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Convolution equation and operators on the Euclidean motion group

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Abstract

Let $G = \mathbb{R}^2 \rtimes SO(2)$ be the Euclidean motion group, let $\mathfrak{g}$ be the Lie algebra of G and let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Then $\mathcal{U}(\mathfrak{g})$ is an infinite dimensional, linear associative and non-commutative algebra consisting of invariant differential operators on G. The Dirac measure on G is represented by δ_G , while the convolution product of functions or measures on G is represented by $*$. Among other notable results, it is demonstrated that for each u in $\mathcal{U}(\mathfrak{g})$, there is a distribution E on G such that the convolution equation $u * E = \delta_G$ is solved by method of convolution. Further more, it is established that the (convolution) operator $A' : C_c^\infty(G) \to C^\infty(G)$, which is defined as $A'f = f * T^n\delta(t)$ extends to a bounded linear operator on $L^2(G)$, for $f \in C_c^\infty(G)$, the space of infinitely differentiable functions on G with compact support. Furthermore, we demonstrate that the left convolution operator L_T denoted as $L_T f = T * f$ commutes with left translation, for $T \in D'(G)$.

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1. Introduction

A solution of the equation of the form $PT = f$, for any distribution f on G, and a solution T (which can also be thought of as a distribution) of the differential operator P are the pivots for the existence of a fundamental solution for differential operators with constant or variable coefficients on a locally compact group. El-Hussein [1] and Battesti [2], in 1954, proved that P with constant coefficient on $\mathbb{R}^n$ has a solution (fundamental) on $\mathbb{R}^n$. A solution $T \in \mathcal{D}'$ of a differential operator P with constant coefficient is referred to as a fundamental solution if $PT = f$ [3]. This result was independently verified by Malgrange in 1955 [2, 4]. Hormander [3], in 1958, also proved that a differential operator P admits a temperate solution; a new proof of Hormander's result was given in 1970 by Atiya [5]. Lewy [6], in 1970, showed that P has a different solution when its coefficients are variables. He showed that for any infinitely differentiable function f on $\mathbb{R}^3$, there is no such T such that the equation $-i\partial_x + \partial_y - 2(x+iy)\partial_z T = f$ is satisfied. A solution of this equation for compactly supported functions was not considered by Lewy [6]. However, El-Hussein [1] proved that any P on the Heisenberg group $\mathbb{H}^n$ has a fundamental solution. He obtained the result by considering the Heisenberg group as $\mathbb{R}^{2n+1}$ with its usual multiplication rather than looking at $\mathbb{H}^n$ as a matrix group. In particular, he proved that for a function f

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on $\mathbb{H}^n$ with compact support, there is a distribution T on $\mathbb{H}^n$ such that the equation $i\partial_x + \partial_y + iy\partial_z T = f$ is satisfied. In this work, we extend the result of El-Hussein [1], in section 3, to Euclidean motion group, for $n = 2$. In section 4, a specified operator for the subspace made up of equivalent classes of right K-invariant functions of the H-valued L^p - space on G denoted by $L^p(G : K, \mathcal{H})$, $1 < p < \infty$, and its adjoint A^* are obtained. We begin with some preliminaries, in section 2, concerning the Euclidean motion group, its Haar measure, Lie algebra, spaces of test functions on $SE(2)$, their topologies and distributions, and Fourier transform of functions on the space of rapidly decreasing functions on G.

2. Preliminaries

2.1. Euclidean motion group

The group $SE(n)$, which is often referred to as the motion group or group of rigid motions, is defined as the semi-direct product of $\mathbb{R}^n$ with $SO(n)$, that is $SE(n) = \mathbb{R}^n \rtimes SO(n)$. Elements of $SE(n)$ may be denoted as $g = (\bar{x}, \xi)$, where $\xi \in SO(n)$ and $\bar{x} \in \mathbb{R}^n$. For any $g_1 = (\bar{x_1}, \xi_1)$ and $g_2 = (\bar{x_2}, \xi_2) \in SE(n)$, multiplication on SE(n) may be defined as

$$g_1 g_2 = (\bar{x_1} + \xi_1 \bar{x_2}, \xi_1 \xi_2),$$

and the inverse is defined as

$$g^{-1} = (-\xi^t \bar{x}, \xi^t).$$

Here ξ^t denotes a transpose. Alternatively, $SE(n)$ could also be identified as a matrix group whose arbitrary element may be identified as $(n+1) \times (n+1)$ matrix of the form

$$H(g) = \begin{pmatrix} \xi & \bar{x} \\ 0^t & 1 \end{pmatrix},$$

where $\xi \in SO(n)$ and $0^t = (0, 0..., 0)$. The explicit matrix form of the element of the group $SE(n)$, for $n = 2$, is given by

$$g((x_1, x_2), \phi) = \begin{pmatrix} cos\phi & -sin\phi & x_1 \\ sin\phi & cos\phi & x_2 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\phi \in [0, 2\pi]$, $(a_1, a_2) \in \mathbb{R}^2$ [4, 7–11]. $SE(2)$ is a non - compact and non- commutative solvable Lie group [12, 13]. For all $n \geq 2$, $SE(n)$ is a group of affine maps induced by orthogonal transformations. It is also referred to as a group of rigid motions on $\mathbb{R}^n$ and is applied in robotic, dynamics and motion planning [9, 14–16].

2.2. Lie algebra of $SE(2)$

The algebra $se(n)$ is regarded as the Lie algebra of $SE(n)$. It is the sub-algebra of $gl(n+1, \mathbb{R})$ which may be written as

$$se(n) = \left\{ X = \begin{pmatrix} Q & M \\ 0 & 0 \end{pmatrix} : Q \in \mathbb{R}^n,\ M \in so(n) \right\}.$$

$gl(n+1, \mathbb{R})$ is made up of real matrices U of order $n+1$ where the Lie bracket $[U, V]$ on it is defined as $[U, V] = UV - VU$ for $U, V \in gl(n+1, \mathbb{R})$; $so(n) = \left\{ U \in gl(n+1, \mathbb{R}) | U + U^t = 0 \right\}$ is the Lie algebra of $SO(n)$; Q is a skew - symmetric matrix; and M is a vector in $\mathbb{R}^n$ [17]. For $n = 2$, $G = \mathbb{R}^2 \rtimes SO(2)$. $SO(2)$ has the Lie algebra $so(2)$ which is generated by $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. $\mathfrak{g} = \mathbb{R}^2 \times so(2)$ with $\mathbb{R}^2$, $X = (1, 0)$ and $Y = (0, 1)$ as generators. That is to say, the generators of $\mathfrak{g}$ are (X, Y, T).

$SE(2)$ is a unimodular Solvable Lie group. A measure(invariant measure)$d\mu_G$ on it is the product of lebesque measure on $\mathbb{R}^2$ and the normalized Haar measure of $SO(2)$.

2.3. Spaces of test functions on $SE(2)$, their topologies and distributions

Here we give a brief description of spaces of distributions and their respective topologies.

2.4. The space $C^\infty(G)$

Given a solvable Lie group G endowed with invariant measure $d\mu(g)$, and $\mathfrak{g}$ its Lie algebra. Lets denote by m the dimension of $\mathfrak{g}$. Fix $\{X_1, ..., X_m\}$ a basis of $\mathfrak{g}$. To each $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{N}^m$, we put $|\alpha| = \alpha_1 + ... + \alpha_n$ and associate a differential operator X^α, which is left invariant, on G acting on $f \in C^\infty(G)$, the space $C^\infty(G)$ of infinitely differentiable functions on G, by

$$X^\alpha f(g) = \frac{\partial^{\alpha_1}}{\partial t_1^{\alpha_1}} ... \frac{\partial^{\alpha_m}}{\partial t_m^{\alpha_m}} f(g\, exp(t_1 X_1)...exp(t_m X_m))|_{t_1 = ... = t_m = 0.}.$$

The space $C^\infty(G)$ may be given a topology defined by a system of seminorms specified as

$$|f|_{\alpha, m} = Sup_{|\alpha| \leq m} |X^\alpha f(g)|.$$

With this topology, C^∞(G) is metrizable, locally convex and complete, hence, it is a Frechet space. This Frechet space may be denoted as $\xi(G)$.

2.5. The space $C_c^\infty(G)$

This space $C_c^\infty(G)$ is the space of complex-valued C^∞ function on G with compact support. For any $\epsilon > 0$, put

$$B_\epsilon = \{(\xi, \theta) \in G : \|\xi\| \leq \epsilon\}$$

and

$$\mathfrak{D}_\epsilon = \mathfrak{D}(B_\epsilon) = \{f \in C_c^\infty(G) : f(\xi, \theta) = 0, if\ \|\xi\| > \epsilon\}.$$

Then $\mathfrak{D}(B_\epsilon)$ is a Frechet space with respect to the family semi norms defined as

$$\left\{ P_\alpha(f) = \|D^\alpha f\|_\infty : \alpha \in \mathbb{N}^3 \right\}.$$

$\mathfrak{D}(G) = \bigcup_{n=1}^\infty \mathfrak{D}(B_n)$ is topologised as the strict inductive limit of $\mathfrak{D}(B_n)$. A linear functional on the topological vector space $\mathfrak{D}(G)$ that is continuous is known as a distribution on G. Then $\mathfrak{D}'(G)$ is the space of distribution on G.

Given a manifold M and a distribution T, T is said to vanish on a subset $V \subset M$, which is open, if $T = 0$. Let $\{U_\alpha\}_{\alpha \in \omega}$ represents the collection of all open sets on which T vanishes and let U stand for the union of $\{U_\alpha\}_{\alpha \in \omega}$. $M - U$, regarded as the complement of M, is the support of T. We denote $\xi'(G)$ a distributions space with compact support.

2.6. The Schwartz space $\mathcal{S}(G)$

Consider the Euclidean motion group $SE(2)$ realised as $\mathbb{R} \rtimes \mathbb{T}$ where $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$. If we choose a system of coordinates (x, y, θ) on G with $x, y \in \mathbb{R}$ and $\theta \in \mathbb{T}$, then a complex - valued C^∞ function f on $G = SE(2)$ is called rapidly decreasing if for any $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}^3$ we have

$$p_{N,\alpha}(f) = Sup_{\theta\in\mathbb{T},\xi\in\mathbb{R}^2} \mid (1 + \|\xi\|^2)^N (D^\alpha f)(\xi, \theta) \mid < +\infty,$$

where

$$D^\alpha = \left(\frac{\partial}{\partial x}\right)^{\alpha_1}\left(\frac{\partial}{\partial y}\right)^{\alpha_2}\left(\frac{\partial}{\partial \theta}\right)^{\alpha_3},$$

$(\alpha = (\alpha_1, \alpha_2, \alpha_3); \xi = (x, y))$. The space of all rapidly decreasing functions on G is denoted by $\mathcal{S} = \mathcal{S}(G)$. Then $\mathcal{S}$ is a Frechet space in the topology given by the family of seminorms $\{P_{N,\alpha} : N \in \mathbb{N}, \alpha \in \mathbb{N}^3\}$.

The space $\mathcal{S}'(G)$ of (continuous) linear functionals on $S(G)$ is referred to as the space of tempered distributions on $G = SE(2)$. This space can be topologised by strong dual topology, which is defined as the topology of uniform convergence on the bounded subsets of $\mathcal{S}(G)$ generated by the seminorms $p_\varphi(u) = |u(\varphi)|$, where $u : \mathcal{S}(G) \to \mathbb{R}$ and $\varphi \in \mathcal{S}(G)$. We close this section by defining the concept of convolution on the space $\mathcal{S}(G)$.

Let $f_1, f_2 \in \mathcal{S}(G)$ or $L^2(G)$. The convolution of f_1 and f_2 is defined as

$$\begin{aligned}(f_1 * f_2)(g) &= \int_G f_1(h) f_2(h^{-1}g) d\mu_G(h) \\ &= \int_G f_1(gh) f_2(h^{-1}) d\mu_G(h).\end{aligned}$$

The convolution operation obeys the associativity property

$$(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3),$$

whenever all the integrals are absolutely convergent [1, 9, 18–20].

2.7. Fourier transform for $SE(2)$

The Fourier transform of the group $SE(2)$ is needed in what follows, and we need the following preparations.

Let $L^2([0, 2\pi], \frac{d\alpha}{2\pi})$, be the space of square integrable functions on $\mathbb{T} \cong [0, 2\pi] \cong SO(2)$. A representation of $SE(2)$ on $L^2([0, 2\pi], \frac{d\alpha}{2\pi})$ is given as

$$U^{(p)}(g)\tilde{\varphi}(X) = e^{-ip(a.X)}\tilde{\varphi}(A^T X)$$

for each $g \in SE(2)$, where $p \in \mathbb{R}^+$ and $X.Y = x_1 y_1 + x_2 y_2$. $A = \begin{pmatrix} cos\phi & -sin\phi \\ sin\phi & cos\phi \end{pmatrix}$, $A^T = \begin{pmatrix} cos\phi & sin\phi \\ -sin\phi & cos\phi \end{pmatrix}$, so that

$$A^T X = \begin{pmatrix} cos\phi & sin\phi \\ -sin\phi & cos\phi \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 cos\phi + x_2 sin\phi \\ -x_s in\phi + x_2 cos\phi \end{pmatrix}.$$

The representation $U^{(p)}(g)$ given is unitary and irreducible [9, 21–23].

The following definition of group Fourier transform may be found in [10, 15, 24, 25].

2.7.1. Definition

The Fourier transform of $f \in S(G)$ (or $f \in L^1(G)$) is a map

$$\mathcal{F}(f) : \mathbb{R}^+_* \to B(L^2(G)),$$

defined as

$$(\mathcal{F} f)(\sigma) = \int_G f(g) U^\sigma(g^{-1}) d\mu(g), \ for\ \sigma > 0,$$

and the inverse is defined as

$$f(g) = \int_0^\infty Tr(U^\sigma_g f(\sigma))\sigma d\sigma,$$

where $g = (\bar{x}, \xi)$ and $d\mu(g)$ stands for a measure on G. $\mathcal{F}(f)(\sigma)$ may also be denoted by $\widehat{f(\sigma)}$ in what follows.

If f and h are integrable functions on $G = SE(2)$, then the Fourier transform for G satisfies the following properties [12, 23] which are required for what follows in section 3.

(i) $\|\widehat{f(\sigma)}\| \leq \|f\|_1$, for any $\sigma > 0$.
(ii) $\widehat{(f_1 * f_2)} = \widehat{f_2}\widehat{f_1}$.
(iii) $(\widehat{f^*})(\sigma) = (\widehat{f(\sigma)})^*$, where $f^*(g) = \overline{f(g^{-1})}$.
(iv) $\int_{SE(2)} |f(g)|^2 d\mu(g) = \int_0^\infty \|\widehat{f(\sigma)}\|_2^2 \sigma d\sigma$, for $f \in L^2(G)$.

Property (ii) is called the convolution property and property (iv) is called the Parseval's equality also known as the Plancherel formula for $SE(2)$.

3. Convolution equation on $SE(2)$

Let the spaces $C^\infty(G)$, $\mathfrak{D}(G)$, $\mathfrak{D}'(G)$ and $\xi'(G)$ be as defined in section 2. Let $U(\mathfrak{g})$ be the enveloping algebra $\mathfrak{g}$ ($= se(n)$) of $G = \mathbb{R}^n \rtimes SO(n)$. For $f \in \mathcal{D}(G)$ (respectively $\mathcal{S}(G)$ or $\xi(G)$), the operator

$$P_u := f \mapsto u * f$$

has the range equals $\xi'(G)$. That is,

$$P_u : \mathcal{D}(G) \ni f \mapsto (P_u f)(g) = u * f(g) \in \xi'(G).$$

The fundamental solution of P_u is a distribution E such that $P_u E = \delta_G$, where δ_G is the Dirac distribution on G supported at $\{e\}$ (= the identity of G). The equation

$$u * E = \delta_G,$$

is called a convolution equation on the group G. Now if $u \in \mathcal{S}(G)$, then the operator

$$P_u : \mathcal{S}(G) \ni f \mapsto P_u f = u * f \in \mathcal{S}(G),$$

is a left multiplication map and is continuous. $\mathcal{S}(G)$ is a Frechet (locally convex) algebra, multiplication in $\mathcal{S}(G)$ is seen to be continuous (jointly).

Let E be a topological vector space and T its subspace. T is said to be barrel if T is absorbing, balanced, closed and convex. E is called a Montel space if it is locally convex, barrel and every closed bounded subset of E is compact [21]. $A \subset E$ is called be pre-compact (or totally bounded) if a finite set $M \subset E$ exists such that $A \subset M + U$, where U is a zero neighborhood in E. An operator $T : E \to E$, which is linear, is said to be pre-compact if $T(V)$ is precompact, V being a zero neighborhood of E. $\mathcal{S}(G)$ is an example of a Montel space and the operator, $P_u : \mathcal{S}(G) \ni f \mapsto P_u f = u * f \in \mathcal{S}(G)$ is pre-compact. The next result is required in the proof of theorem 3.3, our main result.

3.1. Theorem

The Fourier transform of a function $f \in \mathcal{S}(G)$ satisfies the following:

$$\check{u} * f((0,0),1) = \int_G \widehat{(\check{u} * f)}(\sigma) d\mu_G(g) = \int_G \hat{f}(\sigma)\widehat{(\check{u})}(\sigma) d\mu_G(g).$$

Proof.

$$\begin{aligned}\check{u} * f((0,0),1) &= \int_G \widehat{(\check{u} * f)}(\sigma) d\mu_G(g) \\ &= \int_G \int_G (\check{u} * f)(g) U^\sigma(g^{-1}) d\mu_G(g) d\mu_G(g) \\ &= \int_G \int_G \int_G \check{u}(h) f(h^{-1}g) U^\sigma(g^{-1}) d\mu_G(g) d\mu_G(g) \mu_G(g) \\ &= \int_G \int_G \int_G u(h^{-1}) f(h^{-1}g) U^\sigma(g^{-1}) d\mu_G(g) d\mu_G(g) \mu_G(g).\end{aligned}$$

Since $G = SE(2)$ is unimodular, we have

$$\begin{aligned}\int_G f(hg) d_G(g) &= \int_G f(gh) d\mu_G(g) \\ &= \int_G f(g^{-1}) d\mu_G(g) \\ &= \int_G f(g) d\mu_G(g).\end{aligned}$$

Since U^σ is a representation, it follows that $U^\sigma(hg)^{-1} = U^\sigma(g^{-1}h^{-1}) = U^\sigma(g^{-1})U^\sigma(h^{-1})$. Therefore,

$$\int_G \int_G \int_G u(h^{-1}) f(h^{-1}(hg)) U^\sigma(g^{-1}) d\mu_G(g) d\mu_G(g) d\mu_G(g)$$

$$\begin{aligned}&= \int_G \int_G \int_G u(h^{-1}) f(g) U^\sigma(g^{-1}) d\mu_G(g) d\mu_G(g) d\mu_G(g) \\ &= \int_G \int_G \int_G u(h^{-1}) f(h^{-1}(hg)) U^\sigma(g^{-1}) U^\sigma(h^{-1}) d\mu_G(g) d\mu_G(g) d\mu_G(g)\end{aligned}$$

By Fubini's theorem

$$\int_G \int_G \int_G u(h^{-1}) f(h^{-1}(hg)) U^\sigma(g^{-1}) U^\sigma(h^{-1}) d\mu_G(g) d\mu_G(g) d\mu_G(g)$$

$$\begin{aligned}&= \int_G \int_G f(g) U^\sigma(g^{-1}) d\mu_G(g) \Big(\int_G u(h^{-1}) U^\sigma(h^{-1}) d\mu_G(h) \Big) \\ &= \int_G \hat{f}(\sigma) (\check{u})\hat{}(\sigma) d\mu_G(g).\end{aligned}$$

The next proposition is also required for the main result in theorem 3.3.

3.2. Proposition

Let $f, u \in \mathcal{S}(G)$, then

$$(\check{u} * f)\hat{}(\sigma) = \hat{f}(\sigma)(\check{u})\hat{}(\sigma).$$

Proof.

$$\begin{aligned}(\check{u} * f)\hat{}(\sigma) &= \int_G (\check{u} * f)(g) U^\sigma(g^{-1}) d\mu_G(g) \\ &= \int_G \int_G \check{u}(h) f(h^{-1}g)(g) U^\sigma(g^{-1}) d\mu_G(g) \mu_G(g) \\ &= \int_G \int_G u(h^{-1}) f(h^{-1}g)(g) U^\sigma(g^{-1}) d\mu_G(g) \mu_G(g).\end{aligned}$$

Since $SE(2)$ is unimodular,

$$\begin{aligned}\int_G f(hg) d_G(g) &= \int_G f(gh) d\mu_G(g) \\ &= \int_G f(g^{-1}) d\mu_G(g) \\ &= \int_G f(g) d\mu_G(g),\end{aligned}$$

$$\begin{aligned}(\check{u} * f)\hat{}(\sigma) &= \int_G \int_G u(h^{-1}) f(h^{-1}hg)(g) U^\sigma(g^{-1}) d\mu_G(g) d\mu_G(g) \\ &= \int_G \int_G u(h^{-1}) f(g)(g) U^\sigma(g^{-1}) d\mu_G(g) d\mu_G(g).\end{aligned}$$

Since U^σ is a representation, then $U^\sigma(g^{-1}) = U^\sigma((gh)^{-1}) = U^\sigma(h^{-1}g^{-1}) = U^\sigma(h^{-1})U^(g^{-1})$, so that

$$(\check{u} * f)\hat{}(\sigma) = \int_G \int_G u(h^{-1}) f(g)(g) U^\sigma(h^{-1}) U^\sigma(g^{-1}) d\mu_G(g) d\mu_G(g).$$

By Fubini's theorem, we have

$$\begin{aligned}(\check{u} * f)\hat{}(\sigma) &= \int_G f(g) u(g^{-1}) d\mu_G(g) \int_G u(h^{-1})(g) U^\sigma(h^{-1}) d\mu_G(g) \\ &= \hat{f}(\sigma)(\check{u})\hat{}(\sigma).\end{aligned}$$

The main result of this work is stated and presented in theorem 3.3.

3.3. Theorem

For every $u \in U(\mathfrak{g})$, one can find $E \in \mathcal{D}'(G)$ in such a way that

$$u * E = \delta_G,$$

for δ_G being the Dirac measure of G defined at the origin (identity).

Proof. Let $f \in \mathcal{S}(G)$ and let $E \in \mathfrak{D}'(G)$ be defined by

$$\langle E, f \rangle = \int_G \int_{\mathbb{R}^n} \frac{\hat{f}(\sigma)}{(\check{u})\hat{}(\sigma)} \Psi(P_\xi, \xi) d\xi d\mu_G(g), \tag{1}$$

where $\Psi(P_\xi, \xi)$ is the Hormander's function. $\Psi(P_\xi, \xi)$ is infinitely differentiable and has compact support, it is choosen to be analytic [2, 26, 27] and P_ξ is a polynomial of invariant differential operators. Equation (1) is referred to as Hormander's construction. Now

$$\langle \widehat{(u * E}, f \rangle = \langle u * E, \hat{f} \rangle$$

$$= \langle E, \check{u} * \hat{f} \rangle.$$

Using the Hormander's construction in equation (1) above, we have

$$\begin{aligned}
\langle E, \check{u} * \hat{f} \rangle &= \int_G \int_{\mathbb{R}^n} \frac{(\check{u} * \hat{f})\hat{}(\sigma)}{(\check{u})\hat{}(\sigma)} \Psi(P_\xi, \xi) d\xi d\mu_G(g) \\
&= \int_G \int_{\mathbb{R}^n} \frac{(\hat{f})\hat{}(\sigma)(\check{u})\hat{}(\sigma)}{(\check{u})\hat{}(\sigma)} \Psi(P_\xi, \xi) d\xi d\mu_G(g) \\
&= \int_G \int_{\mathbb{R}^n} (\hat{f})\hat{}(\sigma) \Psi(P_\xi, \xi) d\xi d\mu_G(g) \\
&= \int_G \int_{\mathbb{R}^n} (\hat{f})\hat{}(\sigma) \Psi(P_\xi, \xi) d\xi d\mu_G(g) \\
&= \int_G \int_{\mathbb{R}^n} \mathcal{F}(\hat{f})(\sigma) \Psi(P_\xi, \xi) d\xi d\mu_G(g) \\
&= \hat{f}((0,0),1) = \langle \delta_G, \hat{f} \rangle = \langle \hat{\delta}_G, f \rangle,
\end{aligned}$$

δ_G being the measure of G at the identity. We, therefore, have

$$\widehat{u * E}((x,y),\theta) = \widehat{\delta}_G((x,y),\theta),$$

and by invariance,

$$(u * E)((0,0),1) = \delta_G((0,0),1). \tag{2}$$

4. Convolution operators on the euclidean motion group

Some convolution operators on the Euclidean motion group are considered in this section. For $G = SE(2)$ realised as $\mathbb{R}^2 \rtimes \mathbb{T}$, the convolution operators are defined on the space $C_c^\infty(G)$ and take values in the space $C^\infty(G)$. Let $\mathcal{H}$ be a Hilbert space and let $G = SE(n)$ be generated by $R^n \rtimes SO(n)$. A specified operator for the subspace made up of equivalent classes of right K-invariant functions of the H-valued L^p - space on G denoted by $L^p(G : K, \mathcal{H})$, $1 < p < \infty$, and its adjoint A^* are obtained.

The following proposition is needed in what follows.

4.1. Proposition [25]

In the event that f is a rapidly decreasing function on G, then the Fourier transform of f is as follows

$$(\widehat{f(\sigma)}F)(s) = \int_K K_f(\sigma; s, r) F(r) dr, \tag{3}$$

for any $\sigma > 0$ and $F \in L^2(K)$, $K = SO(2) \cong \mathbb{T}$, $r, s \in \mathbb{T}$, where

$$K_f(\sigma; s, r) = \int_{\mathbb{R}^2} f(\xi, rs^{-1}) e^{-i(\xi, r\sigma)} d\mu(\xi). \tag{4}$$

Proof. If F and F' belong to $L^2(K)$ and $g = t(\xi)r$, then $g^{-1} = r^{-1}t(-\xi) = t(-r^{-1}\xi)r^{-1}$ and

$$\begin{aligned}
(\widehat{f(\sigma)}F, F') &= \int_{\mathbb{R}^2} \int_K f(\xi, r)(U^\sigma_{t(-r^{-1}\xi)r^{-1}} F, F') d\mu(\xi) dr \\
&= \int_{\mathbb{R}^2} \int_K \int_K f(\xi, r) e^{-i(sr^{-1}\xi, s\sigma)} F(rs) \overline{F'}(s) d\mu(\xi) dr ds \\
&= \int_{\mathbb{R}^2} \int_K \int_K f(\xi, rs^{-1}) e^{-i(sr^{-1}\xi, s\sigma)} F(r) \overline{F'}(s) d\mu(\xi) dr ds \\
&= \int_K \left\{ \int_K K_f(\sigma; s, r) F(r) dr \right\} \overline{F'}(s) ds.
\end{aligned}$$

Given that F' is arbitrary in $L^2(K)$, equation (3) holds for nearly all s in K. This claim demonstrates $\widehat{f}(\sigma)$ is an integral operator on $L^2(K)$ whose kernel K_f is given by section 2.2 iff $f \in \mathcal{S}(G)$.

4.2. Corollary [25]

As a function of $\xi \in \mathbb{R}^2$, the ordinary Fourier transform of $f(\xi, r)$ is represented by $\widetilde{f}(\zeta, r)$:

$$\widetilde{f}(\zeta, r) = \int_{\mathbb{R}^2} f(\xi, r) e^{-i(\xi, \zeta)} d\mu(\xi).$$

Then the kernel $K_f(\sigma; s, r)$ is given by

$$K_f(\sigma; s, r) = \widetilde{f}(r\sigma, rs^{-1}).$$

It is necessary to review certain definitions, that may be needed in this research, on locally convex spaces. Let E be a vector space over the field of complex numbers. Let E be the union of an increasing sequence of subspaces E_n, $n = 1, 2, ...$, and let each E_n, have a Frechet space structure such that the injection of E_n into E_{n+1} is an isomorphism, i.e, the topology induced by E_{n+1} on E_n is is the same as the topology given on E_n initially. Then, we can define the Hausdorff locally convex space structure on E as follows. Assuming convexity, a subset V of E is a neighbourhood of zero if and only if $V \cap E_n$ is a neighbourhood of zero in the Frechet space E_n for each n = 1,2,... We state that E is an LF-space or, equivalently, a countable strict inductive limit of Frechet spaces when we give it this topology. We may also state that the sequence of Frechet spaces, $\{E_n\}$, is a sequence of definitions of E. If every balanced, convex, bornivorous subset of a locally convex space is a neighbourhood of zero, then the space is said to be bornological. The spaceC $C^\infty(G)$ is a metrizable, locally convex, complete space; on the other hand, the space $C_c^\infty(G)$ is an LF-space. Both spaces are bornological with respect to their respective topologies.

We can now present the main results of this section. For the results presented in propositions 3.2 and 3.3 we take $G = SE(n) = \mathbb{R}^n \rtimes SO(n)$ and $K = SO(n)$. For a Hilbert space $\mathcal{H}$ and for $1 \leq p < \infty$, we denote the subspace formed by equivalent classes of right K-invariant functions of the $\mathcal{H}$-valued L^p-space on G by $L^p(G : K, \mathcal{H})$.

This space is the $\mathcal{H}$-valued L^p on $X = G/K$, lifted to G. We are interested in integral operators of the form

$$A_\varphi(x) = \int_X S(x, y) \varphi(y) dy,$$

where S is the kernel

$$S : X \times X \to L(\mathcal{H}_1, \mathcal{H}_2).$$

Such that

$$S(gx, gy) = U_1^\sigma(g) S(x, y) U_2^\sigma(g^{-1}),$$

for all $g \in G$, with some uniformly bounded representations U_1^σ, U_2^σ of G on the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. By lifting all functions to G we may define an operator A by

$$Af(g) = \int_G U_2^\sigma(h) k(h^{-1}g) U_1^\sigma(h^{-1}) f(h) dh, \tag{5}$$

where $k(g)$ satisfies the identity

$$k(hgh') = U_1^\sigma(h)k(g)U_2^\sigma(h^{-1}), \tag{6}$$

for all $h, h' \in K$. If $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}$ and U_1^σ, U_2^σ are trivial representations, then equation (6) only means that k is left and right invariant (that is, bi-K-invariant) under K. Equation (5) is called a twisted convolution of f and k denoted by $f *_{1,2} k$.

Proposition 4.3 shows L^p- boundedness of the operator A while proposition 4.4 verifies its adjoint.

4.3. Proposition

Let U_1^σ, U_2^σ be representations of G on the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ having uniform bounds M_1, M_2 respectively. Let $k \in L^1(G : K, L(\mathcal{H}_1, \mathcal{H}_2))$ satisfying equation (6). Then for all $1 \le p \le \infty$, A defined by equation (5) is a continuous linear map $L^p(G : K, \mathcal{H}_1) \to L^p(G : K, \mathcal{H}_2)$ with bound not greater than $M_1 M_2 \|k\|_1$.

Proof. Let the operator A be fixed by

$$Af(g) = \int_G U_2^\sigma(h)k(h^{-1}g)U_1^\sigma(h^{-1})f(h)dh.$$

Letting $h = gl$ and $h^{-1}g = (gl)^{-1}g = l^{-1}g^{-1}g = l^{-1}$, the operator A becomes,

$$Af(g) = \int_G U_2^\sigma(gl)k(l^{-1})U_1^\sigma(gl)^{-1}f(gl)dl.$$

By applying the Minkowski inequality,

$$\begin{aligned}
\|Af\|_p &= \left(\int_G |Af(g)|^p dg\right)^{\frac{1}{p}}\\
&\le \int_G \left(\int_G |U_2^\sigma(gl)k(l^{-1})U_1^\sigma(gl)^{-1}f(gl)|^p dg\right)^{\frac{1}{p}} dl\\
&\le \int_G \left(\int_G |U_2^\sigma(gl)|^p dg\right)^{\frac{1}{p}} \left(\int_G |k(l^{-1})|^p dg\right)^{\frac{1}{p}} \left(\int_G |U_1^\sigma(gl)^{-1}|^p dg\right)^{\frac{1}{p}}\Bigg)\\
&\le M_1 M_2 \left(\int_G |k(l^{-1})dg\right)\left(\int_G |f(gl)|^p dg\right)^{\frac{1}{p}} dl\\
&\le M_1 M_2 |k(l^{-1})| \left(\int_G |f(g)|^p dg\right)^{\frac{1}{p}}\\
&= M_1 M_2 \|k\|_1 \|f\|_p.
\end{aligned}$$

We denote by $\langle , \rangle$ the bilinear form concerning $\mathcal{H}$ and its dual $\mathcal{H}'$. This may be a complex bilinear or a Hermitian form, the result that follows hold in either case. The dual of $L^p(G : K, \mathcal{H})$ contains $L^{p'}(G : K, \mathcal{H}')$, here and hereafter, p' is the dual exponent of p, $p' = \frac{p}{p-1}$, and coincides with it if $\mathcal{H}$ is reflexive. We present the following result concerning the adjoint of A.

4.4. Proposition

Let A be as in proposition 3.2. Then, for any $\varphi \in L^{p'}(G : K, \mathcal{H}_2')$, the adjoint A^* of A is given by

$$A^*\varphi(g) = \int_G U_1^\sigma(h)^{*-1}\check{k}(h^{-1}g)^* U_2^\sigma(h^*)\varphi(h)dh,$$

where

$$\check{k}(l) = U_2^\sigma(l)k(l^{-1})U_1^\sigma(h)^{-1}.$$

Proof. For every $f \in L^p(G : K, \mathcal{H})$,

$$\begin{aligned}
\langle f, A^*\varphi\rangle &= \langle Af, \varphi\rangle\\
&= \int_G \left\langle \int_G U_2^\sigma(h)k(h^{-1}g)U_1^\sigma(h)^{-1}f(h)dh, \varphi(g)\right\rangle dg\\
&= \int_G \left\langle f(h), \int_G U_1^\sigma(h)^{*-1}k(h^{-1}g)^* U_2(h)^*\varphi(g)dg\right\rangle dh\\
&= \int_G \left\langle f(h), \int_G U_1^\sigma(g)^{*-1}\check{k}(g^{-1}h)^* U_2^\sigma(g^*)\varphi(g)dg\right\rangle dh.
\end{aligned}$$

By Fubini's theorem, the result follows thus

$$A^*\varphi(g) = \int_G U_1^\sigma(g)^{*-1}\check{k}(g^{-1}h)^* U_2^\sigma(g)^*\varphi(g)dg.$$

The extension of a convolution operator on $G = SE(2)$ to a bounded linear operator on $L^2(G)$ is presented in the next result.

4.5. Theorem

Let $T = \frac{\partial}{\partial t}$. The operator

$$A' : C_c^\infty(G) \to C^\infty(G),$$

defined by $Bf = f * T^n\delta(t)$, extends to a bounded linear operator on $L^2(G)$ and is unitary up to a constant multiple.

Proof. The space $C_c^\infty(G)$ is an LF- space and hence it is bornological. The range space $C^\infty(G)$ is metrizable. Any map between a bornological space and a metrizable space is linear and continuous, therefore A' is linear and continuous. A continuous operator transfers bounded sets to bounded sets, that means A' is continuous and bounded. Following the inclusion property, $C_c^\infty(G) \subset C^\infty(G) \subset L^p(G)$, it means that the range of the map stays inside $L^p(G)$. The induced topology acquired by $C_c^\infty(G)$ from $C^\infty(G)$ is weaker than the strict inductive limit topology of $C_c^\infty(G)$, therefore, the map A' is continuous.

For $f \in C_c^\infty(G)$, $(A'f)(g) = (T^n f * \delta_h)(g)$. Now, $(f * \delta_h)(g) = f(gh^{-1})$. Since $\mathbb{R}^2 \rtimes \mathbb{T}$ is abelian, we can write $f(gh^{-1}) = f((gh)h^{-1}) = f(g)$. Then for $t \in \mathbb{T}$ and $\xi \in \mathbb{R}^2$,

$$(A'f)(g) = T^n f(g) = T^n f(\xi, e^{it}).$$

For a fixed ξ, $(A'f)(g)$ has a compact support in $t \in \mathbb{T}$. Let $\mathcal{F}_c$ denote the Fourier transform of function on $\mathbb{T}$. Then

$$\mathcal{F}_{\mathbb{T}}(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-i\lambda t}dt, \ \lambda \in \mathbb{Z}.$$

We compute $\mathcal{F}_c(A'f)$ as follows.

$$\begin{aligned}
\mathcal{F}_c(A'f) &= \frac{1}{2\pi}\int_0^{2\pi} (A'f)(g)e^{-i\lambda t}dt\\
&= \frac{1}{2\pi}\int_0^{2\pi} T^n f(\xi, e^{it})e^{-i\lambda t}dt\\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[(i)^n f(\xi, e^{it})e^{-i\lambda t} + (-i\lambda)^n f(\xi, e^{it})e^{-i\lambda t}\right]dt\\
&= ((i)^n + (-i\lambda)^n)\frac{1}{2\pi}\int_0^{2\pi} f(\xi, e^{it})e^{-i\lambda t}dt.
\end{aligned}$$

Following proposition 2.1 and Corollary 2.2, $\mathcal{F}_c(A'f)$ can be further expressed as

$$\mathcal{F}_c(A'f) = ((i)^n + (-i\lambda)^n)\int_{\mathbb{R}^2}\mathcal{F}_c(\xi, e^{i\theta})d\mu(\xi)$$
$$= \Big((i)^n + (-i\lambda)^n\Big)\mathcal{F}f(\xi, e^{i\theta}).$$

Let us define

$$S : L^2(\mathbb{R}^2 \rtimes \mathbb{T}) \to L^2(\mathbb{R}^2 \rtimes \mathbb{T}),$$

by $(SF)(\xi, e^{i\theta}) = \Big((i)^n + (-i\lambda)^n\Big)F(\xi, e^{i\theta})$, then $\mathcal{F}_c(A'f) = S\mathcal{F}f$, for $f \in C_c^\infty(G)$. The inclusion relations $C_c^\infty(G) \subset C^\infty(G) \subset L^2(G)$ holds, therefore, $A'f \in L^2(G)$ and $A'f = \mathcal{F}_c^{-1}S\mathcal{F}f$, where $\mathcal{F}_c^{-1}$ is the inverse of $\mathcal{F}_c : L^2(G) \to L^2(G)$. This verifies that A' can be extended to a constant multiple of a unitary operator on $L^2(G)$.

For the remainder of this section, we consider $G = SE(2) = \mathbb{R}^2 \rtimes \mathbb{T}$. Let $T \in \mathcal{D}'(G)$. The continuous linear map

$$L_T : C_c^\infty(G) \ni f \mapsto T * f \in C^\infty(G)$$

is called a left convolution map. To every $T \in \mathcal{D}'(G)$ one can associate a convolution map.

The following theorem characterizes the left convolution map on the Euclidean motion group $SE(2)$.

4.6. Theorem

Let $T \in \mathcal{D}'(G)$. The convolution map L_T is a continuous linear map from $C_c^\infty(G)$ into $C^\infty(G)$ and it commutes with translations. Conversely, if $L : C_c^\infty(G) \ni f \mapsto T * f \in C^\infty(G)$ is a continuous linear map such that

$$L \circ \tau_a = \tau_a \circ L,$$

for every fixed $a \in G$, there is a unique $T \in \mathcal{D}'(G)$ such that

$$L(f) = T * f, \quad \forall f \in C_c^\infty(G).$$

Proof. Let $T \in \mathcal{D}'(G)$ and let a be a fixed element in G, we denote the left translation of a function f in G by τ_a. Then $\tau_a f(g) = f(a^{-1}g)$. Now, put $g' = b^{-1}g$ for a fixed $b \in G$. Then

$$[T * (\tau_a f)](g') = \Big\langle T, (\tau_a f)(b^{-1}g)\Big\rangle$$
$$= \Big\langle T, f(a^{-1}b^{-1}g)\Big\rangle = \Big\langle T, f((ba)^{-1}g)\Big\rangle$$
$$= \Big(T * f\Big)\Big((ba)^{-1}g\Big) = [\tau_a(T * f)](g'),$$

which verifies that the map(convolution) commutes with left translations.

Conversely, suppose that $L : C_c^\infty(G) \ni f \mapsto T * f \in C^\infty(G)$ is a continuous linear map commuting with translations. The map

$$C_c^\infty(G) \ni f \mapsto (Lf)((0,0),1) \in C^\infty(G)$$

defines a continuous linear functional on $C_c^\infty(G)$. Hence, there is a unique $T \in \mathcal{D}'(G)$ such that

$$(Lf)((0,0),1) = \langle T, \check{f}\rangle,$$

where $\check{f}(g) = f(g^{-1})$. We have also that

$$(Lf)((0,0),1) = (T * f)((0,0),1).$$

Since L commutes with left translation and for any $g \in G$, we have

$$(Lf)(g) = [\tau_a(Lf)] = [L(\tau_a f)]((0,0),1)$$
$$= (T * (\tau_a f))((0,0),1) = \langle T, (\tau_a f)(g)\rangle$$
$$= \langle T, f(a^{-1}g)\rangle = (T * \tau_a f)(g),$$

and verifies that L is a convolution map. This completes the proof.

5. Conclusion

In this work, the range of our convolution operator

$$P_u := f \mapsto u * f,$$

has been established to be $\mathcal{E}'(G)$, i.e a distribution space with compact support. Furthermore, two preliminary results concerning convolution of function, distribution and Fourier transform of functions on the $\mathcal{S}SE(2)$ were formulated and proved as theorems 3.1 and proposition 3.2. Theorem 3.3, which is the major result for section 3 was stated and proved. This result is the extension of the work of El-Hussein (see [1]) on the Heisenberg group to the Euclidean motion group using Hormander construction (see [3]). It is established in theorem 4.3 that the (convolution) operator

$$A' : C_c^\infty(G) \to C^\infty(G),$$

which is defined as $A'f = f * T^n\delta(t)$ extends to a bounded linear operator on $L^2(G)$, for $f \in C_c^\infty(G)$. It is further demonstrated that the left convolution operator L_T denoted as $L_T f = T * f$ commutes with left translation, for $T \in D'(G)$. Lastly, the adjoint of a specified operator for the subspace made up of equivalent classes of right K-invariant functions of the H-valued L^p - space on G denoted by $L^p(G : K, \mathcal{H})$, $1 < p < \infty$ is obtained in proposition 4.4 and its L^p boundedness is demonstrated in proposition 4.3.

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