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# Throughflow effect on bi-disperse convection in Rivlin-Ericksen fluid

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**Abstract**

In this investigation, we delve into the influence of throughflow on the phenomenon of bi-disperse convection within Rivlin-Ericksen fluid. In the context of examining bi-disperse convection within this specific type of fluid, the throughflow effect is considered to have a uniform vertical distribution. The primary focus of this study centers on evaluating the system's linear stability. To achieve this, we employ normal mode analysis to compute the Darcy-Rayleigh number at the onset of convection. This Darcy-Rayleigh number is computed for both stationary and oscillatory convection modes. Furthermore, we conduct a comprehensive analysis and present the results in graphical form to illustrate the impact of various parameters, including Peclet number and kinematic viscoelastic parameter, on both stationary and oscillatory convection. Our research findings demonstrate that when Peclet number $\mathrm{Pr}_1 < 0$, it leads to destabilising effect on both stationary and oscillatory convections. Conversely, when Peclet number $\mathrm{Pr}_1 > 0$, it induces stabilising effect on both stationary as well as oscillatory convections.

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## 1. Introduction

Bi-disperse convection refers to the examination of convective flow initiation within materials possessing dual porosity characteristics, which are denoted as bi-disperse porous media. These bi-disperse porous materials, or bi-disperse porous media (BDPM), exhibit distinctive features in the form of two distinct types of pores: macropores and micropores. More specifically, the macropores are identified as the f-phase, representing the fractured phase, while the remaining structural components are designated as the p-phase, signifying the porous phase. Bi-disperse porous media find applications in various fields, including heat pipe evaporators, culinary processes, solar thermal systems, permafrost studies, heat exchangers, computer systems, electronics, nuclear energy conversion, assessment of effective thermal conductivity in sintered porous materials, water filtration using filter tubes, capillary structures, the exploration of subsurface drinking water supply physics, chemistry, chemical engineering, chromatography, coal storage, gas shale storage, hydraulic fracturing of underground rock formations for natural gas extraction and the investigation of landslide phenomena. Chen *et al.* [1] have contributed to this field by conducting both theoretical and experimental studies focused on the characterization of the stagnant thermal conductivity of

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bi-disperse porous media.

Nield and Kuznetsov [2–4] in their contributions, advanced a sophisticated mathematical model that explains the initiation of bi-disperse convection. Their analysis covered the derivation of analytical solutions for fully developed forced convection within a bi-dispersed porous medium. Furthermore, in the case of fully developed convection, their investigation yielded expressions that establish the Nusselt number as a function of the inherent properties of the bi-disperse porous medium (BDPM). Nield and Kuznetsov [4] proved that the $(R_D)_{cri}$ for the onset of bi-disperse convection is higher with respect to $(R_D)_{cri}$ for the single porosity case. Therefore dual-porosity materials are better materials in solving insulation problems and thermal management problems. Consequently, bi-disperse porous materials provide much more materials possibilities to design man-made materials for heat transfer problems. For this reason, recently, the onset of bi-disperse convection has become an interesting topic to investigate for many researchers such as Imani *et al.* [5], Gentile and Straughan [6], Gentile *et al.* [7], Badday *et al.* [8], Falsaperla *et al.* [9] and Capone *et al.* [10].

Rivlin-Ericksen fluid, a type of non-Newtonian fluid, is characterized by its ability to exhibit both viscous and elastic behaviour under flow. This unique rheological property makes it suitable for various industrial applications, including polymer processing, biomedicine and material science. In polymer processing, it is utilized in the manufacturing of products like films, fibers and coatings due to its shear-thinning behaviour. In biomedicine, it finds application in drug delivery systems and tissue engineering for its tunable viscosity and visco-elastic properties. Furthermore, in material science, Rivlin-Ericksen fluids are studied for their role in understanding complex flow phenomena and designing advanced materials with tailored mechanical properties. Several researchers, including Rana and Thakur [11], Chand and Rana [12, 13], Rana [14] and Rana *et al.* [15, 16], have dedicated their research efforts to investigating the properties and behavior of Rivlin-Ericksen fluid.

"Throughflow" typically denotes the movement or transport of a substance, such as water or air, through a medium or material, such as soil or a porous substance. This terminology is commonly employed within the areas of hydrology and soil science. Throughflow represents a critical element of the hydrological cycle, governing the flow of water within earth's ecosystems. It holds significant ecological and environmental ramifications as it facilitates the transportation of nutrients, pollutants and sediments across the landscape. The comprehension of throughflow is of paramount importance in water resource management, enabling an assessment of the influence of land use and land management practices on water quality and availability. The horizontal subsurface movement of water, known as throughflow, typically occurs when the soil reaches a state of complete saturation and it exerts a significant influence on convective instabilities. When a net vertical mass flow (throughflow) occurs across a horizontally heated layer from below, the stability of this motion becomes a fundamental concern. This is especially relevant in applications encompassing cloud physics, astrophysics, hydrological and geophysical investigations, seabed hydrodynamics, subterranean pollution studies and various industrial and technological processes. The investigation of convection in throughflow has attracted attention from multiple researchers, including Sutton [17], Petrolo *et al.* [18] and Capone *et al.* [19, 20]. Notably, Capone *et al.* [20] investigated the impact of throughflow on bi-disperse convection, with a particular focus on single-temperature bi-disperse porous media. Their study based on the utilization of weighted energy method to capture the most suitable description of the underlying physics in this context. Further, there are several authors: Ruo *et al.* [21], Reddy and Ragoju [22] and Awasthi *et al.* [23] who have considered throughflow effect in their respective researches.

Chandrasekhar [24], in his book, conducted all stability analyses using normal mode analysis. Normal mode analysis plays a crucial role in understanding thermal instability by quantifying the growth or decay rates of perturbations and determining the dominant modes that lead to instability. This approach is widely employed in physics, engineering and other scientific disciplines to study the stability and dynamics of systems ranging from mechanical vibrations to electromagnetic waves. Recently, the researchers like: Sharma *et al.* [25, 26], Bains *et al.* [27], Sharma *et al.* [28, 29] and Bains *et al.* [30] have analysed their investigations in different aspects by making use of normal mode analysis.

In this scientific inquiry, we have undertaken a thorough examination, considering the practical applications and the substantial importance of bi-disperse porous media (BDPM) in addressing complex thermal management challenges. Additionally, we recognize the relevance of managing and controlling convective instabilities in the broader context of thermal and engineering sciences. Our investigation is focused on a comprehensive exploration of the impact of throughflow on bi-disperse convection, with a specific emphasis on its manifestation within a viscoelastic fluid, where we have selected the Rivlin-Ericksen fluid as our model system. The analysis of this study is conducted using the normal mode approach, encompassing both stationary and oscillatory modes, to provide a comprehensive understanding of the phenomena under investigation.

## 2. Mathematical formulation and governing the equations

Here, we investigate thermal instability of throughflow effect on bi-disperse porous medium (BDPM) that is saturated with Rivlin-Ericksen fluid. This BDPM is positioned within a spatial domain defined by the planes $z = 0$ and $z = d$. It undergoes uniform heating from its lower boundary, influenced by the gravitational field represented as $\mathbf{g} = (0, 0, -g)$. The variables $T_L$ and $T_U$ represent the temperatures at the lower plane $z = 0$ and the upper plane $z = d$, respectively, with the condition that $T_L > T_U$. Additionally, we make the assumption that the BDPM exhibits a single-temperature characteristic, signifying that temperatures $T^p$ and $T^f$ within the medium are equivalent and designated as $T$ (see Figure 1).

Now, utilizing the Boussinesq approximation, which aligns with the principles of entropy, we can establish the governing equations based on Darcy's model, as per the work of Capone

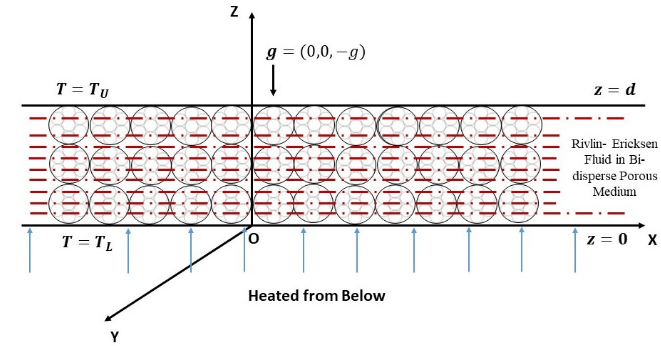


Figure 1. Physical Configuration.

*et al.* [20]:

$$\nabla.\mathbf{q}^f = 0, \tag{1}$$

$$\nabla.\mathbf{q}^P = 0, \tag{2}$$

$$-\frac{1}{K_f}\left(\mu+\mu'\frac{\partial}{\partial t}\right)\mathbf{q}^f-\zeta\left(\mathbf{q}^f-\mathbf{q}^P\right)-\nabla p^f-\rho_0\alpha\mathbf{g}T=0, \tag{3}$$

$$-\frac{1}{K_P}\left(\mu+\mu'\frac{\partial}{\partial t}\right)\mathbf{q}^P-\zeta\left(\mathbf{q}^P-\mathbf{q}^f\right)-\nabla p^P-\rho_0\alpha\mathbf{g}T=0, \tag{4}$$

$$(\rho c)_m\frac{\partial T}{\partial t}+(\rho c)_f\left(\mathbf{q}^P+\mathbf{q}^f\right).\nabla T=k_m\nabla^2T, \tag{5}$$

where $\mathbf{q}^S$ and $p^S$ are seepage velocity and pressure field for $S=\{f,P\}$, respectively, $T$ is temperature, $\mu$ is viscosity of the fluid, $\mu'$ is visco-elasticity of the fluid, $\zeta$ denotes interaction coefficient between the $f-$phase and $p-$phase, $K_S$ denotes permeabilities for $S=\{f,P\}$, $k_m$ is thermal conductivity (following Capone *et al.* [20]).

$$(\rho c)_m=(1-\varphi)(1-\varepsilon)(\rho c)_{sol}+\varphi(\rho c)_f+\varepsilon(1-\varphi)(\rho c)_P,$$
$$k_m=(1-\varphi)(1-\varepsilon)k_{sol}+\varphi k_f+\varepsilon(1-\varphi)k_P,$$

where subscript *sol* referred to solid skeleton.

Given that we are restricting our analysis to a scenario involving a single-temperature bi-disperse porous medium, with both macropores and micropores being saturated by the same fluid, we anticipate that $(\rho c)_f$ and $(\rho c)_P$ will be equivalent [9, 20], hence

$$(\rho c)_m=(1-\varphi)(1-\varepsilon)(\rho c)_{sol}+[\varphi+\varepsilon(1-\varphi)](\rho c)_f.$$

Our objective is to conduct a stability analysis of a vertical throughflow occurring within the layer. Consequently, we can establish the boundary conditions as [20]:

$$\left.\begin{array}{l}\mathbf{q}^P=\left(0,0,Q^P\right),\ \ \mathbf{q}^f=\left(0,0,Q^f\right),\ \ T=T_L\ \text{ at }\ z=0\\ \mathbf{q}^P=\left(0,0,Q^P\right),\ \ \mathbf{q}^f=\left(0,0,Q^f\right),\ \ T=T_U\ \text{ at }\ z=d\end{array}\right\}. \tag{6}$$

## 3. Basic state solutions

The assumed basic state is:

$$\left.\begin{array}{l}\mathbf{q}^f=\left(0,0,q^f\right),\quad \mathbf{q}^P=\left(0,0,q^P\right),\\ T=T_b(z),\quad p^P=p_b^P(z),\quad p^f=p_b^f(z)\end{array}\right\}. \tag{7}$$

Utilizing equation (7) within the preceding equations (1)-(5), we have

$$\frac{dq^f}{dz}=0, \tag{8}$$

$$\frac{dq^P}{dz}=0, \tag{9}$$

$$-\frac{1}{K_f}\left(\mu+\mu'\frac{\partial}{\partial t}\right)\mathbf{q}^f-\zeta\left(\mathbf{q}^f-\mathbf{q}^P\right)-\frac{d}{dz}\left(p_b^f(z)\right)-\rho_0\alpha\mathbf{g}T_b(z)=0, \tag{10}$$

$$-\frac{1}{K_P}\left(\mu+\mu'\frac{\partial}{\partial t}\right)\mathbf{q}^P-\zeta\left(\mathbf{q}^P-\mathbf{q}^f\right)-\frac{d}{dz}\left(p_b^P(z)\right)-\rho_0\alpha\mathbf{g}T_b(z)=0, \tag{11}$$

$$(\rho c)_f\left(\mathbf{q}^P+\mathbf{q}^f\right)\frac{d}{dz}\left(T_b(z)\right)=k_m\frac{d^2}{dz^2}\left(T_b(z)\right). \tag{12}$$

Subsequently incorporating the boundary condition given in (6) into the preceding equations (8) and (9), we arrive at:

$$\mathbf{q}^f=Q^f\hat{k}\quad\text{and}\quad\mathbf{q}^P=Q^P\hat{k}, \tag{13}$$

additionally, by applying boundary condition in equation (6) to equation (12), we have

$$T_b(z)=\frac{T_U-T_Le^{\frac{\bar{Q}d(\rho c)_f}{k_m}}+(T_L-T_U)e^{\frac{\bar{Q}z(\rho c)_f}{k_m}}}{1-e^{\frac{\bar{Q}d(\rho c)_f}{k_m}}}, \tag{14}$$

where $\bar{Q}=Q^f+Q^P$.

Hence, the solutions for the steady state of Rivlin-Ericksen fluid with vertical throughflow are expressed as:

$$\mathbf{q}^f=Q^f\hat{k},\ \mathbf{q}^P=Q^P\hat{k},$$
$$T_b(z)=\frac{T_U-T_Le^{\frac{\bar{Q}d}{\kappa_m}}+(T_L-T_U)e^{\frac{\bar{Q}z}{\kappa_m}}}{1-e^{\frac{\bar{Q}d}{\kappa_m}}}. \tag{15}$$

The outcome expressed in equation (15) corresponds exactly to the results reported by Capone *et al.* [20].

## 4. Perturbation solutions

For the stability analysis, we have incorporated minor disturbances into the basic state as:

$$\left.\begin{array}{l}\mathbf{q}^f=\left(0,0,q^f\right)+\mathbf{q}'^f=\left(0,0,q^f\right)+\left(u'^f,v'^f,w'^f\right),\\ \mathbf{q}^P=\left(0,0,q^P\right)+\mathbf{q}'^P=\left(0,0,q^P\right)+\left(u'^P,v'^P,w'^P\right),\\ T=T_b+T',\quad p^P=p_b^P+\pi'^P,\quad p^f=p_b^f+\pi'^f.\end{array}\right\} \tag{16}$$

Now, incorporating equation (16) into equations (1) – (6) and omitting the prime notation (′) for the sake of convenience, we have:

$$\nabla.\mathbf{q}^f = 0, \quad (17)$$

$$\nabla.\mathbf{q}^P = 0, \quad (18)$$

$$-\frac{1}{K_f}\left(\mu+\mu'\frac{\partial}{\partial t}\right)\mathbf{q}^f - \zeta\left(\mathbf{q}^f-\mathbf{q}^P\right) - \nabla\pi^f + \rho_0\alpha g T\hat{k} = 0, \quad (19)$$

$$-\frac{1}{K_P}\left(\mu+\mu'\frac{\partial}{\partial t}\right)\mathbf{q}^P - \zeta\left(\mathbf{q}^P-\mathbf{q}^f\right) - \nabla\pi^P + \rho_0\alpha g T\hat{k} = 0, \quad (20)$$

$$(\rho c)_m\frac{\partial T}{\partial t} + \frac{(\rho c)_f\left(w^f+w^P\right)(T_L-T_U)}{1-e^{\frac{\bar{Q}d}{\kappa_m}}}\left(\frac{\bar{Q}}{\kappa_m}\right)e^{\frac{\bar{Q}z}{\kappa_m}} + (\rho c)_f\bar{Q}\frac{\partial T}{\partial z} = k_m\nabla^2 T, \quad (21)$$

and perturbed boundary conditions are

$$w^P = w^f = T = 0 \quad \text{at} \quad z=0 \quad \text{and} \quad z=d. \quad (22)$$

Furthermore, we have defined dimensionless variables in the following manner [6, 20]:

$$\left.\begin{array}{c} (x^*,y^*,z^*) = \frac{(x,y,z)}{d}, \quad t^* = \frac{t k_m}{(\rho c)_m d^2}, \\ \mathbf{q}^{P*} = \frac{\mathbf{q}^P d}{\kappa_m}, \quad \mathbf{q}^{f*} = \frac{\mathbf{q}^f d}{\kappa_m}, \\ \pi^{f*} = \frac{\pi^f}{\kappa_m\zeta}, \quad \pi^{P*} = \frac{\pi^P}{\kappa_m\zeta}, \quad T^* = \frac{T}{\sqrt{\frac{\beta\kappa_m\zeta}{\rho_0\alpha g}}} \end{array}\right\}. \quad (23)$$

Applying equation (23) to the previously outlined equations (17) – (22) and removing the star symbol (∗) for ease of representation, we obtain the simplified dimensionless equations as follows:

$$\nabla.\mathbf{q}^f = 0, \quad (24)$$

$$\nabla.\mathbf{q}^P = 0, \quad (25)$$

$$-\psi_1\left(1+F\frac{\partial}{\partial t}\right)\mathbf{q}^f - \left(\mathbf{q}^f-\mathbf{q}^P\right) - \nabla\pi^f + \sqrt{R_D}T\hat{k} = 0, \quad (26)$$

$$-\psi_2\left(1+F\frac{\partial}{\partial t}\right)\mathbf{q}^P - \left(\mathbf{q}^P-\mathbf{q}^f\right) - \nabla\pi^P + \sqrt{R_D}T\hat{k} = 0, \quad (27)$$

$$\frac{\partial T}{\partial t} + \Pr_1\frac{\partial T}{\partial z} + L_1\sqrt{R_D}\left(w^f+w^P\right) = \nabla^2 T, \quad (28)$$

and the dimensionless boundary conditions are

$$w^P = w^f = T = 0 \quad \text{at} \quad z=0 \quad \text{and} \quad z=1, \quad (29)$$

where the dimensionless parameters are [6, 20]:
Kinematic visco-elastic parameter $F = \frac{\mu' k_m}{\mu(\rho c)_m d^2}$, $\psi_1 = \frac{\mu}{\zeta K_f}$, $\psi_2 = \frac{\mu}{\zeta K_P}$, Peclet number $\Pr_1 = \frac{\bar{Q}d}{\kappa_m}$, Darcy-Rayleigh number $R_D = \left(\frac{\rho_0\alpha g\beta d^2}{\zeta\kappa_m}\right)^{\frac{1}{2}}$ and dimensionless basic temperature gradient $L_1 = \frac{\Pr_1 e^{\Pr_1 z}}{1-e^{\Pr_1}}$.

Now operating equations (26) and (27) by $\hat{k}.curl.curl$, we get

$$\psi_1\left(1+F\frac{\partial}{\partial t}\right)\nabla^2 w^f + \nabla^2\left(w^f-w^P\right) - \sqrt{R_D}\nabla_H^2 T = 0, \quad (30)$$

$$\psi_2\left(1+F\frac{\partial}{\partial t}\right)\nabla^2 w^P + \nabla^2\left(w^P-w^f\right) - \sqrt{R_D}\nabla_H^2 T = 0. \quad (31)$$

## 5. Normal mode analysis and dispersion relation

The examination of the system's stability is conducted by utilizing the normal mode technique, in accordance with the approach presented by Chandrasekhar [24]:

$$\left[w^f, w^P, T\right] = \left[W(z), \Phi(z), \Theta(z)\right]\exp\left(ixk_x + iyk_y + nt\right). \quad (32)$$

By incorporating equation (32) in above equations (28) – (31), we have

$$\psi_1(1+nF)\left(D^2-a^2\right)W + \left(D^2-a^2\right)[W-\Phi] + \sqrt{R_D}a^2\Theta = 0, \quad (33)$$

$$\psi_2(1+nF)\left(D^2-a^2\right)\Phi + \left(D^2-a^2\right)[\Phi-W] + \sqrt{R_D}a^2\Theta = 0, \quad (34)$$

$$n\Theta + \Pr_1 D\Theta + \frac{\Pr_1\sqrt{R_D}e^{\Pr_1 z}}{1-e^{\Pr_1}}[W+\Phi] = \left(D^2-a^2\right)\Theta. \quad (35)$$

Furthermore, the boundary conditions are simplified to the form outlined by Chandrasekhar [24] when applying the normal mode analysis:

$$W = D^2W = \Phi = D^2\Phi = \Theta = 0 \ \text{at} \ z=0 \ \text{and} \ z=1, \quad (36)$$

where, $D = \frac{d}{dz}$, $-a^2 = \nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\nabla^2 = D^2 - a^2$.
Consider the trial solutions for $W$, $\Phi$, $\Theta$ of the form following Chandrasekhar [24]:

$$W = W_0\sin(\pi z), \ \Phi = \Phi_0\sin(\pi z), \ \Theta = \Theta_0\sin(\pi z). \quad (37)$$

The solutions presented in equation (37) comply with the boundary conditions as delineated in equation (36).

Substituting (37) into equations (33) – (35) and performing integration for each equation over the range of $z = 0$ and $z = 1$, we obtain:

$$\begin{bmatrix} J[1+\psi_1(1+nF)] & -J & -a^2\sqrt{R_D} \\ -J & J[1+\psi_2(1+nF)] & -a^2\sqrt{R_D} \\ \frac{\pi^2\Pr_1\sqrt{R_D}\left(1+e^{\Pr_1}\right)}{\left(1-e^{\Pr_1}\right)\left(\pi^2+\Pr_1^2\right)} & \frac{\pi^2\Pr_1\sqrt{R_D}\left(1+e^{\Pr_1}\right)}{\left(1-e^{\Pr_1}\right)\left(\pi^2+\Pr_1^2\right)} & 2(J+n) \end{bmatrix}\begin{bmatrix} W_0 \\ \Theta_0 \\ \Phi_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (38)$$

The presence of non-trivial solution within the linear system (38) occurs if and only if

$$R_D = \frac{2J\,(J+n)\left(1-e^{\mathrm{Pr}_1}\right)\left(\pi^2+\mathrm{Pr}_1^2\right)\{1-[1+A_1]\,[1+A_2]\}}{a^2\pi^2\mathrm{Pr}_1\,(1+e^{\mathrm{Pr}_1})\,\{4+A_1+A_2\}}, \tag{39}$$

where $A_1 = \psi_1\,(1+nF)$, $A_2 = \psi_2\,(1+nF)$ and $J = \pi^2 + a^2$.

### 5.1. Stationary convection

In the context of stationary convection, by putting $n = 0$ in equation (39), we get:

$$(R_D)_{sta} = \frac{2J^2\left(1-e^{\mathrm{Pr}_1}\right)\left(\pi^2+\mathrm{Pr}_1^2\right)\{1-(1+\psi_1)\,(1+\psi_2)\}}{a^2\pi^2\mathrm{Pr}_1\,(1+e^{\mathrm{Pr}_1})\,(4+\psi_1+\psi_2)}. \tag{40}$$

The critical value of Darcy-Rayleigh number of above equation (40) exists at $a_{cri} = \pi$.

## 6. Oscillatory convection

For oscillatory convection put $n = in_i$ in equation (39), we have

$$R_D = \frac{2J\,(J+in_i)\left(1-e^{\mathrm{Pr}_1}\right)\left(\pi^2+\mathrm{Pr}_1^2\right)\{1-[1+A_{11}]\,[1+A_{22}]\}}{a^2\pi^2\mathrm{Pr}_1\,(1+e^{\mathrm{Pr}_1})\,\{4+A_{11}+A_{22}\}}, \tag{41}$$

where, $A_{11} = \psi_1\,(1+in_iF)$, $A_{22} = \psi_2\,(1+in_iF)$.

Now, upon partitioning equation (41) into its real and imaginary constituents, we obtain:

$$R_D = \Delta_1 + in_i\Delta_2, \tag{42}$$

where

$$\Delta_1 = \frac{2J\left(1-e^{\mathrm{Pr}_1}\right)\left(\pi^2+\mathrm{Pr}_1^2\right)(B_1+B_2)}{a^2\pi^2\mathrm{Pr}_1\,(1+e^{\mathrm{Pr}_1})\left\{(4+\psi_1+\psi_2)^2+n_i^2F^2(\psi_1+\psi_2)^2\right\}},$$

and

$$\Delta_2 = \frac{2J\left(1-e^{\mathrm{Pr}_1}\right)\left(\pi^2+\mathrm{Pr}_1^2\right)(B_3-B_4)}{a^2\pi^2\mathrm{Pr}_1\,(1+e^{\mathrm{Pr}_1})\left\{(4+\psi_1+\psi_2)^2+n_i^2F^2(\psi_1+\psi_2)^2\right\}},$$

here,

$$B_1 = (4+\psi_1+\psi_2)\left\{J\left[n_i^2F^2\psi_1\psi_2-(\psi_1+\psi_2+\psi_1\psi_2)\right] + n_i^2F(\psi_1+\psi_2+2\psi_1\psi_2)\right\},$$

$$B_2 = n_i^2F\,(\psi_1+\psi_2)\left\{\left[n_i^2F^2\psi_1\psi_2-(\psi_1+\psi_2+\psi_1\psi_2)\right] - JF(\psi_1+\psi_2+2\psi_1\psi_2)\right\},$$

$$B_3 = (4+\psi_1+\psi_2)\left\{\left[n_i^2F^2\psi_1\psi_2-(\psi_1+\psi_2+\psi_1\psi_2)\right] - JF(\psi_1+\psi_2+2\psi_1\psi_2)\right\},$$

and

$$B_4 = F\,(\psi_1+\psi_2)\left\{J\left[n_i^2F^2\psi_1\psi_2-(\psi_1+\psi_2+\psi_1\psi_2)\right] + n_i^2F(\psi_1+\psi_2+2\psi_1\psi_2)\right\}.$$

With oscillatory onset $\Delta_2 = 0$ and $n_i \neq 0$, we get

$$n_i^2 = \frac{P_1-P_2}{P_3}, \tag{43}$$

where

$$P_1 = (\psi_1+\psi_2+\psi_1\psi_2)\,(4+\psi_1+\psi_2-FJ\,(\psi_1+\psi_2)),$$
$$P_2 = FJ\,(4+\psi_1+\psi_2)\,(\psi_1+\psi_2+2\psi_1\psi_2),$$

and

$$P_3 = F^2\left\{\psi_1\psi_2\Big(4+\psi_1+\psi_2-FJ(\psi_1+\psi_2)\Big) - (\psi_1+\psi_2)(\psi_1+\psi_2+2\psi_1\psi_2)\right\}.$$

Then from equation (42), we have

$$(R_D)_{osc} = \frac{2J\left(1-e^{\mathrm{Pr}_1}\right)\left(\pi^2+\mathrm{Pr}_1^2\right)(B_{11}+B_{22})}{a^2\pi^2\mathrm{Pr}_1\,(1+e^{\mathrm{Pr}_1})\left\{(4+\psi_1+\psi_2)^2+n_i^2F^2(\psi_1+\psi_2)^2\right\}}. \tag{44}$$

where

$$B_{11} = (4+\psi_1+\psi_2)\left\{J\left[n_i^2F^2\psi_1\psi_2-(\psi_1+\psi_2+\psi_1\psi_2)\right] + n_i^2F(\psi_1+\psi_2+2\psi_1\psi_2)\right\},$$

$$B_{22} = n_i^2F\,(\psi_1+\psi_2)\left\{\left[n_i^2F^2\psi_1\psi_2-(\psi_1+\psi_2+\psi_1\psi_2)\right] - JF(\psi_1+\psi_2+2\psi_1\psi_2)\right\},$$

here, $n_i^2$ is as represented by above equation (43). The expression found in equation (44) serves as the representation of the Darcy-Rayleigh number specifically for oscillatory convection.

## 7. Results and discussions

In this paper, we have analysed the linear stability of throughflow effect on bi-disperse convection in Rivlin-Ericksen fluid. For the linear stability analysis, we have employed the

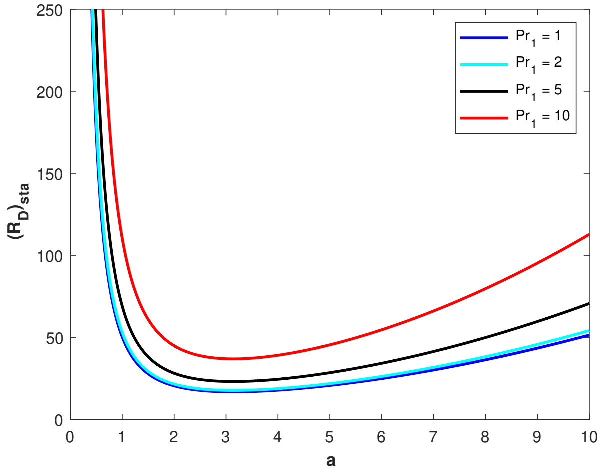


Figure 2. Variation of $(R_D)_{sta}$ w.r.t. $a$, when $\mathrm{Pr}_1 > 0$.

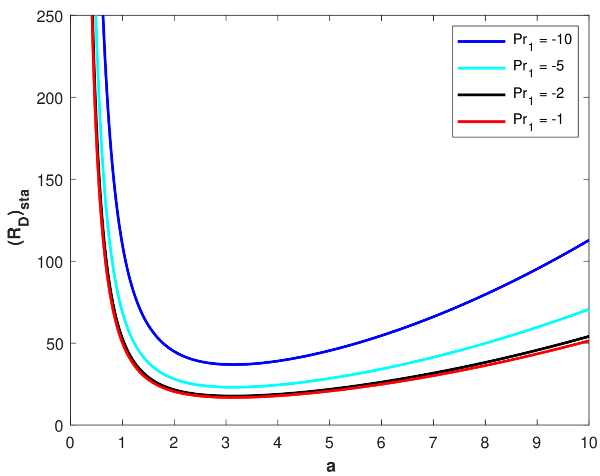


Figure 3. Variation of $(R_D)_{sta}$ w.r.t. $a$, when $\mathrm{Pr}_1 < 0$.

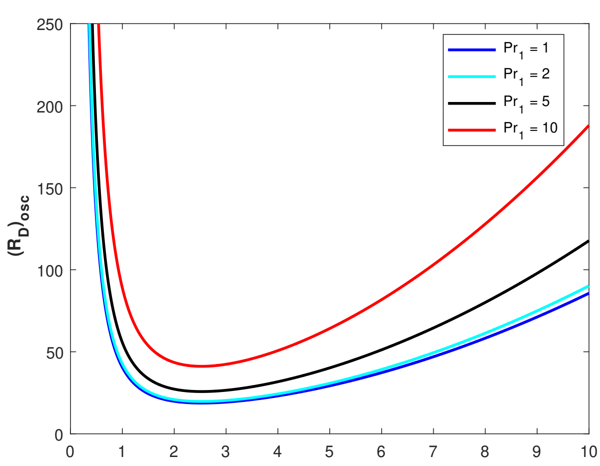


Figure 4. Variation of $(R_D)_{osc}$ w.r.t. $a$, when $\mathrm{Pr}_1 > 0$.

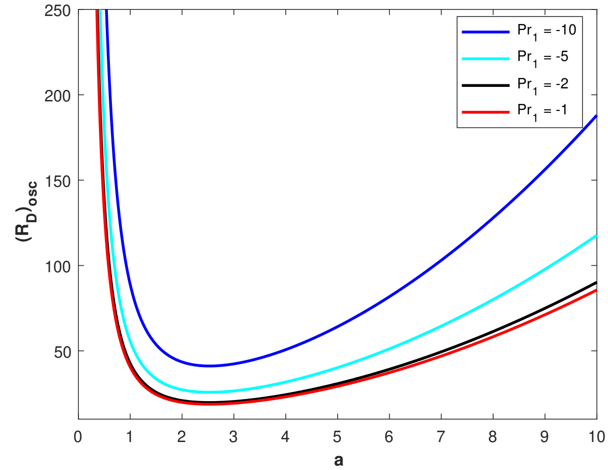


Figure 5. Variation of $(R_D)_{osc}$ w.r.t. $a$, when $\mathrm{Pr}_1 < 0$.

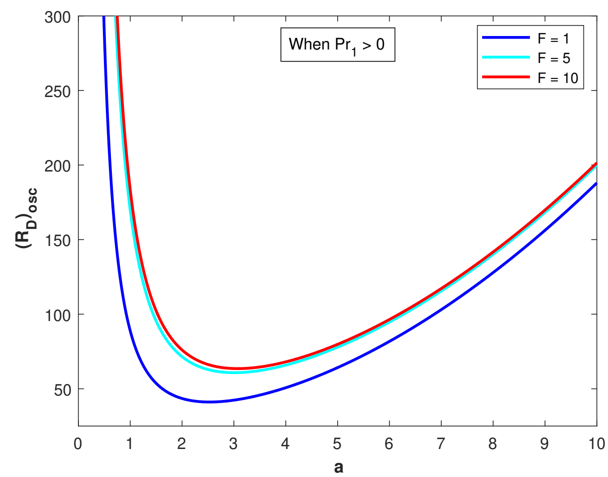


Figure 6. Variation of $(R_D)_{osc}$ w.r.t. $a$ for distinct values of kinematic viscoelastic parameter $F$, when $\mathrm{Pr}_1 > 0$.

normal mode analysis for both stationary as well as oscillatory convections. The effect of Peclet number has been analysed and presented graphically for both stationary as well as oscillatory convections. Further, we have also analysed the effect of kinematic viscoelastic parameter on oscillatory convection.

Figure 2 represents the relationship between the Darcy-Rayleigh number ($R_D$) and the wave number ($a$) in the context of stationary convection, while adhering to the constraint that Peclet number $\mathrm{Pr}_1 > 0$ and holding constant values of $\psi_1$ at 2 and $\psi_2$ at 0.2. The observations derived from Figure 2 lead to the inference that, in the presence of $\mathrm{Pr}_1 > 0$, $(R_D)_{sta}$ exhibits a consistent upward trend with an increasing wave number $a$. Consequently, it can be deduced that the system exerts a stabilising influence on the phenomenon of stationary convection when $\mathrm{Pr}_1$ is greater than zero.

Figure 3 shows a graphical representation of the relationship between the Rayleigh number ($R_D$) and the wave number ($a$) within the context of stationary convection, while following

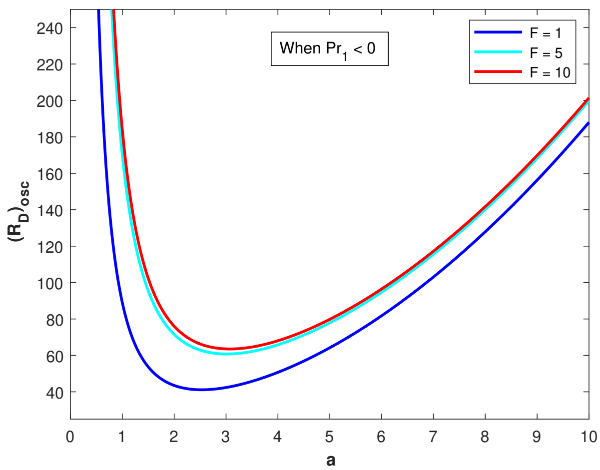


Figure 7. Variation of $(R_D)_{osc}$ w.r.t. $a$ for distinct values of kinematic viscoelastic parameter $F$, when $\mathrm{Pr}_1 < 0$.

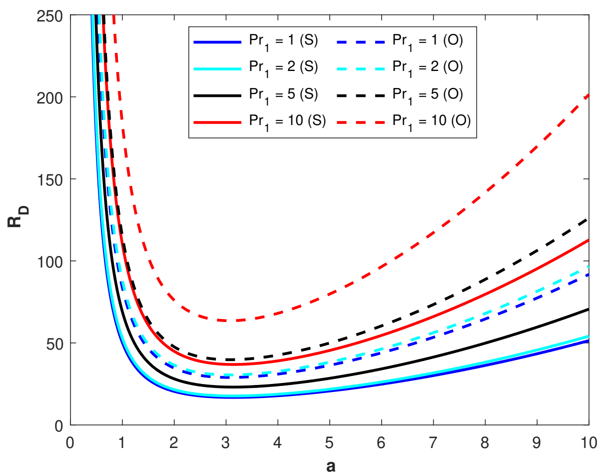


Figure 8. Variation of $R_D$ (for both stationary and oscillatory convection) w.r.t. $a$, when $\mathrm{Pr}_1 > 0$.

the condition that the Peclet number $\mathrm{Pr}_1 < 0$, with constant values of $\psi_1$ set at 2 and $\psi_2$ at 0.2. The analysis of Figure 3 yields the conclusion that, under condition where $\mathrm{Pr}_1$ is less than zero, there is a noticeable decrease in $(R_D)_{sta}$ as the wave number $a$ increases. Consequently, it implies that the system exerts a destabilising influence upon stationary convection when $\mathrm{Pr}_1$ is less than zero.

Figure 4 presents a graphical representation of the dependency of the Darcy-Rayleigh number $(R_D)$ on the wave number $(a)$ in the context of oscillatory convection, while adhering to the condition that Peclet number $\mathrm{Pr}_1$ is greater than zero and holding constant values of $\psi_1$ at 2 and $\psi_2$ at 0.2. Analysis of Figure 4 yields the conclusion that, in the presence of $\mathrm{Pr}_1$ greater than zero, $(R_D)_{osc}$ exhibits a positive linear relationship with the wave number $a$. As a result, it can be inferred that the system exerts a stabilising influence on oscillatory convection under the condition of $\mathrm{Pr}_1$ greater than zero.

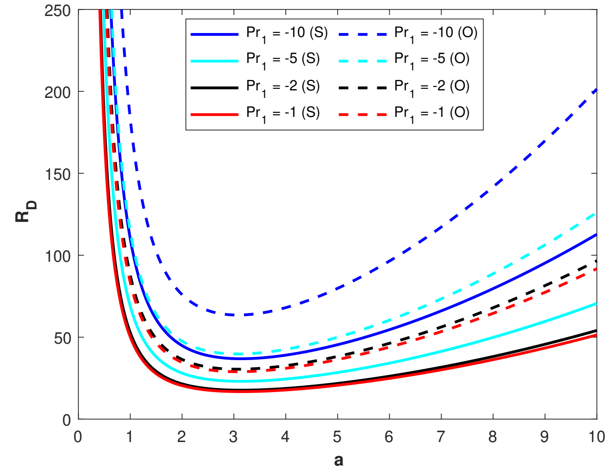


Figure 9. Variation of $R_D$ (for both stationary and oscillatory convection) w.r.t. $a$, when $\mathrm{Pr}_1 < 0$.

Figure 5 represents the graphical relationship between the Darcy-Rayleigh number $(R_D)$ and the wave number $(a)$ within the framework of oscillatory convection, while maintaining the condition that the Peclet number $\mathrm{Pr}_1$ is less than zero, while keeping $\psi_1$ fixed at 2 and $\psi_2$ at 0.2. A thorough examination of Figure 5 leads to the inference that, in scenarios where $\mathrm{Pr}_1 < 0$, there is a consistent decline in $(R_D)_{osc}$ with an increase in the wave number $a$. As a result, it can be deduced that the system induces a destabilizing influence on oscillatory convection when $\mathrm{Pr}_1 < 0$.

In this research study, the kinematic viscoelastic parameter $(F)$ exhibits its effect exclusively in the domain of oscillatory convection, with no noticeable impact on stationary convection. Figure 6 and Figure 7 provide graphical representations of the impact of the kinematic viscoelastic parameter $(F)$ on oscillatory convection. Figure 6 describes the relationship between the Darcy-Rayleigh number $(R_D)_{osc}$ and the wave number $(a)$ when the Peclet number $\mathrm{Pr}_1$ exceeds zero, while Figure 7 illustrates the same relationship when $\mathrm{Pr}_1 < 0$. Examination of both figures reveals a consistent and positive correlation between $(R_D)_{osc}$ and the wave number $a$, across various values of $F = 1, 5, 10$. This leads to the inference that the kinematic viscoelastic parameter $(F)$ exerts a stabilising effect on oscillatory convection, regardless of whether $\mathrm{Pr}_1$ is greater than or less than zero.

In Figure 8, we have conducted an analysis that integrates the concurrent effects of both stationary and oscillatory convection, charting the relationship between the Darcy-Rayleigh number $(R_D)$ and the wave number $(a)$ under the condition where the Peclet number $\mathrm{Pr}_1$ exceeds zero. These investigations were carried out while keeping the parameters $\psi_1$ and $\psi_2$ fixed at 2 and 0.2, respectively. Figure 8 clearly illustrates that $R_D$ increases with the increase in the wave number $a$ for all values of $\mathrm{Pr}_1 > 0$ and this trend holds true for both stationary as well as oscillatory convections. However, it is important to note that within the realm of oscillatory convection, the system seems to exert a more pronounced stabilising effect when $Pr_1 > 0$, as compared to the context of stationary convection.

In Figure 9, we conducted an integrated analysis that accounts for the collective influences of both stationary and oscillatory convections. We examined the variation of the Rayleigh number ($R_D$) with respect to the wave number ($a$) while keeping the Prandtl number ($Pr_1$) less than zero and maintaining fixed values of $\psi_1$ and $\psi_2$ at 2 and 0.2, respectively. The findings depicted in Figure 9 clearly demonstrate a consistent decrease in $R_D$ as the wave number $a$ increases when $Pr_1 < 0$, a pattern that is evident in both stationary and oscillatory convections. Interestingly, within the context of stationary convection, the system appears to exhibit a more pronounced destabilising effect when $Pr_1 < 0$, in contrast to the behaviour observed in oscillatory convection.

## 8. Conclusions

In this research article, we have analysed the linear stability of throughflow effect on bi-disperse convection in Rivlin-Ericksen fluid. The effect of throughflow is considered as vertically constant in this investigation. This study focuses upon the linear stability of the opted system and here, linear stability analysis is acquired through normal mode analysis. This analysis is carried out for both stationary as well as oscillatory convections. The conclusions are based on the graphical analysis of the opted problem.

We have drawn the following conclusions from the graphs as:

1. The Peclet number $Pr_1$ exerts a stabilizing influence on both stationary and oscillatory convection patterns when $Pr_1 > 0$ and the system demonstrates a more pronounced stabilising effect, particularly in the context of oscillatory convection.
2. The Peclet number $Pr_1$ imparts a destabilizing impact on both stationary and oscillatory convection modes when $Pr_1 < 0$, with a more accentuated destabilising effect being evident, particularly in the case of stationary convection.
3. The influence of the kinematic viscoelastic parameter ($F$) is restricted to oscillatory convection and consistently manifests as a stabilising factor, irrespective of whether $Pr_1$ is less than zero or greater than zero.
4. The study of throughflow effect on bi-disperse convection in Rivlin-Ericksen fluids has broad interdisciplinary implications, ranging from industrial processes to environmental and biomedical applications, offering opportunities for innovation and advancement in multiple fields.

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