Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 6 (2024) 2049

Check for updates

Journal of the Nigerian Society of Physical Sciences

Transmuted cosine Topp-Leone G family of distributions: properties and applications

Abdulhameed Ado Osi[a,*], Sani Ibrahim Doguwa[b], Abubakar Yahaya[b], Yahaya Zakari[b], Abubakar Usman[b]

[a]*Department of Statistics, Aliko Dangote University of Science and Technology, Wudil, Nigeria*
[b]*Department of Statistics, Ahmadu Bello University, Zaria, Nigeria*

Abstract

In contemporary data analysis, there is a growing recognition of the need for flexible and adaptable probabilistic models that can effectively capture complex dependencies and tail behaviors in real-life datasets. In response to this demand, this study proposes a new trigonometric generalized family of distribution called the ”transmuted cosine Topp-Leone G family”. Built upon this family, the proposed framework combines the adaptability of the Topp-Leone distribution with the periodicity of the cosine function and the concept of transmutation theory. This combination results in a highly flexible framework that can accurately represent a wide range of real-life phenomena. The study also explores various statistical properties of the introduced family, including survival and hazard functions, as well as moment and moment-generating functions. The model parameters are estimated using the Maximum Likelihood method, and to ensure the reliability and consistency of the estimates, a Monte Carlo simulation is conducted. Additionally, the study examines the impact of the distribution parameters on the shape of the resulting distributions, which can exhibit left-skew, right-skew, increasing, or decreasing patterns. Finally, empirical demonstrations are provided to illustrate the effectiveness of the TrCTLG family models in fitting lifetime datasets.

DOI:10.46481/jnsps.2024.2049

Keywords: Tope-Leone G, Cosine-G, Transmuted-G, Monte-Carlo simulation, Maximum likelihood estimation

Article History :
Received: 27 March 2024
Received in revised form: 20 June 2024
Accepted for publication: 26 July 2024
Published: 02 September 2024

Communicated by: P. Thakur

1. Introduction

In the field of mathematical modeling and statistical analysis, researchers are constantly exploring new probability distributions and their applications. Numerous probability models have been proposed to model real-world phenomena in various disciplines, including technology, medicine, economics, biological studies, environmental sciences, and more. Given the dynamic nature of modern datasets, it has become common practice for researchers to develop generalized families of distributions. These families are typically obtained through parametric transformations to improve the performance of the parent distributions. Some examples of such techniques include the exponentiated G family [1], the Marshall Olkin family [2], the beta-generated family [3], and the Kumaraswamy G family [4].

Ref. [5] introduced a modern approach to generalize probability distributions using trigonometric transformations. This

*Corresponding author: Tel.: +234-806-912-0811.
Email address: aaosi@kustwudil.edu.ng, abuammarosi@gmail.com (Abdulhameed Ado Osi)

approach has gained popularity among statisticians due to its remarkable success in effectively modeling real-world data, surpassing traditional methods. The trigonometric G families offer numerous advantages, such as the simplicity of their functions, and allow for more complex parameters that align with the original distributions. Among the distributions within this framework, the cosine G family has garnered significant interest due to its desirable properties. Within this context, Ref. [6] introduced the cosine Topp-Leone family by merging the cos-G family with the Topp-Leone G family. Ref. [7] proposed a modified cos-G family, which includes the extended cosine Weibull, extended cosine power, and extended cosine generalized half-logistic models. Ref. [8] originated the odd Lomax trigonometric generalized family, with a specific model called the Lomax cosecant Weibull. Ref. [9] provided an expansion of the cosine generalized family, including the extended cosine Weibull, which exhibits various density shapes and hazard rate functions.

The Topp-Leone distribution (TL) [10] is a probability distribution used for modeling lifetime data. It has been widely applied in various fields of study. The TL distribution is known for its versatility and applicability in modeling a wide range of phenomena. An alternative to the exponentiated G distribution called the Topp-Leone G (TL-G) family, was proposed by Ref. [11]. The TL-G family and its several extensions have been extensively studied in research. The Topp-Leone exponential-G family [12] has been proven to be useful for modeling positive real data sets. This family was further extended to the Topp-Leone Gompertz-G distribution [13], showing its capability to handle non-monotonic hazard rate functions and heavy-tailed data. Expanding on this work, Ref. [14] introduced the Topp-Leone odd Lindey G family, which offers flexibility to the model and has demonstrated its effectiveness in real-world data applications. Lastly, Ref. [15] presented the inverted TL distribution, which successfully simulates real datasets and covers a range of hazard function shapes.

The transmuted G class of distributions has been used in various statistical applications, demonstrating its flexibility and usefulness in modeling real datasets. Several studies have introduced and explored the properties of different transmuted G families of distributions. Ref. [16] introduced the transmuted exponentiated G family, which was shown to be flexible and applicable to real datasets. Ref. [17] further expanded on this with the Marshall Olkin transmuted generalized family, and Ref. [18] introduced the transmuted Topp-Leone G family, which also displayed useful properties and applications. These families of distributions provide a range of options for modeling and analyzing data. Ref. [19] extended this further with the transmuted exponential Topp-Leone distribution, which outperformed other distributions when applied to real-life datasets using certain baseline distributions.

In this study, we propose the Transmuted Cosine Topp-Leone Family (TrCTL-G), which combines the flexibility of the TL distribution with the periodicity of the cosine function through transmutation. This family has the potential to explore new approaches for modeling complex phenomena that exhibit both stochastic and oscillatory characteristics.

2. TrCTL-G family

Here, we present the Transmuted Cosine Topp-Leone Family (TrCTL-G). But we first present the Transmuted-G, Top-Leone-G, and Cos-G families.

2.1. Transmuted-G Family

Transmuting refers to the process of introducing an additional parameter to a pre-existing distribution. Let $G(x;\varepsilon)$ and $g(x;\varepsilon)$ represent the cumulative distribution function and probability density function, respectively, of a baseline distribution with parameter ε. The cumulative distribution function (cdf) and probability density function (pdf) of the transmuted G family distribution, as derived by Shaw and Buckley [20], can be expressed as follows:

$$F_{TD}(x;\theta,\varepsilon) = G(x,\varepsilon)\left[(1+\theta) - \theta G(x;\varepsilon)\right], \tag{1}$$

which can also be simplified as:

$$F_{TD}(x;\theta,\varepsilon) = (1+\theta)G(x;\varepsilon) - \theta\left[G(x;\varepsilon)\right]^2. \tag{2}$$

The pdf can be obtained by taking a first derivative of Eq. (1) which becomes:

$$f_{TD}(x;\theta,\varepsilon) = g(x;\varepsilon)\left[(1+\theta) - 2\theta G(x;\varepsilon)\right]. \tag{3}$$

2.2. Cosine-G family

The use of trigonometric transformation to generate new distributions has attracted the interest of many researchers. In their work, Ref. [21] introduced the cos-G family and defined its cumulative distribution function (CDF) and probability density function (PDF) as follows:

$$F_{\cos-G}(x;\varepsilon) = 1 - \cos\left[\frac{\pi}{2}G(x;\varepsilon)\right]; \quad ; x \in \mathrm{R}, \tag{4}$$

$$f_{\cos-G}(x;\varepsilon) = \frac{\pi}{2}g(x;\varepsilon)\sin\left[\frac{\pi}{2}G(x;\varepsilon)\right]. \tag{5}$$

2.3. Transmuted Cosine-G family

Here, we propose the Transmuted Cosine G family by combining the Cos-G family and the Transmuted-G family. The cumulative distribution function (cdf) and probability density function (pdf) of the Transmuted-G family are expressed below:

$$F_{TCG} = (1+\theta)\left[1 - \cos\left[\frac{\pi}{2}G(x;\varepsilon)\right]\right] - \theta\left[1 - \cos\left[\frac{\pi}{2}G(x;\varepsilon)\right]\right]^2. \tag{6}$$

The pdf can be written as:

$$f_{TCG}(x;\theta,\varepsilon) = \frac{\pi}{2}g(x;\varepsilon)\sin\left(\frac{\pi}{2}G(x;\varepsilon)\right) \left[(1+\theta) - 2\theta\left[1 - \cos\left(\frac{\pi}{2}G(x;\varepsilon)\right)\right]\right]. \tag{7}$$

2.4. Topp-Leone-G family

The cumulative distribution function (CDF) and probability density function (PDF) of the Topp-Leone Generalized family, as introduced by Ref. [11], are given by:

$$F_{TLG}(x;\varepsilon) = \{G(x;\varepsilon)^{\alpha}[2-G(x;\varepsilon)]^{\alpha}\} = \left[1-\overline{G}(x;\varepsilon)\right]^{\alpha}, \quad (8)$$

$$f_{TLG}(x;\varepsilon) = 2\alpha g(x;\varepsilon)\overline{G}(x;\varepsilon)\left[1-\overline{G}(x;\varepsilon)^2\right]^{\alpha-1}, \quad (9)$$

where $\overline{G}(x;\varepsilon) = 1-G(x;\varepsilon)$.

By combining the transmuted cosine G in Eq. (6) with the Topp-Leone G in Eq. (8) as a parent distribution, we propose a newly generalized transmuted cosine Topp-Leone G family with the cumulative distribution function (CDF) and probability density function (PDF) given as:

$$F_{TrCTL-G}(x;\theta,\alpha,\varepsilon) = (1+\theta)\left[1-\cos\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)\right)^{\alpha}\right] - \theta\left[1-\cos\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)\right)^{\alpha}\right]^2 \quad (10)$$

$$f_{TrCTL-G}(x;\theta,\alpha,\varepsilon) = (1+\theta)\pi\alpha g(x;\varepsilon)\overline{G}(x;\varepsilon)\left(1-\overline{G}(x;\varepsilon)^2\right)^{\alpha-1} \sin\left[\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)\right)^{\alpha}\right] - 2\theta\pi\alpha g(x;\varepsilon)\overline{G}(x;\varepsilon)\left(1-\overline{G}(x;\varepsilon)^2\right)^{\alpha-1} \sin\left[\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)\right)^{\alpha}\right]\left[1-\cos\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)\right)^{\alpha}\right], \quad (11)$$

where$\overline{G}(x;\varepsilon) = 1-G(x;\varepsilon)$, $\alpha > 0$, $|\theta| < 1$.

The corresponding survival function failure rate or hazard function of the TrCTL-G is derived as follows:

$$\begin{aligned} S(x) &= 1 - F_{TrCLT-G}(x;\theta,\alpha;\varepsilon) \\ S(x) &= 1 - \left[(1+\theta)\left[1-\cos\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)\right)^{\alpha}\right]\right] \\ &\quad - \theta\left[1-\cos\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)^{\alpha}\right]^2\right] \\ h(x) &= \frac{f_{TrCLT-G}(x;\theta,\alpha;\varepsilon)}{1-F_{TrCLT-G}(x;\theta,\alpha;\varepsilon)} \end{aligned} \quad (12)$$

$$h(x) = \frac{\pi\alpha g(x;\varepsilon)\overline{G}(x;\varepsilon)\left(1-\overline{G}(x;\varepsilon)^2\right)^{\alpha-1}\sin\left[\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)\right)^{\alpha}\right] - \left[(1+\theta)-2\theta\left[1-\cos\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)\right)^{\alpha}\right]\right]}{1-\left[(1+\theta)\left[1-\cos\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)\right)^{\alpha}\right]-\theta\left[1-\cos\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)\right)^{\alpha}\right]^2\right]}. \quad (13)$$

3. Mathematical properties

3.1. Quantile function of the TrCTL-G family

We can derive the quantile function of the TrCTL-G family of distribution as follows:

$$u = (1+\theta)\left[1-\cos\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)^{\alpha}\right)\right] - \theta\left[1-\cos\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)^{\alpha}\right)\right]^2, \quad (14)$$

$$\theta\left[1-\cos\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)^{\alpha}\right)\right]^2 - (1+\theta)\left[1-\cos\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)^{\alpha}\right)\right] + u = 0. \quad (15)$$

To solve the nonlinear equation in Eq. (15) the quadratic equation formula is used to obtain:

$$q = G^{-1}\left(1-\left[1-\left[\frac{2}{\pi}\arccos[1 - \left(\frac{(1+\theta)-\sqrt{(1+\theta)^2-4\theta u}}{2\theta}\right)\right]\right]^{\frac{1}{\alpha}}\right]\right), \quad \theta \neq 0 \quad (16)$$

3.2. Useful expansion of the distribution

We provide a helpful representation of the probability density function (pdf) of TrCTL-G in this section. The Taylor series expansion provides the expansions of the sine and cosine functions as follows:

$$\sin t = \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}t^{2j+1}, \quad (17)$$

$$\cos t = \sum_{i=0}^{\infty}\frac{(-1)^i}{(2i)!}t^{2i}. \quad (18)$$

Applying Eq. (17) we have:

$$\left(1-\overline{G}(x;\varepsilon)^2\right)^{\alpha-1}\sin\left[\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)^{\alpha}\right] = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1}\left(1-\overline{G}(x;\varepsilon)^2\right)^{2\alpha(2k+1)-1}. \quad (19)$$

Using binomial expansion

$$(1-y)^a = \sum_{j=0}^{\infty}(-1)^j\binom{a}{j}y^j \qquad |y| < 1. \quad (20)$$

Since $(1-G(x;\varepsilon)) < 1$

$$\overline{G}(x;\varepsilon)\left[1-\left(1-\overline{G}(x;\varepsilon)^2\right)\right]^{2\alpha(2k+1)-1} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{j+k+l}\left(\frac{\pi}{2}\right)^{2k+1}}{(2k+1)!}$$

$$\times\begin{pmatrix} 2\alpha(2k+1)-1 \\ j \end{pmatrix}\begin{pmatrix} 2i+1 \\ l \end{pmatrix}(G(x;\varepsilon))^{l}. \quad (21)$$

Now,

$$(1+\theta)\pi\alpha g(x)\overline{G}(x)\Big(1-\overline{G}(x)^2\Big)^{\alpha-1}\sin\left[\left(\frac{\pi}{2}\left(1-\overline{G}(x)^2\right)\right)^{\alpha}\right]=$$

$$(1+\theta)\pi\alpha\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{j+k+l}\left(\frac{\pi}{2}\right)^{2k+1}}{(2k+1)!}\begin{pmatrix} 2\alpha(2k+1)-1 \\ j \end{pmatrix}$$

$$\begin{pmatrix} 2i+1 \\ l \end{pmatrix}(G(x;))^{l}g(x). \quad (22)$$

Applying series and binomial expansion

$$\left[1-\cos\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)^{\alpha}\right)\right]=$$

$$\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{i+m+n}\begin{pmatrix} 2\alpha i \\ m \end{pmatrix}\begin{pmatrix} 2m \\ n \end{pmatrix}[G(x)]^{n}. \quad (23)$$

The pdf of TrCTL-G can now be reduced to

$$f_{TrTL-G}(x;\alpha,\theta,\varepsilon)=\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}M_{jkl}N_{inm}\mathrm{H}_{n+l}(x), \quad (24)$$

where

$$M_{jkl}=\pi\alpha\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j+k+l}\left(\frac{\pi}{2}\right)^{2k+1}}{(2k+1)!}$$

$$\times\begin{pmatrix} 2\alpha(2k+1)-1 \\ j \end{pmatrix}\begin{pmatrix} 2i+1 \\ l \end{pmatrix}$$

$$N_{inm}=(1+\theta)-2\theta\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{i+m+n}\begin{pmatrix} 2\alpha i \\ m \end{pmatrix}\begin{pmatrix} 2m \\ n \end{pmatrix}.$$

3.3. Moments

The rth moment of the TrCTL-G can be obtained as:

$$\mu_r^{'}=\int_0^{\infty}x^r f_{TrCTL-G}(x)\,dx$$
$$=\int_0^{\infty}x^r\Big(\pi\alpha g(x;\varepsilon)\overline{G}(x;\varepsilon)\left(1-\overline{G}(x;\varepsilon)^2\right)^{\alpha-1}$$
$$\times\sin\left[\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)\right)^{\alpha}\right]$$
$$-\left[(1+\theta)-2\theta\left[1-\cos\left(\frac{\pi}{2}\left(1-\overline{G}(x;\varepsilon)^2\right)\right)^{\alpha}\right]\right]\Big)\,dx.\because. \quad (25)$$

$$\mu_r^{'}=\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}M_{jkl}N_{inm}\int_0^{\infty}x^r\,\mathrm{H}_{n+l}(x)dx. \quad (26)$$

We derive the moment-generating function of the TrCTL-G below

$$M_X(t)=E\left[e^{tx}\right].$$

When applying the exponential expansion (Taylor series expansion) we have, $e^{tx}=\sum_{r=0}^{\infty}\frac{x^r t^r}{r!}$ therefore, $M_X(t)=E\left[\frac{x^r t^r}{r!}\right]=\frac{t^r}{r!}E\left[x^r\right]$

$$\therefore M_X(t)=\frac{t^r}{r!}\mu_r^{'}.$$

3.4. Rényi's entropy of TrCTL-G

A random variable's entropy quantifies the degree of variation or uncertainty it possesses. It has diverse applications across numerous disciplines, including data processing, statistical physics, probability theory, engineering, communication theory, and quantum physics. Suppose X represents a random variable with a probability density function. The Rényi's is defined as:

$$\mathrm{I}_R(\gamma)=\frac{1}{1-\gamma}\log\left[\int_{-\infty}^{\infty}f(x)^{\gamma}dx\right],\gamma\neq 1,\gamma>0. \quad (27)$$

The following is the expression for the TrCTL-G family' Rényi entropy:

$$\mathrm{I}_R(\gamma)=\frac{1}{1-\gamma}\log\left[\int_{-\infty}^{\infty}f_{TrCTL-G}(x)^{\gamma}dx\right],\gamma\neq 1,\gamma>0. \quad (28)$$

Substituting the pdf in Eq.(24), we can derive the Rényi entropy for the TrCTL-G distributions as

$$\mathrm{I}_R(\gamma)=\frac{1}{1-\gamma}\log\left[\int_{-\infty}^{\infty}\left[\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}M_{jkl}N_{inm}\mathrm{H}_{n+l}(x)\right]^{\gamma}dx\right],$$
$$\gamma\neq 1,\gamma>0. \quad (29)$$

3.5. Parameter estimation

The maximum likelihood method is the most commonly used technique for parameter estimation in the literature. It is preferred for its accurate performance and consistent estimation. Therefore, we will focus on estimating the parameters of this family solely using maximum likelihood. Consider a random sample of size n, denoted as $x_1, x_2, ..., x_n$, from the TrCTL-G distribution. We will be estimating a p x 1 vector of parameters, denoted as $\eta=(\alpha,\theta,\epsilon)^T$. The following expression represents the log-likelihood function:

$$\ell=n\log(\pi)+n\log(\theta)+n\log(\alpha)+\sum_{i=1}^{n}\log g(x_i;\varepsilon)+$$
$$\sum_{i=1}^{n}\log(1-G(x_i;\varepsilon))+(\alpha-1)\sum_{i=1}^{n}\log(1-(1-G(x_i;\varepsilon))^2)+$$
$$\sum_{i=1}^{n}\log\left(\sin\left(\frac{\pi}{2}(1-(1-G(x_i;\varepsilon))^2)^{\alpha}\right)\right)$$
$$+\sum_{i=1}^{n}\log\left((1+\theta)-2\theta\left(1-\cos\left(\frac{\pi}{2}(1-(1-G(x_i;\varepsilon))^2)^{\alpha}\right)\right)\right). \quad (30)$$

The score function components $U(\vartheta)$ can be obtained by taking the partial derivative of the log-likelihood function $U(\vartheta)=\left(\delta\ell/_{\delta\alpha},\delta\ell/_{\delta\theta},\delta\ell/_{\delta\varepsilon}\right)^T$ as follows:

$$\delta\ell\delta\alpha=\frac{n}{\alpha}+\sum_{i=1}^{n}\log(1-(1-G(x_i;\varepsilon))^2)$$

$$+\left(\frac{\pi}{2}\sum_{i=1}^{n}(1-(1-G(x_i;\xi)))\right.$$
$$\times\log(1-(\mathcal{G})^2)\cot\left[\frac{\pi}{2}(1-(\mathcal{G})^2)^{\alpha}\right]\Bigg)$$
$$-\left(2\theta\sum_{i=1}^{n}\frac{(1-(\mathcal{G}))\times\log(1-(\mathcal{G})^2)\sin\left[\frac{\pi}{2}(1-(\mathcal{G})^2)^{\alpha}\right]}{(1+\theta)-2\theta\left(1-\cos\left(\frac{\pi}{2}(1-(1-G(x_i;\varepsilon))^2)^{\alpha}\right)\right)}\right), \quad (31)$$

where $\mathcal{G} = 1 - G(x_i;\xi)$.

$$\delta\ell/_{\delta\theta} = \sum_{i=1}^{n}\frac{1-2\left(1-\cos\left[\frac{\pi}{2}(1-(1-G(x_i;\varepsilon))^2)^{\alpha}\right]\right)}{(1+\theta)-2\theta\left(1-\cos\left(\frac{\pi}{2}(1-(1-G(x_i;\varepsilon))^2)^{\alpha}\right)\right)}. \quad (32)$$

$$\frac{\delta\ell}{\delta\varepsilon} = \sum_{i=1}^{n}\frac{g'(x_i;\varepsilon)}{g(x_i;\varepsilon)} - \sum_{i=1}^{n}\frac{G'(x_i;\varepsilon)}{1-G(x_i;\varepsilon)}$$
$$+2(\alpha-1)\sum_{i=1}^{n}\frac{G'(x_i;\varepsilon)}{1-(1-G(x_i;\varepsilon))^2}$$
$$+\alpha\pi\sum_{i=1}^{n}G'(x_i;\varepsilon)(1-G(x_i;\varepsilon))\left(1-(1-G(x_i;\varepsilon))^2\right)^{\alpha-1}$$
$$\cot\left(\frac{\pi}{2}\left(1-(1-G(x_i;\varepsilon))^2\right)^{\alpha}\right)-$$
$$2\theta\alpha\pi\sum_{i=1}^{n}\frac{G'(x_i;\varepsilon)(\mathcal{G}_0)\left(1-(\mathcal{G}_0)^2\right)^{\alpha-1}\sin\left(\frac{\pi}{2}\left(1-(\mathcal{G}_0)^2\right)^{\alpha}\right)}{(1+\theta)-2\theta\left(1-\cos\left(\frac{\pi}{2}(1-(\mathcal{G}_0)^2)^{\alpha}\right)\right)}, \quad (33)$$

where $\mathcal{G}_0 = 1 - G(x_i;\varepsilon)$, $g'(x_i;\varepsilon) = \delta g(x_i;\varepsilon)/_{\delta\varepsilon}$ and $G'(x_i;\varepsilon) = \delta G(x_i;\varepsilon)/_{\delta\varepsilon}$.

MLE estimates can be obtained by setting $U_\alpha = 0$, $U_\theta = 0$, and $U_\varepsilon = 0$ and solving the resulting equations to obtain the maximum likelihood estimators. Moreover, these equations may be solved analytically; however, they may also need to be solved numerically.

4. Some special cases of TrCLT-G

In this section, we present some sub-models of the TrCLT-G distribution, namely the Transmuted Cosine Topp-Leone Weibull (TrCLTW-G) and the Transmuted Cosine Topp-Leone exponential (TrCLTE-G).

4.1. Transmuted cosine Topp-Leone Weibull

The new distribution obtained from the family is obtained by replacing the $G(x)$ and $g(x)$ with the cumulative distribution function and probability density function of the Weibull distribution, respectively. This results in the Transmuted Cosine Topp-Leone Weibull (TrCTLW) distribution, which is presented below with its probability density function.

$$f_{TrCTL-W}(x;\theta,\alpha,\lambda,\beta) = (1+\theta)\pi\alpha\left(\beta\lambda x^{\beta-1}e^{-\lambda x^{\beta}}\right)e^{-\lambda x^{\beta}}$$
$$\left(1-\left(e^{-\lambda x^{\beta}}\right)^2\right)^{\alpha-1}\sin\left[\left(\frac{\pi}{2}\left(1-\left(e^{-\lambda x^{\beta}}\right)^2\right)\right)^{\alpha}\right]-2\theta\pi\alpha\left(\beta\lambda x^{\beta-1}e^{-\lambda x^{\beta}}\right)$$
$$e^{-\lambda x^{\beta}}\left(1-\left(e^{-\lambda x^{\beta}}\right)^2\right)^{\alpha-1}\sin\left[\left(\frac{\pi}{2}\left(1-\left(e^{-\lambda x^{\beta}}\right)^2\right)\right)^{\alpha}\right]$$
$$\left[1-\cos\left(\frac{\pi}{2}\left(1-\left(e^{-\lambda x^{\beta}}\right)^2\right)\right)^{\alpha}\right]. \quad (34)$$

Figure 1 presents the probability density function and hazard graph of the TrCTLW distribution. The probability density function shows a reverse J shape, indicating right skewness, and a nearly symmetrical shape. The hazard function, on the other hand, exhibits increasing and decreasing patterns. These characteristics suggest that the TrCTLW distribution is highly flexible for modeling various lifetime datasets. Figure 2 displays the cumulative distribution function (cdf) and survival plots of the TrCTLW distribution.

4.2. Transmuted cosine Topp-Leone exponential

Suppose $G(x)$ and $g(x)$ are the cdf and the pdf of the exponential distribution. We can derive a new distribution from the family namely the Transmuted Cosine Topp-Leone Exponential (TrCTLE) with the following pdf.

$$f_{TrCTL-E}(x;\theta,\alpha,\lambda) = (1+\theta)\pi\alpha\left(\lambda e^{-\lambda x}\right)e^{-\lambda x}\left(1-\left(e^{-\lambda x}\right)^2\right)^{\alpha-1}$$
$$\sin\left[\left(\frac{\pi}{2}\left(1-\left(e^{-\lambda x}\right)^2\right)\right)^{\alpha}\right]-2\theta\pi\alpha\left(\lambda e^{-\lambda x}\right)e^{-\lambda x}\left(1-\left(e^{-\lambda x}\right)^2\right)^{\alpha-1}$$
$$\sin\left[\left(\frac{\pi}{2}\left(1-\left(e^{-\lambda x}\right)^2\right)\right)^{\alpha}\right]\left[1-\cos\left(\frac{\pi}{2}\left(1-\left(e^{-\lambda x}\right)^2\right)\right)^{\alpha}\right]. \quad (35)$$

Figure 3 presents the pdf and hazard graphs for the TrCTLE distribution with different distribution parameter values. The pdf shows a right-skewed and nearly symmetrical shape. The hazard of the TrCTLE distribution exhibits a shape that increases and then decreases. These characteristics indicate the high flexibility of the TrCTLE distribution in modeling various lifetime datasets. Figure 4 displays the cdf and survival function plots of the TrCTLE.

4.3. Simulation study

Here, we conducted a simulation study to evaluate the accuracy and performance of MLE in estimating the parameters of the TrCTLW distribution. We produced samples of sizes n = 50, 100, 250, 500, and 1000 for 1000 iterations using the quantile function of the TrCTLW distribution.

Two sets of parameter values were assigned: set one with $\theta = -0.5$, $\alpha = 0.9$, $\lambda = 0.1$, $\beta = 1.0$, and set two with $\theta = 0.6$, $\alpha = 1.0$, $\lambda = 0.4$, $\beta = 1.6$. The simulation results compare the actual parameter values with the estimates. Measures such as root mean square errors (RMSE) and average biases (AB) were obtained for both instances and displayed in Table 1.

It is observed that as the sample size increases, both the biases and the root mean square errors decrease, approaching zero. Based on these results, we can conclude that the MLE method is sufficiently accurate in estimating the parameters of the TrCTLW distribution.

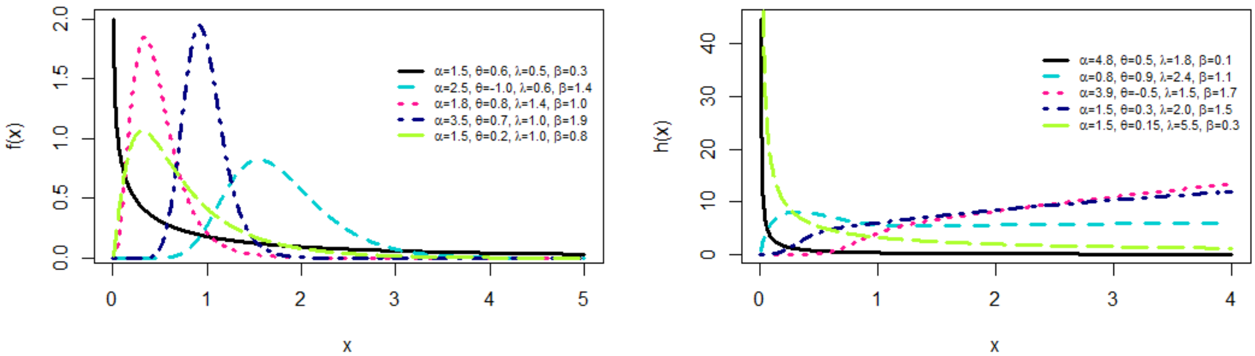

Figure 1: TrCTLW PDF and hazard plots.

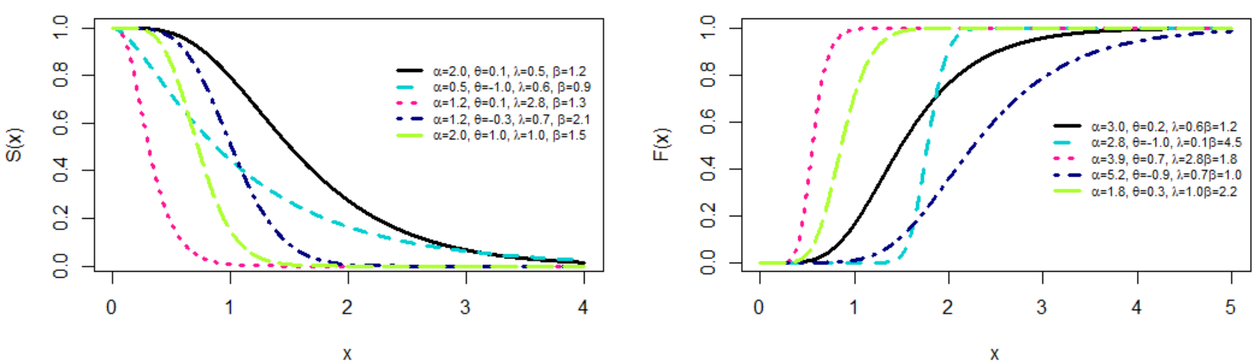

Figure 2: TrCTLW Survival and CDF plots.

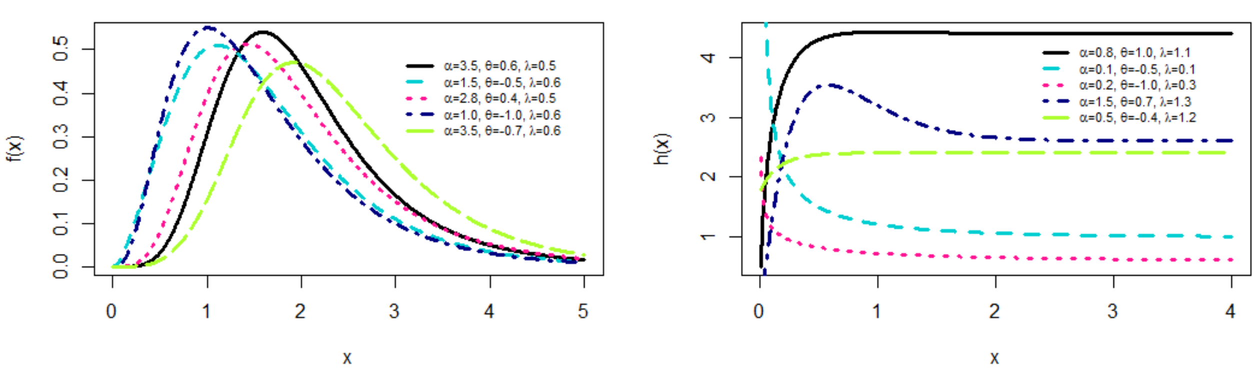

Figure 3: TrCTLE PDF and hazard plots.

5. Applications

In this section, we demonstrate the potential of the TrCTLG family in practical applications by applying it to lifetime data sets. We also investigate the efficiency and effectiveness of the family in fitting real-world data. All computations are performed using the R program. We examine two data sets, comparing the TrCTLW and TrCTLE fitting with the cosine Topp-Leone Weibull (CTLW) and Weibull (WD) distributions. To illustrate the goodness-of-fit, we consider various metrics including the Anderson-Darling statistic (AD), the Kolmogorov-Smirnov statistic (KS), and Cramer-von Mises (CVM). Additionally, we compute the values of negative log-likelihood and

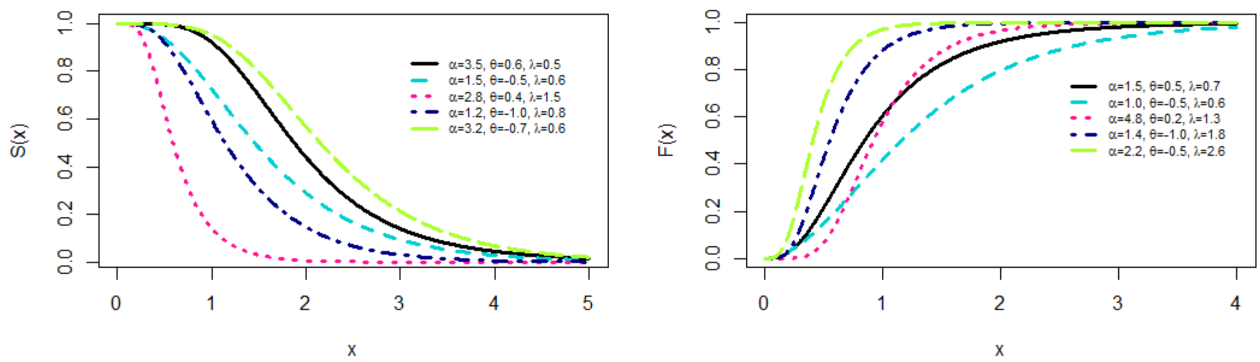

Figure 4: TrCTLE Survival and CDF plots.

Table 1: The outcome of simulation for the model's parameter estimation based on MLE.

		α		θ		λ		β	
SET one									
Actual values	Sample size	AB	RMSE	AB	RMSE	AB	RMSE	AB	RMSE
$\alpha = 0.9$	50	0.0909	0.1637	-0.0090	0.1987	-0.0403	0.0478	-0.0143	0.1189
$\theta = -0.5$	100	0.0878	0.1408	0.0129	0.1358	0.0416	0.0463	-0.0205	0.0962
$\lambda = 0.1$	250	0.0813	0.1180	0.0257	0.0965	-0.0428	0.0454	-0.0241	0.0736
$\beta = 1.0$	500	0.0858	0.1091	0.0283	0.0749	-0.0424	0.0439	-0.0301	0.0600
	1000	0.0850	0.0986	0.0285	0.0574	-0.0425	0.0433	-0.0316	0.0505
SET two		α		θ		λ		β	
Actual values	Sample size	AB	RMSE	AB	RMSE	AB	RMSE	AB	RMSE
$\alpha = 1.0$	50	0.1940	0.2969	-0.0277	0.3040	-0.1449	0.1658	-0.0997	0.2373
$\theta = 0.6$	100	0.1906	0.2549	-0.0289	0.2056	-0.1470	0.1597	-0.1163	0.1930
$\lambda = 0.4$	250	0.1664	0.2087	-0.0098	0.1418	-0.1556	0.1612	-0.1178	0.1560
$\beta = 1.6$	500	0.147	0.1783	0.0094	0.1103	-0.1619	0.1650	-0.1132	0.1396
	1000	0.1329	0.1619	0.0093	0.0812	-0.1653	0.1677	-0.1033	0.1295

Table 2: Model parameters MLE estimates and information criteria for dataset one.

Model	α	θ	λ	β	AIC	CAIC	BIC
TrCTLW	0.3502	-0.6783	2.8620	0.5331	229.16	229.27	244.54
TrCTLE	2.2955	-0.7307	1.4308	-	285.05	285.12	296.59
CTLW	0.3837	-	3.3219	0.3837	233.56	233.63	245.10
WD	-	-	2.7289	1.1329	231.55	231.59	239.24

the information criteria: the Akaike Information Criteria (AIC), the Consistent Akaike Information Criteria (CAIC), and the Bayesian Information Criteria (BIC). The model with the lowest criterion values indicates the best fit, as well as the model with the smallest KS statistic and the greatest p-value.

Data one: This dataset, as shown in Table B.1, includes the total milk production from the first birth of 107 cows of the SINDI breed. The data was collected from [22].

Data two: The Pediatric Oncology Group (POG) presented this dataset of standard-risk acute lymphocytic leukemia in children in May 1981 and published it in [23].

Using dataset one, Table 2 presents the MLE estimates of the TrCTLW, the TrCTLE, the CTLW, and the Weibull distributions as well as the results of the information criteria (AIC, BAIC, and BIC). The TrCTLW distribution demonstrates the lowest values of the information criteria, indicating that it fits the data better. Additionally, Table 3 displays the log-likelihood and the values of KS, AD, and CVM, along with their respective p-values (in parentheses). The proposed distribution appears to be a highly competitive model for this data, as the considered metrics have lower values and the probability value of the Kolmogorov-Smirnov statistics is greater than those of the other models.

Table 3: The test results for the Goodness-of-fit for dataset two.

Model	ℓ	KS (p-value)	AD (p-value)	CVM (p-value)
TrCTLW	-110.58	0.08 (0.015)	2.44 (<0.0001)	0.43 (<0.0001)
TrCTLE	-139.52	0.14 (<0.0001)	7.31 (<0.0001)	1.26 (<0.0001)
CTLW	-113.78	0.09 (0.005)	8.92 (<0.0001)	1.50 (<0.0001)
WD	-113.78	0.10 (0.04)	8.92 (<0.0001)	1.50 (<0.0001)

Using dataset two, Table 4 displays the statistics AIC, CAIC, and BIC for all models studied. When contrasted with

Table 4: the model parameters MLE estimates and information criteria for the dataset two.

Model	α	θ	λ	β	AIC	CAIC	BIC
TrCTLW	0.1246	-0.6058	5.8758	4.7018	-48.97	-48.58	-41.28
TrCTLE	1.6035	-0.6821	2.2111	-	-10.66	-10.42	-2.64
CTLW	0.1485	-	6.8868	5.0200	-48.38	-48.15	-40.37
WD	-	-	2.6012	5.3818	-38.69	-38.58	-33.35

Table 5: The test results for the Goodness-of-fit for dataset two.

Model	ℓ	KS (p-value)	AD (p-value)	CVM (p-value)
TrCTLW	28.49	0.06 (0.84)	0.39 (0.37)	0.06 (0.38)
TrCTLE	8.33	0.12 (0.07)	3.80 (<0.0001)	0.62 (<0.0001)
CTLW	27.19	0.08 (0.56)	0.64 (0.09)	0.10 (0.10)
WD	21.34	0.08 (0.45)	0.64 (0.09)	0.10 (0.11)

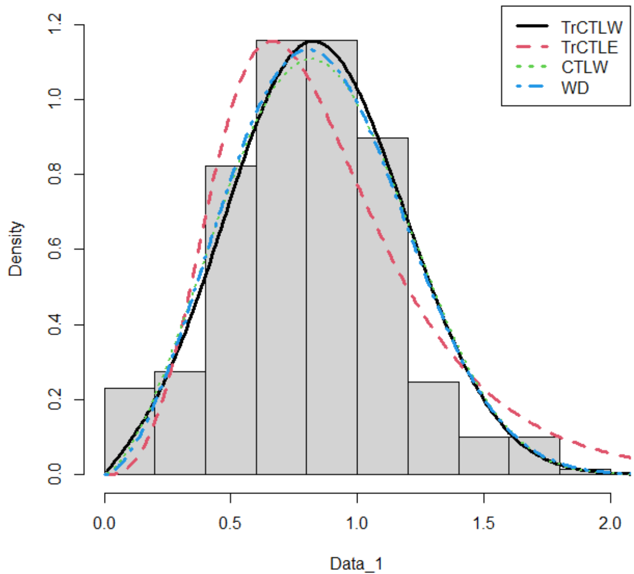

Figure 5: Empirical probability density function plots for the fitted distributions on dataset one.

the values of the other three considered models, the TrCTLW values are smaller than those of the other distributions, indicating that this novel distribution is a highly competitive model for these data. Table 5 shows that TrCTLW has the greatest KS p-value and the lowest KS, AD, and CVM values for dataset two. This illustrates that the TrCTLW distribution performs better in fitting this dataset. However, we demonstrate how well the proposed distribution fits the real data by plotting the empirical pdf of the TrCTLW distribution against other competing models for both datasets one and two, respectively. We display these plots in Figure 5 and Figure 6 which reveal that, in terms of model fitting the TrCTLW superior over other models for both datasets. Based on this, we can conclude that TrCTLW better fits the two data.

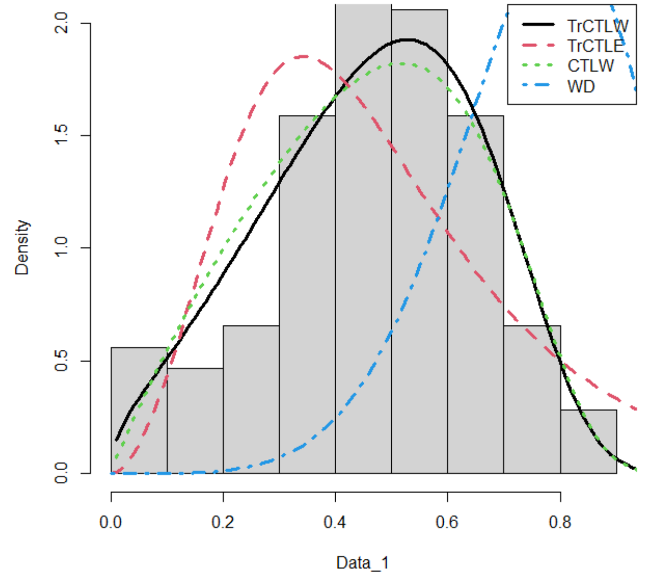

Figure 6: Empirical probability density function plots for the fitted distributions on dataset two.

6. Conclusions

In this study, we introduced a new trigonometric family of distributions called the Transmuted Cosine Topp-Leone G family (TrCTLG). This family combines transmutation techniques with the Cosine Topp-Leone G family to create distributions with unique properties. We derived the corresponding cumulative distribution and probability density functions, as well as their expansions. Additionally, we investigated various statistical characteristics of the TrCTLG family, such as the quantile, hazard, survival functions, entropy, moments, and generating functions. To estimate the model parameters, we employed the maximum likelihood estimation and evaluated the method's performance through Monte Carlo simulation. The results of our study demonstrate the flexibility and consistency of the maximum likelihood estimates. As the sample size increases, the values of AB and RMSE decrease. To further validate our findings, we applied the TrCTLG distribution to two real datasets: a cow milk production dataset and an acute lymphocytic leukemia dataset. Our analysis revealed that the proposed Transmuted Cosine Topp-Leone-Weibull model outperforms other important competitors, including the Cosine Topp-Leone-Weibull and Weibull distributions. Therefore, the practical application of our model provides evidence that it can effectively model lifetime data. Moreover, we believe that the TrCTLG family has the potential for application in other fields of knowledge.

Acknowledgment

The authors would like to thank the reviewers for their insightful feedback in shaping the direction and quality of this

work.

References

[1] R. Gupta & D. Kundu, "Generalized exponential distributions", Australian and New Zealand Journal of Statistics **41** (1999) 173. https://doi.org/10.1111/1467-842X.00072.

[2] A. W. Marshall & I. Olkin, "A new method a parameter to a family for adding with to the exponential and application of distributions Weibull families", Biometrika **84** (1997) 641. https://doi.org/10.1093/biomet/84.3.641.

[3] N. Eugene, C. Lee & F. Famoye, "Beta normal distribution and its applications", communications in statistics - theory and methods, **31** (2002) 497. https://doi.org/http://dx.doi.org/10.1081/STA-120003130.

[4] G. M. Cordeiro & M. de Castro, "A new family of generalized distributions", Journal of Statistical Computation and Simulation **81** (2011) 883. https://doi.org/10.1080/00949650903530745.

[5] D. Kumar, U. Singh & S. K. Singh, "A new distribution using sine function- its application to bladder cancer patients data", J. Stat. Appl. Probab. **4** (2015) 9. http://dx.doi.org/10.12785/jsap/040309.

[6] S. Nanga, S. Nasiru, & J. Dioggban, "Cosine Topp – Leone family of distributions: properties and regression", Research in Mathematics **10** (2023) 2208935. https://doi.org/10.1080/27684830.2023.2208935.

[7] M. Muhammad, R. A. Bantan, L. Liu, C. Chesneau, M. H. Tahir, F. Jamal, & M. Elgarhy, "A new extended cosine–g distributions for lifetime studies", Mathematics **9** (2021) 2758. https://doi.org/10.3390/math9212758.

[8] M. E. Bakr, A. A. Al-Babtain, Z. Mahmood, R. A. Aldallal,S. K. Khosa, M. M. Abd El-Raouf, E. Hussam & A. M. Gemeay, "Statistical modelling for a new family of generalized distributions with real data applications", Mathematical Biosciences and Engineering **19** (2022) 9. https://doi.org/10.3934/mbe.2022404.

[9] Z. Mahmood, T. M Jawa, N. Sayed-Ahmed, E. M. Khalil, A. H. Muse, & A. H. Tolba, "An extended cosine generalized family of distributions for reliability modeling: characteristics and applications with simulation study", Mathematical Problems in Engineering **2022** (2022) 3634698. https://doi.org/10.1155/2022/3634698.

[10] C. W. Topp, & F. C. Leone, "A family of j-shaped frequency functions", Journal of the American Statistical Association **50** (1955) 209. https://doi.org/10.1080/01621459.1955.10501259.

[11] A. Al-shomrani, O. Arif, A. Shawky, S. Hanif & M. Shahbaz, "Topp-Leone family of distributions: some properties and application", Pak. J. Stat. Oper. Res. **7** (2016) 3. http://dx.doi.org/10.18187/pjsor.v12i3.1458.

[12] A. A. Sanusi, S. I. Doguwa, I. Audu, & Y. M. Baraya, "Topp Leone Exponential – G Family of distributions: properties and application", NIPES Journal of Science and Technology Research **2** (2020) 83. https://doi.org/10.37933/nipes/2.4.2020.10

[13] B. Oluyede, S. Chamunorwa, & F. Chipepa, "The Topp-Leone Gompertz-G family of distributions with applications", Journal of Statistics and Management Systems **25** (2022) 1399. https://doi.org/10.1080/09720510.2021.1972623.

[14] H. Reyad, M. Alizadeh, F. Jamal & S. Othman, "The Topp Leone odd Lindley-G family of distributions: properties and applications", Journal of Statistics and Management Systems **21** (2018) 1273. https://doi.org/10.1080/09720510.2018.1495157.

[15] A. S. Hassan, M. Elgarhy, & R. Ragab, "Statistical properties and estimation of inverted topp-leone distribution", J. Stat. Appl. Pro. **331** (2020) 319. http://dx.doi.org/10.18576/jsap/090212.

[16] H. M. Yousof, & A. Z. Afify, "The transmuted exponentiated generalized-g family of distributions", Pakistan Journal of Statistics and Operation Research **11** (2015) 441. http://dx.doi.org/10.18187/pjsor.v11i4.1164.

[17] A. Z. Afify, H. M. Yousof, M. Alizadeh, I. Ghosh, & S.Ray, "The Marshall-Olkin transmuted-G family of distributions", Stochastics and Quality Control **35** (2020) 2. https://doi.org/https://doi.org/10.1515/eqc-2020-0009.

[18] H. M. Yousof, M. A. Jahanshahi, T. G. Ramires, I. Ghosh, G. G. Hamedani & C. Science, "The transmuted Topp-Leone G family of distributions: rheory, characterizations and applications", Journal of Data Science **15** (2017) 723. http://dx.doi.org/10.6339/JDS.201710_15(4).00008.

[19] A. S. Mohammed & F. I. Ugwuowo, "On transmuted exponential-topp leon distribution with monotonic and non-monotonic hazard rates and its applications", RT&A **16** (2021) 197. https://www.google.com/url?sa=t&source=web&rct=j&opi=89978449&url=https://www.gnedenko.net/Journal/2021/042021/RTA_4_2021-15.pdf.

[20] W. T. Shaw, & I. R. Buckley, "The alchemy of probability distributions: beyond Gram-Charlier & Cornish-Fisher expansions, and skew-normal or kurtotic-normal distributions", arXiv. [Online]. https://doi.org/10.48550/arXiv.0901.0434.

[21] L. Souza, W. R. Junior, C. C. Brito, C. Chesneau, T. A. Ferreira, & L. Soares, "General properties for the Cos-G class of distributions with applications", Eurasian Bulletin of Mathematics **2** (2019) 63. https://scholar.google.com/scholar_lookup?title=General+properties+for+the+Cos-G+class+of+distributions+with+applications&author=Souza,+L.&author=Junior,+W.R.O.&author=de+Brito,+C.C.R.&author=Chesneau,+C.&author=Ferreira,+T.A.E.&author=Soares,+L.&publication_year=2019&journal=Eurasian+Bull.+Math.&volume=2&pages=63%E2%80%9379.

[22] G. M. Cordeiro, & R. D. Brito, "The beta power distribution", Brazilian Journal of Probability and Statistics **26** (2012) 88. https://doi.org/10.1214/10-bjps124.

[23] P. W. Gieser, M. N. Chang, P. V. Rao, J. J. Shuster & J. Pullen, "Modelling cure rates using the Gompertz model with covariate information", Stat Med. **17** (1998) 831. https://doi.org/10.1002/(sici)1097-0258(19980430)17:8%3C831::aid-sim790%3E3.0.co;2-g.

Appendix A.

Program developed in R for the plot of the pdf and hazard function of the Transmuted Cosine Topp-Leone Weibull distribution.

```
${#TrCTLW PDF plots
# this is used to resize the current plot window.
resize.win <- function(Width=6, Height=6)
{
# works for windows
dev.off(); # dev.new(width=6, height=6)
windows(record=TRUE, width=Width, height=Height)
}

x=c(0,1)

y=c(0,1)

plot(x,y)

resize.win(25,8)

#=========================

par( mfrow=c(1,2),oma = c(1, 2, 2, 2),mar = c(4, 4, 1, 2))


x<-seq(0,5,length=1000)

alpha=1.5

theta=0.6

lambda=0.5

beta=0.3

Gbar = exp(-lambda*x^beta)

g = beta*lambda*x^(beta-1)*exp(-lambda*x^beta)

H = 1-cos((pi/2)*(1-Gbar^2)^alpha)

h = pi*alpha*g*Gbar*(1-Gbar^2)^(alpha-1)*sin((pi/2)*(1-Gbar^2)^alpha)

f = h*(1+theta-2*theta*H)

fdd13<-f

plot(x,fdd13,type="l",lty=1,lwd=3.,col="black",ylab="f(x)")}$

${#TrCTLW hazard plots

x<-seq(0,4,length=1000)

alpha=4.8

theta=0.5

lambda=1.8

beta=0.1

Gbar = exp(-lambda*x^beta)

g = beta*lambda*x^(beta-1)*exp(-lambda*x^beta)

H = 1-cos((pi/2)*(1-Gbar^2)^alpha)

h = pi*alpha*g*Gbar*(1-Gbar^2)^(alpha-1)*sin((pi/2)*(1-Gbar^2)^alpha)

f = h*(1+theta-2*theta*H)

fdd13<-f

F = (1+theta)*H-theta*H^2

h=fdd13/(1-F)

plot(x,h,type="l",lty=1,lwd=3.,col="black",ylab="h(x)")}$

${#TrCTLW survival plots

par( mfrow=c(1,2),oma = c(1, 2, 2, 2),mar = c(4, 4, 1, 2))

x<-seq(0,4,length=1000)

alpha=2.0

theta=0.1

lambda=0.5

beta =1.2

Gbar = exp(-lambda*x^beta)

g = beta*lambda*x^(beta-1)*exp(-lambda*x^beta)

H = 1-cos((pi/2)*(1-Gbar^2)^alpha)

F = (1+theta)*H-theta*H^2

S = 1-F

plot(x,S,type="l",lty=1,lwd=3.,col="black",ylab="S(x)")}$

${#TrCTLW CDF plots

x<-seq(0,5,length=1000)

alpha=3.0

theta=0.2

lambda=0.6

beta=1.2

Gbar = exp(-lambda*x^beta)

g = beta*lambda*x^(beta-1)*exp(-lambda*x^beta)

H = 1-cos((pi/2)*(1-Gbar^2)^alpha)

F = (1+theta)*H-theta*H^2

plot(x,F,type="l",lty=1,lwd=3.,col="black",ylab="F(x)")}$

Codes for maximum likelihood estimation simulations

${#####simulation of mle

rm(list=ls(all=TRUE))

sink("TrCTL SIMULATION.doc")

####pdf

pdfTrCTL = function(x,alpha,theta,lambda,beta)

{ Gbar = exp(-lambda*x^beta)

g = beta*lambda*x^(beta-1)*exp(-lambda*x^beta)

H = 1-cos((pi/2)*(1-Gbar^2)^alpha)

h = pi*alpha*g*Gbar*(1-Gbar^2)^(alpha-1)*sin((pi/2)*(1-Gbar^2)^alpha)

f = h*(1+theta-2*theta*H)

return(f)

}


###cdf

cdfTrCTL = function(x,alpha,theta,lambda,beta)

{ Gbar = exp(-lambda*x^beta)

H = 1-cos((pi/2)*(1-Gbar^2)^alpha)

F = (1+theta)*H-theta*H^2

return(F)
```

```
}
#likelihood
pdf<-function(par,x){
alpha<-par[1]; theta<-par[2];lambda<-par[3];beta<-par[4]
val<-  -sum(log( pdfTrCTL(x,alpha,theta,lambda,beta)))
val
}


#quantile function
qu<-function(u,alpha,theta,lambda,beta){
T = ((1+theta)-sqrt(((1+theta)^2)-4*theta*u))/(2*theta)
Y =((2/pi)*acos(1-T))^(1/alpha)
P =((1/lambda)*-log(1-Y))^(1/beta)
return(P)
}

#function to do the simulation
eqm<-function(n,alpha1,theta1,lambda1,beta1){
plot(0,0,ylim = c(0,1000))
alpha<-theta<-lambda<-beta<-c();
set.seed(123)
for (i in 1:1000){
u<-runif(n = n,min = 0,max = 1)
data<-qu(u,alpha1,theta1,lambda1,beta1)
hat<-try(optim(c(alpha1,theta1,lambda1,beta1),pdf,x=data,control =
list(maxit = 60)),silent=F)
alpha<-c(alpha,hat$par[1])
theta<-c(theta,hat$par[2])
lambda<-c(lambda,hat$par[3])
beta<-c(beta,hat$par[4])
abline(h=i)
}
means<-c(mean(alpha),mean(theta),mean(lambda),mean(beta))
vars<-c(var(alpha),var(theta),var(lambda),var(beta))
Bias<-means-c(alpha1,theta1,lambda1,beta1)
RMSE<-(vars+Bias^2)^(0.5)
#par( mfrow=c(3,3),oma = c(1, 2, 2, 2),mar = c(4, 4, 1, 2))
#plot(RMSE,lty=1,type="l",xlab="sample size")
result<-new.env()
result$means<-round(means,4)
result$Bias<-round(Bias,4)
result$RMSE<-round(RMSE,4)
return(as.list(result))
}
#eqm(n,alpha,theta,lambda,beta)}$
${#TrCTLW
pdfTrCTLW =  function(par,x){
lambda = par[1]
beta = par[2]
alpha = par[3]
theta = par[4]
Gbar = exp(-lambda*x^beta)
g = beta*lambda*x^(beta-1)*exp(-lambda*x^beta)
H = 1-cos((pi/2)*(1-Gbar^2)^alpha)
h = pi*alpha*g*Gbar*(1-Gbar^2)^(alpha-1)*sin((pi/2)*(1-Gbar^2)^alpha)
f = h*(1+theta-2*theta*H)
return(f)
}


###cdf
cdfTrCTLW =  function(par,x){
lambda = par[1]
beta = par[2]
alpha = par[3]
theta = par[4]
Gbar = exp(-lambda*x^beta)
H = 1-cos((pi/2)*(1-Gbar^2)^alpha)
F = (1+theta)*H-theta*H^2
return(F)
}

set.seed(0)
result_1 = goodness.fit(pdf = pdfTrCTLW, cdf = cdfTrCTLW,
starts = c(0.1,0.1,0.1,-0.1), data = dat, method =
"PSO",
domain = c(0,Inf),mle = NULL, lim_inf = c(0,0,0,0),
lim_sup = c(2,2,2,2), S = 250, prop=0.1, N=50)}$
```

Appendix B.

Table B.1: Lifetime data presentation.

Data	Observation
Dataset one	0.4365, 0.4260, 0.5140, 0.6907, 0.7471, 0.2605, 0.6196, 0.8781, 0.4990, 0.6058, 0.6891, 0.5770, 0.5394, 0.1479, 0.2356, 0.6012, 0.1525, 0.5483, 0.6927, 0.7261, 0.3323, 0.0671, 0.2361, 0.4800, 0.5707, 0.7131, 0.5853, 0.6768, 0.5350, 0.4151, 0.6789, 0.4576, 0.3259, 0.2303, 0.7687, 0.4371, 0.3383, 0.6114, 0.3480, 0.4564, 0.7804, 0.3406, 0.4823, 0.5912, 0.5744, 0.5481, 0.1131, 0.7290, 0.0168, 0.5529, 0.4530, 0.3891, 0.4752, 0.3134, 0.3175, 0.1167, 0.6750, 0.5113, 0.5447, 0.4143, 0.5627, 0.5150, 0.0776, 0.3945, 0.4553, 0.4470, 0.5285, 0.5232, 0.6465, 0.0650, 0.8492, 0.8147, 0.3627, 0.3906, 0.4438, 0.4612, 0.3188, 0.2160, 0.6707, 0.6220, 0.5629, 0.4675, 0.6844, 0.3413, 0.4332, 0.0854, 0.3821, 0.4694, 0.3635, 0.4111, 0.5349, 0.3751, 0.1546, 0.4517, 0.2681, 0.4049, 0.5553, 0.5878, 0.4741, 0.3598, 0.7629, 0.5941, 0.6174, 0.6860, 0.0609, 0.6488, 0.2747
Dataset two	1.3, 1.0, 1.2, 0.9, 1.1, 0.8, 0.5, 1.0, 0.7, 0.5, 1.7, 1.1, 0.8, 0.5, 1.2, 0.8, 1.1, 0.9, 1.2, 0.9, 0.8, 0.6, 0.3, 0.8, 0.6, 0.4, 1.1, 1.1, 0.2, 0.8, 0.5, 1.1, 0.1, 0.8, 1.7, 1.0, 0.8, 1.0, 0.8, 1.0, 0.2, 0.8, 0.4, 1.0, 0.2, 0.8, 1.4, 0.8, 0.5, 1.1, 0.9, 1.3, 0.9, 0.4, 1.4, 0.9, 0.5, 1.7, 0.9, 0.8, 0.8, 1.2, 0.9, 0.8, 0.5, 1.0, 0.6, 0.1, 0.2, 0.5, 0.1, 0.1, 0.9, 0.6, 0.9, 0.6, 1.2, 1.5, 1.1, 1.4, 1.2, 1.7, 1.4, 1.0, 0.7, 0.4,0.9, 0.7, 0.8, 0.7, 0.4, 0.9, 0.6, 0.4, 1.2, 2.0, 0.7, 0.5, 0.9, 0.5, 0.9, 0.7, 0.9, 0.7, 0.4, 1.0, 0.7, 0.9, 0.7, 0.5, 1.3, 0.9, 0.8, 1.0, 0.7, 0.7, 0.6, 0.8, 1.1, 0.9, 0.9, 0.8, 0.8, 0.7, 0.7, 0.4, 0.5, 0.4, 0.9, 0.9 , 0.7, 1.0, 1.0, 0.7, 1.3, 1.0, 1.1, 1.1, 0.9, 1.1, 0.8, 1.0, 0.7, 1.6, 0.8, 0.6, 0.8, 0.6, 1.2,0.9, 0.6, 0.8, 1.0, 0.5, 0.8, 1.0, 1.1, 0.8, 0.8, 0.5, 1.1, 0.8, 0.9, 1.1, 0.8, 1.2, 1.1, 1.2, 1.1, 1.2, 0.2, 0.5, 0.7, 0.2,0.5, 0.6, 0.1, 0.4, 0.6, 0.2, 0.5, 1.1, 0.8, 0.6, 1.1, 0.9, 0.6, 0.3, 0.9, 0.8, 0.8, 0.6, 0.4, 1.2, 1.3, 1.0,0.6, 1.2, 0.9, 1.2, 0.9, 0.5, 0.8, 1.0, 0.7, 0.9, 1.0, 0.1, 0.2, 0.1, 0.1, 1.1, 1.0, 1.1, 0.7, 1.1, 0.7, 1.8, 1.2, 0.9, 1.7, 1.2, 1.3, 1.2, 0.9, 0.7, 0.7, 1.2, 1.0, 0.9, 1.6, 0.8, 0.8, 1.1, 1.1, 0.8, 0.6, 1.0, 0.8, 1.1,0.8, 0.5, 1.5, 1.1, 0.8, 0.6, 1.1, 0.8, 1.1, 0.8, 1.5, 1.1, 0.8, 0.4, 1.0, 0.8, 1.4, 0.9, 0.9, 1.0, 0.9, 1.3, 0.8, 1.0, 0.5, 1.0, 0.7, 0.5, 1.4, 1.2, 0.9, 1.1, 0.9, 1.1, 1.0, 0.9, 1.2, 0.9, 1.2, 0.9, 0.5, 0.9, 0.7, 0.3,1.0, 0.6, 1.0, 0.9, 1.0, 1.1, 0.8, 0.5, 1.1, 0.8, 1.2, 0.8, 0.5, 1.5, 1.5, 1.0, 0.8,1.0, 0.5, 1.7, 0.3, 0.6, 0.6, 0.4, 0.5, 0.5, 0.7, 0.4, 0.5, 0.8, 0.5, 1.3, 0.9, 1.3, 0.9, 0.5, 1.2, 0.9, 1.1, 0.9, 0.5, 0.7, 0.5, 1.1 , 1.1, 0.5, 0.8, 0.6, 1.2, 0.8, 0.4, 1.3, 0.8, 0.5, 1.2, 0.7, 0.5, 0.9, 1.3, 0.8, 1.2, 0.9.