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Understanding normal and restricted normal products in soft directed graphs

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Abstract

This paper extends the framework of soft set theory to directed graphs by introducing and analyzing the concepts of the normal product and restricted normal product of soft directed graphs. Building on Molodtsov's foundational work in developing soft set theory to address uncertainty in data, this study presents new methods for modeling and understanding complex systems where uncertainty plays a significant role. Soft directed graphs, which enhance traditional graph models by incorporating parameters and uncertain relationships, serve as the foundation for this investigation. The normal product, defined as a combination of two soft directed graphs based on their respective parameter sets, and the restricted normal product, which combines soft directed graphs only where their parameter sets intersect, provide a comprehensive framework for these new operations. This paper also establishes the structural properties of these products, ensuring they are well-defined and retain the key features of soft directed graphs. Furthermore, we derive combinatorial identities related to vertex and arc counts, as well as degree sums, offering deeper insights into the composition and behavior of these graph products.

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1. Introduction

Graphs are essential tools for modelling relationships between entities in various real-world scenarios. Their power lies in their ability to simplify complex connections into straightforward representations. In numerous applications, graphs are crucial, providing solutions to problems in areas ranging from social networking to transportation logistics and beyond.

A graph is composed of two primary elements: vertices (or nodes) and edges (or links). Directed graphs, a specific type of graph, feature edges with a direction, indicating a one-way relationship between vertices. In practical applications, graphs are found in many diverse contexts. For instance, social media platforms use graphs to depict connections between users, enabling functions like friend recommendations and social network analysis. Navigation systems, such as Google Maps, use graphs to model road networks, helping users find optimal routes between locations. The internet can be represented as a graph, with web pages serving as vertices and hyperlinks as edges. Blockchains,

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the technology behind cryptocurrencies, employ graphs to represent transaction histories and verify transactions. Additionally, graphs are integral to neural networks, representing the connections between artificial neurons.

The concept of soft sets, introduced by Molodtsov [1], extends the traditional set theory to handle uncertainties. Soft set theory provides a mathematical framework for dealing with imprecise or uncertain information, making it useful in solving problems where traditional mathematical tools fall short. This theory has been successfully applied to various practical problems, with researchers like Maji *et al.* [2, 3], Mohammed[4], and Saleh *et al.* [5, 6] further developing it and using it in decision-making scenarios.

Building on the foundation of soft set theory, researchers such as Thumbakara and George [7, 8] introduced the concept of soft graphs. Soft graphs extend the traditional graph model to incorporate uncertainty, enabling the representation and analysis of uncertain relationships between entities. Later modifications by Akram and Nawas [9–12] introduced variations such as fuzzy soft graphs, which further expanded the applicability of soft graphs. Akram and Zafar [13, 14] studied soft trees and fuzzy soft trees. Advancements in the field of soft graphs have been significant. Researchers like Thenge, Jain, and Reddy [15–17] have contributed to the development of soft graphs, particularly focusing on parameterization, which is essential for practical applications. George, Thumbakara, and Jose have further expanded the domain by introducing concepts such as soft hypergraphs [18], soft directed graphs [19, 20], and soft disemigraphs [21], and thoroughly investigating their properties and applications.

The study of soft graphs has also led to the exploration of graph product operations. Product operations allow the combination of two graphs to create a new graph with specific properties. Additionally, researchers like Baghernejad and Borzooei [22] have demonstrated the utility of soft graphs and soft multigraphs in managing complex systems such as urban traffic flows. Further contributions to the field include the introduction of novel concepts such as Eulerian and Hamiltonian soft graphs [23, 24], graph isomorphism [25], and various product operations on soft graphs [26, 27] and soft directed graphs [28–31]. Additionally, researchers have extended these concepts to soft directed graphs and introduced soft semigraphs [32–36] and soft directed hypergraphs [37], applying principles from soft sets to these structures and defining operations and properties associated with them.

The study of soft graphs represents a significant advancement in graph theory, enabling the representation and analysis of uncertain relationships in complex systems. The application of soft set theory to graphs opens up new possibilities for solving practical problems in diverse fields. In this work, the normal product and restricted normal product of soft directed graphs are introduced and studied.

2. Preliminaries

In this preliminary section, we lay the foundation for comprehending soft sets, directed graphs, and soft directed graphs. Also, we provide a brief overview of topics including the dipart and various types of degrees associated with soft directed graphs.

2.1. Directed Graphs

For preliminaries of directed graphs, we refer to Ref. [38]. "A *directed graph* or *digraph* Ψ^* consists of a non-empty finite set ϱ of elements called *vertices* and a finite set δ of ordered pairs of distinct vertices called *arcs*. We often write $\Psi^* = (\varrho, \delta)$ to represent a directed graph. The number of vertices and arcs in a directed graph Ψ^* are called *order* and *size* respectively. The first vertex w of an arc (w, z) is called its *tail* and the second vertex z is called its *head*. If (w, z) is an arc then z is *adjacent from* w and w is *adjacent to* z. A vertex w is *incident* to an arc a if w is the head or tail of a. A directed graph $\Psi^{**} = (\varrho', \delta')$ is called a *subdigraph* of $\Psi^* = (\varrho, \delta)$ if $\varrho' \subseteq \varrho$ and $\delta' \subseteq \delta$. The *in-degree* of a vertex z denoted by *ideg* z is the number of vertices in Ψ^* from which z is adjacent and *out-degree* of z denoted by *odeg* z is the number of vertices in Ψ^* to which z is adjacent. The sum *ideg* z + *odeg* z is called the *degree* of the vertex z and is denoted by *deg* z. In a directed graph $\Psi^* = (\varrho, \delta)$, $\sum_{z\in\varrho} ideg(z) = \sum_{z\in\varrho} odeg(z)$ =Number of arcs in Ψ^* and $\sum_{z\in\varrho} deg(z) = 2$(Number of arcs in Ψ^*)."

Some directed graph products can be defined in a manner that is similar to how the corresponding graph products are defined [39]. "Let $\Psi_1^* = (\varrho_1, \delta_1)$ and $\Psi_2^* = (\varrho_2, \delta_2)$ be two directed graphs. Their *categorical product* $\Psi_1^* \times \Psi_2^*$ is a directed graph with vertex set $\varrho(\Psi_1^* \times \Psi_2^*) = \varrho_1 \times \varrho_2$ and arc set $\delta(\Psi_1^* \times \Psi_2^*)$, where $((z_1, z_1'), (z_2, z_2'))$ is an arc in $\Psi_1^* \times \Psi_2^*$ if and only if (z_1, z_2) is an arc in Ψ_1^* and (z_1', z_2') is an arc in Ψ_2^*. "

2.2. Soft Set

Molodstov [1] defined soft set as follows: "Let R be a set of parameters and U be an initial universe set. Then a pair (F, R) is called a *soft set* (over U) if and only F is a mapping of R into the power set of U. That is, $F : R \to \mathcal{P}(U)$."

2.3. Soft Directed Graphs

Jose et. al. [19, 20] defined soft directed graph as follows: "Let $\Psi^* = (\varrho, \delta)$ be a directed graph having vertex set ϱ and arc set δ and let $\mathfrak{R}$ be a non-empty set. Let a subset R of $\mathfrak{R}\times\varrho$ be an arbitrary relation from $\mathfrak{R}$ to ϱ. Define a mapping $\gamma : \mathfrak{R} \to \mathcal{P}(\varrho)$ by $\gamma(\varepsilon) = \{u \in \varrho | \varepsilon R u\}$ where $\mathcal{P}(\varrho)$ denotes the powerset of ϱ. The pair $(\gamma, \mathfrak{R})$ is a soft set over ϱ. Define another mapping $\alpha : \mathfrak{R} \to \mathcal{P}(\delta)$ by $\alpha(\varepsilon) = \{(w, z) \in \delta | \{w, z\} \subseteq \gamma(\varepsilon)\}$ where $\mathcal{P}(\delta)$ denotes the powerset of δ. The pair $(\alpha, \mathfrak{R})$ is a soft set over the arc set δ. Then $\Psi = (\Psi^*, \gamma, \alpha, \mathfrak{R})$ is called a soft directed graph if it satisfies the following conditions:

1. $\Psi^* = (\varrho, \delta)$ is a directed graph having vertex set ϱ and arc set δ,
2. $\mathfrak{R}$ is a nonempty set of parameters,
3. $(\gamma, \mathfrak{R})$ *is a soft set over the vertex set* ϱ,
4. $(\alpha, \mathfrak{R})$ is a soft set over the arc set δ,
5. $(\gamma(\varepsilon), \alpha(\varepsilon))$ is a subdigraph of Ψ^* for all $\varepsilon \in \mathfrak{R}$.

If we represent $(\gamma(\varepsilon), \alpha(\varepsilon))$ by $M(\varepsilon)$ then the soft directed graph Ψ is also given by $\{M(\varepsilon) : \varepsilon \in \mathfrak{R}\}$. Then $M(\varepsilon)$ corresponding to a parameter ε in $\mathfrak{R}$ is called a *directed part* or simply *dipart* of the soft directed graph Ψ.

Let $\Psi = (\Psi^*, \gamma, \alpha, \mathfrak{R})$ be a soft directed graph and let $M(\varepsilon)$ be a dipart of Ψ for some $\varepsilon \in \mathfrak{R}$. Let z be a vertex of $M(\varepsilon)$. Then dipart indegree of z in $M(\varepsilon)$ denoted by *ideg* $z[M(\varepsilon)]$ is defined as the number of vertices of $M(\varepsilon)$ from which z is adjacent. That is, *ideg* $z[M(\varepsilon)]$ is the number of arcs of $M(\varepsilon)$ that have z as its head. Similarly, dipart outdegree of z in $M(\varepsilon)$ denoted by *odeg* $z[M(\varepsilon)]$ is defined as the number of vertices of $M(\varepsilon)$ to which z is adjacent. That is, *odeg* $z[M(\varepsilon)]$ is the number of arcs of $M(\varepsilon)$ that have z as its tail. The dipart degree of z in $M(\varepsilon)$ is defined as the sum, *ideg* $z[M(\varepsilon)] +$ *odeg* $z[M(\varepsilon)]$ and is denoted by *deg* $z[M(\varepsilon)]$."

3. Normal Product (or Strong Product) of Soft Directed Graphs

In this section, we define and explore the normal product (or strong product) of two soft directed graphs. We begin with a formal definition of the normal product for two soft directed graphs, Ψ_1 and Ψ_2, constructed from their respective directed graphs Ψ_1^* and Ψ_2^*. Following this, Theorem 1 establishes that $\Psi_1 \boxtimes \Psi_2$ itself forms a soft directed graph of the normal product of Ψ_1^* and Ψ_2^*. Theorem 2 quantifies the vertices and arcs within $\Psi_1 \boxtimes \Psi_2$, offering insight into their structural composition. Finally, Theorem 3 addresses the degree sums of the vertices in $\Psi_1 \boxtimes \Psi_2$, providing important combinatorial identities for the in-degrees and out-degrees.

Definition 3.1. Let $\Psi_1^* = (\varrho_1, \delta_1)$ and $\Psi_2^* = (\varrho_2, \delta_2)$ be two directed graphs and $\Psi_1 = (\Psi_1^*, \gamma_1, \alpha_1, \mathfrak{R}_1) = \{M_1(\varepsilon) : \varepsilon \in \mathfrak{R}_1\}$ and $\Psi_2 = (\Psi_2^*, \gamma_2, \alpha_2, \mathfrak{R}_2) = \{M_2(\varepsilon) : \varepsilon \in \mathfrak{R}_2\}$ be two soft directed graphs of Ψ_1^* and Ψ_2^* respectively. Then the *normal product (or strong product)* of Ψ_1 and Ψ_2, which is represented by $\Psi_1 \boxtimes \Psi_2$ is defined as $\Psi_1 \boxtimes \Psi_2 = \{M_1(\varepsilon_1) \boxtimes M_2(\varepsilon_2) : (\varepsilon_1, \varepsilon_2) \in \mathfrak{R}_1 \times \mathfrak{R}_2\}$. Here $M_1(\varepsilon_1) \boxtimes M_2(\varepsilon_2)$ denotes the normal product (or strong product) of the diparts $M_1(\varepsilon_1)$ of Ψ_1 and $M_2(\varepsilon_2)$ of Ψ_2 which is defined as follows: $M_1(\varepsilon_1) \boxtimes M_2(\varepsilon_2)$ is a directed graph with vertex set $\varrho(M_1(\varepsilon_1) \boxtimes M_2(\varepsilon_2)) = \gamma_1(\varepsilon_1) \times \gamma_2(\varepsilon_2)$ and arc set $\delta(M_1(\varepsilon_1) \boxtimes M_2(\varepsilon_2))$, where $((z_1, z_1'), (z_2, z_2'))$ is an arc in $M_1(\varepsilon_1) \boxtimes M_2(\varepsilon_2)$ if and only if

1. $z_1 = z_2$ and (z_1', z_2') is an arc in $M_2(\varepsilon_2)$ or
2. (z_1, z_2) is an arc in $M_1(\varepsilon_1)$ and $z_1' = z_2'$ or
3. (z_1, z_2) is an arc in $M_1(\varepsilon_1)$ and (z_1', z_2') is an arc in $M_2(\varepsilon_2)$.

Example 1. Let $\Psi_1^* = (\varrho_1, \delta_1)$ be a directed graph which is shown in *Figure* 1.
Let $\mathfrak{R}_1 = \{v_1, v_8\} \subseteq \varrho_1$ be a set of parameters. Define a mapping $\gamma_1 : \mathfrak{R}_1 \to \mathcal{P}(\varrho_1)$ by $\gamma_1(\varepsilon) = \{u \in \varrho_1 \mid u = \varepsilon$ or u is adjacent from $\varepsilon\}, \forall \varepsilon \in \mathfrak{R}_1$. That is, $\gamma_1(v_1) = \{v_1, v_2, v_3\}$ and $\gamma_1(v_8) = \{v_3, v_6, v_8\}$. Here $(\gamma_1, \mathfrak{R}_1)$ is a soft set over ϱ_1. Define another mapping $\alpha_1 : \mathfrak{R}_1 \to \mathcal{P}(\delta_1)$ by $\alpha_1(\varepsilon) = \{(w, z) \in \delta_1 \mid \{w, z\} \subseteq \gamma_1(\varepsilon)\}, \forall \varepsilon \in \mathfrak{R}_1$. That is, $\alpha_1(v_1) = \{(v_1, v_2), (v_1, v_3), (v_2, v_3)\}$ and $\alpha_1(v_8) = \{(v_8, v_3), (v_8, v_6)\}$. Here, $(\alpha_1, \mathfrak{R}_1)$ is a soft set over δ_1.

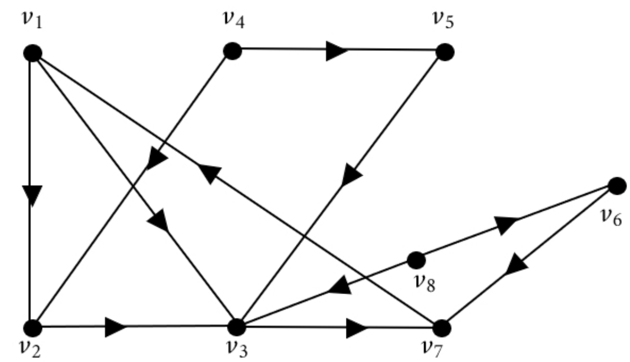

Figure 1. Directed Graph $\Psi_1^* = (\varrho_1, \delta_1)$

Then $M_1(v_1) = (\gamma_1(v_1), \alpha_1(v_1))$ and $M_1(v_8) = (\gamma_1(v_8), \alpha_1(v_8))$ are subgraphs of Ψ_1^* as shown in *Figure* 2. Therefore $\Psi_1 = \{M_1(v_1), M_1(v_8)\}$ is a soft directed graph of Ψ_1^*.

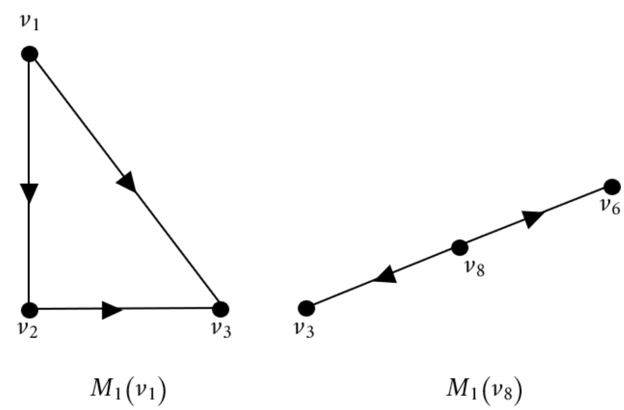

Figure 2. Soft Directed Graph $\Psi_1 = \{M_1(v_1), M_1(v_8)\}$

Let $\Psi_2^* = (\varrho_2, \delta_2)$ be a directed graph which is shown in *Figure* 3. Consider the parameter set $\mathfrak{R}_2 = \{u_2\} \subseteq \varrho_2$. Define a mapping $\gamma_2 : \mathfrak{R}_2 \to \mathcal{P}(\varrho_2)$ by $\gamma_2(\varepsilon) = \{u \in \varrho_2 \mid u = \varepsilon$ or u is adjacent from $\varepsilon\}, \forall \varepsilon \in \mathfrak{R}_2$. That is, $\gamma_2(u_2) = \{u_2, u_3\}$. Here, $(\gamma_2, \mathfrak{R}_2)$ is a soft set over ϱ_2. Define another mapping $\alpha_2 : \mathfrak{R}_2 \to \mathcal{P}(\delta_2)$ by $\alpha_2(\varepsilon) = \{(w, z) \in \delta_2 \mid \{w, z\} \subseteq \gamma_2(\varepsilon)\}, \forall \varepsilon \in \mathfrak{R}_2$. That is, $\alpha_2(u_2) = \{(u_2, u_3)\}$. Here, $(\alpha_2, \mathfrak{R}_2)$ is a soft set over δ_2. Then, $M_2(u_2) = (\gamma_2(u_2), \alpha_2(u_2))$ is a subdigraph of Ψ_2^* as shown in *Figure* 4. Therefore, $\Psi_2 = \{M_2(u_2)\}$ is a soft directed graph of Ψ_2^*.
Then the normal product of these two soft directed graphs Ψ_1 and Ψ_2 is given by $\Psi = \Psi_1 \boxtimes \Psi_2 = \{M_1(v_1) \boxtimes M_2(u_2), M_1(v_8) \boxtimes M_2(u_2)\}$ and is shown in *Figure* 5.

Theorem 3.1. Let $\Psi_1^* = (\varrho_1, \delta_1)$ and $\Psi_2^* = (\varrho_2, \delta_2)$ be two directed graphs and Ψ_1 and Ψ_2 be two soft directed graphs of Ψ_1 and Ψ_2 respectively. Then the normal product $\Psi_1 \boxtimes \Psi_2$ is a soft directed graph of $\Psi_1^* \boxtimes \Psi_2^*$.

Proof. Let $\Psi_1 = (\Psi_1^*, \gamma_1, \alpha_1, \mathfrak{R}_1) = \{M_1(\varepsilon) : \varepsilon \in \mathfrak{R}_1\}$ be a soft directed graph of $\Psi_1^* = (\varrho_1, \delta_1)$ and $\Psi_2 = (\Psi_2^*, \gamma_2, \alpha_2, \mathfrak{R}_2) =$

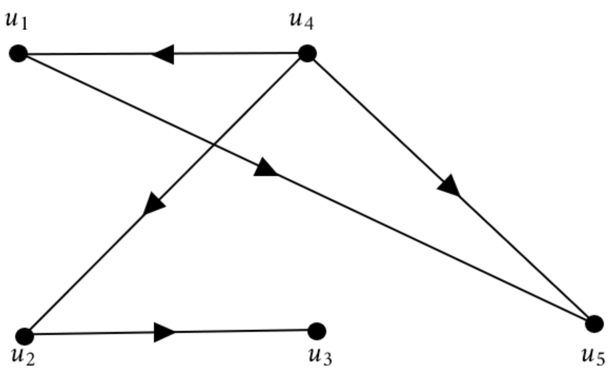

Figure 3. Directed Graph $\Psi_2^* = (\varrho_2, \delta_2)$

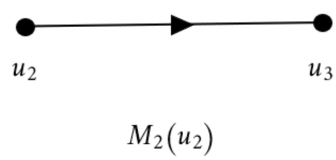

Figure 4. Soft Directed Graph $\Psi_2 = \{M_2(u_2)\}$

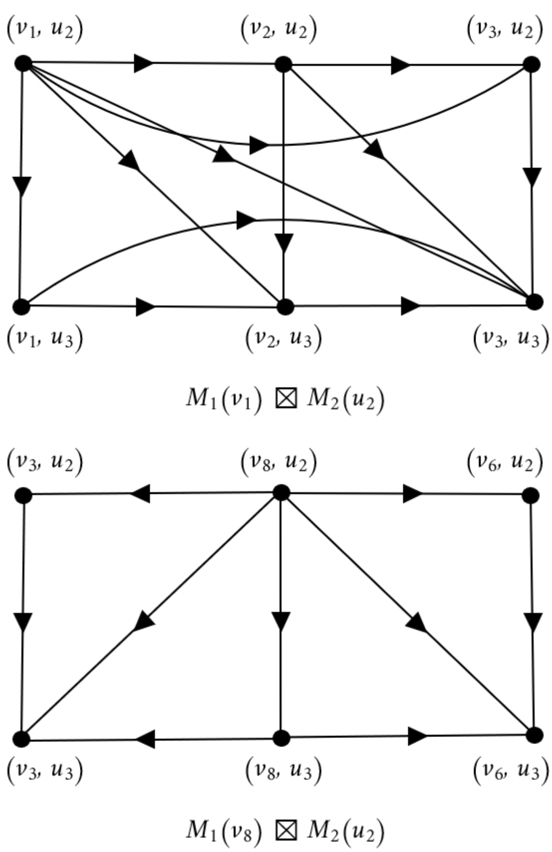

Figure 5. $\Psi = \Psi_1 \boxtimes \Psi_2 = \{M_1(v_1) \boxtimes M_2(u_2), M_1(v_8) \boxtimes M_2(u_2)\}$

$\{M_2(\varepsilon) : \varepsilon \in \mathfrak{R}_2\}$ be a soft directed graph of $\Psi_2^* = (\varrho_2, \delta_2)$. Then the normal product $\Psi_1 \boxtimes \Psi_2$ is defined as $\Psi_1 \boxtimes \Psi_2 = \{M_1(\varepsilon_1) \boxtimes M_2(\varepsilon_2) : (\varepsilon_1, \varepsilon_2) \in \mathfrak{R}_1 \times \mathfrak{R}_2\}$. Here $M_1(\varepsilon_1) \boxtimes M_2(\varepsilon_2)$ denotes the normal product of the diparts $M_1(\varepsilon_1)$ of Ψ_1 and $M_2(\varepsilon_2)$ of Ψ_2, which is defined as follows: $M_1(\varepsilon_1) \boxtimes M_2(\varepsilon_2)$ is a directed graph with vertex set $\varrho(M_1(\varepsilon_1) \boxtimes M_2(\varepsilon_2)) = \gamma_1(\varepsilon_1) \times \gamma_2(\varepsilon_2)$ and arc set $\delta(M_1(\varepsilon_1) \boxtimes M_2(\varepsilon_2))$, where $((z_1, z_1'), (z_2, z_2'))$ is an arc in $M_1(\varepsilon_1) \boxtimes M_2(\varepsilon_2)$ if and only if

1. $z_1 = z_2$ and (z_1', z_2') is an arc in $M_2(\varepsilon_2)$ or
2. (z_1, z_2) is an arc in $M_1(\varepsilon_1)$ and $z_1' = z_2'$ or
3. (z_1, z_2) is an arc in $M_1(\varepsilon_1)$ and (z_1', z_2') is an arc in $M_2(\varepsilon_2)$.

The normal product $\Psi_1^* \boxtimes \Psi_2^*$ of the two directed graphs Ψ_1^* and Ψ_2^* is a directed graph with vertex set $\varrho(\Psi_1^* \boxtimes \Psi_2^*) = \varrho_1 \times \varrho_2$ and arc set $\delta(\Psi_1^* \boxtimes \Psi_2^*)$ where $((z_1, z_1'), (z_2, z_2'))$ is an arc in $\Psi_1^* \boxtimes \Psi_2^*$ if and only if

1. $z_1 = z_2$ and (z_1', z_2') is an arc in Ψ_2^* or
2. (z_1, z_2) is an arc in Ψ_1^* and $z_1' = z_2'$ or
3. (z_1, z_2) is an arc in Ψ_1^* and (z_1', z_2') is an arc in Ψ_2^*.

Let the parameter set be $\mathfrak{R}_{\Psi_1 \boxtimes \Psi_2} = \mathfrak{R}_1 \times \mathfrak{R}_2$. Define a mapping $\gamma_{\Psi_1 \boxtimes \Psi_2}$ from $\mathfrak{R}_{\Psi_1 \boxtimes \Psi_2}$ to $\mathcal{P}[\varrho(\Psi_1^* \boxtimes \Psi_2^*)]$ by $\gamma_{\Psi_1 \boxtimes \Psi_2}(\varepsilon_1, \varepsilon_2) = \gamma_1(\varepsilon_1) \times \gamma_2(\varepsilon_2), \forall (\varepsilon_1, \varepsilon_2) \in \mathfrak{R}_1 \times \mathfrak{R}_2$ where $\mathcal{P}[\varrho(\Psi_1^* \boxtimes \Psi_2^*)]$ denotes the power set of $\varrho(\Psi_1^* \boxtimes \Psi_2^*)$. Then $(\gamma_{\Psi_1 \boxtimes \Psi_2}, \mathfrak{R}_{\Psi_1 \boxtimes \Psi_2})$ is a soft set over $\varrho(\Psi_1^* \boxtimes \Psi_2^*)$. Define another mapping $\alpha_{\Psi_1 \boxtimes \Psi_2}$ from $\mathfrak{R}_{\Psi_1 \boxtimes \Psi_2}$ to $\mathcal{P}[\delta(\Psi_1^* \boxtimes \Psi_2^*)]$ by $\alpha_{\Psi_1 \boxtimes \Psi_2}(\varepsilon_1, \varepsilon_2) = \{((w, z), (c, d)) \in \delta(\Psi_1^* \boxtimes \Psi_2^*) \mid \{(w, z), (c, d)\} \in \gamma_{\Psi_1 \boxtimes \Psi_2}(\varepsilon_1, \varepsilon_2)\}, \forall (\varepsilon_1, \varepsilon_2) \in \mathfrak{R}_1 \times \mathfrak{R}_2$, where $\mathcal{P}[\delta(\Psi_1^* \boxtimes \Psi_2^*)]$ denotes the power set of $\delta(\Psi_1^* \boxtimes \Psi_2^*)$. Then $(\alpha_{\Psi_1 \boxtimes \Psi_2}, \mathfrak{R}_{\Psi_1 \boxtimes \Psi_2})$ is a soft set over $\delta(\Psi_1^* \boxtimes \Psi_2^*)$. Also if we denote $(\gamma_{\Psi_1 \boxtimes \Psi_2}(\varepsilon_1, \varepsilon_2), \alpha_{\Psi_1 \boxtimes \Psi_2}(\varepsilon_1, \varepsilon_2))$ by $M_{\Psi_1 \boxtimes \Psi_2}(\varepsilon_1, \varepsilon_2)$, then $M_{\Psi_1 \boxtimes \Psi_2}(\varepsilon_1, \varepsilon_2)$ is a subdigraph of $\Psi_1^* \boxtimes \Psi_2^*, \forall (\varepsilon_1, \varepsilon_2) \in \mathfrak{R}_1 \times \mathfrak{R}_2$, since $\gamma_1(\varepsilon_1) \times \gamma_2(\varepsilon_2) \subseteq \varrho_1 \times \varrho_2$ and any arc in $\alpha_{\Psi_1 \boxtimes \Psi_2}(\varepsilon_1, \varepsilon_2)$ is also an arc in $\delta(\Psi_1^* \boxtimes \Psi_2^*)$. Then $\Psi_1 \boxtimes \Psi_2$ can be represented by the 4-tuple $(\Psi_1^* \boxtimes \Psi_2^*, \gamma_{\Psi_1 \boxtimes \Psi_2}, \alpha_{\Psi_1 \boxtimes \Psi_2}, \mathfrak{R}_{\Psi_1 \boxtimes \Psi_2})$ and also by $\{M_{\Psi_1 \boxtimes \Psi_2}(\varepsilon_1, \varepsilon_2) : (\varepsilon_1, \varepsilon_2) \in \mathfrak{R}_1 \times \mathfrak{R}_2\}$ and $\Psi_1 \boxtimes \Psi_2$ is a soft directed graph of $\Psi_1^* \boxtimes \Psi_2^*$ since the following conditions are satisfied:

1. $\Psi_1^* \boxtimes \Psi_2^* = (\varrho(\Psi_1^* \boxtimes \Psi_2^*), \delta(\Psi_1^* \boxtimes \Psi_2^*))$ is a directed graph having vertex set $\varrho(\Psi_1^* \boxtimes \Psi_2^*)$ and arc set $\delta(\Psi_1^* \boxtimes \Psi_2^*)$,
2. $\mathfrak{R}_{\Psi_1 \boxtimes \Psi_2} = \mathfrak{R}_1 \times \mathfrak{R}_2$ is the set of parameters which is nonempty,
3. $(\gamma_{\Psi_1 \boxtimes \Psi_2}, \mathfrak{R}_{\Psi_1 \boxtimes \Psi_2})$ is a soft set over $\varrho(\Psi_1^* \boxtimes \Psi_2^*)$,
4. $(\alpha_{\Psi_1 \boxtimes \Psi_2}, \mathfrak{R}_{\Psi_1 \boxtimes \Psi_2})$ is a soft set over $\delta(\Psi_1^* \boxtimes \Psi_2^*)$,
5. $M_{\Psi_1 \boxtimes \Psi_2}(\varepsilon_1, \varepsilon_2) = (\gamma_{\Psi_1 \boxtimes \Psi_2}(\varepsilon_1, \varepsilon_2), \alpha_{\Psi_1 \boxtimes \Psi_2}(\varepsilon_1, \varepsilon_2))$ is a subdigraph of $\Psi_1^* \boxtimes \Psi_2^*, \forall (\varepsilon_1, \varepsilon_2) \in \mathfrak{R}_{\Psi_1 \boxtimes \Psi_2} = \mathfrak{R}_1 \times \mathfrak{R}_2$.

□

Theorem 3.2. The normal product $\Psi_1 \boxtimes \Psi_2$ contains $\sum_{(\varepsilon_i, \varepsilon_j) \in \mathfrak{R}_1 \times \mathfrak{R}_2} |\gamma_1(\varepsilon_i)||\gamma_2(\varepsilon_j)|$ vertices and $\sum_{(\varepsilon_i, \varepsilon_j) \in \mathfrak{R}_1 \times \mathfrak{R}_2} (|\gamma_1(\varepsilon_i)||\alpha_2(\varepsilon_j)| + |\gamma_2(\varepsilon_j)||\alpha_1(\varepsilon_i)| + |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|)$ arcs, if we count the vertices and arcs as many times they appear in different diparts of $\Psi_1 \boxtimes \Psi_2$.

Proof. By definition, $\Psi_1 \boxtimes \Psi_2 = \{M_1(\varepsilon_1) \boxtimes M_2(\varepsilon_2) : (\varepsilon_1, \varepsilon_2) \in \mathfrak{R}_1 \times \mathfrak{R}_2\}$. The parameter set of $\Psi_1 \boxtimes \Psi_2$ is $\mathfrak{R}_1 \times \mathfrak{R}_2$. Consider the dipart $M_1(\varepsilon_i) \boxtimes M_2(\varepsilon_j)$ of $\Psi_1 \boxtimes \Psi_2$ corresponding to the parameter $(\varepsilon_i, \varepsilon_j) \in \Re_1 \times \Re_2$. The vertex set of $M_1(\varepsilon_i) \boxtimes M_2(\varepsilon_j)$ is $\gamma_1(\varepsilon_i) \times \gamma_2(\varepsilon_j)$ which contains $|\gamma_1(\varepsilon_i)||\gamma_2(\varepsilon_j)|$ elements. This is true for all diparts of $\Psi_1 \boxtimes \Psi_2$. Therefore total number of vertices in $\Psi_1 \boxtimes \Psi_2$ is $\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} |\gamma_1(\varepsilon_i)||\gamma_2(\varepsilon_j)|$, if we count the vertices as many times they appear in different diparts of $\Psi_1 \boxtimes \Psi_2$.

Also we know, $((z_q, z_r), (z_s, z_t))$ is an arc in $M_1(\varepsilon_i) \boxtimes M_2(\varepsilon_j)$ if and only

1. $z_q = z_s$ and (z_r, z_t) is an arc in $M_2(\varepsilon_j)$ or
2. (z_q, z_s) is an arc in $M_1(\varepsilon_i)$ and $z_r = z_t$ or
3. (z_q, z_s) is an arc in $M_1(\varepsilon_i)$ and (z_r, z_t) is an arc in $M_2(\varepsilon_j)$.

Now, each arc in $M_1(\varepsilon_i) \boxtimes M_2(\varepsilon_j)$ was made by just one of these three requirements (any two can't be true at the same time). So to get the total number of arcs in $M_1(\varepsilon_i) \boxtimes M_2(\varepsilon_j)$, we add the number of arcs generated by each condition. Consider the first condition for adjacency, i.e.,$z_q = z_s$ and (z_r, z_t) is an arc in $M_2(\varepsilon_j)$. The number of arcs generated by this condition will be the number of arcs in the directed graph $M_2(\varepsilon_j)$ times the number of vertices in the directed graph $M_1(\varepsilon_i)$, which is given by $|\gamma_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|$. Then consider the second condition for adjacency, i.e.,(z_q, z_s) is an arc in $M_1(\varepsilon_i)$ and $z_r = z_t$. The number of arcs generated by this condition will be the number of arcs in the directed graph $M_1(\varepsilon_i)$ times the number of vertices in the directed graph $M_2(\varepsilon_j)$, which is given by $|\gamma_2(\varepsilon_j)||\alpha_1(\varepsilon_i)|$. Finally consider the third condition for adjacency, i.e., (z_q, z_s) is an arc in $M_1(\varepsilon_i)$ and (z_r, z_t) is an arc in $M_2(\varepsilon_j)$. There are $|\alpha_1(\varepsilon_i)|$ arcs in $M_1(\varepsilon_i)$ and $|\alpha_2(\varepsilon_j)|$ arcs in $M_2(\varepsilon_j)$. So we can choose a pair of arcs a_k and a_l such that one is from $M_1(\varepsilon_i)$ and the other is from $M_2(\varepsilon_j)$ in $|\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|$ different ways. Suppose that a_k is the arc (z_q, z_s) in $M_1(\varepsilon_i)$ and a_l is the arc (z_r, z_t) in $M_2(\varepsilon_j)$. Then this pair of arcs gives an arc $((z_q, z_r), (z_s, z_t))$ in $M_1(\varepsilon_i) \boxtimes M_2(\varepsilon_j)$. Hence the number of arcs generated by this condition will be $|\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|$. That is, the total number of arcs in $M_1(\varepsilon_i) \boxtimes M_2(\varepsilon_j) = (|\gamma_1(\varepsilon_i)||\alpha_2(\varepsilon_j)| + |\gamma_2(\varepsilon_j)||\alpha_1(\varepsilon_i)| + |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|)$. This is true for all diparts of $\Psi_1 \boxtimes \Psi_2$. Therefore total number of arcs in $\Psi_1 \boxtimes \Psi_2$ is $\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}(|\gamma_1(\varepsilon_i)||\alpha_2(\varepsilon_j)| + |\gamma_2(\varepsilon_j)||\alpha_1(\varepsilon_i)| + |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|)$, if we count the arcs as many times they appear in different diparts of $\Psi_1 \boxtimes \Psi_2$. □

Example 2. Consider the directed graphs given in Example 1. Here we have, total number of vertices in $\Psi_1 \boxtimes \Psi_2 = 12$ and $\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} |\gamma_1(\varepsilon_i)||\gamma_2(\varepsilon_j)| = (3.2) + (3.2) = 12$. That is, the total number of vertices in $\Psi_1 \boxtimes \Psi_2 = \sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} |\gamma_1(\varepsilon_i)||\gamma_2(\varepsilon_j)|$. Also total number of arcs in $\Psi_1 \boxtimes \Psi_2 = 21$ and $\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}(|\gamma_1(\varepsilon_i)||\alpha_2(\varepsilon_j)| + |\gamma_2(\varepsilon_j)||\alpha_1(\varepsilon_i)| + |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|) = (3.1+2.3+3.1)+(3.1+2.2+2.1) = 21$. That is, total number of arcs in $\Psi_1 \boxtimes \Psi_2 = \sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}(|\gamma_1(\varepsilon_i)||\alpha_2(\varepsilon_j)| + |\gamma_2(\varepsilon_j)||\alpha_1(\varepsilon_i)| + |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|)$.

Theorem 3.3. Let $\Psi_1^* = (\varrho_1, \delta_1)$ and $\Psi_2^* = (\varrho_2, \delta_2)$ be two directed graphs and $\Psi_1 = (\Psi_1^*, \gamma_1, \alpha_1, \Re_1)$ and $\Psi_2 = (\Psi_2^*, \gamma_2, \alpha_2, \Re_2)$ be two soft directed graphs of Ψ_1^* and Ψ_2^* respectively. Then

$$(i) \sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} \sum_{(w,z)\in\gamma_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)} ideg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)] = \sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} \sum_{(w,z)\in\gamma_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)} odeg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)] =$$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} (|\gamma_1(\varepsilon_i)||\alpha_2(\varepsilon_j)| + |\gamma_2(\varepsilon_j)||\alpha_1(\varepsilon_i)| + |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|)$$

$$(ii) \sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} \sum_{(w,z)\in\gamma_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)} deg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)] =$$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} 2(|\gamma_1(\varepsilon_i)||\alpha_2(\varepsilon_j)| + |\gamma_2(\varepsilon_j)||\alpha_1(\varepsilon_i)| + |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|).$$

Proof. (i) Consider any dipart $M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i, \varepsilon_j) = (\gamma_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i, \varepsilon_j), \alpha_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i, \varepsilon_j))$ of $\Psi_1 \boxtimes \Psi_2$ which is given by $M_1(\varepsilon_i) \times M_2(\varepsilon_j)$. By Theorem 3.2, we have number of arcs in $M_1(\varepsilon_i) \times M_2(\varepsilon_j)$ is $(|\gamma_1(\varepsilon_i)||\alpha_2(\varepsilon_j)| + |\gamma_2(\varepsilon_j)||\alpha_1(\varepsilon_i)| + |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|)$. Since the dipart $M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i, \varepsilon_j)$ is a directed graph having $(|\gamma_1(\varepsilon_i)||\alpha_2(\varepsilon_j)| + |\gamma_2(\varepsilon_j)||\alpha_1(\varepsilon_i)| + |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|)$ arcs, we have

$$\sum_{(w,z)\in\gamma_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)} ideg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)] = \sum_{(w,z)\in\gamma_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)} odeg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)] =$$

$$(|\gamma_1(\varepsilon_i)||\alpha_2(\varepsilon_j)| + |\gamma_2(\varepsilon_j)||\alpha_1(\varepsilon_i)| + |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|),$$

since each arc in $M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i, \varepsilon_j)$ contributes 1 each to the sum $\sum_{(w,z)\in\gamma_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)} ideg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)]$ and to the sum $\sum_{(w,z)\in\gamma_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)} odeg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)]$. This is true for all the diparts $M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i, \varepsilon_j)$ of $\Psi_1 \boxtimes \Psi_2$. Hence,

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} \sum_{(w,z)\in\gamma_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)} ideg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)] = \sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} \sum_{(w,z)\in\gamma_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)} odeg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)] =$$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} (|\gamma_1(\varepsilon_i)||\alpha_2(\varepsilon_j)| + |\gamma_2(\varepsilon_j)||\alpha_1(\varepsilon_i)| + |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|).$$

(ii) Since $deg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i, \varepsilon_j)] = ideg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i, \varepsilon_j)] + odeg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i, \varepsilon_j)]$ and by part (i) of this theorem we have,

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} \sum_{(w,z)\in\gamma_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)} deg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)] =$$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} 2(|\gamma_1(\varepsilon_i)||\alpha_2(\varepsilon_j)| + |\gamma_2(\varepsilon_j)||\alpha_1(\varepsilon_i)| + |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|).$$

□

Example 3. Consider the directed graphs given in Example 1. Here we have,

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} \sum_{(w,z)\in\gamma_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)} ideg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)] = 21,$$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} \sum_{(w,z)\in\gamma_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)} odeg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)] = 21,$$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}(|\gamma_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|+|\gamma_2(\varepsilon_j)||\alpha_1(\varepsilon_i)|+|\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|)=$$

$$(3.1+2.3+3.1)+(3.1+2.2+2.1)=21.$$

That is,

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}\ \sum_{(w,z)\in\gamma_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)} ideg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)]=$$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}\ \sum_{(w,z)\in\gamma_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)} odeg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)]=$$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}(|\gamma_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|+|\gamma_2(\varepsilon_j)||\alpha_1(\varepsilon_i)|+|\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|).$$

Also,

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}\ \sum_{(w,z)\in\gamma_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)} deg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)]$$

$$=24+18=42,$$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}2(|\gamma_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|+|\gamma_2(\varepsilon_j)||\alpha_1(\varepsilon_i)|+|\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|)=$$

$$2(3.1+2.3+3.1)+2(3.1+2.2+2.1)=42.$$

That is,

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}\ \sum_{(w,z)\in\gamma_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)} deg(w,z)[M_{\Psi_1\boxtimes\Psi_2}(\varepsilon_i,\varepsilon_j)]=$$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}2(|\gamma_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|+|\gamma_2(\varepsilon_j)||\alpha_1(\varepsilon_i)|+|\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|).$$

4. Restricted Normal Product (or Restricted Strong Product) of Soft Directed Graphs

This section delves into the concept of the restricted normal product (or restricted strong product) of two soft directed graphs, extending the theoretical framework of directed graphs in soft set theory. We start by defining the restricted normal product $\Psi_1 \diamond \Psi_2$, constructed from two soft directed graphs Ψ_1 and Ψ_2 of a base directed graph Ψ^*, ensuring that their parameter sets intersect. Following this, Theorem 4 asserts that the restricted normal product $\Psi_1 \diamond \Psi_2$ is itself a soft directed graph of the normal product of the base graph Ψ^*. Theorem 5 quantifies the vertices and arcs in $\Psi_1 \diamond \Psi_2$, providing a clear combinatorial structure. Finally, Theorem 6 examines the degree sums of the vertices within $\Psi_1 \diamond \Psi_2$.

Definition 4.1. Let $\Psi^* = (\varrho, \delta)$ be a directed graph and $\Psi_1 = (\Psi^*, \gamma_1, \alpha_1, \Re_1) = \{M_1(\varepsilon) : \varepsilon \in \Re_1\}$ and $\Psi_2 = (\Psi^*, \gamma_2, \alpha_2, \Re_2) = \{M_2(\varepsilon) : \varepsilon \in \Re_2\}$ be two soft directed graphs of Ψ^* such that $\Re_1 \cap \Re_2 \neq \phi$. Then the *restricted normal product (or restricted strong product)* of Ψ_1 and Ψ_2, which is represented by $\Psi_1 \diamond \Psi_2$ is defined as $\Psi_1 \diamond \Psi_2 = \{M_1(\varepsilon) \boxtimes M_2(\varepsilon) : \varepsilon \in \Re_1 \cap \Re_2\}$. Here $M_1(\varepsilon) \boxtimes M_2(\varepsilon)$ denotes the normal product of the diparts $M_1(\varepsilon)$ of Ψ_1 and $M_2(\varepsilon)$ of Ψ_2 which is defined as follows: $M_1(\varepsilon) \boxtimes M_2(\varepsilon)$ is a directed graph with vertex set $\varrho(M_1(\varepsilon) \boxtimes M_2(\varepsilon)) = \gamma_1(\varepsilon) \times \gamma_2(\varepsilon)$ and arc set $\delta(M_1(\varepsilon) \boxtimes M_2(\varepsilon))$, where $((z_1, z_1'), (z_2, z_2'))$ is an arc in $M_1(\varepsilon) \boxtimes M_2(\varepsilon)$ if and only if

1. $z_1 = z_2$ and (z_1', z_2') is an arc in $M_2(\varepsilon)$ or
2. (z_1, z_2) is an arc in $M_1(\varepsilon)$ and $z_1' = z_2'$ or
3. (z_1, z_2) is an arc in $M_1(\varepsilon)$ and (z_1', z_2') is an arc in $M_2(\varepsilon)$.

Example 4. Let $\Psi^* = (\varrho, \delta)$ be a directed graph which is shown in *Figure* 6. Let $\Re_1 = \{v_1, v_6\} \subseteq \varrho$ be a set of parameters. Define a mapping $\gamma_1 : \Re_1 \to \mathcal{P}(\varrho)$ by $\gamma_1(\varepsilon) = \{u \in \varrho \mid u = \varepsilon$ or u is adjacent from x or u is adjacent to $\varepsilon\}, \forall \varepsilon \in \Re_1$. That is, $\gamma_1(v_1) = \{v_1, v_2, v_3, v_5\}$ and $\gamma_1(v_6) = \{v_4, v_6, v_7, v_8, v_{10}\}$. Here $(\gamma_1, \Re_1)$ is a soft set over ϱ. Define another mapping $\alpha_1 : \Re_1 \to \mathcal{P}(\delta)$ by $\alpha_1(\varepsilon) = \{(w, z) \in \delta \mid \{w, z\} \subseteq \gamma_1(\varepsilon)\}, \forall \varepsilon \in \Re_1$. That is, $\alpha_1(v_1) = \{(v_1, v_2), (v_3, v_1), (v_5, v_1)\}$ and $\alpha_1(v_6) = \{(v_6, v_{10}), (v_6, v_4), (v_6, v_7), (v_8, v_6), (v_7, v_8)\}$. Here, $(\alpha_1, \Re_1)$ is a soft set over δ. Then $M_1(v_1) = (\gamma_1(v_1), \alpha_1(v_1))$ and $M_1(v_6) = (\gamma_1(v_6), \alpha_1(v_6))$ are subdigraphs of Ψ^* as shown in *Figure* 7. Therefore $\Psi_1 = \{M_1(v_1), M_1(v_6)\}$ is a soft directed graph of Ψ^*.

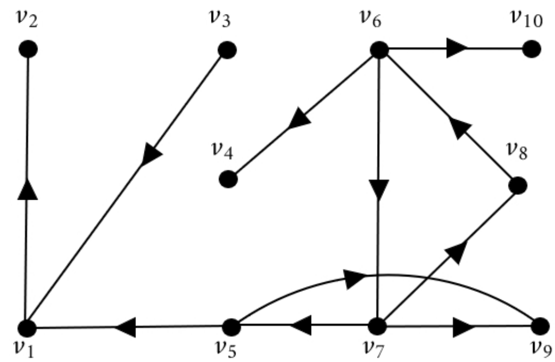

Figure 6. Directed Graph $\Psi^* = (\varrho, \delta)$

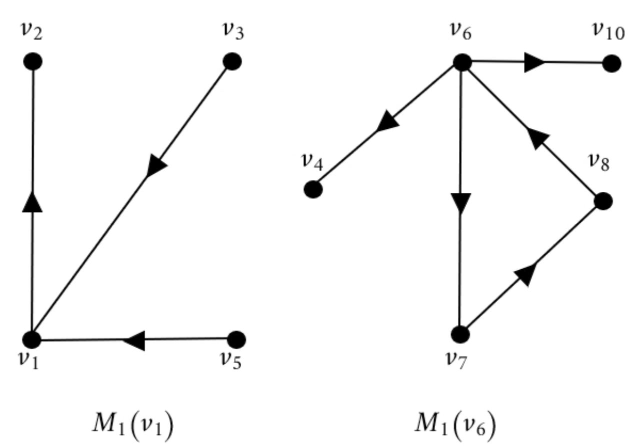

Figure 7. Soft Directed Graph $\Psi_1 = \{M_1(v_1), M_1(v_6)\}$

Consider another parameter set $\Re_2 = \{v_3, v_6\} \subseteq \varrho$. Define a mapping $\gamma_2 : \Re_2 \to \mathcal{P}(\varrho)$ by $\gamma_2(\varepsilon) = \{u \in \varrho \mid u = \varepsilon$ or u is adjacent from $\varepsilon\}, \forall \varepsilon \in \Re_2$. That is, $\gamma_2(v_3) = \{v_1, v_3\}$ and $\gamma_2(v_6) = \{v_4, v_6, v_7, v_{10}\}$. Here, $(\gamma_2, \Re_2)$ is a soft set over ϱ. Define another mapping $\alpha_2 : \Re_2 \to \mathcal{P}(\delta)$ by $\alpha_2(\varepsilon) = \{(w, z) \in \delta \mid \{w, z\} \subseteq \gamma_2(\varepsilon)\}, \forall \varepsilon \in \Re_2$. That is, $\alpha_2(v_3) = \{(v_3, v_1)\}$

and $\alpha_2(v_6) = \{(v_6, v_{10}), (v_6, v_4), (v_6, v_7)\}$. Here, $(\alpha_2, \mathfrak{R}_2)$ is a soft set over δ. Then, $M_2(v_3) = (\gamma_2(v_3), \alpha_2(v_3))$ and $M_2(v_6) = (\gamma_2(v_6), \alpha_2(v_6))$ are subdigraphs of Ψ^* as shown in *Figure* 8. Therefore, $\Psi_2 = \{M_2(v_3), M_2(v_6)\}$ is a soft directed graph of Ψ^*.

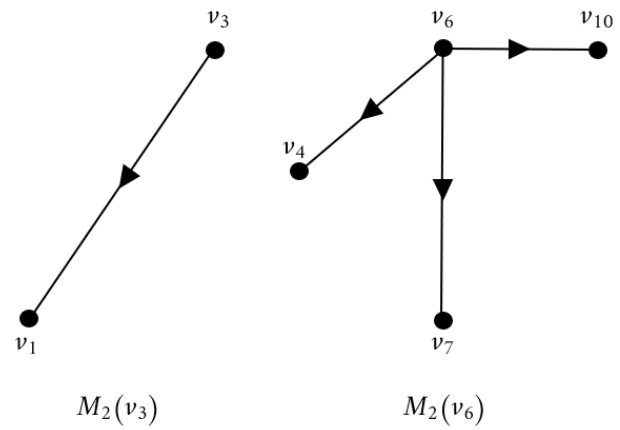

Figure 8. Soft Directed Graph $\Psi_2 = \{M_2(v_3), M_2(v_6)\}$

Then the restricted normal product of these two soft directed graphs Ψ_1 and Ψ_2 is given by $\Psi = \Psi_1 \diamond \Psi_2 = \{M_1(v_6) \boxtimes M_2(v_6)\}$ and is shown in *Figure* 9.

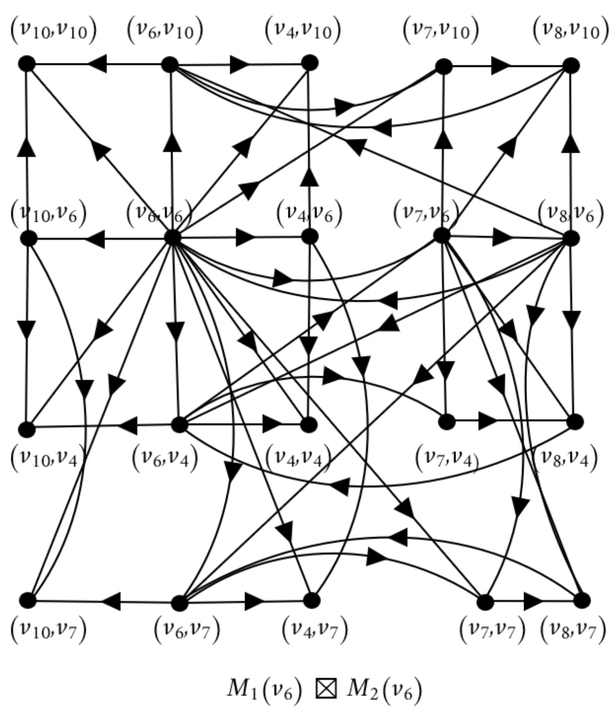

Figure 9. $\Psi = \Psi_1 \diamond \Psi_2 = \{M_1(v_6) \boxtimes M_2(v_6)\}$

Theorem 4.1. Let $\Psi^* = (\varrho, \delta)$ be a directed graph and $\Psi_1 = (\Psi^*, \gamma_1, \alpha_1, \mathfrak{R}_1) = \{M_1(\varepsilon) : \varepsilon \in \mathfrak{R}_1\}$ and $\Psi_2 = (\Psi^*, \gamma_2, \alpha_2, \mathfrak{R}_2) = \{M_2(\varepsilon) : \varepsilon \in \mathfrak{R}_2\}$ be two soft directed graphs of Ψ^* such that $\mathfrak{R}_1 \cap \mathfrak{R}_2 \neq \phi$. Then the restricted normal product $\Psi_1 \diamond \Psi_2$ is a soft directed graph of $\Psi^* \boxtimes \Psi^*$.

Proof. Let $\Psi_1 = (\Psi^*, \gamma_1, \alpha_1, \mathfrak{R}_1) = \{M_1(\varepsilon) : \varepsilon \in \mathfrak{R}_1\}$ and $\Psi_2 = (\Psi^*, \gamma_2, \alpha_2, \mathfrak{R}_2) = \{M_2(\varepsilon) : \varepsilon \in \mathfrak{R}_2\}$ be soft directed graphs of $\Psi^* = (\varrho, \delta)$ such that $\mathfrak{R}_1 \cap \mathfrak{R}_2 \neq \phi$. Then the restricted normal product $\Psi_1 \diamond \Psi_2$ is defined as $\Psi_1 \diamond \Psi_2 = \{M_1(\varepsilon) \boxtimes M_2(\varepsilon) : \varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2\}$. Here $M_1(\varepsilon) \boxtimes M_2(\varepsilon)$ denotes the normal product of the diparts $M_1(\varepsilon)$ of Ψ_1 and $M_2(\varepsilon)$ of Ψ_2 which is defined as follows: $M_1(\varepsilon) \boxtimes M_2(\varepsilon)$ is a directed graph with vertex set $\varrho(M_1(\varepsilon) \boxtimes M_2(\varepsilon)) = \gamma_1(\varepsilon) \times \gamma_2(\varepsilon)$ and arc set $\delta(M_1(\varepsilon) \boxtimes M_2(\varepsilon))$, where $((z_1, z_1'), (z_2, z_2'))$ is an arc in $M_1(\varepsilon) \boxtimes M_2(\varepsilon)$ if and only if

1. $z_1 = z_2$ and (z_1', z_2') is an arc in $M_2(\varepsilon)$ or
2. (z_1, z_2) is an arc in $M_1(\varepsilon)$ and $z_1' = z_2'$ or
3. (z_1, z_2) is an arc in $M_1(\varepsilon)$ and (z_1', z_2') is an arc in $M_2(\varepsilon)$.

The normal product $\Psi^* \boxtimes \Psi^*$ is a directed graph with vertex set $\varrho(\Psi^* \boxtimes \Psi^*) = \varrho \times \varrho$ and arc set $\delta(\Psi^* \boxtimes \Psi^*)$, where $((z_1, z_1'), (z_2, z_2'))$ is an arc in $\Psi^* \boxtimes \Psi^*$ if and only if

1. $z_1 = z_2$ and (z_1', z_2') is an arc in Ψ^* or
2. (z_1, z_2) is an arc in Ψ^* and $z_1' = z_2'$ or
3. (z_1, z_2) as well as (z_1', z_2') are arcs in Ψ^*.

Let the parameter set be $\mathfrak{R}_{\Psi_1 \diamond \Psi_2} = \mathfrak{R}_1 \cap \mathfrak{R}_2$. Define a mapping $\gamma_{\Psi_1 \diamond \Psi_2}$ from $\mathfrak{R}_{\Psi_1 \diamond \Psi_2}$ to $\mathcal{P}[\varrho(\Psi^* \boxtimes \Psi^*)]$ by $\gamma_{\Psi_1 \diamond \Psi_2}(\varepsilon) = \gamma_1(\varepsilon) \times \gamma_2(\varepsilon), \forall \varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2$ where $\mathcal{P}[\varrho(\Psi^* \boxtimes \Psi^*)]$ denotes the power set of $\varrho(\Psi^* \boxtimes \Psi^*)$. Then $(\gamma_{\Psi_1 \diamond \Psi_2}, \mathfrak{R}_{\Psi_1 \diamond \Psi_2})$ is a soft set over $\varrho(\Psi^* \boxtimes \Psi^*)$. Define another mapping $\alpha_{\Psi_1 \diamond \Psi_2}$ from $\mathfrak{R}_{\Psi_1 \diamond \Psi_2}$ to $\mathcal{P}[\delta(\Psi^* \boxtimes \Psi^*)]$ by $\alpha_{\Psi_1 \diamond \Psi_2}(\varepsilon) = \{((w, z), (c, d)) \in \delta(\Psi^* \boxtimes \Psi^*) \mid \{(w, z), (c, d)\} \in \gamma_{\Psi_1 \diamond \Psi_2}\}, \forall \varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2$, where $\mathcal{P}[\delta(\Psi^* \boxtimes \Psi^*)]$ denotes the power set of $\delta(\Psi^* \boxtimes \Psi^*)$. Then $(\alpha_{\Psi_1 \diamond \Psi_2}, \mathfrak{R}_{\Psi_1 \diamond \Psi_2})$ is a soft set over $\delta(\Psi^* \boxtimes \Psi^*)$. Also if we denote $(\gamma_{\Psi_1 \diamond \Psi_2}(\varepsilon), \alpha_{\Psi_1 \diamond \Psi_2}(\varepsilon))$ by $M_{\Psi_1 \diamond \Psi_2}(\varepsilon)$, then $M_{\Psi_1 \diamond \Psi_2}(\varepsilon)$ is a subdigraph of $\Psi^* \boxtimes \Psi^*, \forall \varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2$, since $\gamma_1(\varepsilon) \times \gamma_2(\varepsilon) \subseteq \varrho \times \varrho$ and any arc in $\alpha_{\Psi_1 \diamond \Psi_2}(\varepsilon)$ is also an arc in $\delta(\Psi^* \boxtimes \Psi^*)$. Then $\Psi_1 \diamond \Psi_2$ can be represented by the 4-tuple $(\Psi^* \boxtimes \Psi^*, \gamma_{\Psi_1 \diamond \Psi_2}, \alpha_{\Psi_1 \diamond \Psi_2}, \mathfrak{R}_{\Psi_1 \diamond \Psi_2})$ and also by $\{M_{\Psi_1 \diamond \Psi_2}(\varepsilon) : \varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2\}$ and $\Psi_1 \diamond \Psi_2$ is a soft directed graph of $\Psi^* \boxtimes \Psi^*$ since the following conditions are satisfied:

1. $\Psi^* \boxtimes \Psi^* = (\varrho(\Psi^* \boxtimes \Psi^*), \delta(\Psi^* \boxtimes \Psi^*))$ is a directed graph having vertex set $\varrho(\Psi^* \boxtimes \Psi^*)$ and arc set $\delta(\Psi^* \boxtimes \Psi^*)$,
2. $\mathfrak{R}_{\Psi_1 \diamond \Psi_2} = \mathfrak{R}_1 \cap \mathfrak{R}_2$ is the set of parameters which is nonempty,
3. $(\gamma_{\Psi_1 \diamond \Psi_2}, \mathfrak{R}_{\Psi_1 \diamond \Psi_2})$ is a soft set over $\varrho(\Psi^* \boxtimes \Psi^*)$,
4. $(\alpha_{\Psi_1 \diamond \Psi_2}, \mathfrak{R}_{\Psi_1 \diamond \Psi_2})$ is a soft set over $\delta(\Psi^* \boxtimes \Psi^*)$,
5. $M_{\Psi_1 \diamond \Psi_2}(\varepsilon) = (\gamma_{\Psi_1 \diamond \Psi_2}(\varepsilon), \alpha_{\Psi_1 \diamond \Psi_2}(\varepsilon))$ is a subdigraph of $\Psi^* \boxtimes \Psi^*, \forall \varepsilon \in \mathfrak{R}_{\Psi_1 \diamond \Psi_2} = \mathfrak{R}_1 \cap \mathfrak{R}_2$.

□

Theorem 4.2. The restricted normal product $\Psi_1 \diamond \Psi_2$ contains $\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} |\gamma_1(\varepsilon)||\gamma_2(\varepsilon)|$ vertices and $\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} (|\gamma_1(\varepsilon)||\alpha_2(\varepsilon)| + |\gamma_2(\varepsilon)||\alpha_1(\varepsilon)| + |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|)$ arcs, if we count the vertices and arcs as many times they appear in different diparts of $\Psi_1 \diamond \Psi_2$.

Proof. By definition, $\Psi_1 \diamond \Psi_2 = \{M_1(\varepsilon) \boxtimes M_2(\varepsilon) : \varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2\}$. The parameter set of $\Psi_1 \diamond \Psi_2$ is $\mathfrak{R}_1 \cap \mathfrak{R}_2$. Consider the dipart $M_1(\varepsilon) \boxtimes M_2(\varepsilon)$ of $\Psi_1 \diamond \Psi_2$ corresponding to the parameter $\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2$. The vertex set of $M_1(\varepsilon) \boxtimes M_2(\varepsilon)$ is $\gamma_1(\varepsilon) \times \gamma_2(\varepsilon)$ which contains $|\gamma_1(\varepsilon)||\gamma_2(\varepsilon)|$ elements. This is true for all

diparts of $\Psi_1 \diamond \Psi_2$. Therefore total number of vertices in $\Psi_1 \diamond \Psi_2$ is $\sum_{x \in \mathfrak{R}_1 \cap \mathfrak{R}_2} |\gamma_1(\varepsilon)||\gamma_2(\varepsilon)|$, if we count the vertices as many times they appear in different diparts of $\Psi_1 \diamond \Psi_2$.
Also we know, $((z_q, z_r), (z_s, z_t))$ is an arc in $M_1(\varepsilon) \boxtimes M_2(\varepsilon)$ if and only

1. $z_q = z_s$ and (z_r, z_t) is an arc in $M_2(\varepsilon)$ or
2. (z_q, z_s) is an arc in $M_1(\varepsilon)$ and $z_r = z_t$ or
3. (z_q, z_s) is an arc in $M_1(\varepsilon)$ and (z_r, z_t) is an arc in $M_2(\varepsilon)$.

Now, each arc in $M_1(\varepsilon) \boxtimes M_2(\varepsilon)$ was made by just one of these three requirements (any two can't be true at the same time). So, to get the total number of arcs in $M_1(\varepsilon) \boxtimes M_2(\varepsilon)$, we add the number of arcs generated by each condition. Consider the first condition for adjacency, i.e.,$z_q = z_s$ and (z_r, z_t) is an arc in $M_2(\varepsilon)$. The number of arcs generated by this condition will be the number of arcs in the directed graph $M_2(\varepsilon)$ times the number of vertices in the directed graph $M_1(\varepsilon)$, which is given by $|\gamma_1(\varepsilon)||\alpha_2(\varepsilon)|$. Then consider the second condition for adjacency, i.e.,(z_q, z_s) is an arc in $M_1(\varepsilon)$ and $z_r = z_t$. The number of arcs generated by this condition will be the number of arcs in the directed graph $M_1(\varepsilon)$ times the number of vertices in the directed graph $M_2(\varepsilon)$, which is given by $|\gamma_2(\varepsilon)||\alpha_1(\varepsilon)|$. Finally consider the third condition for adjacency, i.e., (z_q, z_s) is an arc in $M_1(\varepsilon)$ and (z_r, z_t) is an arc in $M_2(\varepsilon)$. There are $|\alpha_1(\varepsilon)|$ arcs in $M_1(\varepsilon)$ and $|\alpha_2(\varepsilon)|$ arcs in $M_2(\varepsilon)$. So we can choose a pair of arcs a_k and a_l such that one is from $M_1(\varepsilon)$ and the other is from $M_2(\varepsilon)$ in $|\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|$ different ways. Suppose that a_k is the arc (z_q, z_s) in $M_1(\varepsilon)$ and a_l is the arc (z_r, z_t) in $M_2(\varepsilon)$. Then this pair of arcs gives an arc $((z_q, z_r), (z_s, z_t))$ in $M_1(\varepsilon) \boxtimes M_2(\varepsilon)$. Hence the number of arcs generated by this condition will be $|\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|$. That is, the total number of arcs in $M_1(\varepsilon) \boxtimes M_2(\varepsilon) = (|\gamma_1(\varepsilon)||\alpha_2(\varepsilon)| + |\gamma_2(\varepsilon)||\alpha_1(\varepsilon)| + |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|)$. This is true for all diparts of $\Psi_1 \diamond \Psi_2$. Therefore total number of arcs in $\Psi_1 \diamond \Psi_2$ is $\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2}(|\gamma_1(\varepsilon)||\alpha_2(\varepsilon)| + |\gamma_2(\varepsilon)||\alpha_1(\varepsilon)| + |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|)$, if we count the arcs as many times they appear in different diparts of $\Psi_1 \diamond \Psi_2$. □

Example 5. Consider the directed graphs given in Example 4. Here we have, total number of vertices in $\Psi_1 \diamond \Psi_2 = 20$ and $\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} |\gamma_1(\varepsilon)||\gamma_2(\varepsilon)| = (5.4) = 20$. That is, the total number of vertices in $\Psi_1 \diamond \Psi_2 = \sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} |\gamma_1(\varepsilon)||\gamma_2(\varepsilon)|$. Also total number of arcs in $\Psi_1 \diamond \Psi_2 = 50$ and $\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2}(|\gamma_1(\varepsilon)||\alpha_2(\varepsilon)| + |\gamma_2(\varepsilon)||\alpha_1(\varepsilon)| + |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|) = (5.3 + 4.5 + 5.3) = 50$. That is, total number of arcs in $\Psi_1 \diamond \Psi_2 = \sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2}(|\gamma_1(\varepsilon)||\alpha_2(\varepsilon)| + |\gamma_2(\varepsilon)||\alpha_1(\varepsilon)| + |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|)$.

Theorem 4.3. Let $\Psi^* = (\varrho, \delta)$ be a directed graph and $\Psi_1 = (\Psi^*, \gamma_1, \alpha_1, \mathfrak{R}_1)$ and $\Psi_2 = (\Psi^*, \gamma_2, \alpha_2, \mathfrak{R}_2)$ be two soft directed graphs of Ψ^*. Then

$$(i) \sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} \sum_{(w,z) \in \gamma_{\Psi_1 \diamond \Psi_2}(\varepsilon)} ideg(w,z)[M_{\Psi_1 \diamond \Psi_2}(\varepsilon)] =$$

$$\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} \sum_{(w,z) \in \gamma_{\Psi_1 \diamond \Psi_2}(\varepsilon)} odeg(w,z)[M_{\Psi_1 \diamond \Psi_2}(\varepsilon)] =$$

$$\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} (|\gamma_1(\varepsilon)||\alpha_2(\varepsilon)| + |\gamma_2(\varepsilon)||\alpha_1(\varepsilon)| + |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|)$$

$$(ii) \sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} \sum_{(w,z) \in \gamma_{\Psi_1 \diamond \Psi_2}(\varepsilon)} deg(w,z)[M_{\Psi_1 \diamond \Psi_2}(\varepsilon)] =$$

$$\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} 2(|\gamma_1(\varepsilon)||\alpha_2(\varepsilon)| + |\gamma_2(\varepsilon)||\alpha_1(\varepsilon)| + |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|).$$

Proof. (i) Consider any dipart $M_{\Psi_1 \diamond \Psi_2}(\varepsilon) = (\gamma_{\Psi_1 \diamond \Psi_2}(\varepsilon), \alpha_{\Psi_1 \diamond \Psi_2}(\varepsilon))$ of $\Psi_1 \diamond \Psi_2$ which is given by $M_1(\varepsilon) \times M_2(\varepsilon)$. By Theorem 4.2, we have number of arcs in $M_1(\varepsilon) \times M_2(\varepsilon)$ is $(|\gamma_1(\varepsilon)||\alpha_2(\varepsilon)| + |\gamma_2(\varepsilon)||\alpha_1(\varepsilon)| + |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|)$. Since the dipart $M_{\Psi_1 \diamond \Psi_2}(\varepsilon)$ is a directed graph having $(|\gamma_1(\varepsilon)||\alpha_2(\varepsilon)| + |\gamma_2(\varepsilon)||\alpha_1(\varepsilon)| + |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|)$ arcs, we have

$$\sum_{(w,z) \in \gamma_{\Psi_1 \diamond \Psi_2}(\varepsilon)} ideg(w,z)[M_{\Psi_1 \diamond \Psi_2}(\varepsilon)] =$$

$$\sum_{(w,z) \in \gamma_{\Psi_1 \diamond \Psi_2}(\varepsilon)} odeg(w,z)[M_{\Psi_1 \diamond \Psi_2}(\varepsilon)] =$$

$$(|\gamma_1(\varepsilon)||\alpha_2(\varepsilon)| + |\gamma_2(\varepsilon)||\alpha_1(\varepsilon)| + |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|),$$

since each arc in $M_{\Psi_1 \diamond \Psi_2}(\varepsilon)$ contributes 1 each to the sums $\sum_{(w,z) \in \gamma_{\Psi_1 \diamond \Psi_2}(\varepsilon)} ideg(w,z)[M_{\Psi_1 \diamond \Psi_2}(\varepsilon)]$ and $\sum_{(w,z) \in \gamma_{\Psi_1 \diamond \Psi_2}(\varepsilon)} odeg(w,z)[M_{\Psi_1 \diamond \Psi_2}(\varepsilon)]$. This is true for all the diparts $M_{\Psi_1 \diamond \Psi_2}(\varepsilon)$ of $\Psi_1 \diamond \Psi_2$. Hence,

$$\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} \sum_{(w,z) \in \gamma_{\Psi_1 \diamond \Psi_2}(\varepsilon)} ideg(w,z)[M_{\Psi_1 \diamond \Psi_2}(\varepsilon)] =$$

$$\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} \sum_{(w,z) \in \gamma_{\Psi_1 \diamond \Psi_2}(\varepsilon)} odeg(w,z)[M_{\Psi_1 \diamond \Psi_2}(\varepsilon)] =$$

$$\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} (|\gamma_1(\varepsilon)||\alpha_2(\varepsilon)| + |\gamma_2(\varepsilon)||\alpha_1(\varepsilon)| + |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|).$$

(ii) Since $deg(w,z)[M_{\Psi_1 \diamond \Psi_2}(\varepsilon)] = ideg(w,z)[M_{\Psi_1 \diamond \Psi_2}(\varepsilon)] + odeg(w,z)[M_{\Psi_1 \diamond \Psi_2}(\varepsilon)]$ and by part (i) of this theorem we have,

$$\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} \sum_{(w,z) \in \gamma_{\Psi_1 \diamond \Psi_2}(\varepsilon)} deg(w,z)[M_{\Psi_1 \diamond \Psi_2}(\varepsilon)] =$$

$$\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} 2(|\gamma_1(\varepsilon)||\alpha_2(\varepsilon)| + |\gamma_2(\varepsilon)||\alpha_1(\varepsilon)| + |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|).$$

□

Example 6. Consider the directed graphs given in Example 4. Here we have,

$$\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} \sum_{(w,z) \in \gamma_{\Psi_1 \diamond \Psi_2}(\varepsilon)} ideg(w,z)[M_{\Psi_1 \diamond \Psi_2}(\varepsilon)] = 50,$$

$$\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} \sum_{(w,z) \in \gamma_{\Psi_1 \diamond \Psi_2}(\varepsilon)} odeg(w,z)[M_{\Psi_1 \diamond \Psi_2}(\varepsilon)] = 50,$$

$$\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} (|\gamma_1(\varepsilon)||\alpha_2(\varepsilon)| + |\gamma_2(\varepsilon)||\alpha_1(\varepsilon)| + |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|)$$

$$= 5.3 + 4.5 + 5.3 = 50.$$

That is,

$$\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} \sum_{(w,z) \in \gamma_{\Psi_1 \diamond \Psi_2}(\varepsilon)} ideg(w,z)[M_{\Psi_1 \diamond \Psi_2}(\varepsilon)] =$$

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2}\ \sum_{(w,z)\in\gamma_{\Psi_1\diamond\Psi_2}(\varepsilon)} odeg(w,z)[M_{\Psi_1\diamond\Psi_2}(\varepsilon)] =$$

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2} (|\gamma_1(\varepsilon)||\alpha_2(\varepsilon)| + |\gamma_2(\varepsilon)||\alpha_1(\varepsilon)| + |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|).$$

Also,

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2}\ \sum_{(w,z)\in\gamma_{\Psi_1\diamond\Psi_2}(\varepsilon)} deg(w,z)[M_{\Psi_1\diamond\Psi_2}(\varepsilon)] = 100,$$

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2} 2(|\gamma_1(\varepsilon)||\alpha_2(\varepsilon)| + |\gamma_2(\varepsilon)||\alpha_1(\varepsilon)| + |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|)$$

$$= 2(5.3 + 4.5 + 5.3) = 100.$$

That is,

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2}\ \sum_{(w,z)\in\gamma_{\Psi_1\diamond\Psi_2}(\varepsilon)} deg(w,z)[M_{\Psi_1\diamond\Psi_2}(\varepsilon)] =$$

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2} 2(|\gamma_1(\varepsilon)||\alpha_2(\varepsilon)| + |\gamma_2(\varepsilon)||\alpha_1(\varepsilon)| + |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|).$$

5. Conclusion

This research has advanced the theoretical framework of graph theory by introducing and thoroughly investigating the concepts of normal and restricted normal products for soft directed graphs. These operations extend the classical notions of graph products into the realm of soft set theory, allowing for the incorporation of uncertainty into graph models. The definitions and theorems presented herein provide a robust foundation for understanding the structural and combinatorial properties of these new graph constructs. The results demonstrate that the normal product and restricted normal product of soft directed graphs preserve the soft directed graph structure, ensuring that these products are well-defined within the context of soft set theory. Moreover, the combinatorial identities related to vertex and arc counts, as well as degree sums, offer valuable insights for further research and practical applications.

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