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A ninth-order first derivative method for numerical integration

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Abstract

In this paper, we present a ninth–order block hybrid method for the numerical solution of stiff and non–stiff systems of first–order differential equations. The method is based on an interpolation and collocation approach which results in a single continuous formulation; from which eight discrete schemes that make the block method were obtained. A convergence analysis of our method illustrated that it is $A(\alpha)$–stable, consistent, and convergent. We applied our method to some numerical examples which showed that the new method not only outperformed a second derivative method of order fourteen in the literature but also compared well with the exact solution.

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1. Introduction

Differential equations have their origins in Chemical Kinetics, Engineering, Physics etc. They may or may not have analytical solutions and there is a need to obtain reliable numerical methods to obtain their approximate solutions. One of the numerical methods for obtaining approximate solutions to ordinary differential equations (ODE) are Linear Multistep Methods (Linear Multistep Method) and most importantly block hybrid methods of various order. It is a known fact that the higher the order of a Linear Multistep Method with a corresponding low error constant, the better the accuracy of the method.

Iavernaro and Mazzia [1] used Linear Multistep Methods for a special class of nonlinear problems. Kirlinger [2] used Linear Multistep Methods in solving both linear and non-linear first-order ODEs. However, the authors did not use off–grid points. Albi *et al*. [3] analyzed high–order linear multistep schemes for the solution of ODE arising from optimal control and hyperbolic partial differential equations. Block methods have the added feature that all the discrete schemes which constitute the block method are obtained from a single continuous formulation and are of uniform order. For more on Linear Multistep Methods and block hybrid methods, see Refs. [4–7].

The authors in Ref. [8] used power series as a basis function with 4-grid and 4-off grid points which are $\{1/2, 3/2, 5/2, 7/2\}$ for the solution of fourth–order initial value problems. Gragg and Stetter [9] used a generalized predictor–corrector method, Ref. [10] used a generalized version of the Runge–Kutta method while Refs. [11, 12] used single–step methods. However, in this work, we used the four off-grid points: $\{3/2, 5/2, 7/2, 9/2\}$ instead besides being a four-and-a-half block hybrid method, see also Refs. [13] and the references therein. Kamoh *et al*. [14] used a shifted Legendre polynomial

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to derive a ninth-order block hybrid method for the solution of ODE using the following off-grid points: $\{9/2, 13/3, 14/3\}$. However, our off-grid points are different and the problems considered in that paper were non–stiff while the present work can numerically integrate both stiff and non-stiff ODEs. In [15], the authors used Hermite polynomial as basis functions in deriving the numerical schemes (see also the works of Fotta *et al.* [16], Ayinde and Ibijola [17], Odejide and Adeniran [18] and Akinola *et al.* [19]).

Besides, Ismail *et al.* [20] developed a two–point, fifth order explicit multistep block method by using Taylor Series interpolation polynomial in solving neural delay differential equation of Pantograph type. However, in this work, our method is also two–step and order nine. In a related development, Ref. [21] derived a Predictor–Corrector method PE(CE)^m where m is the number of iterations of the method, for solving first–order single linear Neural Differential Differential Equations with constant and Pantograph delay types. Here, our method is self–starting and does not depend on predictors or correctors to start (see also Refs. [22, 23]).

Yakubu *et al.* [24] used a second derivative approach in deriving a block hybrid method of orders seven and fourteen with the off-grid points $\{1/2, 3/2, 5/2\}$ for the numerical integration of both linear and non-linear initial-valued first-order differential equations. Nevertheless, we show in this present work with numerical examples that this second derivative method can be circumvented by presenting a ninth-order Block Hybrid Method; based on the first derivative for the solution of linear and non-linear stiff and non-stiff initial valued first–order differential systems of equations. This article is structured as follows: in Section 2, we presented the new block hybrid method, and showed that the order is uniformly nine, analyzed zero stability of the method, convergence analysis was done as well as the region of absolute stability is plotted. This is then followed in Section 3, where we implemented the methods in some numerical experiments. Results are presented by tables and figures which enforces the validity of the new method. In this paper, all norms are the two norms.

2. Materials and methods

To derive the continuous formulation, we used the method in Ref. [25], where a k–step linear multistep collocation method

$$\sum_{j=0}^{k} \alpha_j(x) y_{n+j} = h \sum_{j=0} \beta_j(x) f_{n+j}, \quad \alpha_0 \neq 0, \ \ \beta_0 \neq 0,$$

(in our case $k = 4$) with m collocation points defined as:

$$y(x) = \sum_{j=0}^{t-1} \alpha_j(x) y_{n+j} + h \sum_{j=0}^{m-1} \beta_j(x) f_{n+j}. \qquad (1)$$

The $\alpha_j(x)$'s and $\beta_j(x)$'s are

$$\alpha_j(x) = \sum_{j=0}^{t+m-1} \alpha_{j,i+1} x^i,$$

and

$$h\beta_j(x) = h \sum_{j=0}^{t+m-1} \beta_{j,i+1} x^i,$$

for $j = 0, 1, 2, \cdots$, where t and m are the number of interpolation and collocation points respectively. To derive both the discrete and continuous formulations of the new hybrid method, we took $t = 2$, $m = 8$ such that equation (1) becomes:

$$\begin{aligned} y(x) =& \alpha_0(x) y_n + \alpha_1(x) y_{n+1} + h\Big[\beta_0(x) f_n + \beta_1(x) f_{n+1} + \beta_{\frac{3}{2}}(x) f_{n+\frac{3}{2}} \\ &+ \beta_2(x) f_{n+2} + \beta_{\frac{5}{2}}(x) f_{n+\frac{5}{2}} + \beta_3(x) f_{n+3} + \beta_{\frac{7}{2}}(x) f_{n+\frac{7}{2}} \\ &+ \beta_4(x) f_{n+4}\Big]. \end{aligned} \qquad (2)$$

The matrix **D** used in deriving the continuous scheme is shown in equation (21). We moved D to the appendix because of formatting issues. We need the following result to prove the non–singularity of the matrix **D**.

Theorem 2.1. *The determinant of* **D** *is* $\det(\mathbf{D}) = -\frac{2751503792625}{128} h^{37}$.

Proof: Perform elementary row operations on **D** after replacing $x_n = x_{n+1} - h$, $x_{n+2} = x_{n+1} + h$, $x_{n+3} = x_{n+1} + 2h$, $x_{n+4} = x_{n+1} + 3h$, $x_{n+\frac{3}{2}} = x_{n+1} + \frac{h}{2}$, $x_{n+\frac{5}{2}} = x_{n+1} + \frac{3h}{2}$ and $x_{n+\frac{7}{2}} = x_{n+1} + \frac{5h}{2}$. Therefore,

$$\det(\mathbf{D}) = -\frac{2751503792625\, h^{37}}{128}.$$

■

Theorem 2.2. *Let ε be a small but positive real number, then* $h > [\frac{128\varepsilon}{2751503792625}]^{\frac{1}{37}}$.

Proof: Since $\varepsilon > 0$, then let $\frac{2751503792625 h^{37}}{128} > \varepsilon$. The result is immediate by routine algebraic simplifications. ■

In the next important result, we give a necessary condition for **D** to be non-singular using the above results.

Corollary 2.1. : *If $h > 0.19846$, then $\det(\mathbf{D}) \neq 0$ and* **D** *is non-singular.*

Proof: Let $\frac{2751503792625 h^{37}}{128} > 2^{-52}$ where `macheps`$= 2^{-52}$ is machine epsilon. Hence

$$h^{37} > \frac{128(2^{-52})}{2751503792625},$$

and

$$h > \left[\frac{2^{-45}}{2751503792625}\right]^{\frac{1}{37}} = 0.19846.$$

■

We defined $\omega = x_{n+1} - x$ such that $x = x_{n+1} - \omega$ and after inverting **D**, the first row of the inverse gives the following continuous coefficients respectively:

$$\alpha_0 = -(160\omega^9 + 1710h\omega^8 + 6840h^2\omega^7 + 11550h^3\omega^6$$

$$+ 2772h^4\omega^5 - 16695h^5\omega^4 - 21060h^6\omega^3 - 8100h^7\omega^2)/22823h^9,$$

$$\alpha_1 =(160\omega^9 + 1710h\omega^8 + 6840h^2\omega^7 + 11550h^3\omega^6 + 2772h^4\omega^5 - 16695h^5\omega^4 - 21060h^6\omega^3 - 8100h^7\omega^2 + 22823h^9)/22823h^9,$$

$$\beta_0 = -(294400\omega^9 + 3214869h\omega^8 + 13407228h^2\omega^7 + 25246025h^3\omega^6 + 15165423h^4\omega^5 - 16819593h^5\omega^4 - 28685457h^6\omega^3 - 11822895h^7\omega^2)/172541880h^8,$$

$$\beta_1 = -(1590800\omega^9 + 16043109h\omega^8 + 57599412h^2\omega^7 + 72339449h^3\omega^6 - 46248972h^4\omega^5 - 184442433h^5\omega^4 - 90846388h^6\omega^3 + 87694083h^7\omega^2 + 86270940h^8\omega)/86270940h^8,$$

$$\beta_{\frac{3}{2}} =772000\omega^9 + 7292184h\omega^8 + 23143464h^2\omega^7 + 18983720h^3\omega^6 - 38387664h^4\omega^5 - 66654168h^5\omega^4 + 7662024h^6\omega^3 + 47188440h^7\omega^2)/21567735h^8,$$

$$\beta_2 = -(87792\omega^9 + 778516h\omega^8 + 2201144h^2\omega^7 + 1065372h^3\omega^6 - 4454065h^4\omega^5 - 4767119h^5\omega^4 + 2343585h^6\omega^3 + 2744775h^7\omega^2)/(1917132h^8),$$

$$\beta_{\frac{5}{2}} =(802240\omega^9 + 6656808h\omega^8 + 16767696h^2\omega^7 + 3592960h^3\omega^6 - 35946624h^4\omega^5 - 27153336h^5\omega^4 + 18379696h^6\omega^3 + 16900560h^7\omega^2)/21567735h^8,$$

$$\beta_3 = -(324208\omega^9 + 2506407h\omega^8 + 5643612h^2\omega^7 + 78659h^3\omega^6 - 12212424h^4\omega^5 - 7708155h^5\omega^4 + 6212988h^6\omega^3 + 5154705h^7\omega^2)/17254188h^8,$$

$$\beta_{\frac{7}{2}} =(12960\omega^9 + 92864h\omega^8 + 188872h^2\omega^7 - 23016h^3\omega^6 - 414512h^4\omega^5 - 233968h^5\omega^4 + 211272h^6\omega^3 + 165528h^7\omega^2)/2396415h^8,$$

$$\beta_4 = -(117280\omega^9 + 774147h\omega^8 + 1453332h^2\omega^7 - 320705h^3\omega^6 - 3240237h^4\omega^5 - 1693209h^5\omega^4 + 1657447h^6\omega^3 + 1251945h^7\omega^2)/172541880h^8.$$

The continuous formulation is obtained by substituting the above into equation (2). The resulting continuous formulation of the new block hybrid method is presented in equation (22) of the appendix. We evaluated the continuous formulation (22) at $\omega = -\frac{h}{2}$ to yield:

$$y_{n+\frac{3}{2}} = \frac{27895280}{1635952640}y_n + \frac{1608057360}{1635952640}y_{n+1} + \frac{h}{1635952640}\Big[5627045f_n + 364063338f_{n+1} + 608019008f_{n+\frac{3}{2}} - 202077720f_{n+2} + 98873280f_{n+\frac{5}{2}} - 35861530f_{n+3} + 8059392f_{n+\frac{7}{2}} - 831213f_{n+4}\Big]. \quad (3)$$

Evaluating the continuous formulation at $\omega = -h$ yields

$$y_{n+2} = \frac{466830}{43135470}y_n + \frac{42668640}{43135470}y_{n+1} + \frac{h}{43135470}\Big[90797f_n + 8365792f_{n+1} + 27240704f_{n+\frac{3}{2}} + 8030520f_{n+2} - 12032f_{n+\frac{5}{2}} - 158080f_{n+3} + 50688f_{n+\frac{7}{2}} - 6089f_{n+4}\Big]. \quad (4)$$

We evaluated the continuous formulation at $\omega = -\frac{3h}{2}$ to obtain

$$y_{n+\frac{5}{2}} = \frac{4980528}{327190528}y_n + \frac{322210000}{327190528}y_{n+1} + \frac{h}{327190528}\Big[1006785f_n + 68879650f_{n+1} + 188193600f_{n+\frac{3}{2}} + 160461000f_{n+2} + 86484160f_{n+\frac{5}{2}} - 11135250f_{n+3} + 2073600f_{n+\frac{7}{2}} - 197225f_{n+4}\Big]. \quad (5)$$

Evaluating the continuous formulation at $\omega = -2h$, yields

$$y_{n+3} = \frac{15680}{1597610}y_n + \frac{1581930}{1597610}y_{n+1} + \frac{h}{1597610}\Big[2971f_n + 306666f_{n+1} + 1001920f_{n+\frac{3}{2}} + 604530f_{n+2} + 998784f_{n+\frac{5}{2}} + 307450f_{n+3} - 12096f_{n+\frac{7}{2}} + 675f_{n+4}\Big]. \quad (6)$$

We evaluated the continuous formulation of the scheme at $\omega = -\frac{5h}{2}$ to obtain

$$y_{n+\frac{7}{2}} = \frac{27270000}{1262020608}y_n + \frac{1234750608}{1262020608}y_{n+1} + \frac{h}{1262020608}\Big[5740525f_n + 290633210f_{n+1} + 668046400f_{n+\frac{3}{2}} + 698985000f_{n+2} + 489137600f_{n+\frac{5}{2}} + 805057750f_{n+3} + 230146560f_{n+\frac{7}{2}} - 5425525f_{n+4}\Big]. \quad (7)$$

Evaluating the continuous formulation at $\omega = -3h$ results in:

$$y_{n+4} = -\frac{25515}{798805}y_n + \frac{824320}{798805}y_{n+1} + \frac{h}{798805}\Big[-6210f_n + 49856f_{n+1} + 746496f_{n+\frac{3}{2}} - 89640f_{n+2} + 911360f_{n+\frac{5}{2}} + 8640f_{n+3} + 635904f_{n+\frac{7}{2}} + 114494f_{n+4}\Big]. \quad (8)$$

The continuous formulation was evaluated at $\omega = -\frac{7h}{2}$ to yield:

$$y_{n+\frac{9}{2}} = \frac{379099280}{233707520}y_n - \frac{145391760}{233707520}y_{n+1}$$

$$+\frac{h}{233707520}\Big[87823827f_n+1192665222f_{n+1} -2512835136f_{n+\frac{3}{2}}+4218684120f_{n+2}-4109579712f_{n+\frac{5}{2}} +3023948970f_{n+3}-1189665792f_{n+\frac{7}{2}}+486034101f_{n+4}\Big]. \quad (9)$$

We differentiated the continuous formulation (22) at $\omega=-\frac{7h}{2}$ and obtained the discrete scheme

$$y_{n+1}=y_n+\frac{h}{2041200}\Big[473977f_n+6190578f_{n+1} -14256264f_{n+\frac{3}{2}}+21960504f_{n+2}-22333032f_{n+\frac{5}{2}} +15056670f_{n+3}-6504408f_{n+\frac{7}{2}}+1635759f_{n+4} -182584f_{n+\frac{9}{2}}\Big]. \quad (10)$$

We present the New Block Hybrid Method (consisting of the above 8 discrete schemes) beginning with y_{n+1} as follows:

$$y_{n+1}-y_n-\frac{h}{2041200}\Big[473977f_n+6190578f_{n+1}-14256264f_{n+\frac{3}{2}} +21960504f_{n+2}-22333032f_{n+\frac{5}{2}}+15056670f_{n+3} -6504408f_{n+\frac{7}{2}}+1635759f_{n+4}-182584f_{n+\frac{9}{2}}\Big]=0,$$

$$y_{n+\frac{3}{2}}-\frac{27895280}{1635952640}y_n-\frac{1608057360}{1635952640}y_{n+1} -\frac{h}{1635952640}\Big[5627045f_n+364063338f_{n+1} +608019008f_{n+\frac{3}{2}}-202077720f_{n+2}+98873280f_{n+\frac{5}{2}} -35861530f_{n+3}+8059392f_{n+\frac{7}{2}}-831213f_{n+4}\Big]=0,$$

$$y_{n+2}-\frac{466830}{43135470}y_n-\frac{42668640}{43135470}y_{n+1}-\frac{h}{43135470}\Big[90797f_n +8365792f_{n+1}+27240704f_{n+\frac{3}{2}}+8030520f_{n+2} -12032f_{n+\frac{5}{2}}-158080f_{n+3}+50688f_{n+\frac{7}{2}}-6089f_{n+4}\Big]=0,$$

$$y_{n+\frac{5}{2}}-\frac{4980528}{327190528}y_n-\frac{322210000}{327190528}y_{n+1} -\frac{h}{327190528}\Big[1006785f_n+68879650f_{n+1}+188193600f_{n+\frac{3}{2}} +160461000f_{n+2}+86484160f_{n+\frac{5}{2}}-11135250f_{n+3} +2073600f_{n+\frac{7}{2}}-197225f_{n+4}\Big]=0,$$

$$y_{n+3}-\frac{15680}{1597610}y_n-\frac{1581930}{1597610}y_{n+1}-\frac{h}{1597610}\Big[2971f_n +306666f_{n+1}+1001920f_{n+\frac{3}{2}}+604530f_{n+2}+998784f_{n+\frac{5}{2}} +307450f_{n+3}-12096f_{n+\frac{7}{2}}+675f_{n+4}\Big]=0, \quad (11)$$

$$y_{n+\frac{7}{2}}-\frac{27270000}{1262020608}y_n-\frac{1234750608}{1262020608}y_{n+1} -\frac{h}{1262020608}\Big[5740525f_n+290633210f_{n+1} +668046400f_{n+\frac{3}{2}}+698985000f_{n+2}+489137600f_{n+\frac{5}{2}} +805057750f_{n+3}+230146560f_{n+\frac{7}{2}}-5425525f_{n+4}\Big]=0,$$

$$y_{n+4}+\frac{25515}{798805}y_n-\frac{824320}{798805}y_{n+1}-\frac{h}{798805}\Big[-6210f_n +49856f_{n+1}+746496f_{n+\frac{3}{2}}-89640f_{n+2}+911360f_{n+\frac{5}{2}} +8640f_{n+3}+635904f_{n+\frac{7}{2}}+114494f_{n+4}\Big]=0,$$

$$y_{n+\frac{9}{2}}-\frac{379099280}{233707520}y_n+\frac{145391760}{233707520}y_{n+1} -\frac{h}{233707520}\Big[87823827f_n+1192665222f_{n+1} -2512835136f_{n+\frac{3}{2}}+4218684120f_{n+2}-4109579712f_{n+\frac{5}{2}} +3023948970f_{n+3}-1189665792f_{n+\frac{7}{2}}+486034101f_{n+4}\Big]=0.$$

The nonlinear system of eight equations in eight unknowns above can be expressed as $\mathbf{F}(\mathbf{y})=\mathbf{0}$ where

$$\mathbf{y}=\left[y_{n+1},y_{n+\frac{3}{2}},y_{n+2},y_{n+\frac{5}{2}},y_{n+3},y_{n+\frac{7}{2}},y_{n+4},y_{n+\frac{9}{2}}\right]^T, \quad (12)$$

are the unknowns. This will be discussed further under the implementation and algorithm subsection below.

2.1. Convergence analysis

In this section, we examine the order, error constant, zero stability and convergence of the new block hybrid method given by equations (3)–(10) in this paper as follows with

$$\boldsymbol{\alpha}_0=-\begin{bmatrix}\frac{2041200}{182584}\\ \frac{27895280}{1635952640}\\ \frac{466830}{43135470}\\ \frac{4980528}{327190528}\\ \frac{15680}{1597610}\\ \frac{27270000}{1262020608}\\ -\frac{25515}{798805}\\ \frac{379099280}{233707520}\end{bmatrix},\ \boldsymbol{\alpha}_1=-\begin{bmatrix}-\frac{2041200}{182584}\\ \frac{1608057360}{1635952640}\\ \frac{42668640}{43135470}\\ \frac{322210000}{327190528}\\ \frac{1581930}{1597610}\\ \frac{1234750608}{1262020608}\\ \frac{824320}{798805}\\ -\frac{145391760}{233707520}\end{bmatrix},\ \boldsymbol{\alpha}_{\frac{3}{2}}=\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix},\ \boldsymbol{\alpha}_2=\begin{bmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix},$$

$$\alpha_{\frac{5}{2}}=\begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix},\ \alpha_3=\begin{bmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix},\ \alpha_{\frac{7}{2}}=\begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix},\ \alpha_4=\begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix},\ \alpha_{\frac{9}{2}}=\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{bmatrix}.$$

In the same vein,

$$\beta_0=\begin{bmatrix}\frac{473977}{182584}\\ \frac{5627045}{1635952640}\\ \frac{90797}{43135470}\\ \frac{1006785}{327190528}\\ \frac{2971}{1597610}\\ \frac{5740525}{1262020608}\\ -\frac{6210}{798805}\\ \frac{87823827}{233707520}\end{bmatrix},\ \beta_1=\begin{bmatrix}\frac{6190578}{182584}\\ \frac{364063338}{1635952640}\\ \frac{8365792}{43135470}\\ \frac{68879650}{327190528}\\ \frac{306666}{1597610}\\ \frac{290633210}{1262020608}\\ \frac{49856}{798805}\\ \frac{1192665222}{233707520}\end{bmatrix},\ \beta_{\frac{3}{2}}=\begin{bmatrix}-\frac{14256264}{182584}\\ \frac{608019008}{1635952640}\\ \frac{27240704}{43135470}\\ \frac{188193600}{327190528}\\ \frac{1001920}{1597610}\\ \frac{668046400}{1262020608}\\ \frac{746496}{798805}\\ -\frac{2512835136}{233707520}\end{bmatrix},$$

$$\beta_2=\begin{bmatrix}\frac{21960504}{182584}\\ -\frac{202077720}{1635952640}\\ \frac{8030520}{43135470}\\ \frac{160461000}{327190528}\\ \frac{604530}{1597610}\\ \frac{698985000}{1262020608}\\ -\frac{89640}{798805}\\ \frac{4218684120}{233707520}\end{bmatrix},\ \beta_{\frac{5}{2}}=\begin{bmatrix}-\frac{22333032}{182584}\\ \frac{98873280}{1635952640}\\ -\frac{12032}{43135470}\\ \frac{86484160}{327190528}\\ \frac{998784}{1597610}\\ \frac{489137600}{1262020608}\\ \frac{911360}{798805}\\ -\frac{4109579712}{233707520}\end{bmatrix},\ \beta_3=\begin{bmatrix}\frac{15056670}{182584}\\ -\frac{35861530}{1635952640}\\ -\frac{158080}{43135470}\\ -\frac{11135250}{327190528}\\ \frac{307450}{1597610}\\ \frac{805057750}{1262020608}\\ \frac{8640}{798805}\\ \frac{3023948970}{233707520}\end{bmatrix},$$

$$\beta_{\frac{7}{2}}=\begin{bmatrix}\frac{6504408}{182584}\\ \frac{8059392}{1635952640}\\ \frac{50688}{43135470}\\ \frac{2073600}{327190528}\\ -\frac{12096}{1597610}\\ \frac{230146560}{1262020608}\\ \frac{635904}{798805}\\ \frac{1189665792}{233707520}\end{bmatrix},\ \beta_4=\begin{bmatrix}\frac{1635759}{182584}\\ -\frac{831213}{1635952640}\\ -\frac{6089}{43135470}\\ -\frac{197225}{327190528}\\ \frac{675}{1597610}\\ -\frac{5425525}{1262020608}\\ \frac{114494}{798805}\\ \frac{486034101}{233707520}\end{bmatrix},\ \beta_{\frac{9}{2}}=\begin{bmatrix}-1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}.$$

Theorem 2.3. *Each discrete scheme in equations (3)–(10) that constitutes the new block hybrid method is of order nine.*

Proof. We substituted the above vectors in the following and after some algebraic simplifications, we obtained

$$C_0=\alpha_0+\alpha_1+\alpha_{\frac{3}{2}}+\alpha_2+\alpha_{\frac{5}{2}}+\alpha_3+\alpha_{\frac{7}{2}}+\alpha_4+\alpha_{\frac{9}{2}}=0,$$

$$\begin{aligned}C_1=&\left[\alpha_1+\left(\frac{3}{2}\right)\alpha_{\frac{3}{2}}+2\alpha_2+\left(\frac{5}{2}\right)\alpha_{\frac{5}{2}}+3\alpha_3+\left(\frac{7}{2}\right)\alpha_{\frac{7}{2}}\right.\\&\left.+4\alpha_4+\left(\frac{9}{2}\right)\alpha_{\frac{9}{2}}\right]-\left[\beta_0+\beta_1+\beta_{\frac{3}{2}}+\beta_2+\beta_{\frac{5}{2}}+\beta_3\right.\\&\left.+\beta_{\frac{7}{2}}+\beta_4+\beta_{\frac{9}{2}}\right]=0,\end{aligned}$$

$$\begin{aligned}C_2=&\frac{1}{2!}\left[\alpha_1+\left(\frac{3}{2}\right)^2\alpha_{\frac{3}{2}}+2^2\alpha_2+\left(\frac{5}{2}\right)^2\alpha_{\frac{5}{2}}+3^2\alpha_3+\left(\frac{7}{2}\right)^2\alpha_{\frac{7}{2}}\right.\\&\left.+4^2\alpha_4+\left(\frac{9}{2}\right)^2\alpha_{\frac{9}{2}}\right]-\left[\beta_1+\left(\frac{3}{2}\right)\beta_{\frac{3}{2}}+2\beta_2+\left(\frac{5}{2}\right)\beta_{\frac{5}{2}}\right.\\&\left.+3\beta_3+\left(\frac{7}{2}\right)\beta_{\frac{7}{2}}+4\beta_4+\left(\frac{9}{2}\right)\beta_{\frac{9}{2}}\right]=0,\end{aligned}$$

$$\begin{aligned}C_3=&\frac{1}{3!}\left[\alpha_1+\left(\frac{3}{2}\right)^3\alpha_{\frac{3}{2}}+2^3\alpha_2+\left(\frac{5}{2}\right)^3\alpha_{\frac{5}{2}}+3^3\alpha_3+\left(\frac{7}{2}\right)^3\alpha_{\frac{7}{2}}\right.\\&\left.+4^3\alpha_4+\left(\frac{9}{2}\right)^3\alpha_{\frac{9}{2}}\right]-\frac{1}{2!}\left[\beta_1+\left(\frac{3}{2}\right)^2\beta_{\frac{3}{2}}+2^2\beta_2+\left(\frac{5}{2}\right)^2\beta_{\frac{5}{2}}\right.\\&\left.+3^2\beta_3+\left(\frac{7}{2}\right)^2\beta_{\frac{7}{2}}+4^2\beta_4+\left(\frac{9}{2}\right)^2\beta_{\frac{9}{2}}\right]=0,\end{aligned}$$

$$\begin{aligned}C_4=&\frac{1}{4!}\left[\alpha_1+\left(\frac{3}{2}\right)^4\alpha_{\frac{3}{2}}+2^4\alpha_2+\left(\frac{5}{2}\right)^4\alpha_{\frac{5}{2}}+3^4\alpha_3+\left(\frac{7}{2}\right)^4\alpha_{\frac{7}{2}}\right.\\&\left.+4^4\alpha_4+\left(\frac{9}{2}\right)^4\alpha_{\frac{9}{2}}\right]-\frac{1}{3!}\left[\beta_1+\left(\frac{3}{2}\right)^3\beta_{\frac{3}{2}}+2^3\beta_2+\left(\frac{5}{2}\right)^3\beta_{\frac{5}{2}}\right.\\&\left.+3^3\beta_3+\left(\frac{7}{2}\right)^3\beta_{\frac{7}{2}}+4^3\beta_4+\left(\frac{9}{2}\right)^3\beta_{\frac{9}{2}}\right]=0,\end{aligned}$$

$$C_5=\frac{1}{5!}\left[\alpha_1+\left(\frac{3}{2}\right)^5\alpha_{\frac{3}{2}}+2^5\alpha_2+\left(\frac{5}{2}\right)^5\alpha_{\frac{5}{2}}+3^5\alpha_3+\left(\frac{7}{2}\right)^5\alpha_{\frac{7}{2}}\right.$$

Table 1: The order (second column) and error constants $C_{10} \neq 0$ (third column) of the new block hybrid method.

y_{n+i}	Order	Error constant $C_{10} \neq 0$
y_{n+1}	9	2.023×10^{-03}
$y_{n+\frac{3}{2}}$	9	6.411×10^{-07}
y_{n+2}	9	2.660×10^{-07}
$y_{n+\frac{5}{2}}$	9	6.296×10^{-07}
y_{n+3}	9	6.161×10^{-08}
$y_{n+\frac{7}{2}}$	9	1.565×10^{-06}
y_{n+4}	9	$- 6.478\times10^{-06}$
$y_{n+\frac{9}{2}}$	9	2.841×10^{-04}

$$+ 4^5\alpha_4 + \left(\frac{9}{2}\right)^5 \alpha_{\frac{9}{2}}\bigg] - \frac{1}{4!}\bigg[\beta_1 + \left(\frac{3}{2}\right)^4 \beta_{\frac{3}{2}} + 2^4\beta_2 + \left(\frac{5}{2}\right)^4 \beta_{\frac{5}{2}} + 3^4\beta_3 + \left(\frac{7}{2}\right)^4 \beta_{\frac{7}{2}} + 4^4\beta_4 + \left(\frac{9}{2}\right)^4 \beta_{\frac{9}{2}}\bigg] = 0,$$

$$C_6 = \frac{1}{6!}\bigg[\alpha_1 + \left(\frac{3}{2}\right)^6 \alpha_{\frac{3}{2}} + 2^6\alpha_2 + \left(\frac{5}{2}\right)^6 \alpha_{\frac{5}{2}} + 3^6\alpha_3 + \left(\frac{7}{2}\right)^6 \alpha_{\frac{7}{2}} + 4^6\alpha_4 + \left(\frac{9}{2}\right)^6 \alpha_{\frac{9}{2}}\bigg] - \frac{1}{5!}\bigg[\beta_1 + \left(\frac{3}{2}\right)^5 \beta_{\frac{3}{2}} + 2^5\beta_2 + \left(\frac{5}{2}\right)^5 \beta_{\frac{5}{2}} + 3^5\beta_3 + \left(\frac{7}{2}\right)^5 \beta_{\frac{7}{2}} + 4^5\beta_4 + \left(\frac{9}{2}\right)^5 \beta_{\frac{9}{2}}\bigg] = 0,$$

$$C_7 = \frac{1}{7!}\bigg[\alpha_1 + \left(\frac{3}{2}\right)^7 \alpha_{\frac{3}{2}} + 2^7\alpha_2 + \left(\frac{5}{2}\right)^7 \alpha_{\frac{5}{2}} + 3^7\alpha_3 + \left(\frac{7}{2}\right)^7 \alpha_{\frac{7}{2}} + 4^7\alpha_4 + \left(\frac{9}{2}\right)^7 \alpha_{\frac{9}{2}}\bigg] - \frac{1}{6!}\bigg[\beta_1 + \left(\frac{3}{2}\right)^6 \beta_{\frac{3}{2}} + 2^6\beta_2 + \left(\frac{5}{2}\right)^6 \beta_{\frac{5}{2}} + 3^6\beta_3 + \left(\frac{7}{2}\right)^6 \beta_{\frac{7}{2}} + 4^6\beta_4 + \left(\frac{9}{2}\right)^6 \beta_{\frac{9}{2}}\bigg] = 0,$$

$$C_8 = \frac{1}{8!}\bigg[\alpha_1 + \left(\frac{3}{2}\right)^8 \alpha_{\frac{3}{2}} + 2^8\alpha_2 + \left(\frac{5}{2}\right)^8 \alpha_{\frac{5}{2}} + 3^8\alpha_3 + \left(\frac{7}{2}\right)^8 \alpha_{\frac{7}{2}} + 4^8\alpha_4 + \left(\frac{9}{2}\right)^8 \alpha_{\frac{9}{2}}\bigg] - \frac{1}{7!}\bigg[\beta_1 + \left(\frac{3}{2}\right)^7 \beta_{\frac{3}{2}} + 2^7\beta_2 + \left(\frac{5}{2}\right)^7 \beta_{\frac{5}{2}} + 3^7\beta_3 + \left(\frac{7}{2}\right)^7 \beta_{\frac{7}{2}} + 4^7\beta_4 + \left(\frac{9}{2}\right)^7 \beta_{\frac{9}{2}}\bigg] = 0,$$

$$C_9 = \frac{1}{9!}\bigg[\alpha_1 + \left(\frac{3}{2}\right)^9 \alpha_{\frac{3}{2}} + 2^9\alpha_2 + \left(\frac{5}{2}\right)^9 \alpha_{\frac{5}{2}} + 3^9\alpha_3 + \left(\frac{7}{2}\right)^9 \alpha_{\frac{7}{2}} + 4^9\alpha_4 + \left(\frac{9}{2}\right)^9 \alpha_{\frac{9}{2}}\bigg] - \frac{1}{8!}\bigg[\beta_1 + \left(\frac{3}{2}\right)^8 \beta_{\frac{3}{2}} + 2^8\beta_2 + \left(\frac{5}{2}\right)^8 \beta_{\frac{5}{2}} + 3^8\beta_3 + \left(\frac{7}{2}\right)^8 \beta_{\frac{7}{2}} + 4^8\beta_4 + \left(\frac{9}{2}\right)^8 \beta_{\frac{9}{2}}\bigg] = 0.$$

Therefore, $C_0 = C_1 = C_2 = \ldots = C_9 = 0$ and $C_{10} \neq 0$ are as tabulated in the last column of Table 1. Hence, the proof. □

In Table 1, we present the order and error constants of the New Block Hybrid Method.

In the next section, we present a discussion on the order of convergence of the method.

2.2. Order of convergence

In this section, we examine the order of convergence of the newly derived block hybrid method. We begin by presenting this well–known definition of Lipschitz condition which will be used shortly.

Definition 2.1. : *A function $f(x, y)$ satisfies the Lipschitz condition on a domain $D \subseteq \mathbb{R}^2$, if there exists a constant $L > 0$ such that*

$$|f(x, y) - f(x, y^*)| \leq L|y - y^*|,$$

for all $(x, y), (x, y^*) \in D$.

In the same vein, we define the convergence as used in Ref. [21] .

Definition 2.2. : *A Linear Multistep Method is said to be convergent if, for all initial value problems subject to the above Lipschitz conditions*

$$\lim_{h\to 0} y_n = y^*(x_n),$$

for all solutions $\{y_n\}$ of the method.

Assuming the exact solution of equation (10) is:

$$y^*_{n+1} = y^*_n + \frac{h}{2041200}\bigg[473977 f_n + 6190578 f_{n+1} - 14256264 f_{n+\frac{3}{2}} + 21960504 f_{n+2} - 22333032 f_{n+\frac{5}{2}} + 15056670 f_{n+3} - 6504408 f_{n+\frac{7}{2}} + 1635759 f_{n+4} - 182584 f_{n+\frac{9}{2}}\bigg] + \frac{189145}{93483008} h^{10} y^{*(10)}(\xi_n) + R_{10}, \tag{13}$$

where R_{10} is the remainder term. Assuming the exact solution of equation (3) is:

$$y^*_{n+\frac{3}{2}} = \frac{27895280}{1635952640} y^*_n + \frac{1608057360}{1635952640} y^*_{n+1} + \frac{h}{1635952640}\bigg[5627045 f_n + 364063338 f_{n+1} + 608019008 f_{n+\frac{3}{2}} - 202077720 f_{n+2} + 98873280 f_{n+\frac{5}{2}} - 35861530 f_{n+3} + 8059392 f_{n+\frac{7}{2}} - 831213 f_{n+4}\bigg] + \frac{2684971}{4188038758400} h^{10} y^{*(10)}(\xi_n) + R_{10}. \tag{14}$$

Assuming the exact solution of y_{n+2} is:

$$y^*_{n+2} = \frac{466830}{43135470} y^*_n + \frac{42668640}{43135470} y^*_{n+1} + \frac{h}{43135470}\bigg[90797 f_n + 8365792 f_{n+1} + 27240704 f_{n+\frac{3}{2}} + 8030520 f_{n+2} - 12032 f_{n+\frac{5}{2}} - 158080 f_{n+3} + 50688 f_{n+\frac{7}{2}} - 6089 f_{n+4}\bigg] + \frac{22031}{82820102400} h^{10} y^{*(10)}(\xi_n) + R_{10}, \tag{15}$$

and so on and so forth.

Subtract equation (10) from equation (13) to obtain:

$$y^*_{n+1} - y_{n+1} = y^*_n + \frac{h}{2041200}\bigg[473977 f_n + 6190578 f_{n+1}$$

$$- 14256264f_{n+\frac{3}{2}} + 21960504f_{n+2} - 22333032f_{n+\frac{5}{2}} + 15056670f_{n+3} - 6504408f_{n+\frac{7}{2}} + 1635759f_{n+4} - 182584f_{n+\frac{9}{2}}\Big] - y_n - \frac{h}{2041200}\Big[473977f_n + 6190578f_{n+1} - 14256264f_{n+\frac{3}{2}} + 21960504f_{n+2} - 22333032f_{n+\frac{5}{2}} + 15056670f_{n+3} - 6504408f_{n+\frac{7}{2}} + 1635759f_{n+4} - 182584f_{n+\frac{9}{2}}\Big] + \frac{189145}{93483008}h^{10}y^{*(10)}(\xi_n) + R_{10}. \tag{16}$$

Expanding the above,

$$\begin{aligned}
y^*_{n+1} - y_{n+1} &= y^*_n - y_n + \frac{473977h}{2041200}[f(x_n, y^*_n) - f(x_n, y_n)] \\
&+ \frac{6190578h}{2041200}\Big[f(x_{n+1}, y^*_{n+1}) - f(x_{n+1}, y_{n+1})\Big] \\
&- \frac{14256264h}{2041200}\Big[f(x_{n+\frac{3}{2}}, y^*_{n+\frac{3}{2}}) - f(x_{n+\frac{3}{2}}, y_{n+\frac{3}{2}})\Big] \\
&+ \frac{21960504h}{2041200}\Big[f(x_{n+2}, y^*_{n+2}) - f(x_{n+2}, y_{n+2})\Big] \\
&- \frac{22333032h}{2041200}\Big[f(x_{n+\frac{5}{2}}, y^*_{n+\frac{5}{2}}) - f(x_{n+\frac{5}{2}}, y_{n+\frac{5}{2}})\Big] \\
&+ \frac{15056670h}{2041200}\Big[f(x_{n+3}, y^*_{n+3}) - f(x_{n+3}, y_{n+3})\Big] \\
&- \frac{6504408h}{2041200}\Big[f(x_{n+\frac{7}{2}}, y^*_{n+\frac{7}{2}}) - f(x_{n+\frac{7}{2}}, y_{n+\frac{7}{2}})\Big] \\
&+ \frac{1635759h}{2041200}\Big[f(x_{n+4}, y^*_{n+4}) - f(x_{n+4}, y_{n+4})\Big] \\
&- \frac{182584h}{2041200}\Big[f(x_{n+\frac{9}{2}}, y^*_{n+\frac{9}{2}}) - f(x_{n+\frac{9}{2}}, y_{n+\frac{9}{2}})\Big] \\
&+ \frac{189145}{93483008}h^{10}y^{*(10)}(\xi_n) + R_{10}.
\end{aligned} \tag{17}$$

Let $d_n = y^*_n - y_n$, $d_{n+\frac{3}{2}} = y^*_{n+\frac{3}{2}} - y_{n+\frac{3}{2}}$, $d_{n+1} = y^*_{n+1} - y_{n+1}$, etc in line with Refs. [20, 21]. Taking the absolute value of both sides, using the triangle inequality and imposing appropriate Lipschitz conditions yield

$$\begin{aligned}
|d_{n+1}| \le \Big(1 + \frac{473977h}{2041200}L\Big)|d_n| &+ \frac{343921h}{113400}L|d_{n+1}| \\
+ \frac{594011h}{85050}L|d_{n+\frac{3}{2}}| &+ \frac{101669h}{9450}L|d_{n+2}| \\
+ \frac{310181h}{28350}L|d_{n+\frac{5}{2}}| &+ \frac{501889h}{68040}L|d_{n+3}| \\
+ \frac{30113h}{9450}L|d_{n+\frac{7}{2}}| &+ \frac{181751h}{226800}L|d_{n+4}| \\
+ \frac{22823h}{255150}L|d_{n+\frac{9}{2}}| &+ \frac{189145}{93483008}h^{10}\Big|y^{*(10)}(\xi_n)\Big| + O(h^{11}).
\end{aligned}$$

This simplifies to

$$\begin{aligned}
\Big(1 - \frac{343921h}{113400}L\Big)|d_{n+1}| &\le \Big(1 + \frac{473977h}{2041200}L\Big)|d_n| \\
+ \frac{594011h}{85050}L|d_{n+\frac{3}{2}}| + \frac{101669h}{9450}L|d_{n+2}| &+ \frac{310181h}{28350}L|d_{n+\frac{5}{2}}| + \frac{501889h}{68040}L|d_{n+3}| + \frac{30113h}{9450}L|d_{n+\frac{7}{2}}| \\
+ \frac{181751h}{226800}L|d_{n+4}| + \frac{22823h}{255150}L|d_{n+\frac{9}{2}}| &+ \frac{189145}{93483008}h^{10}\Big|y^{*(10)}(\xi_n)\Big| + O(h^{11}).
\end{aligned}$$

As $h \to 0$, except $|d_n|$ every other term on the right–hand side tends to zero while on the left–hand side, we are left with $|d_{n+1}|$ and by the definition of convergence,

$$\lim_{h\to 0} y_{n+1} = y^*_{n+1}.$$

Thus, $|d_{n+1}| \le |d_n|$ and $y^*_{n+1} - y^*_n \le y_{n+1} - y_n$ is satisfied. The convergence of y_{n+1} is established.

Similarly, if we subtract equation (4) from equation (15)

$$\begin{aligned}
y^*_{n+2} - y_{n+2} &= \frac{42668640}{43135470}[y^*_{n+1} - y_{n+1}] + \frac{466830}{43135470}[y^*_n - y_n] \\
&+ \frac{90797h}{43135470}[f(x_n, y^*_n) - f(x_n, y_n)] \\
&+ \frac{8365792h}{43135470}\Big[f(x_{n+1}, y^*_{n+1}) - f(x_{n+1}, y_{n+1})\Big] \\
&+ \frac{27240704h}{43135470}\Big[f(x_{n+\frac{3}{2}}, y^*_{n+\frac{3}{2}}) - f(x_{n+\frac{3}{2}}, y_{n+\frac{3}{2}})\Big] \\
&+ \frac{8030520h}{43135470}\Big[f(x_{n+2}, y^*_{n+2}) - f(x_{n+2}, y_{n+2})\Big] \\
&- \frac{12032h}{43135470}\Big[f(x_{n+\frac{5}{2}}, y^*_{n+\frac{5}{2}}) - f(x_{n+\frac{5}{2}}, y_{n+\frac{5}{2}})\Big] \\
&- \frac{158080h}{43135470}\Big[f(x_{n+3}, y^*_{n+3}) - f(x_{n+3}, y_{n+3})\Big] \\
&+ \frac{50688h}{43135470}\Big[f(x_{n+\frac{7}{2}}, y^*_{n+\frac{7}{2}}) - f(x_{n+\frac{7}{2}}, y_{n+\frac{7}{2}})\Big] \\
&- \frac{6089h}{43135470}\Big[f(x_{n+4}, y^*_{n+4}) - f(x_{n+4}, y_{n+4})\Big] \\
&+ \frac{22031}{82820102400}h^{10}y^{*(10)}(\xi_n) + R_{10}.
\end{aligned} \tag{18}$$

Taking the absolute value of both sides and after some routine simplifications

$$\begin{aligned}
\Big(1 - \frac{89228h}{479283}L\Big)|d_{n+2}| &\le \Big(\frac{1}{L}\frac{42668640}{43135470} + \frac{8365792h}{43135470}\Big)L|d_{n+1}| \\
&+ \Big(\frac{466830}{43135470} + \frac{90797h}{43135470}L\Big)|d_n| + \frac{13620352\,h}{21567735}L|d_{n+\frac{3}{2}}| \\
&+ \frac{6016h}{21567735}L|d_{n+\frac{5}{2}}| + \frac{15808h}{4313547}L|d_{n+3}| + \frac{2816h}{2396415}L|d_{n+\frac{7}{2}}| \\
&+ \frac{6089h}{43135470}L|d_{n+4}| + \frac{22031}{82820102400}h^{10}\Big|y^{*(10)}(\xi_n)\Big| + O(h^{11}).
\end{aligned}$$

Taking the limit of both sides as $h \to 0$,

$$\begin{aligned}
|d_{n+2}| &\le \Big(\frac{42668640}{43135470}\Big)|d_{n+1}| + \Big(\frac{466830}{43135470}\Big)|d_n| \\
&= 0.9892|d_{n+1}| + 0.010822|d_n| \\
&\le |d_n|,
\end{aligned}$$

and by the definition of convergence,

$$\lim_{h\to 0} y_{n+2} = y^*_{n+2}.$$

Thus, $|d_{n+2}| \leq |d_n|$ and $y^*_{n+2} - y^*_n \leq y_{n+2} - y_n$ is satisfied. The convergence of y_{n+2} is established in agreement with Ref. [21].

Similarly, it can be shown that the other y_{n+i}'s for $i \in \{\frac{3}{2}, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}\}$ converge. Therefore, we have shown that the new block hybrid method is convergent. The remainder term R_{10} which is

$$R_{p+1} = C_{p+2} h^{p+2} y^{*(p+2)}(\xi) = O(h^{11}).$$

Following Ref. [20], $p + 2 \geq 1$ is the order of convergence. Hence, the order of convergence of the new method is 11.

2.3. Region of absolute stability

In this section, we plotted the region of absolute stability of the new method using the stability polynomial. To get the stability polynomial which will be used in plotting the region of absolute stability of the new method, we rewrite the new block hybrid method in Linear Multistep form as follows:

$$\begin{bmatrix} \frac{255150}{22823} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \\ y_{n+\frac{7}{2}} \\ y_{n+4} \\ y_{n+\frac{9}{2}} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{255150}{22823} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{49813}{2921344} & \frac{2871531}{2921344} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{247}{22823} & \frac{22576}{22823} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{44469}{2921344} & \frac{2876875}{2921344} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{224}{22823} & \frac{22599}{22823} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{63125}{2921344} & \frac{2858219}{2921344} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{729}{22823} & \frac{23552}{22823} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{4738741}{2921344} & -\frac{1817397}{2921344} \end{bmatrix} \begin{bmatrix} y_{n-7} \\ y_{n-6} \\ y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_n \\ y_{n+1} \end{bmatrix}$$

$$+ h \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{473977}{182584} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5627045}{1635952640} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{90797}{43135470} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1006785}{327190528} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2971}{1597610} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5740525}{1262020608} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{6210}{798805} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{87823827}{233707520} \end{bmatrix} \begin{bmatrix} f_{n-7} \\ f_{n-6} \\ f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}$$

$$+ h \begin{bmatrix} \beta_1 & \beta_{\frac{3}{2}} & \beta_2 & \beta_{\frac{5}{2}} & \beta_3 & \beta_{\frac{7}{2}} & \beta_4 & \beta_{\frac{9}{2}} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \\ f_{n+\frac{7}{2}} \\ f_{n+4} \\ f_{n+\frac{9}{2}} \end{bmatrix},$$

where

$$P = \begin{bmatrix} \frac{255150}{22823} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{255150}{22823} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{49813}{2921344} & \frac{2871531}{2921344} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{247}{22823} & \frac{22576}{22823} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{44469}{2921344} & \frac{2876875}{2921344} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{224}{22823} & \frac{22599}{22823} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{63125}{2921344} & \frac{2858219}{2921344} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{729}{22823} & \frac{23552}{22823} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{4738741}{2921344} & -\frac{1817397}{2921344} \end{bmatrix},$$

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{473977}{182584} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5627045}{1635952640} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{90797}{43135470} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1006785}{327190528} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2971}{1597610} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5740525}{1262020608} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{6210}{798805} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{87823827}{233707520} \end{bmatrix},$$

$$S = \begin{bmatrix} \beta_1 & \beta_{\frac{3}{2}} & \beta_2 & \beta_{\frac{5}{2}} & \beta_3 & \beta_{\frac{7}{2}} & \beta_4 & \beta_{\frac{9}{2}} \end{bmatrix}.$$

The stability polynomial is defined as

$$\rho(w,z) = \det(Pw - Q - Rz - Swz),$$

where $y' = \lambda y$ is the test problem $z = \lambda h$ and $w = \exp(i\theta)$ with $i = \sqrt{-1}, 0 \leq \theta \leq 2\pi$. Hence,

$$\begin{aligned} \rho(w,z) &= |Pw - Q - Rz - Swz| \\ &= \Big[(45654581448000w^8 - 5706822681000w^7)z^8 \\ &- (479445572793600w^8 + 271660491424350w^7 \\ &+ 2516235750w^6)z^7 + (2795424991500960w^8 \\ &+ 2127232111618515w^7 - 3200361509707 5w^6)z^6 \\ &- (11035803283477560w^8 + 2899801484257566w^7 \\ &+ 332878936639830w^6)z^5 + (30760767369642240w^8 \\ &+ 3068020123111457 5w^7 - 2010734128970455w^6)z^4 \\ &- (59509351269997440w^8 + 11562711533656878w^7 \\ &+ 7715950397115282w^6)z^3 + (74510463960629760w^8 \\ &+ 86891574321745760w^7 - 19189891950192320w^6)z^2 \\ &- (51110991900702720w^8 + 38468683257356520w^7 \\ &+ 28467553549460760w^6)z + (11687572850688000w^8 \\ &+ 7644272853168000w^7 \\ &- 19331845703856000w^6)\Big]/1045445718876160. \end{aligned}$$

The region of absolute stability of the method is defined as

$$E(w,z) = \{z \in \mathbb{C} : \rho(w,z) = 1, |z| \leq 1\}.$$

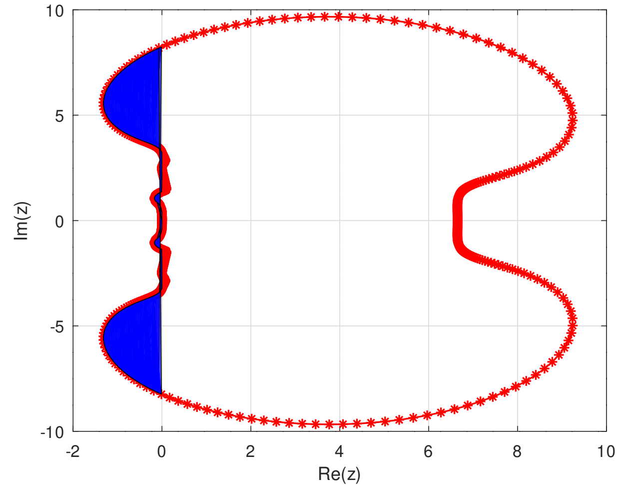

Figure 1: Region of absolute stability of the New Block Hybrid Method. The stability region is shaded in blue. This shows that the method is $A(86°)-$stable.

The stability region is obtained by finding the roots of the stability polynomial and plotting accordingly. This is as shown in Figure 1. We need the following definition of $A(\alpha)$-stability to describe the nature of the region of absolute stability of the new method.

Definition 2.3. : *A numerical method is $A(\alpha)$–stable (Ref. [31]), where $\alpha \in [0, \frac{\pi}{2}]$, if its region of absolute stability contains the wedge*

$$W_\alpha = \{z = \lambda h \in \mathbb{C}| - \alpha < \arg z - \pi < \alpha\}.$$

Using the above definition, we computed the value of $\alpha = 86°$. Therefore, the New Block Hybrid Method is $A(86°)-$stable. The region of absolute stability of the New Block Hybrid Method is shown in Figure 1 and the stability region is shaded in blue.

2.4. Zero stability

In this section, we examine the zero stability of the New Block Hybrid Method. This is done by re-writing the block method in the following manner:

$$\begin{bmatrix} \frac{2041200}{182584} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1608057360}{1635952640} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{42668640}{43135470} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{322210000}{327190528} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1581930}{1597610} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1234750608}{1262020608} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{824320}{798805} & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{145391760}{233707520} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \\ y_{n+\frac{7}{2}} \\ y_{n+4} \\ y_{n+\frac{9}{2}} \end{bmatrix} =$$

$$\begin{bmatrix} 0&0&0&0&0&0&0&\frac{2041200}{182584}\\ 0&0&0&0&0&0&0&\frac{27895280}{1635952640}\\ 0&0&0&0&0&0&0&\frac{466830}{43135470}\\ 0&0&0&0&0&0&0&\frac{4980528}{327190528}\\ 0&0&0&0&0&0&0&\frac{15680}{1597610}\\ 0&0&0&0&0&0&0&\frac{27270000}{1262020608}\\ 0&0&0&0&0&0&0&-\frac{25515}{798805}\\ 0&0&0&0&0&0&0&\frac{379099280}{233707520} \end{bmatrix}\begin{bmatrix} y_{n-7}\\ y_{n-6}\\ y_{n-5}\\ y_{n-4}\\ y_{n-3}\\ y_{n-2}\\ y_{n-1}\\ y_n \end{bmatrix} + hR\mathbf{g} + hS\mathbf{f},$$

where

$$\mathbf{f} = \left[f_{n+1},\ f_{n+\frac{3}{2}},\ f_{n+2},\ f_{n+\frac{5}{2}},\ f_{n+3},\ f_{n+\frac{7}{2}},\ f_{n+4},\ f_{n+\frac{9}{2}}\right]^T,$$

and

$$\mathbf{g} = \left[f_{n-7},\ f_{n-6},\ f_{n-5},\ f_{n-4},\ f_{n-3},\ f_{n-2},\ f_{n-1}, f_n\right]^T.$$

The characteristic polynomial is:

$$\rho(r) = \det(rP_1 - Q_1)$$

$$= \det\left(\begin{bmatrix} \frac{2041200}{182584}r&0&0&0&0&0&0&0\\ -\frac{1608057360}{1635952640}r&r&0&0&0&0&0&0\\ -\frac{42668640}{43135470}r&0&r&0&0&0&0&0\\ -\frac{322210000}{327190528}r&0&0&r&0&0&0&0\\ -\frac{1581930}{1597610}r&0&0&0&r&0&0&0\\ -\frac{1234750608}{1262020608}r&0&0&0&0&r&0&0\\ -\frac{824320}{798805}r&0&0&0&0&0&r&0\\ \frac{145391760}{233707520}r&0&0&0&0&0&0&r \end{bmatrix} - \begin{bmatrix} 0&0&0&0&0&0&0&\frac{2041200}{182584}\\ 0&0&0&0&0&0&0&\frac{27895280}{1635952640}\\ 0&0&0&0&0&0&0&\frac{466830}{43135470}\\ 0&0&0&0&0&0&0&\frac{4980528}{327190528}\\ 0&0&0&0&0&0&0&\frac{15680}{1597610}\\ 0&0&0&0&0&0&0&\frac{27270000}{1262020608}\\ 0&0&0&0&0&0&0&-\frac{25515}{798805}\\ 0&0&0&0&0&0&0&\frac{379099280}{233707520} \end{bmatrix}\right)$$

$$= \frac{255150r^8 - 255150r^7}{22823}.$$

The characteristic equation $\frac{255150r^8-255150r^7}{22823} = 0$, $r^7(r-1) = 0$. The roots are r equals zero of multiplicities seven and $r = 1$.

Lemma 2.1. *The new block hybrid method is consistent and zero stable. Hence, convergent.*

Proof. The new block hybrid method is consistent because the order of each of the discrete schemes that constitute the block is 9 which is greater than one (see, for example, Henrici [32]). Zero stability is established by the above because the roots of the characteristic polynomial (zero *i.e.*, multiplicity seven and one as shown above) have modulus zero which is less than or equal to one and those of modulus one (i.e, $r = 1$) is distinct. Therefore, because the method is both consistent and zero stable, it is convergent. □

2.5. Implementation and algorithm

In this section, we present an implementation of Newton's method to the nonlinear system of eight equations in eight unknowns of equation (11) *i.e.,* $\mathbf{F}(\mathbf{y}) = \mathbf{0}$, where

$$\mathbf{y} = \left[y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}, y_{n+\frac{5}{2}}, y_{n+3}, y_{n+\frac{7}{2}}, y_{n+4}, y_{n+\frac{9}{2}}\right]^T, \quad (19)$$

are the unknowns and the corresponding algorithm is given. We are only interested in the unknown y_{n+1}.

An application of Newton's method to $\mathbf{F}(\mathbf{y}) = \mathbf{0}$ is as follows

$$\mathbf{G}(\mathbf{y}^{(n)})\Delta\mathbf{y}^{(n)} = -\mathbf{F}(\mathbf{y}^{(n)}), \quad (20)$$

where $\mathbf{G}(\mathbf{y}^{(n)})$ is the Jacobian of $\mathbf{F}(\mathbf{y})$ and $\mathbf{y}^{(n+1)} = \mathbf{y}^{(n)} + \Delta\mathbf{y}^{(n)}$ for $n = 0, 1, 2, 3, \cdots$ until there is convergence. The partial derivatives are:

$$\frac{\partial f_{n+1}}{\partial y_{n+1}} = \frac{\partial f(x_n + h, y(x_n + h))}{\partial y_{n+1}},$$
$$\frac{\partial f_{n+\frac{3}{2}}}{\partial y_{n+\frac{3}{2}}} = \frac{\partial f(x_n + \frac{3}{2}h, y(x_n + \frac{3}{2}h))}{\partial y_{n+\frac{3}{2}}},$$
$$\frac{\partial f_{n+2}}{\partial y_{n+2}} = \frac{\partial f(x_n + 2h, y(x_n + 2h))}{\partial y_{n+2}},$$

e.t.c. The starting values are the initial values of the corresponding systems of ODE under consideration, so there is no need for predictors and correctors. In addition to this, we present the Newton-based algorithm for the new block hybrid method.

Algorithm 2.1. *Choose the tol and step size h such that the Jacobian is non-singular. From the given system of differential equations $y' = f(x, y)$ and for $n = 0, 1, 2, \cdots$. Compute the following partial derivatives:*

$$\frac{\partial f_{n+1}}{\partial y_{n+1}}, \frac{\partial f_{n+\frac{3}{2}}}{\partial y_{n+\frac{3}{2}}}, \frac{\partial f_{n+2}}{\partial y_{n+2}}, \frac{\partial f_{n+\frac{5}{2}}}{\partial y_{n+\frac{5}{2}}}, \frac{\partial f_{n+3}}{\partial y_{n+3}}, \frac{\partial f_{n+\frac{7}{2}}}{\partial y_{n+\frac{7}{2}}}, \frac{\partial f_{n+4}}{\partial y_{n+4}}$$

1. *Evaluate $\mathbf{F}(\mathbf{y}^{(n)})$ using equation (11).*
2. *Find the LU factorization of $\mathbf{G}(\mathbf{y}^{(n)})$.*

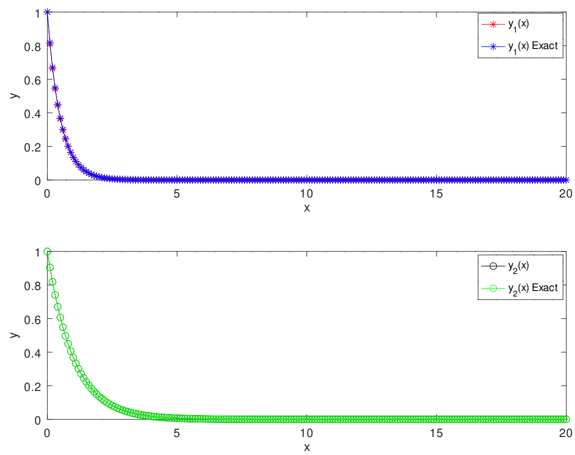

Figure 2: Comparing the performance of the New Block Hybrid Method with the exact solution on Example 3.1.

3. *Solve the linear system* $\mathbf{G}(\mathbf{y}^{(n)})\Delta\mathbf{y}^{(n)} = -\mathbf{F}(\mathbf{y}^{(n)})$, *for* $\Delta\mathbf{y}^{(n)}$.
4. *Update* $\mathbf{y}^{(n+1)} = \mathbf{y}^{(n)} + \Delta\mathbf{y}^{(n)}$.
5. *Continue until Newton's method converges.*

Output: y_{n+1}.

The stopping criteria for the algorithm are $\|\Delta\mathbf{y}^{(n)}\| < \texttt{tol}$ and $\|\mathbf{F}(\mathbf{y}^{(n)})\| < \texttt{tol}$, where `tol` is some user–defined tolerance. The time complexity of the new method is the cost of the LU factorization of the Jacobian in step two at each iteration which is of $O(n^3)$. However, for larger systems, one can resort to iterative methods at a cost of $O(n)$ which is much cheaper for solving the linear system in step 3 of the above algorithm.

3. Results and discussion

In this section, we compare the performances of the New Block Hybrid Method with the seventh-order (recognized as Method 18) and fourteen-order (referred to as Method 21) block hybrid method of Yakubu *et al*. [24]. Throughout this section, all the results are self explanatory.

Example 3.1. *We consider the Kaps problem in Ref. [26] which is a non–linear stiff problem:*

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} -1002y_1(x) - 1000y_2^2(x) \\ y_1(x) - y_2(x) - y_2^2(x) \end{bmatrix},$$

such that $y_1(0) = 1, y_2(0) = 1$.

The exact solution is $y_1(x) = \exp(-2x)$, $y_2(x) = \exp(-x)$ *and* $h = 0.1$.

The results of numerical experiment for this example is as shown in Table A1 and Figure 2. The computed stiffness ratio of this problem at the root is 1002.

Example 3.2. *The following non-linear stiff problem was studied by Gear in Ref. [28]:*

$$y_1'(x) = -0.013y_1 - 1000y_1y_3$$

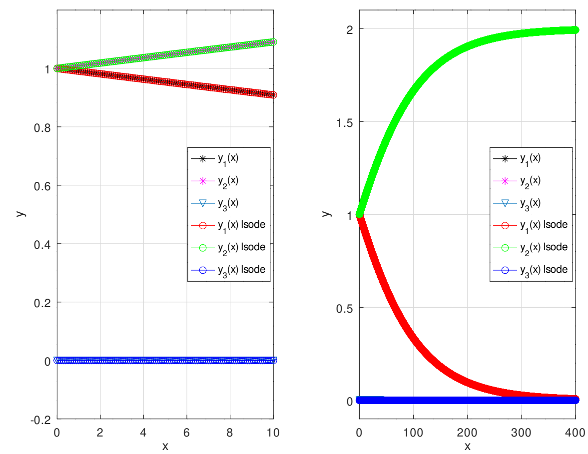

Figure 3: The left Figure shows a plot of the solution on the interval $[0, 10]$ on the x axis while the right figure was on the interval $[0, 400]$ on Example 3.2.

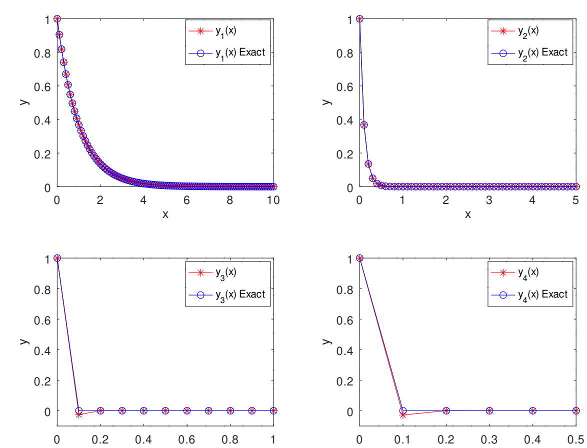

Figure 4: Comparing the performance of the New Block Hybrid Method with the exact solution on Example 3.3.

$$\begin{aligned} y_2'(x) &= -2500y_2y_3 \\ y_3'(x) &= -0.013y_1 - 1000y_1y_3 - 2500y_2y_3, \end{aligned}$$

such that

$$\begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

We applied the new block hybrid method with a fixed step size of $h = 0.1$, and compared the approximations with `octave lsode` [29] and the results are shown in Figure 3. This shows that the new scheme performed considerably well in comparison to known programs in the literature. We found the maximum absolute eigenvalue of the Jacobian of the system at the root to be 3998.5, the corresponding minimum absolute eigenvalue was 8.1042×10^{-16}, thereby resulting in a stiffness ratio of 4.9338×10^{18}.

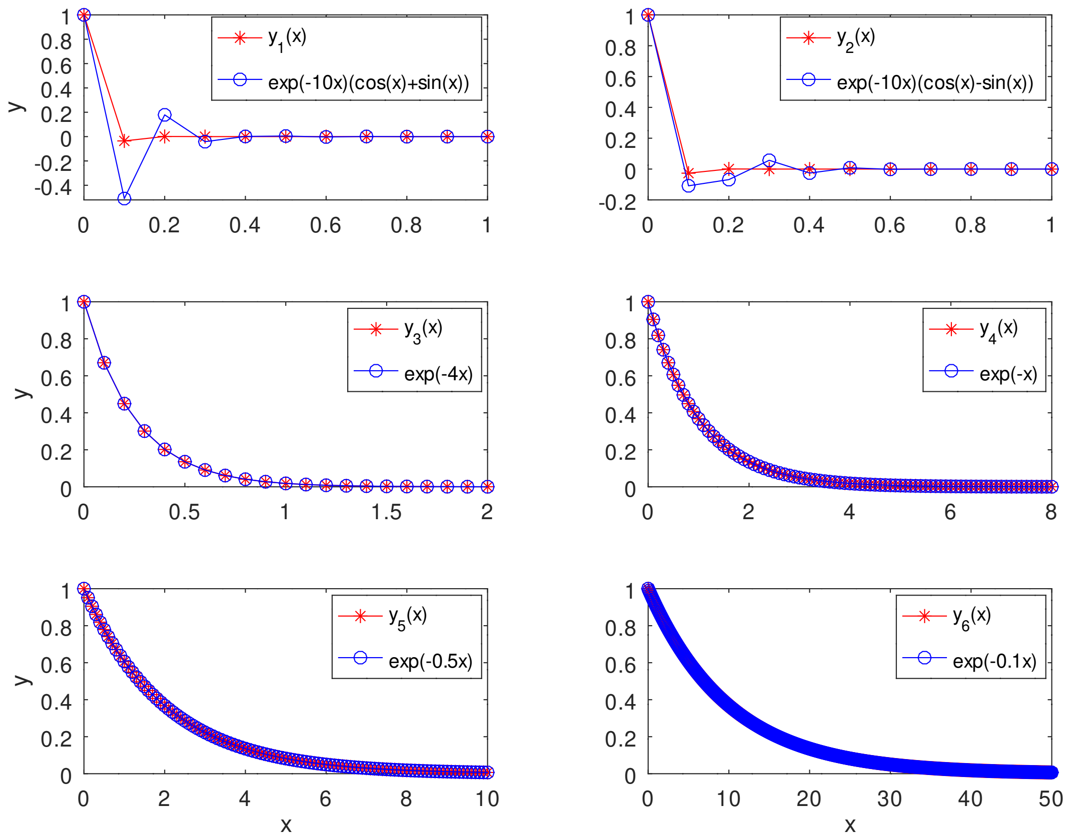

Figure 5: Comparing the performance of the new block hybrid method with the exact solution on Example 3.4.

Example 3.3. *The linear stiff IVP [27]:*

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \\ y_4'(x) \end{bmatrix} = \begin{bmatrix} -y_1 \\ -10y_2 \\ -100y_3 \\ -1000y_4 \end{bmatrix}, \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

with a step size of $h = 0.1$.

The exact solution of the IVPs is:

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} e^{-x} \\ e^{-10x} \\ e^{-100x} \\ e^{-1000x} \end{bmatrix}.$$

Results are shown in Figure 4 and Table A2. The stiffness ratio of the system of differential equations is 1000.

Example 3.4. *We consider the linear stiff IVPs studied by Fatunla in Ref. [30]:*

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \\ y_4'(x) \\ y_5'(x) \\ y_6'(x) \end{bmatrix} = \begin{bmatrix} -10y_1 + 100y_2 \\ -100y_1 - 10y_2 \\ -4y_3 \\ -y_4 \\ -0.5y_5 \\ -0.1y_6 \end{bmatrix}, \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \\ y_5(0) \\ y_6(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

We used a fixed step size of $h = 0.1$. By the nature of the matrix, we considered only four components y_1, y_2, y_3 and y_4. The results of numerical experiments are compared with the exact in Figure 5 and recent results in the literature in Table A3. The stiffness ratio of the system of differential equations is 1005.

In Table 2, we present the computational time (`cputime`) as well as the number of iterations in Newton's method in all numerical experiments.

Aside from solving the above examples using the new block hybrid method, we tried solving the same examples with the same starting guesses using only y_{n+1} and obtained the results as shown in Table 3. It was observed that this also gave good results which were all most at par with the block method albeit the block method performed far better and is highly recommended for the numerical integration of linear and non-linear differential equations.

It is pertinent to mention that due to the "big gap" in the absolute errors between the y_i's, there is a need to buttress this point. These big gaps are because we only used the scheme involving y_{n+1} alone instead of the whole block hybrid method. However, we make the following explanations for academic purposes. In Example 3.1, the exact solution is $y_1(x) = \exp(-2x), y_2(x) = \exp(-x)$. This implies that $y_1(x)$ will

Table 2: Computational times of the numerical integration of the IVPs on Examples 3.1, 3.2, 3.3 and 3.4. For $x = 5$ and $x = 50$, there were 50 and 500 number of iterations in Newton's method, respectively.

Examples	x	cputime
Example 3.1	5	0.2173
	50	1.8539
Example 3.2	5	0.2666
	50	2.6114
Example 3.3	5	0.2423
	50	2.2058
Example 3.4	5	0.3416
	50	3.3937

Table 3: Absolute errors of the numerical integration of the IVPs in Examples 3.1, 3.2 and 3.4 using equation (10).

Examples	Size	x	y_i	New method
Example 3.1	2×2	5	y_1	$2.64615455331249\times10^{-05}$
			y_2	$1.78144787057234\times10^{-03}$
Example 3.1	2×2	50	y_1	$3.97220625567371\times10^{-42}$
			y_2	$1.81946223485350\times10^{-21}$
Example 3.3	4×4	5	y_1	$1.78060428041512\times10^{-03}$
			y_2	$8.88178226825138\times10^{-16}$
			y_3	$8.51855127949365\times10^{-53}$
			y_4	0
Example 3.3	4×4	50	y_1	$1.81926143035958\times10^{-21}$
			y_2	0
			y_3	0
			y_4	0
Example 3.4	6×6	5	y_1	$9.65312078222632\times10^{-23}$
			y_2	$2.74021282732718\times10^{-23}$
			y_3	$3.38936239129207\times10^{-08}$
			y_4	$1.64739096130308\times10^{-03}$
			y_5	$4.96950254397107\times10^{-03}$
			y_6	$1.52305949362452\times10^{-03}$
Example 3.4	6×6	50	y_1	0
			y_2	0
			y_3	$8.62943100710741\times10^{-74}$
			y_4	$1.81926143035958\times10^{-21}$
			y_5	$1.15422964948989\times10^{-11}$
			y_6	$1.69429182203972\times10^{-04}$

decay twice as fast as $y_2(x)$. This is confirmed in row four column five and row five, column five where the absolute errors are $3.9722062556737173\times10^{-42}$ and $1.8194622348535021\times10^{-21}$ respectively. We did not bother to do the same for Example 3.2 because the exact solution is not known. We expected in theory that $y_4(x)$ should decay 10 times faster than $y_3(x)$, $y_3(x)$ should decay 10 times faster than $y_2(x)$ and $y_2(x)$ should decay 10 times faster than $y_1(x)$. The computational results confirm the theoretical underpinning. That is why there is a "big gap" in the absolute errors between the y_i's. In addition, from the exact solution of Example 3.4:

$$
\begin{aligned}
y_1(x) &= \exp(-10x)(\cos(100x) + \sin(100x)),\\
y_2(x) &= \exp(-10x)(\cos(100x) - \sin(100x)),\\
y_3(x) &= \exp(-4x),\\
y_4(x) &= \exp(-x),\\
y_5(x) &= \exp(-0.5x),\\
y_6(x) &= \exp(-0.1x),
\end{aligned}
$$

and by the presence of the exponential function in each of the solutions, it is not difficult to see that $y_1(x)$ decays faster than $y_2(x)$, $y_2(x)$ decays faster than $y_3(x)$ and so on. This explains the big gap in the absolute errors of Example 3.4 as shown from the fourteenth row (downwards) and last column of Table 3.

4. Conclusion

In this work, we derived and implemented a uniform-ninth-order block hybrid method for the numerical integration of linear and non–linear first-order ODEs. The method was shown to be zero–stable, consistent, convergent having a region of absolute stability which depicts, A(α)–stability. A further theoretical analysis of the method showed that the order of convergence of the method is eleven. Results of computational experiments showed that the new method outperformed recent results in the literature and compared favourably with the exact solution.

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APPENDIX A.

D Matrix

$$\mathbf{D} = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 & x_{n+1}^9 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 & 9x_n^8 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 & 8x_{n+1}^7 & 9x_{n+1}^8 \\ 0 & 1 & 2x_{n+\frac{3}{2}} & 3x_{n+\frac{3}{2}}^2 & 4x_{n+\frac{3}{2}}^3 & 5x_{n+\frac{3}{2}}^4 & 6x_{n+\frac{3}{2}}^5 & 7x_{n+\frac{3}{2}}^6 & 8x_{n+\frac{3}{2}}^7 & 9x_{n+\frac{3}{2}}^8 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 & 8x_{n+2}^7 & 9x_{n+2}^8 \\ 0 & 1 & 2x_{n+\frac{5}{2}} & 3x_{n+\frac{5}{2}}^2 & 4x_{n+\frac{5}{2}}^3 & 5x_{n+\frac{5}{2}}^4 & 6x_{n+\frac{5}{2}}^5 & 7x_{n+\frac{5}{2}}^6 & 8x_{n+\frac{5}{2}}^7 & 9x_{n+\frac{5}{2}}^8 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 & 8x_{n+3}^7 & 9x_{n+3}^8 \\ 0 & 1 & 2x_{n+\frac{7}{2}} & 3x_{n+\frac{7}{2}}^2 & 4x_{n+\frac{7}{2}}^3 & 5x_{n+\frac{7}{2}}^4 & 6x_{n+\frac{7}{2}}^5 & 7x_{n+\frac{7}{2}}^6 & 8x_{n+\frac{7}{2}}^7 & 9x_{n+\frac{7}{2}}^8 \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & 7x_{n+4}^6 & 8x_{n+4}^7 & 9x_{n+4}^8 \end{bmatrix}. \quad (21)$$

Table A1: Comparing the absolute errors of the new block hybrid method with those of [24] on Example 3.1.

x	y_i	Method 18 Ref. [24]	Method 21 Ref. [24]	New Method
5	y_1	$1.223052805026881 \times 10^{-03}$	$1.228938367083599 \times 10^{-03}$	$4.8405800671225103 \times 10^{-07}$
	y_2	$1.290570363021715 \times 10^{-06}$	$1.800318343625484 \times 10^{-06}$	$5.2809007554041609 \times 10^{-08}$
50	y_1	$3.320709446422848 \times 10^{-05}$	$3.325679258575631 \times 10^{-05}$	$3.9663889639351048 \times 10^{-46}$
	y_2	$9.887815172193726 \times 10^{-08}$	$5.804723043345561 \times 10^{-07}$	$1.5085805782915302 \times 10^{-27}$

Table A2: Comparing the absolute errors of the New Block Hybrid Scheme with those of Ref. [24] on Example 3.3.

x	y_i	Method 18 Ref. [24]	Method 21 Ref. [24]	New Method
5	y_1	0	0	$4.8806444996607468\times10^{-15}$
	y_2	0	0	$2.4700101997280476\times10^{-25}$
	y_3	0	0	$4.3679599334235881\times10^{-80}$
	y_4	$1.110223024625157\times10^{-16}$	$1.110223024625157\times10^{-16}$	$2.4544793415072880\times10^{-78}$
50	y_1	$2.220446049250313\times10^{-16}$	$2.220446049250313\times10^{-16}$	$1.5428128255672359\times10^{-33}$
	y_2	$4.440892098500626\times10^{-16}$	$4.440892098500626\times10^{-16}$	0
	y_3	$1.110223024625157\times10^{-16}$	$1.110223024625157\times10^{-16}$	0
	y_4	$2.220446049250313\times10^{-16}$	$1.110223024625157\times10^{-16}$	0

Table A3: Comparison of the absolute errors of the New Block Hybrid Method with those of [24] on Example 3.4.

x	y_i	Method 18 in Ref. [24]	Method 21 in Ref. [24]	New Method
5	y_1	$2.024105327791403\times10^{-10}$	$2.220446049250313\times10^{-16}$	$2.6069389501515157 \times 10^{-22}$
	y_2	$4.056337835067758 \times 10^{-10}$	$1.318389841742373\times10^{-16}$	$8.0250935335644924 \times 10^{-23}$
	y_3	0	0	$8.6430510337936034 \times 10^{-16}$
	y_4	0	0	$4.8806444996607468 \times 10^{-15}$
50	y_1	$1.721994824510631\times10^{-09}$	$3.330669073875470\times10^{-16}$	0
	y_2	$1.453979242560521\times10^{-09}$	$7.771561172376096\times10^{-16}$	0
	y_3	$4.440892098500626\times10^{-16}$	$4.440892098500626\times10^{-16}$	$5.8030927546064915 \times 10^{-093}$
	y_4	0	$1.110223024625157\times10^{-16}$	$1.399919732281278 6\times10^{-033}$

APPENDIX B.

The continuous formulation

The continuous formulation of the proposed hybrid block method is

$$
\begin{aligned}
y(x) = &-\frac{(1209600\omega^9 + 12927600h\omega^8 + 51710400h^2\omega^7 + 87318000h^3\omega^6 + 20956320h^4\omega^5 - 126214200h^5\omega^4 - 159213600h^6\omega^3 - 61236000h^7\omega^2))}{172541880h^9} y_n \\
&\frac{(-1209600\omega^9 - 12927600h\omega^8 - 51710400h^2\omega^7 - 87318000h^3\omega^6 - 20956320h^4\omega^5 + 126214200h^5\omega^4 + 159213600h^6\omega^3 + 61236000h^7\omega^2 - 172541880h^9))}{172541880h^9} y_{n+1} \\
&+ \Big[(3181600hf_{n+1} + 294400f_nh))\omega^9 + (32086218h^2f_{n+1} + 3214869f_nh^2))\omega^8 + (115198824h^3f_{n+1} + 13407228f_nh^3))\omega^7 \\
&+ (144678898h^4f_{n+1} + 25246025f_nh^4))\omega^6 + (15165423f_nh^5 - 92497944h^5f_{n+1}))\omega^5 + (-368884866h^6f_{n+1} - 16819593f_nh^6))\omega^4 \\
&+ (-181692776h^7f_{n+1} - 28685457f_nh^7))\omega^3 + (175388166h^8f_{n+1} - 11822895f_nh^8))\omega^2 + 172541880h^9f_{n+1}\omega\Big]/172541880h^9 \\
&+ \frac{(-6176000h\omega^9 - 58337472h^2\omega^8 - 185147712h^3\omega^7 - 151869760h^4\omega^6 + 307101312h^5\omega^5 + 533233344h^6\omega^4 - 61296192h^7\omega^3 - 377507520h^8\omega^2))}{172541880h^9} f_{n+\frac{3}{2}} \\
&+ \frac{(7901280h\omega^9 + 70066440h^2\omega^8 + 198102960h^3\omega^7 + 95883480h^4\omega^6 - 400865850h^5\omega^5 - 429040710h^6\omega^4 + 210922650h^7\omega^3 + 247029750h^8\omega^2)}{172541880h^9} f_{n+2} \\
&+ \frac{(-6417920h\omega^9 - 53254464h^2\omega^8 - 134141568h^3\omega^7 - 28743680h^4\omega^6 + 287572992h^5\omega^5 + 217226688h^6\omega^4 - 147037568h^7\omega^3 - 135204480h^8\omega^2))}{172541880h^9} f_{n+\frac{5}{2}} \\
&+ \frac{(3242080h\omega^9 + 25064070h^2\omega^8 + 56436120h^3\omega^7 + 786590h^4\omega^6 - 122124240h^5\omega^5 - 77081550h^6\omega^4 + 62129880h^7\omega^3 + 51547050h^8\omega^2))}{172541880h^9} f_{n+3} \\
&+ \frac{(-933120h\omega^9 - 6686208h^2\omega^8 - 13598784h^3\omega^7 + 1657152h^4\omega^6 + 29844864h^5\omega^5 + 16845696h^6\omega^4 - 15211584h^7\omega^3 - 11918016h^8\omega^2))}{172541880h^9} f_{n+\frac{7}{2}} \\
&+ \frac{(117280h\omega^9 + 774147h^2\omega^8 + 1453332h^3\omega^7 - 320705h^4\omega^6 - 3240237h^5\omega^5 - 1693209h^6\omega^4 + 1657447h^7\omega^3 + 1251945h^8\omega^2))}{172541880h^9} f_{n+4}.
\end{aligned}
\tag{22}
$$