parameters which would capture current realities. For this reason, we use five - first dose vaccinated, second dose vaccinated, booster dose vaccinated, active cases, and recovered case compartments to determine/estimate parameter for the $SSVEIHQR-D$ model.

This research uses the COVID-19 data provided by Ministry of Health [43] from 14 February 2023 to 23rd September 2023 for data fitting and parameters estimations. The simulation was carried out using the ”lsqcurvefit” package on MATLAB. The estimated parameters are give in Table 2 while Figure 2 shows the simulation graph of the data fitting of the $SSVEIHQR-D$ model. The estimated values for the infection force parameters with respect to the highly immune, weak immune, first dose vaccinated, second dose vaccinated, and booster dose vaccinated are $\beta_1 = 0.731379771325$, $\beta_2 = 0.256876730917$, $\beta_3 = 0.996573290939$, $\beta_4 = 0.420669910335$, $\beta_5 = 0.849366396106$ respectively, while the basic reproduction number with respect to these estimated parameters is $R_0 = 0.038295930703259$.

The Herd Immunity Threshold (HIT)

Having estimated the Basic reproduction number of the SSVEIHQR-D model, we can now estimate the threshold of herd immunity for the population. According to the WHO "Herd immunity, also known as population immunity, is the indirect protection from an infectious disease that happens when a population is immune either through vaccination or immunity developed through previous infection" [11].

6. Uncertainty and sensitivity analysis

Uncertainty analysis (UA) is the process of determining and quantifying possible errors that could be associated with an estimated parameter. It aims at quantifying the outcome of a parameter based on the variability of the input. While sensitivity analysis (SA) is the process of determining how a given parameter influences/affects a dependent variable with respect to a given set of assumptions. SA enables researchers to determine how much a parameter influence a dependent variable in a model. Both UA and SA are necessary in determining the behaviour of a model. In order to capture the uncertainty of estimated parameters confidence interval for each parameter was estimated which is then used to determine the confidence interval for the basic reproduction number, and the sampling-based method of sensitivity analysis which is known as Latin Hypercube Sampling with partial rank correlation coefficient index (LHS-PRCC) was adopted, since the aim is to determine the influence of each parameter on the basic reproduction number. Refer to Marino *et al.* [44] for more details on this method.

The sensitivity indices of the basic reproduction number of the *SSVEIHQR-D* model was derived by the LHS-PRCC method using a uniform distribution of 5000 samples of each parameter as inputs. Figure (3) is a pictorial representation of the PRCC. The bars indicate the magnitude of the influence of the parameters in the dynamics of the disease. Thus a slight change in a parameter with higher magnitude could likely result to a significant change in the dynamics of the disease. Indices with positive signs indicate direct relationship with the reproduction number i.e an increase on such parameters will result to an increase in the value of the reproduction number. In the same way, those with negative signs indicate an inverse relationship with the reproduction number, i.e increase on the value of such parameters will result in decrease in the value of the reproduction number. The list of the sensitivity indices is given in Table 3. Figure (4 and 5) show the relationship of each parameter with the basic reproduction number R_0 by using a uniform distribution of 5000 samples as input.

7. Optimal control

This section analyses the outcome of control mechanisms incorporated for the prevention of the viral spread of the COVID-19. Optimal control strategy has proven to be effective in minimizing the spread of malicious objects. Thus, the need to find a means of reducing the number of infectious persons as well as the cost of appropriating such mechanisms becomes expedient. Following the preventing measures and guidelines given by the World Health Organization (WHO) and the Center for Disease Control (CDC): such as the use of face mask, frequent washing of hands and use of hand sanitizers; observation of social distancing, etc. This research work assumes that a fraction (φ) of the population abide by the WHO preventive measures with $p(t)$ as the successful level of compliance and $v(t)$ as the successfully completed vaccination. Thus, two control parameter which are classified into four are incorporated into the model: $p_h(t)$ and $p_w(t)$ are the fraction that met the successful level of preventive measures compliance at a time (t) among high and weak immune persons respectively, while $v_h(t)$ and $v_h(t)$ are the successfully vaccinated persons among the high and weak immune persons at a time t respectively. Now, removing the vaccine compartments (since we have assigned control parameters to them) from Equation (3) and incorporating the control parameters we have

$$\begin{aligned}
\frac{dS_h}{dt} &= \mu\theta - (1 - \varphi_h p_h(t))\beta_1(I_s + I_a)S_h - (v_h(t) + \xi_1)S_h \\
\frac{dS_w}{dt} &= \mu(1 - \theta) + uS_h - (1 - \varphi_w p_w(t))\beta_2(I_s + I_a)S_w \\
&\quad - (v_w(t) + \xi_2)S_w \\
\frac{dE}{dt} &= (1 - \varphi_h p_h(t))\beta_1(I_s + I_a)S_h + (1 - \varphi_w p_w(t))\beta_2(I_s + I_a)S_w \\
&\quad - \xi_6 E \\
\frac{dI_s}{dt} &= \alpha k E - \xi_7 I_s \\
\frac{dI_a}{dt} &= \alpha(1 - k)E - \xi_8 I_a \\
\frac{dH}{dt} &= \rho I_s - \xi_9 H \\
\frac{dQ}{dt} &= \sigma_s I_s + \sigma_a I_a - \xi_{10} Q \\
\frac{dR}{dt} &= v_h(t)S_h + v_w(t)S_w + \phi_a I_a + \phi_q Q + \phi_h H - \mu R \\
\frac{dD}{dt} &= \delta_s I_s + \delta_h H + \delta_q Q.
\end{aligned} \tag{26}$$

The goal now is to examine analytically the given control problem with the aim of minimizing the objective functional.

To optimize the control variables $p_h(t)$, $p_w(t)$, $v_h(t)$, and $v_w(t)$ which is assumed to be Lebesque measurable on $[0, t_f]$ (t_f being the stopping time), the control set is define as:

$$\begin{aligned}
\Omega = \{(p_h(t), p_w(t), v_h(t), v_w(t)) \in L^1(0, t_f) \times L^1(0, t_f) : 0 \le p_h(t) \\
\le p_h^{max}, 0 \le p_w(t) \le p_w^{max}, 0 \le v_h(t) \le v_h^{max}, 0 \le v_w(t) \\
\le v_w^{max}\},
\end{aligned} \tag{27}$$

where p_h^{max}, v_w^{max}, v_h^{max} and v_w^{max} denote the upper bound for successful compliance to COVID-19 preventive measure and successfully completed dosage of the COVID-19 vaccination.

Following the steps of Adewole *et al.* [22] and Marsden *et al.* [45] the following functional is formulated:

$$J(p_h, p_h, v_h, v_w) = \min_{p_h, p_h, v_h, v_h} \int_0^{t_f} \Big[\eta_1 E(t) + \eta_2 I_s(t) + \eta_3 I_a(t) + \frac{1}{2}\left(\eta_4 p_h^2(t) + \eta_5 p_w^2(t) + \eta_6 v_h^2(t) + \eta_7 v_w^2(t)\right)\Big] dt. \quad (28)$$

The aim is to minimize the total number of infectious persons while keeping the cost associated with the control mechanism minimum. In other words, the target is to find an optimal control $(p_h^*(t), p_w^*(t), v_h^*(t), (v_w^*(t))$ such that

$$J(p_h^*(t), p_w^*(t), v_h^*(t), (v_w^*(t)) = \min_{p_h^*, p_w^*, v_h^*, v_w^* \in \Omega} J(p_h, p_w, v_h, v_w). \quad (29)$$

Quadratic terms are incorporated (Equation (28)) to show nonlinear costs arising due to high intervention cases [45, 46]. The terms given in the integrand in Equation (28) are explained below:

- The term $\eta_1 E(t) + \eta_2 I_s(t) + \eta_3 I_a(t)$ represent the cost deploying and maintaining mechanism for monitoring infected persons at all stage
- The term $\eta_4 p_h^2(t)$ and $\eta_5 p_h^2(t)$ represent the cost of observing the WHO's preventing guidelines by the strong immune and weak immune respectively.
- The term $\eta_6 v_h^2(t)$ and $\eta_7 v_h^2(t)$ represent the cost of the vaccination for the strong and weak immune respectively at time t

7.1. Theoretical analysis of the optimal control

Based on the standard optimal control theorem in [47], the following theorem establish the existence of an optimal control for the proposed COVID-19 model.

theorem 7.1. There exist $p_h(t)$, $p_w(t)$, $v_h(t)$ and $v_w(t) \in \Omega$ such that the given objective functional Equation (28) is minimized.

proof. Observe that the control set as define in Equation (27) is closed since it contains all of its limit points, as a result it is bounded; the set is also convex. The right hand side of the set of equations in Equation (26) and the integrand of the objective functional in Equation (27) are continuously differentiable. Hence, these satisfied the conditions for the existence of global optimal control theorem based on theorem [47].

To determine the necessary conditions for the optimal control for the spread of the disease in Malaysia, the Pontryagin's Minimum Principle (PMP) [48] is applied. Applying this principle the system of equations (Equation (26)) and Equation (28) changes from a problem of minimizing the objective functional (Equation (28)) subject to the state variable equation (Equation (26)), into a problem of minimizing a point-wise Hamiltonian ($\mathcal{L}$) with respect to the control $p_h(t)$, $p_w(t)$, $v_h(t)$ and $v_w(t)$. The Hamiltonian ($\mathcal{L}$) is given by:

$$\begin{aligned}
\mathcal{L} = \; & \eta_1 E(t) + \eta_2 I_s(t) + \eta_3 I_a(t) + \frac{1}{2}\Big(\eta_4 p_h^2(t) + \eta_5 p_w^2(t) \\
& + \eta_6 v_h^2(t) + \eta_7 v_w^2(t)\Big) \\
& + \lambda_1 \left[\mu\theta - (1 - \varphi_h p_h(t))\beta_1 (I_s + I_a) S_h - (v_h(t) \right. \\
& \left. + \xi_1) S_h\right] \\
& + \lambda_2 \left[\mu(1-\theta) + u S_h - (1 - \varphi_w p_w(t))\beta_2 (I_s + I_a) S_w \right. \\
& \left. - (v_w(t) + \xi_2) S_w\right] \\
& + \lambda_3 \left[(1 - \varphi_h p_h(t))\beta_1 (I_s + I_a) S_h + (1 - \varphi_w p_w(t)) \right. \\
& \left. \cdot\beta_2 (I_s + I_a) S_w - \xi_6 E\right] \\
& + \lambda_4 \left[\alpha k E - \xi_7 I_s\right] + \lambda_5 [\alpha(1-k)E - \xi_8 I_a] \\
& + \lambda_6 [\rho I_s - \xi_9 H] \\
& + \lambda_7 \left[\sigma_s I_s + \sigma_a I_a - \xi_{10} Q\right] + \lambda_8 [v_h(t) S_h + v_w(t) S_w \\
& + \phi_a I_a + \phi_q Q + \phi_h H - \mu R] \\
& \lambda_9 [\delta_s I_s + \delta_h H + \delta_q Q],
\end{aligned} \quad (30)$$

where $\lambda_1, \lambda_2, \cdots, \lambda_9$ are the adjoint functions associated with the variable of the equations (Equation (26)). By applying the PMP the following theorem is obtained:

Theorem 7.2. Given an optimal control points $p_h(t)$, $p_w(t)$, $v_h(t)$ and $v_w(t)$ and a corresponding solution $(S_h^*, S_w^*, E^*, I_s^*, I_a^*, H^*, Q^*, R^*)$ of Equation (26) that minimize $J(p_h(t) p_w(t), v_h(t), v_w(t))$ over the set Ω, then there exist adjoint functions such that:

$$\begin{aligned}
-\frac{d\lambda_1}{dt} &= \frac{\partial \mathcal{L}}{\partial S_h} = (\lambda_3 - \lambda_1)(1 - \varphi_h p_h(t))\beta_1 (I_s + I_a) \\
&\quad + (\lambda_8 - \lambda_1) v_h(t) + \lambda_1 \xi_1 + \lambda_2 u \\
-\frac{d\lambda_2}{dt} &= \frac{\partial \mathcal{L}}{\partial S_w} = (\lambda_3 - \lambda_2)(1 - \varphi_w p_w(t))\beta_1 (I_s + I_a) \\
&\quad + (\lambda_8 - \lambda_2) v_w(t) + \lambda_2 \xi_2 \\
-\frac{d\lambda_3}{dt} &= \frac{\partial \mathcal{L}}{\partial E} = \eta_2 + \lambda_3 \xi_6 + \lambda_4 \alpha k + \lambda_5 \alpha(1-k) \\
-\frac{d\lambda_4}{dt} &= \frac{\partial \mathcal{L}}{\partial I_s} = \eta_2 + (\lambda_3 - \lambda_1)(1 - \varphi_h p_h(t))\beta_1 S_h \\
&\quad + (\lambda_3 - \lambda_2)(1 - \varphi_w p_w(t))\beta_2 S_w \\
&\quad - \lambda_4 \xi_7 + \lambda_6 \rho + \lambda_7 \delta_s + \lambda_9 \delta_s \\
-\frac{d\lambda_5}{dt} &= \frac{\partial \mathcal{L}}{\partial I_a} = \eta_3 + (\lambda_3 - \lambda_1)(1 - \varphi_h p_h(t))\beta_1 S_h \\
&\quad + (\lambda_3 - \lambda_2)(1 - \varphi_w p_w(t))\beta_2 S_w \\
&\quad - \lambda_5 \xi_8 + \lambda_7 \sigma_a + \lambda_8 \phi_a \\
-\frac{d\lambda_6}{dt} &= \frac{\partial \mathcal{L}}{\partial H} = -\lambda_6 \xi_9 + \lambda_8 \phi_h + \lambda_9 \delta_h \\
-\frac{d\lambda_7}{dt} &= \frac{\partial \mathcal{L}}{\partial Q} = -\lambda_7 \xi_9 + \lambda_8 \phi_q + \lambda_9 \delta_0 \\
-\frac{d\lambda_8}{dt} &= \frac{\partial \mathcal{L}}{\partial R} = -\lambda_8 \mu \\
-\frac{d\lambda_9}{dt} &= \frac{\partial \mathcal{L}}{\partial D}) = 0.
\end{aligned} \quad (31)$$

With transversality conditions $\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = \lambda_4(t_f) = \lambda_5(t_f) = \lambda_6(t_f) = \lambda_7(t_f) = \lambda_8(t_f) = \lambda_9(t_f) = 0$ and

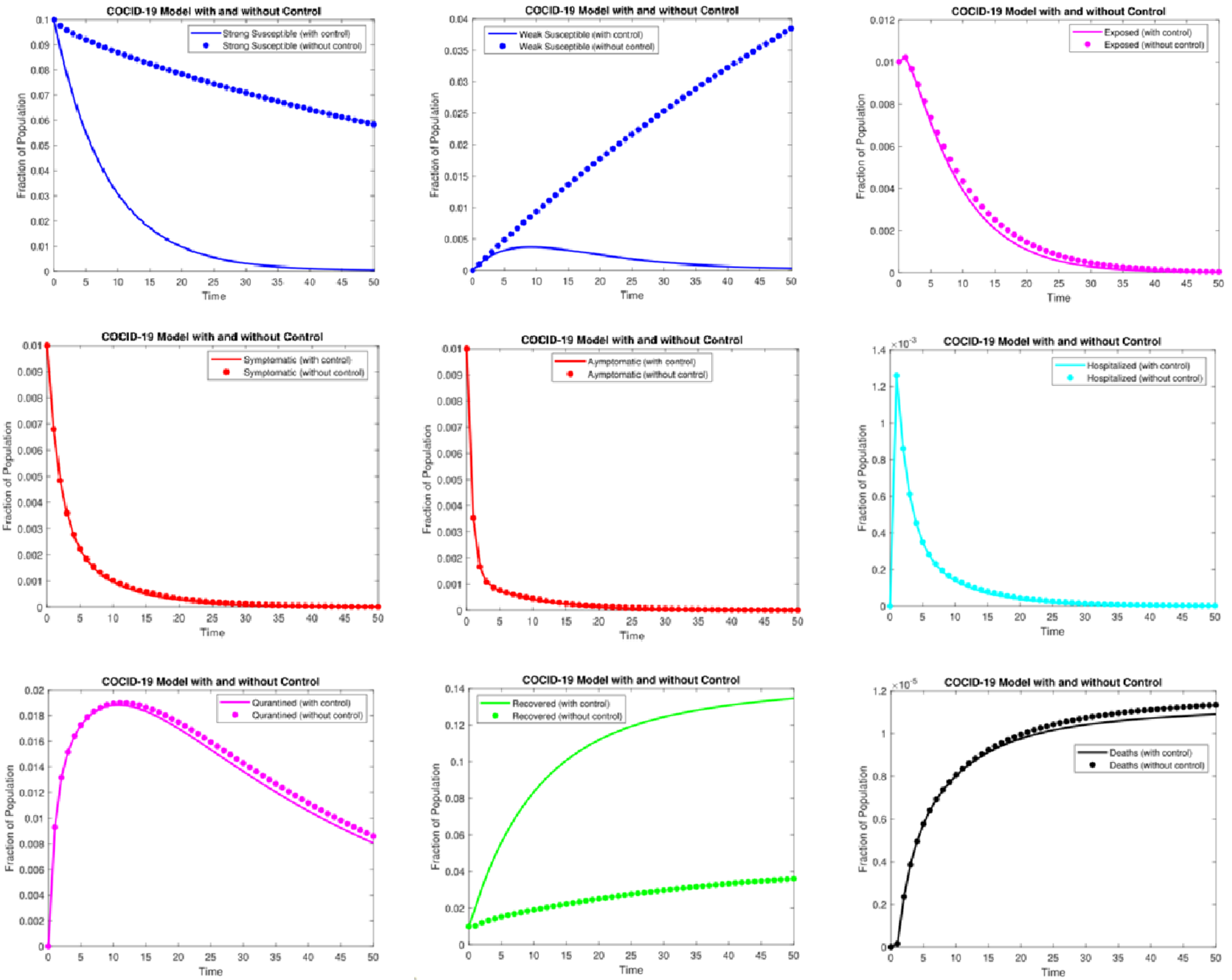

Figure 6: The $SSEEIHQR-D$ model with two control $p(t)$ and $v(t)$: preventive guidelines and vaccination. **a** Strong immune susceptible (S_h), **b** weaker immune susceptible (S_w), **c** Exposed (E), **d** Symptomatic (I_s), **e** Asymptomatic (I_a), **f** Hospitalized (H), **g** Quarantined (Q), **h** Recovered (R), and **i** Deaths (D).

$N^* = S_h^* + S_w^* + E^* + I_h^* + I_a^* + H^* + Q^* + R^*$; satisfying the following optimality condition:

$$\begin{aligned} p_h^*(t) &= min\left(max\left(0, \frac{(\lambda_3-\lambda_1)\varphi_h\beta_1(I_s+I_a)S_h^*}{\eta_4}\right), p_h^{max}\right) \\ p_w^*(t) &= min\left(max\left(0, \frac{(\lambda_3-\lambda_1)\varphi_w\beta_2(I_s+I_a)S_w^*}{\eta_5}\right), p_w^{max}\right) \\ v_h^*(t) &= min\left(max\left(0, \frac{(\lambda_1-\lambda_8)S_h^*}{\eta_6}\right), v_h^{max}\right) \\ v_w^*(t) &= min\left(max\left(0, \frac{(\lambda_2-\lambda_8)S_w^*}{\eta_7}\right), v_w^{max}\right). \end{aligned} \tag{32}$$

Proof. By applying the Pontryagin's Minimum Principle, the system of adjoint functions Equation (31) is obtained from the Hamiltonian function Equation (30), subject to the transversality condition $\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = \lambda_4(t_f) = \lambda_5(t_f) = \lambda_6(t_f) = \lambda_7(t_f) = \lambda_8(t_f) = \lambda_9(t_f) = 0$. Now, evaluating at the control pair $(p_h(t), (p_w(t), (v_h(t), v_w(t))$ subject to the state variables, and taking into consideration the optimality condition: $\frac{\partial\mathcal{L}}{\partial p_h} = 0$, $\frac{\partial\mathcal{L}}{\partial p_w} = 0$, $\frac{\partial\mathcal{L}}{\partial v_h} = 0$, and $\frac{\partial\mathcal{L}}{\partial v_w} = 0$, on the set $\{t : 0 < p_h(t) < p_h^{max}\}$, $\{t : 0 < p_w(t) < p_w^{max}\}$, $\{t : 0 < v_h(t) < v_h^{max}\}$ and $\{t : 0 < v_w(t) < v_w^{max}\}$, we obtain:

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial p_h} &= \eta_4 p_h(t) + \lambda_1\varphi_h\beta_1(I_s+I_a)S_h - \lambda_3\varphi_h\beta_1(I_s+I_a)S_h \\ &= 0 \\ \Rightarrow p_h(t) &= \frac{(\lambda_3-\lambda_1)\varphi_h\beta_1(I_s+I_a)S_h}{\eta_4}, \\ \frac{\partial\mathcal{L}}{\partial p_w} &= \eta_5 p_w(t) + \lambda_2\varphi_w\beta_2(I_s+I_a)S_w - \lambda_3\varphi_w\beta_2(I_s+I_a)S_w \\ &= 0 \\ \Rightarrow p_w(t) &= \frac{(\lambda_3-\lambda_2)\varphi_w\beta_2(I_s+I_a)S_w}{\eta_5}, \end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial v_h} = \eta_6 v_h(t) - \lambda_1 S_h + \lambda_8 S_h = 0$$
$$\Rightarrow v_h(t) = \frac{(\lambda_1 - \lambda_8) S_h}{\eta_6},$$

and

$$\frac{\partial \mathcal{L}}{\partial v_w} = \eta_7 v_w(t) - \lambda_2 S_w + \lambda_8 S_w = 0$$
$$\Rightarrow v_w(t) = \frac{(\lambda_2 - \lambda_8) S_w}{\eta_7}.$$

Showing that the optimality condition only holds in the interior of the control set.

8. Results and discussion

The result from the model analysis for the proposed $SSVEIHQR - D$ show that the system for the model have a non-negative solution and the COVID-19-free equilibrium point was obtained. The *Next Generation Matrix* as seen in [34] was used to obtain the reproduction number R_0. The model analysis also show that the COVID-19-free equilibrium point of model is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$. To illustrate the main result of the proposed model and the combined impact of the spreading of COVID-19 in Malaysia, we carried out some numerical simulations, by collecting the daily COVID-19 data published by Ministry of Health [43] and some parameters from available corresponding literature. Recent data were collected for 220 days for fittings and parameters estimations. Among the parameters that were estimated are the infection force parameters with respect to the highly immune, weak immune, first dose vaccinated, second dose vaccinated, and booster dose vaccinated, which are $\beta_1 = 0.731379771325$, $\beta_2 = 0.256876730917$, $\beta_3 = 0.996573290939$, $\beta_4 = 0.420669910335$, $\beta_5 = 0.849366396106$ respectively, and the basic reproduction number with respect to these estimated parameters is $R_0 = 0.038295930703259$. This value is significantly different from previously estimated/assumed values for R_0. The substantial decrease in the value of R_0 is expected due to the preventive mechanism and vaccination programs introduced against the spread of the virus. The decline in the value of R_0 shows the validity of the $SSVEIHQR - D$ model as it corresponds to real life situation outcome for the COVID-19 pandemic at the moment.

Also, we calculated the herd immunity threshold which normally takes value from 0% to 100%, but ours was far less than the range, because the basic reproduction number is much less than 1 (i.e. stability), which means that, on average, each infected individual is infecting less than one other person, this indicates that the COVID-19 new cases will naturally decline and the virus will eventually die out of the population. Thus, parameters estimated from the fittings could be used for further analysis. Next is the simulations of the uncertainty and sensitivity analysis of the basic reproduction number. We carried out a global sensitivity analysis using 5000 uniformly distributed samples of each parameter of the $SSVEIHQR-D$ model within a 99% interval of confidence for each parameter as obtained from the fittings and other literature in order to capture uncertainty associated with estimations.

These samples were obtained through sampling based method known as the Latin Hypercube Sampling with partial rank correlation coefficientnindex (LHS-PRCC). A bar chat representing the magnitude/influence of each parameter against the basic reproduction number R_0 is given in Figure 3. The bars on the positive axis implies positive influence on the R_0 while the bars on the negative axis represent negative influence. The length of each bar denote the magnitude of the influnce of the parameter. It can be observed from the PRCC chat that the parameter with the highest influence on R_0 is the infectious force of the weak immune susceptible individuals. The graphical/pictorial representation of the 5000 samples of each parameters against the R_0 is given in Figure 4-5. Finally is the simulation of the optimal control analysis. The essence of the optimal control analysis is to get a picture of the dynamics of the impact of incorporated control mechanism for curbing the spread of COVID-19 and to juxtapose the outcome of the impact with the model.

In reality, control strategies adopted toward minimizing the spread of the virus can be classified into two: preventive guidelines as stipulated by the WHO such as wearing of face mask,frequent use of hand sanitizer and washing of hands, observation of social distancing; and the introduction of vaccination programme; as a result two control parameters were incorporated into the model. For easy apprehension, the simulation for each compartment was plotted on a different plane with the scenarios when there is control and when there is no control. These simulations are given in Figure **??**. There are two main compartment of interest here: the susceptible compartment and the recovered compartment, these are the main classes of target. It can be observed immediately from the simulations given in Figure **??** the impact of the control parameters on the susceptible class and the recovered class, the rate of becoming susceptible reduced so much, while the rate of recovery increased very significantly. This is as a result of the fact that the fraction of persons who did not become exposed and those who have completed the vaccination are moved into the recovered class. A juxtaposition between the recovered class in the data fittings in **??** and the recovered class obtained from the simulation control in Figure **??** gives a similar line graph. This reveals the impact of incorporating control mechanism in tracking and reducing the viral spread of the COVID-19 virus in the the Malaysia population.

9. Summary and conclusion

In order to capture in a model the peculiarity of having two distinct susceptible classes and its dynamics along the three stages of the vaccination programs introduced by the WHO, which have not been taken into consideration by other researchers; and to obtain the best fit values of parameters that propagates the virus by fitting real life situation data to a model and give a comprehensive interpretation to the discrepancies between the results of a proposed model and real life outcomes, taking Malaysia as a case study, a new twelve compartmental *SSVEIHQR-D* epidemiological model was developed.

From the theoretical analysis of the proposed *SSVEIHQR-D* model, we were able to obtain the characteristics of the threshold parameter R_0 with the help of the *Next Generation Matrix* as given in [34], and established that if $R_0 < 1$ the system is locally asymptotically stable and unstable if otherwise. The "lsqcurvefit" package in MATLAB was used for the estimation of parameters and data fittings, and the basic reproduction number obtained with respect to the estimated parameters is $R_0 = 0.038295930703259$. This value of R_0 corresponds to the current state (stability) of the pandemic, it also reveal the impact of control mechanisms (the value could be said to be more realistic when compared with previous R_0 estimated by other researchers). Using the estimated R_0 we calculated the herd immunity threshold which shows that the COVID-19 new cases will naturally decline and the virus will eventually die out of the population. We carried out uncertain and sensitivity analysis from which we discovered that one parameter u - *the rate of strong immune becoming weak* out of 24 parameters highly influences the R_0 negatively, while five parameters, namely: infection force for the weak immune β_2, quarantined rate for symptomatic σ_s, quarantined rate for asymptomatic σ_s, hospitalized rate for the symptomatic ρ, recovery rate for the quarantined ϕ_q highly influence the R_0 positively.

For the optimal control strategy for tracking and reducing the viral spread of the COVID-19 virus in Malaysia four control parameters were introduced: $p_h(t)$, $p_w(t)$, $v_h(t)$, and $v_w(t)$ into the model which represent the fraction of strong immune that met the successful level of preventive measures, the fraction of weak immune that met the successful level of preventive measures, the strong immune that have successfully completed vaccination dose and the weak immune that have successfully completed vaccination dose rate respectively. The simulation result shows the impact of the control mechanisms over the period, and juxtaposing of the result with the data fittings simulation shows significant similarity, this shows that optimal control analysis is an effective way of minimizing the spread of COVID-19 in the population.

Since having a basic reproduction R_0 that is less than 1 implies the virus will naturally die out of the population over time, it becomes necessary to ensure that this value remains less than 1. To ensure this, we recommend that the government should not relent encouraging the populace to continue observing the preventive guidelines as stipulated by the WHO and MOH; and to also acknowledge the effort of the populace so far for their patience and sacrifices in fighting against the virus. Also, we recommend that researchers should be encouraged to estimating the value of the R_0 and other parameters time to time, to monitor its value in order that immediate action be taken whenever it becomes grater than 1. The result from he sensitivity analysis show that the rate of moving from the strong immune group to weaker immune negatively influence the basic reproduction number, so there is need to make the value lesser, to achieve government and health organization should encourage sensitization program about building a strong immune system against COVID-19; and allocation strategies should be devised for priotizing the vaccination of individuals with weaker immune. To prepare against potential effect of new variants of the virus, government, business and medical organization should encourage research on the efficiency of existing vaccines against future potential variants.

Finally, one major limitation or weakness of this research work which could affect the accuracy of the fittings and estimated parameter is the accuracy of available data. For instance, it is possible that there are unreported cases of infection that the Ministry of Health (MOH) is unaware of. Therefore, we hold the view that the data related to these new cases may lack the necessary accuracy for the purposes of model fittings and estimation. Instead we used the vaccination data since people have to visit a medical center for vaccination, which we consider to be a more reliable and accurate dataset. Hence, when using the recovered data for simulating the fittings and estimations, we made some adjustments to our assumptions, retaining only those related to recovery from the hospitalized and quarantined, while discarding others.

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APPENDIX A.

E^* is the solution of a third degree polynomial $a_0z^3 + a_1z^2 + a_2z^1 + a_3$, where,

$$
\begin{aligned}
a_0 = {}& \xi_7\xi_8(-\beta_2\beta_3\beta_5\xi_1\xi_4\xi_6 I - \beta_2\beta_4\beta_5\xi_1\xi_3\xi_6 I - \beta_3\beta_4\beta_5\xi_1\xi_2\xi_6 I \\
& - \beta_1\beta_2\beta_3\xi_4\xi_5\xi_6 I - \beta_1\beta_2\beta_4\xi_3\xi_5\xi_6 I - \beta_1\beta_2\beta_5\xi_3\xi_4\xi_6 I \\
& - \beta_1\beta_3\beta_4\xi_2\xi_5\xi_6 I - \beta_2\beta_3\beta_4\xi_1\xi_5\xi_6 I - \alpha\mu\beta_1\beta_3\beta_4\xi_5\xi_7 I \\
& - \alpha\mu\beta_1\beta_3\beta_5\xi_4\xi_7 I - \alpha\mu\beta_1\beta_4\beta_5\tau_2\xi_7 I - \alpha\mu\beta_1\beta_4\beta_5\xi_3\xi_7 I \\
& - \alpha\mu\beta_3\beta_4\beta_5\xi_1\xi_7 I + \alpha\mu\theta\beta_3\beta_4\beta_5\xi_1\xi_7 I - \alpha\mu\theta u\beta_3\beta_4\beta_5\xi_7 I \\
& - \alpha\mu\theta\beta_2\beta_3\beta_4\xi_5\xi_7 I - \alpha\mu\theta\beta_2\beta_3\beta_5\xi_4\xi_7 I - \alpha\mu\theta\beta_2\beta_4\beta_5\tau_1\xi_7 I \\
& - \alpha\mu\theta\beta_2\beta_4\beta_5\xi_3\xi_7 I - \alpha\mu\theta\beta_3\beta_4\beta_5\xi_2\xi_7 I + \alpha k\mu\beta_1\beta_3\beta_4\xi_5\xi_7 I \\
& + \alpha k\mu\beta_1\beta_3\beta_4\xi_5\xi_8 I + \alpha k\mu\beta_1\beta_3\beta_5\xi_4\xi_7 I + \alpha k\mu\beta_1\beta_3\beta_5\xi_4\xi_8 I \\
& + \alpha k\mu\beta_1\beta_4\beta_5\tau_2\xi_7 I + \alpha k\mu\beta_1\beta_4\beta_5\tau_2\xi_8 I + \alpha k\mu\beta_1\beta_4\beta_5\xi_3\xi_7 I \\
& + \alpha k\mu\beta_1\beta_4\beta_5\xi_3\xi_8 I + \alpha k\mu\beta_3\beta_4\beta_5\xi_1\xi_7 I + \alpha k\mu\beta_3\beta_4\beta_5\xi_1\xi_8 I \\
& + \alpha\mu\theta\beta_1\beta_3\beta_4\xi_5\xi_7 I + \alpha\mu\theta\beta_1\beta_3\beta_5\xi_4\xi_7 I + \alpha\mu\theta\beta_1\beta_4\beta_5\tau_2\xi_7 I \\
& + \alpha\mu\theta\beta_1\beta_4\beta_5\xi_3\xi_7 I - \alpha k\mu\theta\beta_1\beta_4\beta_5\tau_2\xi_7 I - \alpha k\mu\theta\beta_1\beta_4\beta_5\tau_2\xi_8 I \\
& - \alpha k\mu\theta\beta_1\beta_4\beta_5\xi_3\xi_7 I - \alpha k\mu\theta\beta_1\beta_4\beta_5\xi_3\xi_8 I - \alpha k\mu\theta\beta_3\beta_4\beta_5\xi_1\xi_7 I \\
& - \alpha k\mu\theta\beta_3\beta_4\beta_5\xi_1\xi_8 I + \alpha k\mu\theta\beta_2\beta_3\beta_4\xi_5\xi_7 I + \alpha k\mu\theta\beta_2\beta_3\beta_4\xi_5\xi_8 I \\
& + \alpha k\mu\theta\beta_2\beta_3\beta_5\xi_4\xi_7 I + \alpha k\mu\theta\beta_2\beta_3\beta_5\xi_4\xi_8 I + \alpha k\mu\theta\beta_2\beta_4\beta_5\tau_1\xi_7 I \\
& + \alpha k\mu\theta\beta_2\beta_4\beta_5\tau_1\xi_8 I + \alpha k\mu\theta\beta_2\beta_4\beta_5\xi_3\xi_7 I + \alpha k\mu\theta\beta_2\beta_4\beta_5\xi_3\xi_8 I \\
& + \alpha k\mu\theta\beta_3\beta_4\beta_5\xi_2\xi_7 I + \alpha k\mu\theta\beta_3\beta_4\beta_5\xi_2\xi_8 I - \alpha k\mu\theta\beta_1\beta_3\beta_4\xi_5\xi_7 I \\
& - \alpha k\mu\theta\beta_1\beta_3\beta_4\xi_5\xi_8 I - \alpha k\mu\theta\beta_1\beta_3\beta_5\xi_4\xi_7 I - \alpha k\mu\theta\beta_1\beta_3\beta_5\xi_4\xi_8 I \\
& + \alpha k\mu\theta u\beta_3\beta_4\beta_5\xi_7 I + \alpha k\mu\theta u\beta_3\beta_4\beta_5\xi_8 I))
\end{aligned}
$$

$$
\begin{aligned}
a_1 = {}& \xi_7\xi_8(\alpha k\mu\theta u\beta_3\beta_4\xi_5\xi_7 + \alpha k\mu\theta u\beta_3\beta_4\xi_5\xi_8 + \alpha k\mu\theta u\beta_3\beta_5\xi_4\xi_7 \\
& + \alpha k\mu\theta u\beta_3\beta_5\xi_4\xi_8 + \alpha k\mu\theta u\beta_4\beta_5\tau_2\xi_7 + \alpha k\mu\theta u\beta_4\beta_5\tau_2\xi_8 \\
& + \alpha k\mu\theta u\beta_4\beta_5\xi_3\xi_7 + \alpha k\mu\theta u\beta_4\beta_5\xi_3\xi_8 - \alpha k\mu\theta\beta_1\beta_3\xi_4\xi_5\xi_7 \\
& - \alpha k\mu\theta\beta_1\beta_3\xi_4\xi_5\xi_8 - \alpha k\mu\theta\beta_1\beta_4\tau_2\xi_5\xi_7 - \alpha k\mu\theta\beta_1\beta_4\tau_2\xi_5\xi_8 \\
& - \alpha k\mu\theta\beta_1\beta_4\xi_3\xi_5\xi_7 - \alpha k\mu\theta\beta_1\beta_4\xi_3\xi_5\xi_8 - \alpha k\mu\theta\beta_1\beta_5\tau_2\tau_3\xi_7 \\
& + \alpha k\mu\theta\beta_2\beta_3\xi_4\xi_5\xi_8 + \alpha k\mu\theta\beta_2\beta_4\tau_1\xi_5\xi_7 + \alpha k\mu\theta\beta_2\beta_4\tau_1\xi_5\xi_8 \\
& + \alpha k\mu\theta\beta_2\beta_4\xi_3\xi_5\xi_7 + \alpha k\mu\theta\beta_2\beta_4\xi_3\xi_5\xi_8 + \alpha k\mu\theta\beta_2\beta_5\tau_1\tau_3\xi_7 \\
& + \alpha k\mu\theta\beta_2\beta_5\tau_1\tau_3\xi_8 + \alpha k\mu\theta\beta_2\beta_5\tau_1\xi_4\xi_7 + \alpha k\mu\theta\beta_2\beta_5\tau_1\xi_4\xi_8 \\
& + \alpha k\mu\theta\beta_2\beta_5\xi_3\xi_4\xi_7 + \alpha k\mu\theta\beta_2\beta_5\xi_3\xi_4\xi_8 - \alpha k\mu\theta\beta_3\beta_4\xi_1\xi_5\xi_7 \\
& - \alpha k\mu\theta\beta_3\beta_4\xi_1\xi_5\xi_8 + \alpha k\mu\theta\beta_3\beta_4\xi_2\xi_5\xi_7 + \alpha k\mu\theta\beta_3\beta_4\xi_2\xi_5\xi_8 \\
& - \alpha k\mu\theta\beta_3\beta_5\xi_1\xi_4\xi_7 - \alpha k\mu\theta\beta_3\beta_5\xi_1\xi_4\xi_8 + \alpha k\mu\theta\beta_3\beta_5\xi_2\xi_4\xi_7 \\
& + \alpha k\mu\theta\beta_3\beta_5\xi_2\xi_4\xi_8 + \alpha k\mu\theta\beta_4\beta_5\tau_1\xi_2\xi_7 + \alpha k\mu\theta\beta_4\beta_5\tau_1\xi_2\xi_8 \\
& - \alpha k\mu\theta\beta_4\beta_5\tau_2\xi_1\xi_7 - \alpha k\mu\theta\beta_4\beta_5\tau_2\xi_1\xi_8 - \alpha k\mu\theta\beta_4\beta_5\xi_1\xi_3\xi_7 \\
& - \alpha k\mu\theta\beta_4\beta_5\xi_1\xi_3\xi_8 + \alpha k\mu\theta\beta_4\beta_5\xi_2\xi_3\xi_7 + \alpha k\mu\theta\beta_4\beta_5\xi_2\xi_3\xi_8 \\
& + \alpha k\mu\beta_1\beta_3\xi_4\xi_5\xi_7 + \alpha k\mu\beta_1\beta_3\xi_4\xi_5\xi_8 + \alpha k\mu\beta_1\beta_4\tau_2\xi_5\xi_7 \\
& + \alpha k\mu\beta_1\beta_4\tau_2\xi_5\xi_8 + \alpha k\mu\beta_1\beta_4\xi_3\xi_5\xi_7 + \alpha k\mu\beta_1\beta_4\xi_3\xi_5\xi_8 \\
& + \alpha k\mu\beta_1\beta_5\tau_2\tau_3\xi_7 + \alpha k\mu\beta_1\beta_5\tau_2\tau_3\xi_8 + \alpha k\mu\beta_1\beta_5\tau_2\xi_4\xi_7 \\
& + \alpha k\mu\beta_1\beta_5\tau_2\xi_4\xi_8 + \alpha k\mu\beta_1\beta_5\xi_3\xi_4\xi_7 + \alpha k\mu\beta_1\beta_5\xi_3\xi_4\xi_8 \\
& + \alpha k\mu\beta_3\beta_4\xi_1\xi_5\xi_7 + \alpha k\mu\beta_3\beta_4\xi_1\xi_5\xi_8 + \alpha k\mu\beta_3\beta_5\xi_1\xi_4\xi_7 \\
& + \alpha k\mu\beta_3\beta_5\xi_1\xi_4\xi_8 + \alpha k\mu\beta_4\beta_5\tau_2\xi_1\xi_7 + \alpha k\mu\beta_4\beta_5\tau_2\xi_1\xi_8 \\
& + \alpha k\mu\beta_4\beta_5\xi_1\xi_3\xi_7 + \alpha k\mu\beta_4\beta_5\xi_1\xi_3\xi_8 - \alpha\mu\theta u\beta_3\beta_4\xi_5\xi_7 \\
& - \alpha\mu\theta u\beta_3\beta_5\xi_4\xi_7 - \alpha\mu\theta u\beta_4\beta_5\tau_2\xi_7 - \alpha\mu\theta u\beta_4\beta_5\xi_3\xi_7 \\
& + \alpha\mu\theta\beta_1\beta_3\xi_4\xi_5\xi_7 + \alpha\mu\theta\beta_1\beta_4\tau_2\xi_5\xi_7 + \alpha\mu\theta\beta_1\beta_4\xi_3\xi_5\xi_7 \\
& + \alpha\mu\theta\beta_1\beta_5\tau_2\tau_3\xi_7 + \alpha\mu\theta\beta_1\beta_5\tau_2\xi_4\xi_7 + \alpha\mu\theta\beta_1\beta_5\xi_3\xi_4\xi_7 \\
& - \alpha\mu\theta\beta_2\beta_3\xi_4\xi_5\xi_7 - \alpha\mu\theta\beta_2\beta_4\tau_1\xi_5\xi_7 - \alpha\mu\theta\beta_2\beta_4\xi_3\xi_5\xi_7 \\
& - \alpha\mu\theta\beta_2\beta_5\tau_1\tau_3\xi_7 - \alpha\mu\theta\beta_2\beta_5\tau_1\xi_4\xi_7 - \alpha\mu\theta\beta_2\beta_5\xi_3\xi_4\xi_7 \\
& + \alpha\mu\theta\beta_3\beta_4\xi_1\xi_5\xi_7 - \alpha\mu\theta\beta_3\beta_4\xi_2\xi_5\xi_7 + \alpha\mu\theta\beta_3\beta_5\xi_1\xi_4\xi_7 \\
& - \alpha\mu\theta\beta_3\beta_5\xi_2\xi_4\xi_7 - \alpha\mu\theta\beta_4\beta_5\tau_1\xi_2\xi_7 + \alpha\mu\theta\beta_4\beta_5\tau_2\xi_1\xi_7 \\
& + \alpha\mu\theta\beta_4\beta_5\xi_1\xi_3\xi_7 - \alpha\mu\theta\beta_4\beta_5\xi_2\xi_3\xi_7 - \alpha\mu\beta_1\beta_3\xi_4\xi_5\xi_7 \\
& - \alpha\mu\beta_1\beta_4\tau_2\xi_5\xi_7 - \alpha\mu\beta_1\beta_4\xi_3\xi_5\xi_7 - \alpha\mu\beta_1\beta_5\tau_2\tau_3\xi_7 \\
& - \alpha\mu\beta_1\beta_5\tau_2\xi_4\xi_7 - \alpha\mu\beta_1\beta_5\xi_3\xi_4\xi_7 - \alpha\mu\beta_3\beta_5\xi_1\xi_4\xi_7 \\
& - \alpha\mu\beta_4\beta_5\tau_2\xi_1\xi_7 - \alpha\mu\beta_4\beta_5\xi_1\xi_3\xi_7 - \beta_1\beta_2\xi_3\xi_4\xi_5\xi_6 \\
& - \beta_1\beta_3\xi_2\xi_4\xi_5\xi_6 - \beta_1\beta_4\xi_2\xi_3\xi_5\xi_6 - \beta_1\beta_5\xi_2\xi_3\xi_4\xi_6 \\
& - \beta_2\beta_3\xi_1\xi_4\xi_5\xi_6 - \beta_2\beta_4\xi_1\xi_3\xi_5\xi_6 - \beta_2\beta_5\xi_1\xi_3\xi_4\xi_6 - \beta_3\beta_4\xi_1\xi_2\xi_5\xi_6 \\
& - \beta_3\beta_5\xi_1\xi_2\xi_4\xi_6 - \beta_4\beta_5\xi_1\xi_2\xi_3\xi_6)
\end{aligned}
$$

$$
\begin{aligned}
a_2 = {}& \xi_7\xi_8(\mathrm{I}\beta_1\xi_2\xi_3\xi_4\xi_5\xi_6 + \mathrm{I}\beta_2\xi_1\xi_3\xi_4\xi_5\xi_6 + \mathrm{I}\beta_3\xi_1\xi_2\xi_4\xi_5\xi_6 \\
& + \mathrm{I}\beta_4\xi_1\xi_2\xi_3\xi_5\xi_6 + \mathrm{I}\beta_5\xi_1\xi_2\xi_3\xi_4\xi_6 - \mathrm{I}\xi_7\alpha\mu k\beta_4\xi_1\xi_3\xi_5 \\
& - \mathrm{I}\alpha k\mu\beta_4\xi_1\xi_3\xi_5\xi_8 - \mathrm{I}\xi_7\alpha\mu k\beta_5\tau_2\tau_3\xi_1 - \mathrm{I}\alpha k\mu\beta_5\tau_2\tau_3\xi_1\xi_8 \\
& - \mathrm{I}\xi_7\alpha\mu k\beta_5\tau_2\xi_1\xi_4 - \mathrm{I}\alpha k\mu\beta_5\tau_2\xi_1\xi_4\xi_8 - \mathrm{I}\xi_7\alpha\mu k\beta_5\xi_1\xi_3\xi_4 \\
& - \mathrm{I}\alpha k\mu\beta_5\xi_1\xi_3\xi_4\xi_8 - \mathrm{I}\xi_7\alpha\mu\theta\beta_1\tau_2\tau_3\tau_4 - \mathrm{I}\xi_7\alpha\mu\theta\beta_1\tau_2\tau_3\xi_5 \\
& - \mathrm{I}\xi_7\alpha\mu\theta\beta_1\tau_2\xi_4\xi_5 - \mathrm{I}\xi_7\alpha\mu\theta\beta_1\xi_3\xi_4\xi_5 - \mathrm{I}\xi_7\alpha\mu\theta\beta_3\xi_1\xi_4\xi_5 \\
& - \mathrm{I}\xi_7\alpha\mu\theta\beta_4\tau_2\xi_1\xi_5 - \mathrm{I}\xi_7\alpha\mu\theta\beta_4\xi_1\xi_3\xi_5 - \mathrm{I}\xi_7\alpha\mu\theta\beta_5\tau_2\tau_3\xi_1 \\
& - \mathrm{I}\xi_7\alpha\mu\theta\beta_5\tau_2\xi_1\xi_4 - \mathrm{I}\xi_7\alpha\mu\theta\beta_5\xi_1\xi_3\xi_4 + \mathrm{I}\xi_7\alpha\mu\theta\beta_2\tau_1\xi_4\xi_5 \\
& + \mathrm{I}\xi_7\alpha\mu\theta\beta_2\xi_3\xi_4\xi_5 + \mathrm{I}\xi_7\alpha\mu\theta\beta_3\xi_2\xi_4\xi_5 + \mathrm{I}\xi_7\alpha\mu\theta\beta_4\tau_1\xi_2\xi_5 \\
& + \mathrm{I}\xi_7\alpha\mu\theta\beta_4\xi_2\xi_3\xi_5 + \mathrm{I}\xi_7\alpha\mu\theta\beta_5\tau_1\tau_3\xi_2 + \mathrm{I}\xi_7\alpha\mu\theta\beta_5\tau_1\xi_2\xi_4 \\
& + \mathrm{I}\xi_7\alpha\mu\theta\beta_5\xi_2\xi_3\xi_4 - \mathrm{I}\xi_7\alpha\mu k\beta_1\tau_2\tau_3\tau_4 - \mathrm{I}\alpha k\mu\beta_1\tau_2\tau_3\tau_4\xi_8 \\
& - \mathrm{I}\xi_7\alpha\mu k\beta_1\tau_2\tau_3\xi_5 - \mathrm{I}\alpha k\mu\beta_1\tau_2\tau_3\xi_5\xi_8 - \mathrm{I}\xi_7\alpha\mu k\beta_1\tau_2\xi_4\xi_5 \\
& - \mathrm{I}\alpha k\mu\beta_1\tau_2\xi_4\xi_5\xi_8 - \mathrm{I}\xi_7\alpha\mu k\beta_1\xi_3\xi_4\xi_5 - \mathrm{I}\alpha k\mu\beta_1\xi_3\xi_4\xi_5\xi_8 \\
& - \mathrm{I}\xi_7\alpha\mu k\beta_3\xi_1\xi_4\xi_5 - \mathrm{I}\alpha k\mu\beta_3\xi_1\xi_4\xi_5\xi_8 - \mathrm{I}\xi_7\alpha\mu k\beta_4\tau_2\xi_1\xi_5 \\
& - \mathrm{I}\alpha k\mu\beta_4\tau_2\xi_1\xi_5\xi_8 + \mathrm{I}\xi_7\alpha\mu\theta u\beta_3\xi_4\xi_5 + \mathrm{I}\xi_7\alpha\mu\theta u\beta_4\tau_2\xi_5 \\
& + \mathrm{I}\xi_7\alpha\mu\theta u\beta_4\xi_3\xi_5 + \mathrm{I}\xi_7\alpha\mu\theta u\beta_5\tau_2\tau_3 + \mathrm{I}\xi_7\alpha\mu\theta u\beta_5\tau_2\xi_4 \\
& + \mathrm{I}\xi_7\alpha\mu\theta u\beta_5\xi_3\xi_4 + \mathrm{I}\xi_7\alpha\mu\theta\beta_2\tau_1\tau_3\tau_4 + \mathrm{I}\xi_7\alpha\mu\theta\beta_2\tau_1\tau_3\xi_5 \\
& - \mathrm{I}\alpha k\mu\theta\beta_5\tau_1\xi_2\xi_4\xi_8 - \mathrm{I}\xi_7\alpha\mu k\theta\beta_5\xi_2\xi_3\xi_4 - \mathrm{I}\alpha k\mu\theta\beta_5\xi_2\xi_3\xi_4\xi_8 \\
& - \mathrm{I}\alpha k\mu\theta u\beta_4\xi_3\xi_5\xi_8 - \mathrm{I}\xi_7\alpha\mu k\theta u\beta_5\tau_2\tau_3 - \mathrm{I}\alpha k\mu\theta u\beta_5\tau_2\tau_3\xi_8 \\
& - \mathrm{I}\xi_7\alpha\mu k\theta u\beta_5\tau_2\xi_4 - \mathrm{I}\alpha k\mu\theta u\beta_5\tau_2\xi_4\xi_8 - \mathrm{I}\xi_7\alpha\mu k\theta u\beta_5\xi_3\xi_4 \\
& - \mathrm{I}\alpha k\mu\theta u\beta_5\xi_3\xi_4\xi_8 - \mathrm{I}\xi_7\alpha\mu k\theta\beta_2\tau_1\tau_3\tau_4 - \mathrm{I}\alpha k\mu\theta\beta_2\tau_1\tau_3\tau_4\xi_8 \\
& - \mathrm{I}\xi_7\alpha\mu k\theta\beta_2\tau_1\tau_3\xi_5 - \mathrm{I}\alpha k\mu\theta\beta_2\tau_1\tau_3\xi_5\xi_8 - \mathrm{I}\xi_7\alpha\mu k\theta\beta_2\tau_1\xi_4\xi_5
\end{aligned}
$$

$$
\begin{aligned}
&- \mathrm{I}\alpha k\mu\theta\beta_2\tau_1\xi_4\xi_5\xi_8 - \mathrm{I}\xi_7\alpha\mu k\theta\beta_2\xi_3\xi_4\xi_5 - \mathrm{I}\alpha k\mu\theta\beta_2\xi_3\xi_4\xi_5\xi_8\\
&- \mathrm{I}\xi_7\alpha\mu k\theta\beta_3\xi_2\xi_4\xi_5 - \mathrm{I}\alpha k\mu\theta\beta_3\xi_2\xi_4\xi_5\xi_8 - \mathrm{I}\xi_7\alpha\mu k\theta\beta_4\tau_1\xi_2\xi_5\\
&- \mathrm{I}\alpha k\mu\theta\beta_4\tau_1\xi_2\xi_5\xi_8 - \mathrm{I}\xi_7\alpha\mu k\theta\beta_4\xi_2\xi_3\xi_5 - \mathrm{I}\alpha k\mu\theta\beta_4\xi_2\xi_3\xi_5\xi_8\\
&- \mathrm{I}\xi_7\alpha\mu k\theta\beta_5\tau_1\tau_3\xi_2 - \mathrm{I}\alpha k\mu\theta\beta_5\tau_1\tau_3\xi_2\xi_8 - \mathrm{I}\xi_7\alpha\mu k\theta\beta_5\tau_1\xi_2\xi_4\\
&+ \mathrm{I}\xi_7\alpha\mu k\theta\beta_4\tau_2\xi_1\xi_5 + \mathrm{I}\alpha k\mu\theta\beta_4\tau_2\xi_1\xi_5\xi_8 + \mathrm{I}\xi_7\alpha\mu k\theta\beta_4\xi_1\xi_3\xi_5\\
&+ \mathrm{I}\alpha k\mu\theta\beta_4\xi_1\xi_3\xi_5\xi_8 + \mathrm{I}\xi_7\alpha\mu k\theta\beta_5\tau_2\tau_3\xi_1 + \mathrm{I}\alpha k\mu\theta\beta_5\tau_2\tau_3\xi_1\xi_8\\
&+ \mathrm{I}\xi_7\alpha\mu k\theta\beta_5\tau_2\xi_1\xi_4 + \mathrm{I}\alpha k\mu\theta\beta_5\tau_2\xi_1\xi_4\xi_8 + \mathrm{I}\xi_7\alpha\mu k\theta\beta_5\xi_1\xi_3\xi_4\\
&+ \mathrm{I}\alpha k\mu\theta\beta_5\xi_1\xi_3\xi_4\xi_8 - \mathrm{I}\xi_7\alpha\mu k\theta u\beta_3\xi_4\xi_5 - \mathrm{I}\alpha k\mu\theta u\beta_3\xi_4\xi_5\xi_8\\
&- \mathrm{I}\xi_7\alpha\mu k\theta u\beta_4\tau_2\xi_5 - \mathrm{I}\alpha k\mu\theta u\beta_4\tau_2\xi_5\xi_8 - \mathrm{I}\xi_7\alpha\mu k\theta u\beta_4\xi_3\xi_5\\
&+ \mathrm{I}\xi_7\alpha\mu\beta_1\tau_2\tau_3\tau_4 + \mathrm{I}\xi_7\alpha\mu\beta_1\tau_2\tau_3\xi_5 + \mathrm{I}\xi_7\alpha\mu\beta_1\tau_2\xi_4\xi_5\\
&+ \mathrm{I}\xi_7\alpha\mu\beta_1\xi_3\xi_4\xi_5 + \mathrm{I}\xi_7\alpha\mu\beta_3\xi_1\xi_4\xi_5 + \mathrm{I}\xi_7\alpha\mu\beta_4\tau_2\xi_1\xi_5\\
&+ \mathrm{I}\xi_7\alpha\mu\beta_4\xi_1\xi_3\xi_5 + \mathrm{I}\xi_7\alpha\mu\beta_5\tau_2\tau_3\xi_1 + \mathrm{I}\xi_7\alpha\mu\beta_5\tau_2\xi_1\xi_4\\
&+ \mathrm{I}\xi_7\alpha\mu\beta_5\xi_1\xi_3\xi_4 + \mathrm{I}\xi_7\alpha\mu k\theta\beta_1\tau_2\tau_3\tau_4 + \mathrm{I}\alpha k\mu\theta\beta_1\tau_2\tau_3\tau_4\xi_8\\
&+ \mathrm{I}\xi_7\alpha\mu k\theta\beta_1\tau_2\tau_3\xi_5 + \mathrm{I}\alpha k\mu\theta\beta_1\tau_2\tau_3\xi_5\xi_8 + \mathrm{I}\xi_7\alpha\mu k\theta\beta_1\tau_2\xi_4\xi_5\\
&+ \mathrm{I}\alpha k\mu\theta\beta_1\tau_2\xi_4\xi_5\xi_8 + \mathrm{I}\xi_7\alpha\mu k\theta\beta_1\xi_3\xi_4\xi_5 + \mathrm{I}\alpha k\mu\theta\beta_1\xi_3\xi_4\xi_5\xi_8\\
&+ \mathrm{I}\xi_7\alpha\mu k\theta\beta_3\xi_1\xi_4\xi_5 + \mathrm{I}\alpha k\mu\theta\beta_3\xi_1\xi_4\xi_5\xi_8)
\end{aligned}
$$

$$
\begin{aligned}
a_3 = {}& \xi_7\xi_8(\alpha k\mu\theta\beta_1\beta_3\beta_4\beta_5\xi_7 + \alpha k\mu\theta\beta_1\beta_3\beta_4\beta_5\xi_8 - \alpha k\mu\theta\beta_2\beta_3\beta_4\beta_5\xi_7\\
&- \alpha k\mu\theta\beta_2\beta_3\beta_4\beta_5\xi_8 - \alpha k\mu\beta_1\beta_3\beta_4\beta_5\xi_7 - \alpha k\mu\beta_1\beta_3\beta_4\beta_5\xi_8\\
&- \alpha\mu\theta\beta_1\beta_3\beta_4\beta_5\xi_7 + \alpha\mu\theta\beta_2\beta_3\beta_4\beta_5\xi_7 + \alpha\mu\beta_1\beta_3\beta_4\beta_5\xi_7\\
&+ \beta_1\beta_2\beta_3\beta_4\xi_5\xi_6 + \beta_1\beta_2\beta_3\beta_5\xi_4\xi_6 + \beta_1\beta_2\beta_4\beta_5\xi_3\xi_6\\
&+ \beta_1\beta_3\beta_4\beta_5\xi_2\xi_6 + \beta_2\beta_3\beta_4\beta_5\xi_1\xi_6)
\end{aligned}
$$

APPENDIX B.

From equation 17: $Det(J_{(EEP)} - \lambda I)(f(\lambda)(\lambda+\mu)(\lambda+l_{10})(\lambda+l_9)) = 0$; $f(\lambda) = b_0\lambda^8 + b_1\lambda^7 + b_2\lambda^6 + b_3\lambda^5 + b_4\lambda^b + b_5\lambda^3 + b_6\lambda^2 + b_7\lambda^1 + b_8$, where b_i for $i = 1, 2, \cdots 8$ are:

$$b_1 = l_5 + l_4 + l_3 + l_8 + l_2 + l_6 + l_7 + l_1$$

$$
\begin{aligned}
b_2 = {}& -Ad_1 - Ad_2 + l_1l_2 + l_1l_3 + l_1l_4 + l_1l_5 + l_1l_6 + l_1l_7 + l_1l_8 + l_2l_3\\
&+ l_2l_4 + l_2l_5 + l_2l_6 + l_2l_7 + l_2l_8 + l_3l_4 + l_3l_5 + l_3l_6 + l_3l_7\\
&+ l_3l_8 + l_4l_5 + l_4l_6 + l_4l_7 + l_4l_8 + l_5l_6 + l_5l_7 + l_5l_8 + l_6l_7 + l_6l_8 + l_7l_8
\end{aligned}
$$

$$
\begin{aligned}
b_3 = {}& l_6l_7l_1 - Ad_1l_1 - Ad_1l_2 + l_1l_2l_6 + l_1l_2l_7\\
&+ l_2l_6l_7 - I\beta_2^2S_wd_1 - Ad_1l_5 - Ad_2l_5 - Ad_1l_4 - Ad_2l_4\\
&+ l_1l_2l_4 + l_1l_3l_4 + l_1l_4l_6 + l_1l_4l_7 + l_1l_4l_8\\
&+ l_2l_3l_4 + l_2l_4l_6 + l_2l_4l_7 + l_2l_4l_8 + l_3l_4l_6 + l_3l_4l_7\\
&+ l_3l_4l_8 + l_4l_6l_7 + l_4l_6l_8 + l_4l_7l_8 + l_1l_2l_5 + l_1l_3l_5\\
&+ l_1l_4l_5 + l_1l_5l_6 + l_1l_5l_7 + l_1l_5l_8\\
&+ l_2l_3l_5 + l_2l_4l_5 + l_2l_5l_6 + l_2l_5l_7 + l_2l_5l_8\\
&+ l_3l_4l_5 + l_3l_5l_6 + l_3l_5l_7 + l_3l_5l_8 + l_4l_5l_6 + l_4l_5l_7\\
&+ l_4l_5l_8 + l_5l_6l_7 + l_5l_6l_8 + l_5l_7l_8 + l_1l_3l_6\\
&+ l_1l_3l_7 + l_1l_3l_8 + l_2l_3l_6 + l_2l_3l_7 + l_2l_3l_8 + l_3l_6l_7 + l_3l_6l_8\\
&+ l_3l_7l_8 - Ad_1l_3 - Ad_2l_3 + l_1l_2l_3 - I\beta_1^2S_hd_1\\
&+ l_2l_7l_8 + l_6l_7l_8 - d_2Al_2 - d_2Al_7 - d_2Al_1 - Ad_1l_8 + l_1l_2l_8\\
&+ l_1l_6l_8 + l_1l_7l_8 + l_2l_6l_8 - I\beta_3^2V_1d_1\\
&- I\beta_3^2V_1d_2 - IS_w\beta_2^2d_2 - IS_h\beta_1^2d_2 - I\beta_5^2V_bd_1\\
&- I\beta_5^2V_bd_2 - I\beta_4^2V_2d_1 - I\beta_4^2V_2d_2
\end{aligned}
$$

$$
\begin{aligned}
b_4 = {}& -IV_2\beta_4\beta_5d_1\tau_4 - IV_2\beta_4\beta_5d_2\tau_4 - IV_1\beta_3\beta_4d_1\tau_3\\
&- IV_1\beta_3\beta_4d_2\tau_3 - IS_h\beta_1^2d_1l_2 - I\beta_2^2S_wd_1l_1 - IV_1\beta_3^2d_2l_5\\
&- IV_2\beta_4^2d_1l_5 - IV_2\beta_4^2d_2l_5 - IV_b\beta_5^2d_1l_1 - IV_b\beta_5^2d_1l_2\\
&- IV_b\beta_5^2d_1l_3 - IV_b\beta_5^2d_1l_4 - IV_b\beta_5^2d_1l_8 - IV_b\beta_5^2d_2l_1\\
&- IV_b\beta_5^2d_2l_2 - IV_b\beta_5^2d_2l_3 - IV_b\beta_5^2d_2l_4 - IV_b\beta_5^2d_2l_7\\
&- IS_h\beta_1^2d_1l_4 - IS_h\beta_1^2d_2l_4 - IS_w\beta_2^2d_1l_4 - IS_w\beta_2^2d_2l_4\\
&- IV_1\beta_3^2d_1l_4 - IV_1\beta_3^2d_2l_4 - IV_2\beta_4^2d_1l_1 - IV_2\beta_4^2d_1l_2\\
&- IV_2\beta_4^2d_1l_3 - IV_2\beta_4^2d_1l_8 - IV_2\beta_4^2d_2l_1 - IV_2\beta_4^2d_2l_2\\
&- IV_2\beta_4^2d_2l_3 - IV_2\beta_4^2d_2l_7 - IS_h\beta_1^2d_2l_7 - IS_h\beta_1^2d_2l_2\\
&- IS_h\beta_1^2d_1l_8 - IS_w\beta_2^2d_2l_1 - IS_w\beta_2^2d_1l_8 - IS_w\beta_2^2d_2l_7\\
&- IS_h\beta_1^2d_1l_3 - IS_h\beta_1^2d_2l_3 - IS_w\beta_2^2d_1l_3 - IS_w\beta_2^2d_2l_3\\
&- IV_1\beta_3^2d_1l_1 - IV_1\beta_3^2d_1l_2 - IV_1\beta_3^2d_1l_8 - IV_1\beta_3^2d_2l_1\\
&- IV_1\beta_3^2d_2l_2 - IV_1\beta_3^2d_2l_7 - Ad_1l_1l_5 - Ad_1l_2l_5\\
&- Ad_1l_3l_5 - Ad_1l_4l_5 - Ad_1l_5l_8 - Ad_2l_1l_5 - Ad_2l_2l_5\\
&- Ad_2l_3l_5 - Ad_2l_4l_5 - Ad_2l_5l_7 + l_1l_2l_3l_5 + l_1l_2l_4l_5\\
&+ l_1l_2l_5l_6 + l_1l_2l_5l_7 + l_1l_2l_5l_8 + l_1l_3l_4l_5 + l_1l_3l_5l_6\\
&+ l_1l_3l_5l_7 + l_1l_3l_5l_8 + l_1l_4l_5l_6 + l_1l_4l_5l_7 + l_1l_4l_5l_8\\
&+ l_1l_5l_6l_7 + l_1l_5l_6l_8 + l_1l_5l_7l_8 + l_2l_3l_4l_5 + l_2l_3l_5l_6\\
&+ l_2l_3l_5l_7 + l_2l_3l_5l_8 + l_2l_4l_5l_6 + l_2l_4l_5l_7 + l_2l_4l_5l_8\\
&+ l_2l_5l_6l_7 + l_2l_5l_6l_8 + l_2l_5l_7l_8 + l_3l_4l_5l_6 + l_3l_4l_5l_7\\
&+ l_3l_4l_5l_8 + l_3l_5l_6l_7 + l_3l_5l_6l_8 + l_3l_5l_7l_8 + l_4l_5l_6l_7\\
&+ l_4l_5l_6l_8 + l_4l_5l_7l_8 + l_5l_6l_7l_8 - Iu\beta_1S_hd_1\beta_2\\
&- IS_h\beta_1^2d_1l_5 - IS_h\beta_1^2d_2l_5 - IS_w\beta_2^2d_1l_5 - IS_w\beta_2^2d_2l_5\\
&- IV_1\beta_3^2d_1l_5 - Ad_1l_1l_2 + l_1l_2l_6l_7 + l_1l_2l_6l_8 + l_1l_2l_7l_8\\
&+ l_1l_6l_7l_8 + l_2l_6l_7l_8 - Ad_1l_1l_8 - Ad_1l_2l_8 - Ad_2l_1l_2\\
&- Ad_2l_1l_7 - Ad_2l_2l_7 - Ad_1l_1l_3 - Ad_1l_2l_3\\
&- Ad_1l_3l_8 - Ad_2l_1l_3 - Ad_2l_2l_3 - Ad_2l_3l_7 + l_1l_2l_3l_6\\
&+ l_1l_2l_3l_7 + l_1l_2l_3l_8 + l_1l_3l_6l_7 + l_1l_3l_6l_8 + l_1l_3l_7l_8 + l_2l_3l_6l_7\\
&+ l_2l_3l_6l_8 + l_2l_3l_7l_8 + l_3l_6l_7l_8 - IS_h\beta_1\beta_3d_1\tau_1\\
&- IS_h\beta_1\beta_3d_2\tau_1 - IS_w\beta_2\beta_3d_1\tau_2 - IS_w\beta_2\beta_3d_2\tau_2\\
&- IS_hu\beta_1\beta_2d_2 - Ad_1l_1l_4 - Ad_1l_2l_4 - Ad_1l_3l_4\\
&- Ad_1l_4l_8 - Ad_2l_1l_4 - Ad_2l_2l_4 - Ad_2l_3l_4\\
&- Ad_2l_4l_7 + l_1l_2l_3l_4 + l_1l_2l_4l_6 + l_1l_2l_4l_7 + l_1l_2l_4l_8\\
&+ l_1l_3l_4l_6 + l_1l_3l_4l_7 + l_1l_3l_4l_8 + l_1l_4l_6l_7 + l_1l_4l_6l_8\\
&+ l_1l_4l_7l_8 + l_2l_3l_4l_6 + l_2l_3l_4l_7 + l_2l_3l_4l_8 + l_2l_4l_6l_7\\
&+ l_2l_4l_6l_8 + l_2l_4l_7l_8 + l_3l_4l_6l_7 + l_3l_4l_6l_8 + l_3l_4l_7l_8 + l_4l_6l_7l_8
\end{aligned}
$$

$$
\begin{aligned}
b_5 = {}& -IV_1\beta_3^2d_1l_1l_4 - IV_1\beta_3^2d_1l_2l_4 - IV_1\beta_3^2d_1l_4l_8\\
&- IV_1\beta_3^2d_2l_1l_4 - IV_1\beta_3^2d_2l_2l_4 - IV_1\beta_3^2d_2l_4l_7 - IV_2\beta_4^2d_1l_1l_2\\
&- IV_2\beta_4^2d_1l_1l_3 - IV_2\beta_4^2d_1l_1l_8 - IV_2\beta_4^2d_1l_2l_3 - IV_2\beta_4^2d_1l_2l_8
\end{aligned}
$$

$$- IV_2\beta_4^2 d_1 l_3 l_8 - IV_2\beta_4^2 d_2 l_1 l_2 - IV_2\beta_4^2 d_2 l_1 l_3 - IV_2\beta_4^2 d_2 l_1 l_7$$
$$- IV_2\beta_4^2 d_2 l_2 l_3 - IV_2\beta_4^2 d_2 l_2 l_7 - IV_2\beta_4^2 d_2 l_3 l_7 - IS_h\beta_1^2 d_1 l_2 l_4$$
$$- IS_h\beta_1^2 d_1 l_3 l_4 - IS_h\beta_1^2 d_1 l_4 l_8 - IS_h\beta_1^2 d_2 l_2 l_4 - IS_h\beta_1^2 d_2 l_3 l_4$$
$$- IS_h\beta_1^2 d_2 l_4 l_7 - IS_w\beta_2^2 d_1 l_1 l_4 - IS_w\beta_2^2 d_1 l_3 l_4 - IS_w\beta_2^2 d_1 l_4 l_8$$
$$- IS_w\beta_2^2 d_2 l_1 l_4 - IS_w\beta_2^2 d_2 l_3 l_4 - IS_w\beta_2^2 d_2 l_4 l_7 - Ad_1 l_1 l_2 l_3$$
$$- Ad_1 l_1 l_3 l_8 - Ad_1 l_2 l_3 l_8 - Ad_2 l_1 l_2 l_3 - Ad_2 l_1 l_3 l_7 - Ad_2 l_2 l_3 l_7$$
$$+ l_1 l_2 l_3 l_6 l_7 + l_1 l_2 l_3 l_6 l_8 + l_1 l_2 l_3 l_7 l_8 + l_1 l_3 l_6 l_7 l_8 + l_2 l_3 l_6 l_7 l_8$$
$$- Ad_1 l_1 l_2 l_4 - Ad_1 l_1 l_3 l_4 - Ad_1 l_1 l_4 l_8 - Ad_1 l_2 l_3 l_4 - Ad_1 l_2 l_4 l_8$$
$$- Ad_1 l_3 l_4 l_8 - Ad_2 l_1 l_2 l_4 - Ad_2 l_1 l_3 l_4 - Ad_2 l_1 l_4 l_7 - Ad_2 l_2 l_3 l_4$$
$$- Ad_2 l_2 l_4 l_7 - Ad_2 l_3 l_4 l_7 + l_1 l_2 l_3 l_4 l_6 + l_1 l_2 l_3 l_4 l_7 + l_1 l_2 l_3 l_4 l_8$$
$$+ l_1 l_2 l_4 l_6 l_7 + l_1 l_2 l_4 l_6 l_8 + l_1 l_2 l_4 l_7 l_8 + l_1 l_3 l_4 l_6 l_7 + l_1 l_3 l_4 l_6 l_8$$
$$+ l_1 l_3 l_4 l_7 l_8 + l_1 l_4 l_6 l_7 l_8 + l_2 l_3 l_4 l_6 l_7 + l_2 l_3 l_4 l_6 l_8 + l_2 l_3 l_4 l_7 l_8$$
$$+ l_2 l_4 l_6 l_7 l_8 + l_3 l_4 l_6 l_7 l_8 + l_1 l_2 l_6 l_7 l_8 - Ad_1 l_1 l_2 l_8 - Ad_2 l_1 l_2 l_7$$
$$- IS_h\beta_1^2 d_1 l_2 l_8 - IS_h\beta_1^2 d_2 l_2 l_7 - IS_w\beta_2^2 d_1 l_1 l_8 - IS_w\beta_2^2 d_2 l_1 l_7$$
$$- IV_1\beta_3\beta_5 d_2\tau_3\tau_4 - IV_1\beta_3\beta_5 d_1\tau_3\tau_4 - IS_h\beta_1\beta_4 d_1\tau_1\tau_3$$
$$- IS_h\beta_1\beta_4 d_2\tau_1\tau_3 - IS_w\beta_2\beta_4 d_1\tau_2\tau_3 - IS_w\beta_2\beta_4 d_2\tau_2\tau_3$$
$$- IS_h u\beta_1\beta_2 d_1 l_4 - IS_h u\beta_1\beta_2 d_2 l_4 - IS_h\beta_1\beta_3 d_1 l_4\tau_1$$
$$- IS_h\beta_1\beta_3 d_2 l_4\tau_1 - IS_w\beta_2\beta_3 d_1 l_4\tau_2 - IS_w\beta_2\beta_3 d_2 l_4\tau_2$$
$$- IV_1\beta_3\beta_4 d_1 l_1\tau_3 - IV_1\beta_3\beta_4 d_1 l_2\tau_3 - IV_1\beta_3\beta_4 d_1 l_8\tau_3$$
$$- IV_1\beta_3\beta_4 d_2 l_1\tau_3 - IV_1\beta_3\beta_4 d_2 l_2\tau_3 - IV_1\beta_3\beta_4 d_2 l_7\tau_3$$
$$- IS_h u\beta_1\beta_2 d_1 l_5 - IS_h u\beta_1\beta_2 d_2 l_5 - IS_h\beta_1\beta_3 d_1 l_5\tau_1$$
$$- IS_h\beta_1\beta_3 d_2 l_5\tau_1 - IS_w\beta_2\beta_3 d_1 l_5\tau_2 - IS_w\beta_2\beta_3 d_2 l_5\tau_2$$
$$- IV_1\beta_3\beta_4 d_1 l_5\tau_3 - IS_h\beta_1^2 d_1 l_2 l_3 - IS_h\beta_1^2 d_1 l_3 l_8$$
$$- IS_h\beta_1^2 d_2 l_2 l_3 - IS_h\beta_1^2 d_2 l_3 l_7 - IS_w\beta_2^2 d_1 l_1 l_3$$
$$- IS_w\beta_2^2 d_1 l_3 l_8 - IS_w\beta_2^2 d_2 l_1 l_3 - IS_w\beta_2^2 d_2 l_3 l_7$$
$$- IV_1\beta_3^2 d_1 l_1 l_2 - IV_1\beta_3^2 d_1 l_1 l_8 - IV_1\beta_3^2 d_1 l_2 l_8$$
$$- IV_1\beta_3^2 d_2 l_1 l_2 - IV_1\beta_3^2 d_2 l_1 l_7 - IV_1\beta_3^2 d_2 l_2 l_7$$
$$- Ad_1 l_1 l_2 l_5 - Ad_1 l_1 l_3 l_5 - Ad_1 l_1 l_4 l_5 - Ad_1 l_1 l_5 l_8 - Ad_1 l_2 l_3 l_5$$
$$- Ad_1 l_2 l_4 l_5 - Ad_1 l_2 l_5 l_8 - Ad_1 l_3 l_4 l_5 - Ad_1 l_3 l_5 l_8 - Ad_1 l_4 l_5 l_8$$
$$- Ad_2 l_1 l_2 l_5 - Ad_2 l_1 l_3 l_5 - Ad_2 l_1 l_4 l_5 - Ad_2 l_1 l_5 l_7 - Ad_2 l_2 l_3 l_5$$
$$- Ad_2 l_2 l_4 l_5 - Ad_2 l_2 l_5 l_7 - Ad_2 l_3 l_4 l_5 - Ad_2 l_3 l_5 l_7 - Ad_2 l_4 l_5 l_7$$
$$+ l_1 l_2 l_3 l_4 l_5 + l_1 l_2 l_3 l_5 l_6 + l_1 l_2 l_3 l_5 l_7 + l_1 l_2 l_3 l_5 l_8 + l_1 l_2 l_4 l_5 l_6$$
$$+ l_1 l_2 l_4 l_5 l_7 + l_1 l_2 l_4 l_5 l_8 + l_1 l_2 l_5 l_6 l_7 + l_1 l_2 l_5 l_6 l_8 + l_1 l_2 l_5 l_7 l_8$$
$$+ l_1 l_3 l_4 l_5 l_6 + l_1 l_3 l_4 l_5 l_7 + l_1 l_3 l_4 l_5 l_8 + l_1 l_3 l_5 l_6 l_7 + l_1 l_3 l_5 l_6 l_8$$
$$+ l_1 l_3 l_5 l_7 l_8 + l_1 l_4 l_5 l_6 l_7 + l_1 l_4 l_5 l_6 l_8 + l_1 l_4 l_5 l_7 l_8 + l_1 l_5 l_6 l_7 l_8$$
$$+ l_2 l_3 l_4 l_5 l_6 + l_2 l_3 l_4 l_5 l_7 + l_2 l_3 l_4 l_5 l_8 + l_2 l_3 l_5 l_6 l_7 + l_2 l_3 l_5 l_6 l_8$$
$$+ l_2 l_3 l_5 l_7 l_8 + l_2 l_4 l_5 l_6 l_7 + l_2 l_4 l_5 l_6 l_8 + l_2 l_4 l_5 l_7 l_8 + l_2 l_5 l_6 l_7 l_8$$
$$+ l_3 l_4 l_5 l_6 l_7 + l_3 l_4 l_5 l_6 l_8 + l_3 l_4 l_5 l_7 l_8 + l_3 l_5 l_6 l_7 l_8 + l_4 l_5 l_6 l_7 l_8$$
$$- IS_h u \cdots$$

$$b_5 = \cdots \beta_1\beta_3 d_2\tau_2 - IS_h u\beta_1\beta_3 d_1\tau_2 - IS_h u\beta_1\beta_2 d_1 l_8$$
$$- IS_h u\beta_1\beta_2 d_2 l_7 - IS_h u\beta_1\beta_2 d_1 l_3 - IS_h u\beta_1\beta_2 d_2 l_3$$
$$- IS_h\beta_1\beta_3 d_1 l_2\tau_1 - IS_h\beta_1\beta_3 d_1 l_8\tau_1 - IS_h\beta_1\beta_3 d_2 l_2\tau_1$$
$$- IS_h\beta_1\beta_3 d_2 l_7\tau_1 - IS_w\beta_2\beta_3 d_1 l_1\tau_2 - IS_w\beta_2\beta_3 d_1 l_8\tau_2$$
$$- IS_w\beta_2\beta_3 d_2 l_1\tau_2 - IS_w\beta_2\beta_3 d_2 l_7\tau_2 - IS_h\beta_1^2 d_1 l_2 l_5$$
$$- IS_h\beta_1^2 d_1 l_4 l_5 - IS_h\beta_1^2 d_1 l_5 l_8 - IS_h\beta_1^2 d_2 l_2 l_5$$
$$- IS_h\beta_1^2 d_2 l_3 l_5 - IS_h\beta_1^2 d_2 l_4 l_5 - IS_h\beta_1^2 d_2 l_5 l_7$$
$$- IS_w\beta_2^2 d_1 l_1 l_5 - IS_w\beta_2^2 d_1 l_3 l_5 - IS_w\beta_2^2 d_1 l_4 l_5$$
$$- IS_w\beta_2^2 d_1 l_5 l_8 - IS_w\beta_2^2 d_2 l_1 l_5 - IS_w\beta_2^2 d_2 l_3 l_5$$
$$- IS_w\beta_2^2 d_2 l_4 l_5 - IS_w\beta_2^2 d_2 l_5 l_7 - IV_1\beta_3^2 d_1 l_1 l_5$$
$$- IV_1\beta_3^2 d_1 l_2 l_5 - IV_1\beta_3^2 d_1 l_4 l_5 - IV_1\beta_3^2 d_1 l_5 l_8 - IV_1\beta_3^2 d_2 l_1 l_5$$
$$- IV_1\beta_3^2 d_2 l_2 l_5 - IV_1\beta_3^2 d_2 l_4 l_5 - IV_1\beta_3^2 d_2 l_5 l_7 - IV_2\beta_4^2 d_1 l_1 l_5$$
$$- IV_2\beta_4^2 d_1 l_2 l_5 - IV_2\beta_4^2 d_1 l_3 l_5 - IV_2\beta_4^2 d_1 l_5 l_8 - IV_2\beta_4^2 d_2 l_1 l_5$$
$$- IV_2\beta_4^2 d_2 l_2 l_5 - IV_2\beta_4^2 d_2 l_3 l_5 - IV_2\beta_4^2 d_2 l_5 l_7 - IV_b\beta_5^2 d_1 l_1 l_2$$
$$- IV_b\beta_5^2 d_1 l_1 l_3 - IV_b\beta_5^2 d_1 l_1 l_4 - IV_b\beta_5^2 d_1 l_1 l_8 - IV_b\beta_5^2 d_1 l_2 l_3$$
$$- IV_b\beta_5^2 d_1 l_2 l_4 - IV_b\beta_5^2 d_1 l_2 l_8 - IV_b\beta_5^2 d_1 l_3 l_4 - IV_b\beta_5^2 d_1 l_3 l_8 - IV_b\beta_5^2 d_1 l_4 l_8$$
$$- IV_b\beta_5^2 d_2 l_1 l_2 - IV_b\beta_5^2 d_2 l_1 l_3 - IV_b\beta_5^2 d_2 l_1 l_4 - IV_b\beta_5^2 d_2 l_1 l_7$$
$$- IV_b\beta_5^2 d_2 l_2 l_3 - IV_b\beta_5^2 d_2 l_2 l_4 - IV_b\beta_5^2 d_2 l_2 l_7 - IV_b\beta_5^2 d_2 l_3 l_4$$
$$- IV_b\beta_5^2 d_2 l_3 l_7 - IV_b\beta_5^2 d_2 l_4 l_7 - IV_1\beta_3\beta_4 d_2 l_5\tau_3 - IV_2\beta_4\beta_5 d_1 l_1\tau_4$$
$$- IV_2\beta_4\beta_5 d_1 l_2\tau_4 - IV_2\beta_4\beta_5 d_1 l_3\tau_4 - IV_2\beta_4\beta_5 d_1 l_8\tau_4 - IV_2\beta_4\beta_5 d_2 l_1\tau_4$$
$$- IV_2\beta_4\beta_5 d_2 l_2\tau_4 - IV_2\beta_4\beta_5 d_2 l_3\tau_4 - IV_2\beta_4\beta_5 d_2 l_7\tau_4$$

$$b_6 = -Ad_1 l_1 l_2 l_3 l_4 - Ad_1 l_1 l_2 l_4 l_8 - Ad_1 l_1 l_3 l_4 l_8 - Ad_1 l_2 l_3 l_4 l_8$$
$$- Ad_2 l_1 l_2 l_3 l_4 - Ad_2 l_1 l_2 l_4 l_7 - Ad_2 l_1 l_3 l_4 l_7 - Ad_2 l_2 l_3 l_4 l_7$$
$$+ l_1 l_2 l_3 l_4 l_6 l_7 + l_1 l_2 l_3 l_4 l_6 l_8 + l_1 l_2 l_3 l_4 l_7 l_8 + l_1 l_2 l_4 l_6 l_7 l_8$$
$$+ l_1 l_3 l_4 l_6 l_7 l_8 + l_2 l_3 l_4 l_6 l_7 l_8 - IS_h u\beta_1\beta_2 d_1 l_3 l_8 - IS_h u\beta_1\beta_2 d_2 l_3 l_7$$
$$- IS_h u\beta_1\beta_3 d_1 l_8\tau_2 - IS_h u\beta_1\beta_3 d_2 l_7\tau_2 - IS_h\beta_1\beta_3 d_1 l_2 l_8\tau_1$$
$$- IS_h\beta_1\beta_3 d_2 l_2 l_7\tau_1 - IS_w\beta_2\beta_3 d_1 l_1 l_8\tau_2 - IS_w\beta_2\beta_3 d_2 l_1 l_7\tau_2$$
$$- IS_h\beta_1\beta_5 d_1\tau_1\tau_3\tau_4 - IS_h\beta_1\beta_5 d_2\tau_1\tau_3\tau_4 - IS_w\beta_2\beta_5 d_1\tau_2\tau_3\tau_4$$
$$- IS_w\beta_2\beta_5 d_2\tau_2\tau_3\tau_4 - IS_h\beta_1\beta_3 d_1 l_4 l_8\tau_1 - IS_h\beta_1\beta_3 d_2 l_2 l_4\tau_1$$
$$- IS_h\beta_1\beta_3 d_2 l_4 l_7\tau_1 - IS_h\beta_1\beta_4 d_1 l_2\tau_1\tau_3 - IS_h\beta_1\beta_4 d_1 l_8\tau_1\tau_3$$
$$- IS_h\beta_1\beta_4 d_2 l_2\tau_1\tau_3 - IS_h\beta_1\beta_4 d_2 l_7\tau_1\tau_3 - IS_w\beta_2\beta_3 d_1 l_1 l_4\tau_2$$
$$- IS_w\beta_2\beta_3 d_1 l_4 l_8\tau_2 - IS_w\beta_2\beta_3 d_2 l_1 l_4\tau_2 - IS_w\beta_2\beta_3 d_2 l_4 l_7\tau_2$$
$$- IS_w\beta_2\beta_4 d_1 l_1\tau_2\tau_3 - IS_w\beta_2\beta_4 d_1 l_8\tau_2\tau_3 - IS_w\beta_2\beta_4 d_2 l_1\tau_2\tau_3$$
$$- IS_w\beta_2\beta_4 d_2 l_7\tau_2\tau_3 - IV_1\beta_3\beta_4 d_1 l_1 l_2\tau_3 - IV_1\beta_3\beta_4 d_1 l_1 l_8\tau_3$$
$$- IV_1\beta_3\beta_4 d_1 l_2 l_8\tau_3 - IV_1\beta_3\beta_4 d_2 l_1 l_2\tau_3 - IV_1\beta_3\beta_4 d_2 l_1 l_7\tau_3$$
$$- IV_1\beta_3\beta_4 d_2 l_2 l_7\tau_3 - IS_h u\beta_1\beta_2 d_1 l_3 l_4 - IS_h u\beta_1\beta_2 d_1 l_4 l_8$$
$$- IS_h u\beta_1\beta_2 d_2 l_3 l_4 - IS_h u\beta_1\beta_2 d_2 l_4 l_7 - IS_h u\beta_1\beta_3 d_1 l_4\tau_2$$
$$- IS_h u\beta_1\beta_3 d_2 l_4\tau_2 - IS_h\beta_1\beta_3 d_1 l_2 l_4\tau_1 - IS_h u\beta_1\beta_4 d_1\tau_2\tau_3$$
$$- IS_h u\beta_1\beta_4 d_2\tau_2\tau_3 - IS_h\beta_1^2 d_1 l_2 l_3 l_4 - IS_h\beta_1^2 d_1 l_2 l_4 l_8$$
$$- IS_h\beta_1^2 d_1 l_3 l_4 l_8 - IS_h\beta_1^2 d_2 l_2 l_3 l_4 - IS_h\beta_1^2 d_2 l_2 l_4 l_7$$
$$- IS_h\beta_1^2 d_2 l_3 l_4 l_7 - IS_w\beta_2^2 d_1 l_1 l_3 l_4 - IS_w\beta_2^2 d_1 l_1 l_4 l_8$$
$$- IS_w\beta_2^2 d_1 l_3 l_4 l_8 - IS_w\beta_2^2 d_2 l_1 l_3 l_4 - IS_w\beta_2^2 d_2 l_1 l_4 l_7$$
$$- IS_w\beta_2^2 d_2 l_3 l_4 l_7 - IV_1\beta_3^2 d_1 l_1 l_2 l_4 - IV_1\beta_3^2 d_1 l_1 l_4 l_8$$
$$- IV_1\beta_3^2 d_1 l_2 l_4 l_8 - IV_1\beta_3^2 d_2 l_1 l_2 l_4 - IV_1\beta_3^2 d_2 l_1 l_4 l_7$$
$$- IV_1\beta_3^2 d_2 l_2 l_4 l_7 - IV_2\beta_4^2 d_1 l_1 l_2 l_3 - IV_2\beta_4^2 d_1 l_1 l_2 l_8$$
$$- IV_2\beta_4^2 d_1 l_1 l_3 l_8 - IV_2\beta_4^2 d_1 l_2 l_3 l_8 - IV_2\beta_4^2 d_2 l_1 l_2 l_3$$
$$- IV_2\beta_4^2 d_2 l_1 l_2 l_7 - IV_2\beta_4^2 d_2 l_1 l_3 l_7 - IV_2\beta_4^2 d_2 l_2 l_3 l_7$$

$$- IS_h\beta_1^2 d_1 l_2 l_3 l_5 - IS_h\beta_1^2 d_1 l_2 l_4 l_5 - IS_h\beta_1^2 d_1 l_2 l_5 l_8$$
$$- IS_h\beta_1^2 d_1 l_3 l_4 l_5 - IS_h\beta_1^2 d_1 l_3 l_5 l_8 - IS_h\beta_1^2 d_1 l_4 l_5 l_8$$
$$- IS_h\beta_1^2 d_2 l_2 l_3 l_5 - IS_h\beta_1^2 d_2 l_2 l_4 l_5 - IS_h\beta_1^2 d_2 l_2 l_5 l_7$$
$$- IS_h\beta_1^2 d_2 l_3 l_4 l_5 - IS_h\beta_1^2 d_2 l_3 l_5 l_7 - IS_h\beta_1^2 d_2 l_4 l_5 l_7$$
$$- IS_w\beta_2^2 d_1 l_1 l_3 l_5 - IS_w\beta_2^2 d_1 l_1 l_4 l_5 - IS_w\beta_2^2 d_1 l_1 l_5 l_8$$
$$- IS_w\beta_2^2 d_1 l_3 l_4 l_5 - IS_w\beta_2^2 d_1 l_3 l_5 l_8 - IS_w\beta_2^2 d_1 l_4 l_5 l_8$$
$$- IS_w\beta_2^2 d_2 l_1 l_3 l_5 - IS_w\beta_2^2 d_2 l_1 l_4 l_5 - IS_w\beta_2^2 d_2 l_1 l_5 l_7$$
$$- IS_w\beta_2^2 d_2 l_3 l_4 l_5 - IS_w\beta_2^2 d_2 l_3 l_5 l_7 - IS_w\beta_2^2 d_2 l_4 l_5 l_7$$
$$- IV_1\beta_3^2 d_1 l_1 l_2 l_5 - IV_1\beta_3^2 d_1 l_1 l_4 l_5 - IV_1\beta_3^2 d_1 l_1 l_5 l_8$$
$$- IV_1\beta_3^2 d_1 l_2 l_4 l_5 - IV_1\beta_3^2 d_1 l_2 l_5 l_8 - IV_1\beta_3^2 d_1 l_4 l_5 l_8$$
$$- IV_1\beta_3^2 d_2 l_1 l_2 l_5 - IV_1\beta_3^2 d_2 l_1 l_4 l_5 - IV_1\beta_3^2 d_2 l_1 l_5 l_7$$
$$- IV_1\beta_3^2 d_2 l_2 l_4 l_5 - IV_1\beta_3^2 d_2 l_2 l_5 l_7 - IV_1\beta_3^2 d_2 l_4 l_5 l_7$$
$$- IV_2\beta_4^2 d_1 l_1 l_2 l_5 - IV_2\beta_4^2 d_1 l_1 l_3 l_5 - IV_2\beta_4^2 d_1 l_1 l_5 l_8$$
$$- IV_2\beta_4^2 d_1 l_2 l_3 l_5 - IV_2\beta_4^2 d_1 l_2 l_5 l_8 - IV_2\beta_4^2 d_1 l_3 l_5 l_8$$
$$- IV_2\beta_4^2 d_2 l_1 l_2 l_5 - IV_2\beta_4^2 d_2 l_1 l_3 l_5 - IV_2\beta_4^2 d_2 l_1 l_5 l_7$$
$$- IV_2\beta_4^2 d_2 l_2 l_3 l_5 - IV_2\beta_4^2 d_2 l_2 l_5 l_7 - IV_2\beta_4^2 d_2 l_3 l_5 l_7$$
$$- IV_b\beta_5^2 d_1 l_1 l_2 l_3 - IV_b\beta_5^2 d_1 l_1 l_2 l_4 - IV_b\beta_5^2 d_1 l_1 l_2 l_8$$
$$- IV_b\beta_5^2 d_1 l_1 l_3 l_4 - IV_b\beta_5^2 d_1 l_1 l_3 l_8 - IV_b\beta_5^2 d_1 l_1 l_4 l_8$$
$$- IV_b\beta_5^2 d_1 l_2 l_3 l_4 - IV_b\beta_5^2 \cdots$$

$$b_6 = \cdots d_1 l_2 l_3 l_8 - Ad_1 l_1 l_2 l_3 l_5 - Ad_1 l_1 l_2 l_4 l_5 - Ad_1 l_1 l_2 l_5 l_8$$
$$- Ad_1 l_1 l_3 l_4 l_5 - Ad_1 l_1 l_3 l_5 l_8 - Ad_1 l_1 l_4 l_5 l_8 - Ad_1 l_2 l_3 l_4 l_5$$
$$- Ad_1 l_2 l_3 l_5 l_8 - Ad_1 l_2 l_4 l_5 l_8 - Ad_1 l_3 l_4 l_5 l_8 - Ad_2 l_1 l_2 l_3 l_5$$
$$- Ad_2 l_1 l_2 l_4 l_5 - Ad_2 l_1 l_2 l_5 l_7 - Ad_2 l_1 l_3 l_4 l_5 - Ad_2 l_1 l_3 l_5 l_7$$
$$- Ad_2 l_1 l_4 l_5 l_7 - Ad_2 l_2 l_3 l_4 l_5 - Ad_2 l_2 l_3 l_5 l_7 - Ad_2 l_2 l_4 l_5 l_7$$
$$- Ad_2 l_3 l_4 l_5 l_7 + l_1 l_2 l_3 l_4 l_5 l_6 + l_1 l_2 l_3 l_4 l_5 l_7 + l_1 l_2 l_3 l_4 l_5 l_8$$
$$+ l_1 l_2 l_3 l_5 l_6 l_7 + l_1 l_2 l_3 l_5 l_6 l_8 + l_1 l_2 l_3 l_5 l_7 l_8 + l_1 l_2 l_4 l_5 l_6 l_7$$
$$+ l_1 l_2 l_4 l_5 l_6 l_8 + l_1 l_2 l_4 l_5 l_7 l_8 + l_1 l_2 l_5 l_6 l_7 l_8 + l_1 l_3 l_4 l_5 l_6 l_7$$
$$+ l_1 l_3 l_4 l_5 l_6 l_8 + l_1 l_3 l_4 l_5 l_7 l_8 + l_1 l_3 l_5 l_6 l_7 l_8 + l_1 l_4 l_5 l_6 l_7 l_8$$
$$+ l_2 l_3 l_4 l_5 l_6 l_7 + l_2 l_3 l_4 l_5 l_6 l_8 + l_2 l_3 l_4 l_5 l_7 l_8 + l_2 l_3 l_5 l_6 l_7 l_8$$
$$+ l_2 l_4 l_5 l_6 l_7 l_8 + l_3 l_4 l_5 l_6 l_7 l_8 - Ad_1 l_1 l_2 l_3 l_8 - Ad_2 l_1 l_2 l_3 l_7$$
$$+ l_1 l_2 l_3 l_6 l_7 l_8 - IS_h u\beta_1\beta_2 d_1 l_3 l_5 - IS_h u\beta_1\beta_2 d_1 l_4 l_5$$
$$- IS_h u\beta_1\beta_2 d_1 l_5 l_8$$
$$- IS_h u\beta_1\beta_2 d_2 l_3 l_5 - IS_h u\beta_1\beta_2 d_2 l_4 l_5 - IS_h u\beta_1\beta_2 d_2 l_5 l_7$$
$$- IS_h u\beta_1\beta_3 d_1 l_5 \tau_2 - IS_h u\beta_1\beta_3 d_2 l_5 \tau_2 - IS_h\beta_1\beta_3 d_1 l_2 l_5 \tau_1$$
$$- IS_h\beta_1\beta_3 d_1 l_4 l_5 \tau_1 - IS_h\beta_1\beta_3 d_1 l_5 l_8 \tau_1 - IS_h\beta_1\beta_3 d_2 l_2 l_5 \tau_1$$
$$- IS_h\beta_1\beta_3 d_2 l_4 l_5 \tau_1 - IS_h\beta_1\beta_3 d_2 l_5 l_7 \tau_1 - IS_h\beta_1\beta_4 d_1 l_5 \tau_1 \tau_3$$
$$- IS_h\beta_1\beta_4 d_2 l_5 \tau_1 \tau_3 - IS_w\beta_2\beta_3 d_1 l_1 l_5 \tau_2 - IS_w\beta_2\beta_3 d_1 l_4 l_5 \tau_2$$
$$- IS_w\beta_2\beta_3 d_1 l_5 l_8 \tau_2 - IS_w\beta_2\beta_3 d_2 l_1 l_5 \tau_2 - IS_w\beta_2\beta_3 d_2 l_4 l_5 \tau_2$$
$$- IS_w\beta_2\beta_3 d_2 l_5 l_7 \tau_2 - IS_w\beta_2\beta_4 d_1 l_5 \tau_2 \tau_3 - IS_w\beta_2\beta_4 d_2 l_5 \tau_2 \tau_3$$
$$- IV_1\beta_3\beta_4 d_1 l_1 l_5 \tau_3 - IV_1\beta_3\beta_4 d_1 l_2 l_5 \tau_3 - IV_1\beta_3\beta_4 d_1 l_5 l_8 \tau_3$$
$$- IV_1\beta_3\beta_4 d_2 l_1 l_5 \tau_3 - IV_1\beta_3\beta_4 d_2 l_2 l_5 \tau_3 - IV_1\beta_3\beta_4 d_2 l_5 l_7 \tau_3$$
$$- IV_1\beta_3\beta_5 d_1 l_1 \tau_3 \tau_4 - IV_1\beta_3\beta_5 d_1 l_2 \tau_3 \tau_4 - IV_1\beta_3\beta_5 d_1 l_8 \tau_3 \tau_4$$
$$- IV_1\beta_3\beta_5 d_2 l_1 \tau_3 \tau_4 - IV_1\beta_3\beta_5 d_2 l_2 \tau_3 \tau_4 - IV_1\beta_3\beta_5 d_2 l_7 \tau_3 \tau_4$$
$$- IV_2\beta_4\beta_5 d_1 l_1 l_2 \tau_4 - IV_2\beta_4\beta_5 d_1 l_1 l_3 \tau_4 - IV_2\beta_4\beta_5 d_1 l_1 l_8 \tau_4$$
$$- IV_2\beta_4\beta_5 d_1 l_2 l_3 \tau_4 - IV_2\beta_4\beta_5 d_1 l_2 l_8 \tau_4 - IV_2\beta_4\beta_5 d_1 l_3 l_8 \tau_4$$
$$- IV_2\beta_4\beta_5 d_2 l_1 l_2 \tau_4 - IV_2\beta_4\beta_5 d_2 l_1 l_3 \tau_4 - IV_2\beta_4\beta_5 d_2 l_1 l_7 \tau_4$$
$$- IV_2\beta_4\beta_5 d_2 l_2 l_3 \tau_4 - IV_2\beta_4\beta_5 d_2 l_2 l_7 \tau_4 - IV_2\beta_4\beta_5 d_2 l_3 l_7 \tau_4$$
$$- IV_b\beta_5^2 d_1 l_2 l_4 l_8 - IV_b\beta_5^2 d_1 l_3 l_4 l_8 - IV_b\beta_5^2 d_2 l_1 l_2 l_3$$
$$- IV_b\beta_5^2 d_2 l_1 l_2 l_4 - IV_b\beta_5^2 d_2 l_1 l_2 l_7 - IV_b\beta_5^2 d_2 l_1 l_3 l_4$$
$$- IV_b\beta_5^2 d_2 l_1 l_3 l_7 - IV_b\beta_5^2 d_2 l_1 l_4 l_7 - IV_b\beta_5^2 d_2 l_2 l_3 l_4$$
$$- IV_b\beta_5^2 d_2 l_2 l_3 l_7 - IV_b\beta_5^2 d_2 l_2 l_4 l_7 - IV_b\beta_5^2 d_2 l_3 l_4 l_7$$
$$- IS_h\beta_1^2 d_1 l_2 l_3 l_8 - IS_h\beta_1^2 d_2 l_2 l_3 l_7 - IS_w\beta_2^2 d_1 l_1 l_3 l_8$$
$$- IS_w\beta_2^2 d_2 l_1 l_3 l_7 - IV_1\beta_3^2 d_1 l_1 l_2 l_8 - IV_1\beta_3^2 d_2 l_1 l_2 l_7$$

$$b_7 = -Ad_1 l_1 l_2 l_3 l_4 l_5 - Ad_1 l_1 l_2 l_3 l_5 l_8 - Ad_1 l_1 l_2 l_4 l_5 l_8$$
$$- Ad_1 l_1 l_3 l_4 l_5 l_8 - Ad_1 l_2 l_3 l_4 l_5 l_8 - Ad_2 l_1 l_2 l_3 l_4 l_5 - Ad_2 l_1 l_2 l_3 l_5 l_7$$
$$- Ad_2 l_1 l_2 l_4 l_5 l_7 - Ad_2 l_1 l_3 l_4 l_5 l_7 - Ad_2 l_2 l_3 l_4 l_5 l_7 + l_1 l_2 l_3 l_4 l_5 l_6 l_7$$
$$+ l_1 l_2 l_3 l_4 l_5 l_6 l_8 + l_1 l_2 l_3 l_4 l_5 l_7 l_8 + l_1 l_2 l_3 l_5 l_6 l_7 l_8 + l_1 l_2 l_4 l_5 l_6 l_7 l_8$$
$$+ l_1 l_3 l_4 l_5 l_6 l_7 l_8 + l_2 l_3 l_4 l_5 l_6 l_7 l_8 - IS_h\beta_1^2 d_1 l_2 l_3 l_4 l_8$$
$$- IS_h\beta_1^2 d_2 l_2 l_3 l_4 l_7 - IS_w\beta_2^2 d_1 l_1 l_3 l_4 l_8 - IS_w\beta_2^2 d_2 l_1 l_3 l_4 l_7$$
$$- IV_1\beta_3^2 d_1 l_1 l_2 l_4 l_8 - IV_1\beta_3^2 d_2 l_1 l_2 l_4 l_7 - IV_2\beta_4^2 d_1 l_1 l_2 l_3 l_8$$
$$- IV_2\beta_4^2 d_2 l_1 l_2 l_3 l_7 - IS_h u\beta_1\beta_5 d_1 \tau_2 \tau_3 \tau_4 - IS_h u\beta_1\beta_5 d_2 \tau_2 \tau_3 \tau_4$$
$$- IS_h u\beta_1\beta_2 d_1 l_3 l_4 l_5 - IS_h u\beta_1\beta_2 d_1 l_3 l_5 l_8 - IS_h u\beta_1\beta_2 d_1 l_4 l_5 l_8$$
$$- IS_h u\beta_1\beta_2 d_2 l_3 l_4 l_5 - IS_h u\beta_1\beta_2 d_2 l_3 l_5 l_7 - IS_h u\beta_1\beta_2 d_2 l_4 l_5 l_7$$
$$- IS_h u\beta_1\beta_3 d_1 l_4 l_5 \tau_2 - IS_h u\beta_1\beta_3 d_1 l_5 l_8 \tau_2 - IS_h u\beta_1\beta_3 d_2 l_4 l_5 \tau_2$$
$$- IS_h u\beta_1\beta_3 d_2 l_5 l_7 \tau_2 - IS_h u\beta_1\beta_4 d_1 l_5 \tau_2 \tau_3 - IS_h u\beta_1\beta_4 d_2 l_5 \tau_2 \tau_3$$
$$- IS_h\beta_1\beta_3 d_1 l_2 l_4 l_5 \tau_1 - IS_h\beta_1\beta_3 d_1 l_2 l_5 l_8 \tau_1 - IS_h\beta_1\beta_3 d_1 l_4 l_5 l_8 \tau_1$$
$$- IS_h\beta_1\beta_3 d_2 l_2 l_4 l_5 \tau_1 - IS_h\beta_1\beta_3 d_2 l_2 l_5 l_7 \tau_1 - IS_h\beta_1\beta_3 d_2 l_4 l_5 l_7 \tau_1$$
$$- IS_h\beta_1\beta_4 d_1 l_2 l_5 \tau_1 \tau_3 - IS_h\beta_1\beta_4 d_1 l_5 l_8 \tau_1 \tau_3 - IS_h\beta_1\beta_4 d_2 l_2 l_5 \tau_1 \tau_3$$
$$- IS_h\beta_1\beta_4 d_2 l_5 l_7 \tau_1 \tau_3 - Il_2 S_h\beta_1\beta_5 d_1 \tau_1 \tau_3 \tau_4 - Il_8 S_h\beta_1\beta_5 d_1 \tau_1 \tau_3 \tau_4$$
$$- Il_2 S_h\beta_1\beta_5 d_2 \tau_1 \tau_3 \tau_4 - Il_7 S_h\beta_1\beta_5 d_2 \tau_1 \tau_3 \tau_4 - IS_w\beta_2\beta_3 d_1 l_1 l_4 l_5 \tau_2$$
$$- IS_w\beta_2\beta_3 d_1 l_1 l_5 l_8 \tau_2 - IS_w\beta_2\beta_3 d_1 l_4 l_5 l_8 \tau_2 - IS_w\beta_2\beta_3 d_2 l_1 l_4 l_5 \tau_2$$
$$- IS_w\beta_2\beta_3 d_2 l_1 l_5 l_7 \tau_2 - IS_w\beta_2\beta_3 d_2 l_4 l_5 l_7 \tau_2 - IS_w\beta_2\beta_4 d_1 l_1 l_5 \tau_2 \tau_3$$
$$- IS_w\beta_2\beta_4 d_1 l_5 l_8 \tau_2 \tau_3 - IS_w\beta_2\beta_4 d_2 l_1 l_5 \tau_2 \tau_3 - IS_w\beta_2\beta_4 d_2 l_5 l_7 \tau_2 \tau_3$$
$$- Il_1 S_w\beta_2\beta_5 d_1 \tau_2 \tau_3 \tau_4 - Il_8 S_w\beta_2\beta_5 d_1 \tau_2 \tau_3 \tau_4 - Il_1 S_w\beta_2\beta_5 d_2 \tau_2 \tau_3 \tau_4$$
$$- Il_7 S_w\beta_2\beta_5 d_2 \tau_2 \tau_3 \tau_4 - IV_1\beta_3\beta_4 d_1 l_1 l_2 l_5 \tau_3 - IV_1\beta_3\beta_4 d_1 l_1 l_5 l_8 \tau_3$$
$$- IV_1\beta_3\beta_4 d_1 l_2 l_5 l_8 \tau_3 - IV_1\beta_3\beta_4 d_2 l_1 l_2 l_5 \tau_3 - IV_1\beta_3\beta_4 d_2 l_1 l_5 l_7 \tau_3$$
$$- IV_1\beta_3\beta_4 d_2 l_2 l_5 l_7 \tau_3 - IV_1\beta_3\beta_5 d_1 l_1 l_2 \tau_3 \tau_4 - IV_1\beta_3\beta_5 d_1 l_1 l_8 \tau_3 \tau_4$$
$$- IV_1\beta_3\beta_5 d_1 l_2 l_8 \tau_3 \tau_4 - IV_1\beta_3\beta_5 d_2 l_1 l_2 \tau_3 \tau_4 - IV_1\beta_3\beta_5 d_2 l_1 l_7 \tau_3 \tau_4$$
$$- IV_1\beta_3\beta_5 d_2 l_2 l_7 \tau_3 \tau_4 - IV_2\beta_4\beta_5 d_1 l_1 l_2 l_3 \tau_4 - IV_2\beta_4\beta_5 d_1 l_1 l_2 l_8 \tau_4$$
$$- IV_2\beta_4\beta_5 d_1 l_1 l_3 l_8 \tau_4 - IV_2\beta_4\beta_5 d_1 l_2 l_3 l_8 \tau_4 - IV_2\beta_4\beta_5 d_2 l_1 l_2 l_3 \tau_4$$
$$- IV_2\beta_4\beta_5 d_2 l_1 l_2 l_7 \tau_4 - IV_2\beta_4\beta_5 d_2 l_1 l_3 l_7 \tau_4 - IV_2\beta_4\beta_5 d_2 l_2 l_3 l_7 \tau_4$$
$$- IS_h\beta_1^2 d_1 l_2 l_3 l_4 l_5 - IS_h\beta_1^2 d_1 l_2 l_3 l_5 l_8 - IS_h\beta_1^2 d_1 l_2 l_4 l_5 l_8$$
$$- IS_h\beta_1^2 d_1 l_3 l_4 l_5 l_8 - IS_h\beta_1^2 d_2 l_2 l_3 l_4 l_5 - IS_h\beta_1^2 d_2 l_2 l_3 l_5 l_7$$
$$- IS_h\beta_1^2 d_2 l_2 l_4 l_5 l_7 - IS_h\beta_1^2 d_2 l_3 l_4 l_5 l_7 - IS_w\beta_2^2 d_1 l_1 l_3 l_4 l_5$$
$$- IS_w\beta_2^2 d_1 l_1 l_3 l_5 l_8 - IS_w\beta_2^2 d_1 l_1 l_4 l_5 l_8 - IS_w\beta_2^2 d_1 l_3 l_4 l_5 l_8$$
$$- IS_w\beta_2^2 d_2 l_1 l_3 l_4 l_5 - IS_w\beta_2^2 d_2 l_1 l_3 l_5 l_7 - IS_w\beta_2^2 d_2 l_1 l_4 l_5 l_7$$

$$
\begin{aligned}
&- IS_w\beta_2^2 d_2 l_3 l_4 l_5 l_7 - IV_1\beta_3^2 d_1 l_1 l_2 l_4 l_5 - IV_1\beta_3^2 d_1 l_1 l_2 l_5 l_8 \\
&- IV_1\beta_3^2 d_1 l_1 l_4 l_5 l_8 - IV_1\beta_3^2 d_1 l_2 l_4 l_5 l_8 - IV_1\beta_3^2 d_2 l_1 l_2 l_4 l_5 \\
&- IV_1\beta_3^2 d_2 l_1 l_2 l_5 l_7 - IV_1\beta_3^2 d_2 l_1 l_4 l_5 l_7 - IV_1\beta_3^2 d_2 l_2 l_4 l_5 l_7 \\
&- IV_2\beta_4^2 d_1 l_1 l_2 l_3 l_5 - IV_2\beta_4^2 d_1 l_1 l_2 l_5 l_8 - IV_2\beta_4^2 d_1 l_1 l_3 l_5 l_8 \\
&- IV_2\beta_4^2 d_1 l_2 l_3 l_5 l_8 - IV_2\beta_4^2 d_2 l_1 l_2 l_3 l_5 - IV_2\beta_4^2 d_2 l_1 l_2 l_5 l_7 \\
&- IV_2\beta_4^2 d_2 l_1 l_3 l_5 l_7 - IV_2\beta_4^2 d_2 l_2 l_3 l_5 l_7 - IV_b\beta_5^2 d_1 l_1 l_2 l_3 l_4 \\
&- IV_b\beta_5^2 d_1 l_1 l_2 l_3 l_8 - IV_b\beta_5^2 d_1 l_1 l_2 l_4 l_8 - IV_b\beta_5^2 d_1 l_1 l_3 l_4 l_8 \\
&- IV_b\beta_5^2 d_1 l_2 l_3 l_4 l_8 - IV_b\beta_5^2 d_2 l_1 l_2 l_3 l_4 - Ad_1 l_1 l_2 l_3 l_4 l_8 \\
&- Ad_2 l_1 l_2 l_3 l_4 l_7 + l_1 l_2 l_3 l_4 l_6 l_7 l_8 - IV_b\beta_5^2 d_2 l_1 l_2 l_3 l_7 \\
&- IV_b\beta_5^2 d_2 l_1 l_2 l_4 l_7 - IV_b\beta_5^2 d_2 l_1 l_3 l_4 l_7 - IV_b\beta_5^2 d_2 l_2 l_3 l_4 l_7 \\
&- IS_h u\beta_1\beta_2 d_1 l_3 l_4 l_8 - IS_h u\beta_1\beta_2 d_2 l_3 l_4 l_7 - IS_h u\beta_1\beta_3 d_1 l_4 l_8 \tau_2 \\
&- IS_h u\beta_1\beta_3 d_2 l_4 l_7 \tau_2 - Il_8 S_h u\beta_1\beta_4 d_1 \tau_2\tau_3 - Il_7 S_h u\beta_1\beta_4 d_2 \tau_2\tau_3 \\
&- IS_h\beta_1\beta_3 d_1 l_2 l_4 l_8 \tau_1 - IS_h\beta_1\beta_3 d_2 l_2 l_4 l_7 \tau_1 - IS_h\beta_1\beta_4 d_1 l_2 l_8 \tau_1\tau_3 \\
&- IS_h\beta_1\beta_4 d_2 l_2 l_7 \tau_1\tau_3 - IS_w\beta_2\beta_3 d_1 l_1 l_4 l_8 \tau_2 - IS_w\beta_2\beta_3 d_2 l_1 l_4 l_7 \tau_2 \\
&- IS_w\beta_2\beta_4 d_1 l_1 l_8 \tau_2\tau_3 - IS_w\beta_2\beta_4 d_2 l_1 l_7 \tau_2\tau_3 - IV_1\beta_3\beta_4 d_1 l_1 l_2 l_8 \tau_3 \\
&- IV_1\beta_3\beta_4 d_2 l_1 l_2 l_7 \tau_3
\end{aligned}
$$

$$
\begin{aligned}
b_8 = &-IS_h\beta_1^2 d_1 l_2 l_3 l_4 l_5 l_8 - IS_h\beta_1^2 d_2 l_2 l_3 l_4 l_5 l_7 - IS_w\beta_2^2 d_1 l_1 l_3 l_4 l_5 l_8 \\
&- IS_w\beta_2^2 d_2 l_1 l_3 l_4 l_5 l_7 - IV_1\beta_3^2 d_1 l_1 l_2 l_4 l_5 l_8 - IV_1\beta_3^2 d_2 l_1 l_2 l_4 l_5 l_7 \\
&- IV_2\beta_4^2 d_1 l_1 l_2 l_3 l_5 l_8 - IV_2\beta_4^2 d_2 l_1 l_2 l_3 l_5 l_7 - IV_b\beta_5^2 d_1 l_1 l_2 l_3 l_4 l_8 \\
&- IV_b\beta_5^2 d_2 l_1 l_2 l_3 l_4 l_7 - Ad_1 l_1 l_2 l_3 l_4 l_5 l_8 - Ad_2 l_1 l_2 l_3 l_4 l_5 l_7 \\
&+ l_1 l_2 l_3 l_4 l_5 l_6 l_7 l_8 - IS_h u\beta_1\beta_2 d_1 l_3 l_4 l_5 l_8 - IS_h u\beta_1\beta_2 d_2 l_3 l_4 l_5 l_7 \\
&- IS_h u\beta_1\beta_3 d_1 l_4 l_5 l_8 \tau_2 - IS_h u\beta_1\beta_3 d_2 l_4 l_5 l_7 \tau_2 - IS_h u\beta_1\beta_4 d_1 l_5 l_8 \tau_2\tau_3 \\
&- IS_h u\beta_1\beta_4 d_2 l_5 l_7 \tau_2\tau_3 - IS_h u\beta_1\beta_5 d_1 l_8 \tau_2\tau_3\tau_4 \\
&- IS_h u\beta_1\beta_5 d_2 l_7 \tau_2\tau_3\tau_4 - IS_h\beta_1\beta_3 d_1 l_2 l_4 l_5 l_8 \tau_1 \\
&- IS_h\beta_1\beta_3 d_2 l_2 l_4 l_5 l_7 \tau_1 - IS_h\beta_1\beta_4 d_1 l_2 l_5 l_8 \tau_1\tau_3 \\
&- IS_h\beta_1\beta_4 d_2 l_2 l_5 l_7 \tau_1\tau_3 - IS_h\beta_1\beta_5 d_1 l_2 l_8 \tau_1\tau_3\tau_4 \\
&- IS_h\beta_1\beta_5 d_2 l_2 l_7 \tau_1\tau_3\tau_4 - IS_w\beta_2\beta_3 d_1 l_1 l_4 l_5 l_8 \tau_2 \\
&- IS_w\beta_2\beta_3 d_2 l_1 l_4 l_5 l_7 \tau_2 - IS_w\beta_2\beta_4 d_1 l_1 l_5 l_8 \tau_2\tau_3 \\
&- IS_w\beta_2\beta_4 d_2 l_1 l_5 l_7 \tau_2\tau_3 - IS_w\beta_2\beta_5 d_1 l_1 l_8 \tau_2\tau_3\tau_4 - IS_w\beta_2\beta_5 d_2 l_1 l_7 \tau_2\tau_3\tau_4 \\
&- IV_1\beta_3\beta_4 d_1 l_1 l_2 l_5 l_8 \tau_3 - IV_1\beta_3\beta_4 d_2 l_1 l_2 l_5 l_7 \tau_3 - IV_1\beta_3\beta_5 d_1 l_1 l_2 l_8 \tau_3\tau_4 \\
&- IV_1\beta_3\beta_5 d_2 l_1 l_2 l_7 \tau_3\tau_4 - IV_2\beta_4\beta_5 d_1 l_1 l_2 l_3 l_8 \tau_4 - IV_2\beta_4\beta_5 d_2 l_1 l_2 l_3 l_7 \tau_4
\end{aligned}
$$