Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 7 (2025) 2291

Journal of the Nigerian Society of Physical Sciences

Optimizing discrete dutch auctions with time considerations: a strategic approach for lognormal valuation distributions

Raja Aqib Shamim [a,b], Majid Khan Majahar Ali [a,*]

[a] School of Mathematical Sciences, Universiti Sains Malaysia, 11800, Pulau Penang, Malaysia
[b] Department of Mathematics, University of Kotli, 11100, Azad Jammu and Kashmir, Pakistan

Abstract

This research deviates from usual studies in auction literature primarily focused on maximizing expected revenue. Instead, we concentrate on the strategic design of discrete Dutch auctions in the context of bidder emotional attachment, wherein valuations follow a lognormal distribution. Our objective is to attain an optimal balance between the auctioned object's selling price and the auction duration, ultimately maximizing the auctioneer's expected revenue per unit of time. Our proposed models exhibit significantly higher average revenues per unit of time than counterparts neglecting time considerations and emotional attachment of the bidders. This achievement results from strategically reducing auction durations, enabling more auctions within the allotted time. This intentional trade-off ensures the marginal revenue decrease in shorter auctions is surpassed by the substantial increase in overall revenues from heightened auction frequency. Numerical results emphasize the utility of our modified discrete Dutch auction design, particularly in scenarios with a large number of bidders. Furthermore, increasing skewness in valuation distributions correlates with higher revenue per unit of time. Complete knowledge of the number of participating bidders is crucial, leading to a noticeable elevation in the auctioneer's expected revenue per unit of time. However, the predictability of auction outcomes may be challenging, underscoring the nuanced nature of auction dynamics.

DOI:10.46481/jnsps.2025.2291

Keywords: Auctions, Lognormal distribution, Nonlinear programming, Discrete Dutch auction, Revenue

Article History :
Received: 06 August 2024
Received in revised form: 31 October 2024
Accepted for publication: 11 November 2024
Available online: 02 January 2025

Communicated by: B. J. Falaye

1. Introduction

The Dutch auction is also called Descending Price Auction or Reverse Clock Auction. Contrary to well-known auction formats, e.g., English Auction, First Price Auction, Second Price Auction, etc., in Dutch auction, the price of an item decreases through a predetermined time interval established by the auctioneer, commonly shown on the display screen. When the auction starts, the price is kept high so that nobody is willing to buy but after a predefined time interval, the price falls until a bidder is willing to buy the item. That bidder will bid for the item by calling out 'mine' or by stopping the Dutch clock and will take the item by paying his bid value [1]. The auctioneer also sets the reserve price for the item after which the auction will be terminated and the object will go unsold.

In ascending price auction formats, the bidders continue to

*Corresponding author: Tel.: +60 14 9543 405
Email addresses: raja.aqib5@student.usm.my (Raja Aqib Shamim), majidkhanmajaharali@usm.my (Majid Khan Majahar Ali)

bid again and again by raising their bid value but in Dutch auction, the first bidder who bids will stop the clock and the item will be sold to him so it results in faster transactions [2]. This auction format is widely used to sell perishable goods whose value decreases over time, e.g., cut flowers, fish, concert tickets, plane tickets, seasonal items, etc. The name of the Dutch auction came from a flower auction market in the Netherlands [3]. Dutch Auction is also used for after-Chrismas sales [4], tobacco sales [5], initial public offerings [6], airline booking [7], and repurchasing the shares [8]. In addition to these studies, auctions have recently been examined in various contexts, including renewable energy market design [9], future investment cost estimation [10], edge computing applications [11], blockchain integration [12, 13], and resource management and pricing [14], among others.

In literature, the valuation of the bidders is usually taken as a continuous variable in a specified interval [15–17]. In this case, auctioneers choose small price decrements as the valuations are assumed to be continuous, leading to long auctions. Such kinds of auctions are preferable to sell unique objects like paintings, antiques, art, etc. but using this kind of bidders' valuation is not suitable for faster transactions in the case of perishable goods or services that need to be sold quickly [3, 18]. Many auctions, in reality, last for a very short time duration, for example, in Italian fish markets, almost fifteen transactions are completed in a minute, and at Royal Flora Holland, each auction lasts on average for four seconds [19, 20].

In another study by Li *et al.* [3], the bidders' valuations in discrete Dutch auctions are taken to follow different kinds of distribution including standard uniform distribution, exponential distribution, and truncated normal distribution, and the auctioneer's revenue per unit of time is maximized by developing a nonlinear program (NLP) and solving it subject to the inequality constraints. Other than Li *et al.*[3], there are only a limited number of studies in the literature that consider the discrete bid levels in Dutch auctions and those studies prove that the expected revenue of the auctioneer increases by considering the discrete settings [15, 16, 21, 22]. In literature, it is traditional to consider that the auctioneer works in the seller's interest and he/she wants to maximize the revenue of the seller [23, 24] but in reality, this is not exactly the case. Normally, the auctioneers run multiple auctions for different sellers simultaneously, and therefore, auctioneers usually prefer auctions to last for less time and go through a larger number of transactions rather than going through very long auctions with comparatively higher revenue and a smaller number of transactions. It refers that auctioneers want to maximize their revenue per unit of time instead of maximizing the revenue of the seller only [3]. The auctioneers prefer to go to the next auction and save time even if they expect a higher bid, and it assures the fact that the fast Dutch auctions are practiced in reality. Chipty *et al.* [25], in their studies observed almost 400 auctioneers from the National Auctioneers Association and reported that 92 percent of the auctioneers in outcry English auctions would prefer to go to the next auction to save time instead of waiting for the higher bid.

Bidders' emotions significantly influence the bidding process, thereby impacting the auctioneer's revenue. Adam *et al.* [26] conducted a series of experiments demonstrating that external factors and emotional states can lead to higher bids, ultimately increasing auctioneer revenue. In another study, Adam *et al.* [27] introduced the "emotional bidding framework," which outlined how human emotions affect electronic auctions through six key propositions. While the framework emphasized the importance of immediate emotional responses, that lacked empirical evaluation due to limited data on real-time emotional experiences of bidders. Later advancements in NeuroIS research, however, then allow for a comprehensive assessment of these propositions using neurophysiological evidence. In Ref. [28], the author builds on this foundation, synthesizing insights from previous work, refining the framework, and identifying new directions for future research that address the remaining gaps in understanding the emotional dynamics of bidding. Ku *et al.* [29] describe competitive arousal as driven by elements like rivalry, time pressure, social facilitation, and the first-mover advantage. These factors highlight how emotional influences can play a powerful role in urgent, competitive decision-making, offering insights into their significant impact in various auction environments.

In this research, we considered for-profit discrete Dutch auctions to maximize the auctioneers' expected revenue per unit of time when the bidders are emotionally attached to the items to be auctioned off or they want to get those items even if they need to bid higher. For instance, in case, if a market supplier participates in the auction when he/she has already taken the responsibility of providing a certain item/items to the market, he/she will not be willing to lose the item while waiting for the bid level to decrease enough until another bidder bids and wins the auction. In such cases, the valuations of the bidders can best be depicted by lognormal distribution instead of the standard normal distribution.

We propose a novel framework for optimizing discrete Dutch auctions by incorporating lognormal valuation distributions, which effectively capture the skewness in bidders' behavior, particularly in scenarios where emotional attachment to auctioned items significantly impacts their willingness to pay. Previous studies, such as Li *et al.* [3], have focused on maximizing auction revenue per unit of time within time-sensitive auction frameworks. However, our work extends this approach by addressing situations where bidders' valuations deviate from symmetric assumptions, specifically when psychological factors lead to higher valuations for some bidders compared to the majority.

To model this asymmetry, we employ lognormal distributions, denoted as $LN(\mu, \sigma^2)$, as they better represent the right-skewed nature of bidder valuations in these emotionally charged auctions. In our analysis, we fixed the scale parameter at $\mu = 0$ and examined three cases of the shape parameter σ, reflecting different levels of skewness: (1) near symmetry with $\sigma = 0.1$, (2) moderate skewness with $\sigma = 0.2$, and (3) significant skewness with $\sigma = 0.3$. We then compared our results with the standard normal distribution $N(0, 1)$, often assumed in the literature for symmetric valuations.

Our research not only highlights the importance of accounting for asymmetry in bidder behavior but also explores the impact of auction duration on revenue maximization. We demonstrate that, under specific conditions, reducing the duration of auctions in the presence of emotionally attached bidders can yield higher overall revenue, a finding that contributes significantly to the optimization of real-world, time-based auctions. This nuanced integration of bidder dynamics into the auction model marks a substantial advancement over existing literature, providing new insights into the interplay between valuation asymmetry and auction efficiency.

Another aim of this study is to compare the auction outcomes when the number of bidders is fixed and when it is a random variable. Such kinds of studies are very limited in the literature until now. In earlier literature, before the 80s, the number of bidders was often considered to be given but in recent studies, it is considered as a random variable [5, 30, 31]. Here, in our study, we compared the auctioneer's expected revenue per unit of time when the valuations of the bidders follow the lognormal distribution for two models; a model for a fixed number of bidders, and a model for a random number of bidders. Thus the effect of the information about the number of bidders is also investigated.

Furthermore, we determined the optimal number of bid levels necessary to optimize the auctioneer's anticipated revenue per unit of time in scenarios where certain bidders exhibit emotional attachments to the auctioned items, leading to bid valuations following a lognormal distribution. We also conducted a comparative analysis with instances where bidder valuations adhere to the standard normal distribution.

The rest of the paper is organized as follows. Section 2 details the mathematical formulation of four distinct models based on the bidders' valuations, whether lognormal or normal, and the fixed or random number of bidders. The first model in Section 2.1 refers to a discrete Dutch auction model with a fixed number of bidders when the valuations of the bidders follow the lognormal distribution. The second model in Section 2.2 is the case when the valuations of the bidders follow the standard normal distribution and the number of bidders is fixed. In Section 2.3, a discrete Dutch auction model when the number of bidders is a Poisson variable and the valuations of the bidders follow the lognormal distribution is developed and discussed. The model is then modified for the case when the valuations of the bidders follow the standard normal distribution in Section 2.4. The major findings and comparisons are made and discussed in Section 3. Finally, the significance of the research and future work suggestions are given in Section 4.

2. Model Formulation

In this paper, we developed an NLP to describe the discrete Dutch auction in an independent private value (IPV) setting with symmetric information. In the IPV setting, each bidder is sure about the value he/she gives to the object auctioned off and his valuation doesn't depend on the valuation of other participants, nor does any other bidder know his/her valuation. We aim to maximize the auctioneer's revenue per unit of time when all the bidders are rational to bid as per their valuations and are risk-neutral. There are n number of bidders participating in the auction and the valuation of bidder j, $j = 1, 2, 3, \ldots, n$ is v_j which is an independent and identically distributed continuous random variable having probability density function f and cumulative density function F.

Suppose that we are required to set the bid levels $l_1, l_2, \ldots, l_m$ where $m \geq 1$ and the object that is to be auctioned has a reserve price u, then we have $0 \leq u \leq l_1 \leq l_2 \leq \ldots \leq l_m$. When the auction unfolds, the price of the item will be comparatively high, say l_{m+1}, so that nobody is willing to buy at this point. If s is the Dutch clock speed then after each $s > 0$ seconds, the price falls sequentially as l_{m+1}, l_m, ..., l_2, l_1. If no bidder bids higher than l_{i+1}, and there exist $q \geq 1$ number of bidders whose valuations are in the interval $[l_i, l_{i+1})$, and the rest of the $n-q$ bidders' valuations lie below l_i, $i = 1, 2, \ldots, m$, then the selling price will be l_i (refer to Figure 1). If two or more bidders' valuations lie in the interval $[l_i, l_{i+1})$, then the first bidder who stops the clock will be the winner.

2.1. Model with a fixed number of bidders and lognormal valuations

In this section, a discrete Dutch auction is modeled as an NLP when the number of bidders is fixed. Let us suppose that we have n number of bidders and the probability that the object is sold at the bid level l_i, $i = 1, 2, \ldots, m$, is $P(l_i)$, then $P(l_i)$ is given by the binomial distribution as below;

$$\begin{aligned} P(l_i) &= \sum_{q=1}^{n} \binom{n}{q} F(l_i)^{n-q} [F(l_{i+1}) - F(l_i)]^q, \\ &= \left[\sum_{q=1}^{n} \binom{n}{q} F(l_i)^{n-q} [F(l_{i+1}) - F(l_i)]^q \right] \\ &\quad + F(l_i) - F(l_i), \\ &= \left[\sum_{q=0}^{n} \binom{n}{q} F(l_i)^{n-q} [F(l_{i+1}) - F(l_i)]^q \right] - F(l_i). \end{aligned} \tag{1}$$

Using the binomial expansion, $\sum_{q=0}^{n} \binom{n}{q} a^{n-q} b^q = (a+b)^n$, within the square brackets(1) becomes;

$$P(l_i) = F(l_{i+1})^n - F(l_i)^n. \tag{2}$$

As we are interested in finding the revenue per unit of time, we calculated the auction duration D_1. The auction duration is the product of the clock speed s and the number of expected bid levels $E(m)$ that the auction lasts. Here, $E(m)$ is the sum of the expected number of bid levels when the item goes unsold and the number of bid levels required to sell the object.

Therefore, the auction duration D_1 is given by;

$$\begin{aligned} D_1 &= sE(m), \\ &= s \left[\sum_{i=1}^{m} (m + 2 - i) P(l_i) + (m+1) \left(1 - \sum_{i=1}^{m} P(l_i) \right) \right], \end{aligned}$$

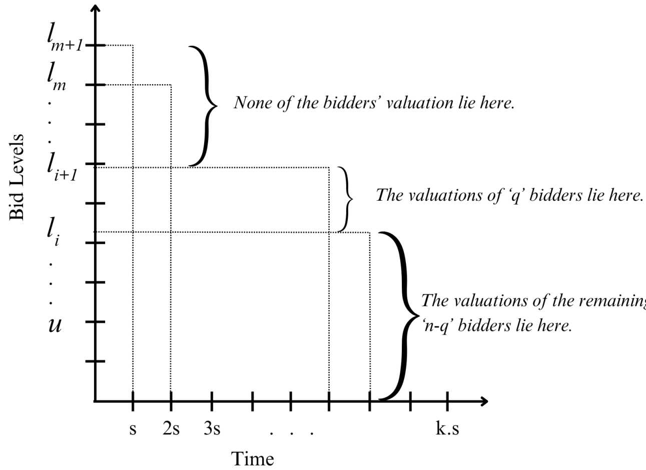


Figure 1. Discrete Dutch auction mechanism.

$$\begin{aligned} D_1 &= s\left[\sum_{i=1}^{m}(m+2-i)\left[F(l_{i+1})^n - F(l_i)^n\right]\right. \\ &\quad \left. +(m+1)\left(1-\sum_{i=1}^{m}\left[F(l_{i+1})^n - F(l_i)^n\right]\right)\right], \\ &= s\left[(1+m)\left(1+F(l_1)^n - F(l_{1+m})^n\right)\right. \\ &\quad \left. +\sum_{i=1}^{m}(2-i+m)\left(F(l_{i+1})^n - F(l_i)^n\right)\right]. \end{aligned} \tag{3}$$

Now, the auctioneer's expected revenue per unit of time, Z_1, is obtained by dividing the expected selling price by the auction duration D_1 and it follows as under;

$$\begin{aligned} Z_1 &= \frac{\sum_{i=1}^{m} l_i P(l_i) + u\left(1-\sum_{i=1}^{m} P(l_i)\right)}{D_1}, \\ &= \frac{\left[\begin{array}{l}\sum_{i=1}^{m} l_i\left[F(l_{i+1})^n - F(l_i)^n\right] \\ +u\left(1-\sum_{i=1}^{m}\left[F(l_{i+1})^n - F(l_i)^n\right]\right)\end{array}\right]}{\left[\begin{array}{l}s\left[(1+m)\left(1+F(l_1)^n - F(l_{1+m})^n\right)\right. \\ \left.+\sum_{i=1}^{m}(2-i+m)\left(F(l_{i+1})^n - F(l_i)^n\right)\right]\end{array}\right]}, \\ &= \frac{\left[\begin{array}{l}u+uF(l_1)^n - uF(l_{m+1})^n \\ +\sum_{i=1}^{m} l_i\left[F(l_{i+1})^n - F(l_i)^n\right]\end{array}\right]}{\left[\begin{array}{l}s\left[(1+m)\left(1+F(l_1)^n - F(l_{1+m})^n\right)\right. \\ \left.+\sum_{i=1}^{m}(2-i+m)\left(F(l_{i+1})^n - F(l_i)^n\right)\right]\end{array}\right]}. \end{aligned} \tag{4}$$

Hence, our required model can be formulated as an NLP given below, where $l_1, l_2, \ldots, l_m$ are decision variables and m, n, s and u are the parameters.

Maximize

$$Z_1 = \frac{\left[\begin{array}{l}u+uF(l_1)^n - uF(l_{m+1})^n \\ +\sum_{i=1}^{m} l_i\left[F(l_{i+1})^n - F(l_i)^n\right]\end{array}\right]}{\left[\begin{array}{l}s\left[(1+m)\left(1+F(l_1)^n - F(l_{1+m})^n\right)\right. \\ \left.+\sum_{i=1}^{m}(2-i+m)\left(F(l_{i+1})^n - F(l_i)^n\right)\right]\end{array}\right]},$$

subject to the constraints;

$$\begin{aligned} & l_{i+1} \geq l_i, \quad i = 1, 2, \ldots, m \quad , \\ & l_1 \geq u. \end{aligned} \tag{5}$$

When certain bidders are emotionally attached to the auctioned item, their valuations are best modeled by a lognormal distribution, which is characterized by a right-skewed shape. This indicates that the emotionally attached bidders have relatively higher valuations compared to the majority. In this context, the bidders' valuations are treated as lognormally distributed positive random variables, with their natural logarithms following a normal distribution with mean μ and variance σ^2. It follows that $f(l_i) = \frac{1}{l_i\sigma\sqrt{2\pi}}\exp\left(-\frac{(ln l_i-\mu)^2}{2\sigma^2}\right)$ and $F(l_i) = \frac{1}{2}\,\mathrm{erfc}\left(-\frac{ln l_i-\mu}{\sigma\sqrt{2}}\right)$ where erfc is the complementary error function defined by $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}\,dt$.

Hence the modified NLP we get is as under;

Maximize

$$Z_1 = \frac{\left[\begin{array}{l}u+u\left[\frac{1}{2}\mathrm{erfc}\left(-\frac{ln l_1-\mu}{\sigma\sqrt{2}}\right)\right]^n - u\left[\frac{1}{2}\mathrm{erfc}\left(-\frac{ln l_{m+1}-\mu}{\sigma\sqrt{2}}\right)\right]^n \\ +\sum_{i=1}^{m} l_i\left\{\left[\frac{1}{2}\mathrm{erfc}\left(-\frac{ln l_{i+1}-\mu}{\sigma\sqrt{2}}\right)\right]^n - \left[\frac{1}{2}\mathrm{erfc}\left(-\frac{ln l_i-\mu}{\sigma\sqrt{2}}\right)\right]^n\right\}\end{array}\right]}{\left[\begin{array}{l}s\left[(1+m)\left(1+\left[\frac{1}{2}\mathrm{erfc}\left(-\frac{ln l_1-\mu}{\sigma\sqrt{2}}\right)\right]^n - \left[\frac{1}{2}\mathrm{erfc}\left(-\frac{ln l_{m+1}-\mu}{\sigma\sqrt{2}}\right)\right]^n\right)\right. \\ \left.+\sum_{i=1}^{m}(2-i+m)\left\{\left[\frac{1}{2}\mathrm{erfc}\left(-\frac{ln l_{i+1}-\mu}{\sigma\sqrt{2}}\right)\right]^n - \left[\frac{1}{2}\mathrm{erfc}\left(-\frac{ln l_i-\mu}{\sigma\sqrt{2}}\right)\right]^n\right\}\right]\end{array}\right]},$$

subject to;

$$\begin{aligned} l_{i+1} &\geq l_i, \quad i = 1, 2, \ldots, m \quad , \\ l_1 &\geq u, \end{aligned} \tag{6}$$

where n, m, s, u, μ, and σ are parameters and $l_1, l_2, \ldots, l_{m+1}$ are decision variables.

In the above NLP (6), each of the constraints is a convex set being a half-space. The convexity of the feasible region is obvious by the fact that the intersection of the convex sets is a convex set. Moreover, the objective function is concave. Hence, the optimal solution of NLP (6) exists [32].

Numerical analysis

In this section, we solved the NLP (6) as a function of parameters m, n, u, μ, σ, and s subject to the inequality constraints. The clock speed s is taken as 1 second because by taking $s = k$ seconds, the revenue per unit of time will simply decrease by k times [33]. In our discussion, we took the number of bidders, n, from the set $\{2, 5, 10, 20, \ldots, 100\}$ and the salvage value, u, from the set $\{0, 0.1, \ldots, 0.8\}$. Furthermore, the number of bid levels goes up to 6, *i.e.*, $m \in \{1, 2, \ldots, 6\}$, μ is taken as 0, and $\sigma \in \{0.1, 0.2, 0.3\}$. With the different combinations of n, u, m, σ, and μ with $s = 1$ seconds, the NLP (6) is set up and solved using the software R.

Before proceeding with the discussion, we first outline the rationale behind selecting the aforementioned parameters. The values of m, n, u, and s have been justified based on the findings of Li *et al.* [3]. In our study, we model bidders' valuations using a lognormal distribution, where μ and σ represent the mean and standard deviation of the natural logarithm of the variable, respectively. We have chosen $\mu = 0$, which places the median of the distribution at 1. This serves as a neutral baseline, facilitating comparison across different levels of variability in the bidders' valuations, while allowing for a clear interpretation of how valuations are distributed around the median. The parameter σ, controlling the spread of the distribution, allows us to capture different degrees of variability in the bidders' emotional responses. We explore three distinct cases: $\sigma = 0.1$, $\sigma = 0.2$, and $\sigma = 0.3$, representing increasing levels of dispersion. A lower σ value of 0.1 implies that the majority of bidders have similar valuations close to the median, suggesting minimal emotional attachment to the auctioned item. As σ increases to 0.2 and 0.3, the distribution becomes more spread out, indicating a wider range of valuations among bidders as shown in Figure 2. This greater spread signifies a stronger emotional attachment, as some bidders place substantially higher value on the object, distinguishing themselves from the rest. By varying σ, we effectively capture the impact of emotional attachment on bidding behavior, with higher values of σ reflecting increased emotional investment and a broader disparity in bidders' valuations. To start the discussion, a special case of auctioneer's expected revenue per unit of time for the item that has no salvage value, *i.e.*, $u = 0$ for $s = 1$, $\mu = 0$, and $\sigma \in \{0.1, 0.2, 0.3\}$ is summarized in Table 1, where $Z^*_{1,m=1}$, $Z^*_{1,m=2}$, $Z^*_{1,m=3}$, $Z^*_{1,m=4}$, $Z^*_{1,m=5}$ and $Z^*_{1,m=6}$ denote the auctioneer's maximum expected revenue per unit of time at $m = 1$, $m = 2$, $m = 3$, $m = 4$, $m = 5$ and $m = 6$ respectively. The boldfaced value in each row represents the maximum value of Z^*_1 among all the values of m for a particular value of n. The case for $n = 70$ is depicted in Figure 3 where Z^*_1 increases until it reaches its highest value at $m = 3$ for both $\sigma = 0.1$ and 0.2 and at $m = 4$ for $\sigma = 0.3$, and then decreases a little for higher values of m. A close look at Table 1 shows that, in most of the cases, the highest expected revenue per unit of time exists at either $m = 3$ or $m = 4$). In a few cases, it also lies at $m = 5$ or $m = 6$ (but still the value is only slightly greater than the case for $m = 3$ or 4) and these exceptions are logical and are due to the lognormal distribution of the valuations of the bidders. If the bidders follow the lognormal distribution, the expected revenue per unit of time will not be predictable due to the asymmetry of the distribution and the same happens in the real-world auctions. The maximization of the revenue at $m = 3$ or $m = 4$ is in agreement with the fact that most of the transactions in real-world fast Dutch auctions are completed in less than a minute [20, 34] or even in four seconds [19]. In literature, most of the studies reveal that the revenue of the auctioneer increases with the number of bid levels [15, 16, 22] but in real-world scenarios it doesn't happen for fast oral Dutch auctions. Li *et al.* [3], in their studies, revealed the fact that the auctioneers in reality are interested in revenue maximization per unit of time rather than per unit auction and that's why the fast oral Dutch auctions are commonly used in reality. In our new modified model, we encountered real-world Dutch auctions when some of the bidders are emotionally attached to the items to be auctioned off and their valuations follow the lognormal distribution.

Lastly, if we look closely at Table 1(a) with $\sigma = 0.1$ when the distribution of the valuations of the bidders is close to symmetry, Table 1(b) with $\sigma = 0.2$ when the distribution of the valuations of the bidders is slightly skewed, and Table 1(c) with $\sigma = 0.3$ when the distribution of the valuations of the bidders is highly skewed, we observe that if σ increases, the corresponding auctioneer's maximum expected revenue per unit of time increases for each value of n except for the case when $n = 2$. It shows that with the increase in skewness of the distribution of the valuations of the bidders, the expected revenue per unit of time increases. This result is logical because when the skewness of the distribution of the valuations of the bidders increases, the higher number of participants will have comparatively higher valuations than the majority of the participants, and hence they are supposed to bid higher which results in higher revenue per unit of time.

In Table 2, the optimal number of bid levels m^* along with corresponding maximum expected revenue per unit of time Z^*_{1,m^*} of the auctioneer for different values of the salvage value $u \in \{0, 0.1, \ldots, 0.8\}$ and the different values of shape parameter of the distribution of the valuations of the bidders $\sigma \in \{0.1, 0.2, 0.3\}$ are presented. From Table 2, it is evident that Z^*_{1,m^*} increases with the increase in the number of participants n in the auction for all values of u and σ with some exceptions when $u = 0.1$ or 0.2. The fact of increasing the maximum revenue per unit of time with the increase in the number of participants in the auction for $u \in \{0, 0.3, 0.4, \ldots, 0.8\}$ makes intuitive sense because the higher the number of participants, the

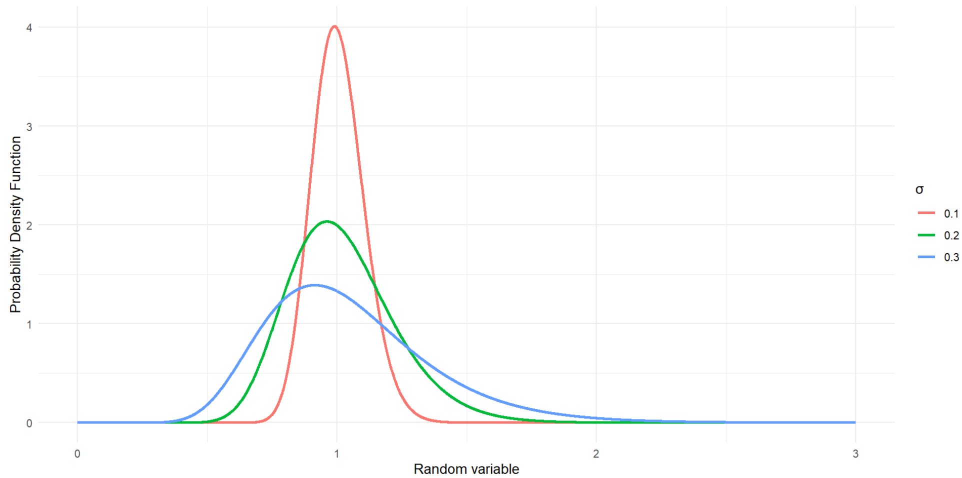


Figure 2. Probabilty density function of lognormal distribution for $\mu = 0$ and $\sigma = 0.1, 0.2, 0.3$.

higher is the competition and hence the selling price [3]. Now, the maximum expected revenue per unit of time for $u = 0.1$ and 0.2 is also increasing with the increase in n for each value of σ until n reaches a specific value ($n = 60$ in most of the cases) and then it doesn't follow the increasing pattern. The value of Z^*_{1,m^*} drops abruptly and then either goes constant (for $\sigma = 0.1, 0.2$) or increases again and then drops and goes constant (for $u = 0.1$ and $\sigma = 0.3$). The unusual behavior in the auctioneer's maximum expected revenue per unit of time makes sense when we take into account the bidders' valuations following the lognormal distribution. This means predicting revenue isn't always straightforward; it tends to follow an abnormal pattern due to asymmetry in the valuations of the bidders. In real situations, emotional connections and various factors make it tricky to predict auctioneers' maximum revenue per unit of time, adding to the unpredictability of the pattern.

Moreover, in most cases, a maximum of four bid levels are required (i.e., $m^* \leq 4$) to maximize the auctioneer's expected revenue per unit of time except for a few cases when five of six bid levels are needed (still the value is very close to three or four bid levels' value). Also, with the increase in the value of u, the maximum number of bid levels required to maximize the auctioneer's expected revenue per unit of time reduces up to one regardless of the number of participants. For instance, for $u = 0.8$ in Table 2, only one bid level is required to maximize the auctioneer's expected revenue per unit of time for each value of n and σ. It makes logical sense because, in reality, most of the discrete Dutch auctions need very little time to finish and hence the required number of bid levels to maximize the auctioneer's expected revenue per unit of time is four at highest in most of the cases. If the salvage value is higher, it will likely lead to increased competition, and it makes sense for there to be only one reasonable bid level. Finally, with the increase in the salvage value u, the maximum expected revenue increases in most of the cases until $u = 0.6$, and when the value of u increases 0.6, the maximum expected revenue falls except in some cases. These exceptional cases are obvious due to the unpredictability of the revenue when the valuations of the bidders follow the lognormal distribution. When $\sigma = 0.1$, for $n \in \{70, 80, 90, 100\}$, the highest expected revenue per unit of time is observed at $u = 0.8$. When $\sigma = 0.2$ and 0.3, for $n \in \{2, 5\}$, the highest expected revenue per unit of time is at $u = 0.4$ and for $n = 100$ it lies at $u = 0.8$.

2.2. Model with a fixed number of bidders and standard normal valuations

Li *et al.* [3] developed a discrete Dutch auction model when the valuations of the bidders follow the truncated normal distribution to maximize the auctioneer's expected revenue per unit of time. In this section, we considered the valuations of the bidders following the standard normal distribution $N(\mu, \sigma^2)$ where $\mu = 0$ and $\sigma = 1$ to maximize the auctioneer's expected revenue per unit of time. The modified discrete Dutch auction model for the fixed number of bidders is formulated and its properties are explored below.

If the valuations of the bidders follow the standard normal distribution, then the PDF and CDF are respectively $f(l_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}l_i^2}$ and $F(l_i) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{l_i}{\sqrt{2}}\right)\right]$ where erf is the error function given by $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$. Hence, the NLP (5) in this case becomes;

Maximize

$$Z_2 = \frac{\left[\begin{array}{l} u + u\left[\frac{1}{2}\left(1 + \text{erf}\left(\frac{l_1}{\sqrt{2}}\right)\right)\right]^n - u\left[\frac{1}{2}\left(1 + \text{erf}\left(\frac{l_{m+1}}{\sqrt{2}}\right)\right)\right]^n \\ + \sum\limits_{i=1}^{m} l_i \left\{\left[\frac{1}{2}\left(1 + \text{erf}\left(\frac{l_{i+1}}{\sqrt{2}}\right)\right)\right]^n - \left[\frac{1}{2}\left(1 + \text{erf}\left(\frac{l_i}{\sqrt{2}}\right)\right)\right]^n\right\} \end{array}\right]}{\left[\begin{array}{l} s\left[(1+m)\left\{1 + \left[\frac{1}{2}\left(1 + \text{erf}\left(\frac{l_1}{\sqrt{2}}\right)\right)\right]^n - \left[\frac{1}{2}\left(1 + \text{erf}\left(\frac{l_{m+1}}{\sqrt{2}}\right)\right)\right]^n\right\}\right. \\ \left. + \sum\limits_{i=1}^{m}(2 - i + m)\left\{\left[\frac{1}{2}\left(1 + \text{erf}\left(\frac{l_{i+1}}{\sqrt{2}}\right)\right)\right]^n - \left[\frac{1}{2}\left(1 + \text{erf}\left(\frac{l_i}{\sqrt{2}}\right)\right)\right]^n\right\}\right] \end{array}\right]},$$

subject to the constraints;

$$\begin{aligned} & l_{i+1} \geq l_i, \quad i = 1, 2, \ldots, m \quad , \\ & l_1 \geq u \quad , \end{aligned} \tag{7}$$

Table 1. Auctioneer's maximum expected revenue per unit of time for a fixed number of bidders whose valuations follow lognormal distribution vs. number of bidders and number of bid levels with $u = 0$, $s = 1$, $\mu = 0$, and $\sigma \in \{0.1, 0.2, 0.3\}$.

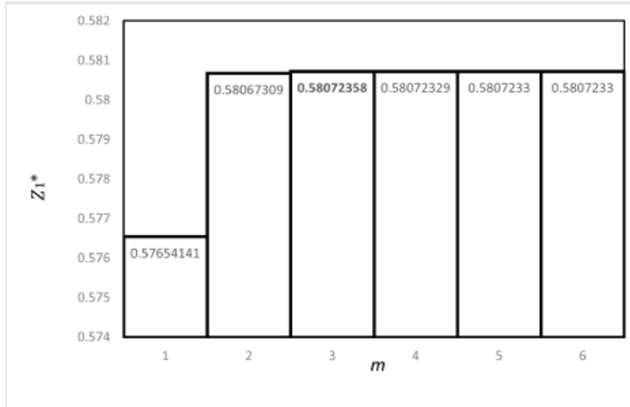
n	$Z^*_{1,m=1}$	$Z^*_{1,m=2}$	$Z^*_{1,m=3}$	$Z^*_{1,m=4}$	$Z^*_{1,m=5}$	$Z^*_{1,m=6}$
2	0.44148617	0.44968853	0.4499683	**0.44997542**	0.44997526	0.44997525
5	0.48486967	0.49159576	**0.49176554**	0.4917639	0.49176485	0.49176391
10	0.51296263	0.51877695	**0.51889514**	0.51889396	0.51889401	0.518894
20	0.53781077	0.54289311	**0.54297772**	0.5429771	0.54297713	0.54297703
30	0.55111072	0.55583872	**0.55590941**	0.5559089	0.55590892	0.5559089
40	0.56007542	0.56458081	**0.56464349**	0.56464307	0.56464307	0.56464311
50	0.56678627	0.57113383	**0.57119118**	0.57119082	0.57119082	0.57119082
60	0.57212406	0.57635151	0.57638507	**0.57640466**	0.57640466	0.57640466
70	0.57654141	0.58067309	**0.58072358**	0.58072329	0.5807233	0.5807233
80	0.5803007	0.58435346	0.5843814	0.58438247	0.58438682	**0.58440131**
90	0.58356717	0.58755326	0.58758334	0.58758156	0.58757942	**0.58759917**
100	0.58645137	0.59038007	**0.59042458**	0.59040362	0.59042438	0.59042435
(a) For $\sigma = 0.1$						
2	0.41010832	0.42412031	0.42492038	0.42495232	**0.42495318**	0.42495309
5	0.4869579	0.50013472	0.5007457	**0.50076599**	0.50076542	0.50076552
10	0.54090918	0.55327198	0.55375629	**0.55376978**	0.55376943	0.55376943
20	0.59146377	0.60302571	0.60341027	0.60341855	**0.60341907**	0.60341907
30	0.61963358	0.63076329	0.63110141	0.63110551	0.63110538	**0.6311062**
40	0.63906082	0.64990481	0.65021459	0.65021661	0.65021266	**0.65021807**
50	0.65383591	0.66447093	**0.66476108**	0.66476026	0.66476003	0.66475935
60	0.66572991	0.67620245	**0.67647794**	0.67647656	0.67647624	0.67647656
70	0.67566798	0.68600891	**0.68627289**	0.68627126	0.6862713	0.68627133
80	0.68419332	0.69442446	**0.69467909**	0.69467767	0.69467762	0.69467743
90	0.6916516	0.70178908	**0.70203589**	0.70203399	0.7020335	0.70203459
100	0.69827611	0.7083323	**0.70857246**	0.70857021	0.70857118	0.70857046
(b) For $\sigma = 0.2$						
2	0.39086159	0.40880794	0.41001047	**0.41005332**	0.41005308	0.41005297
5	0.49781529	0.51683671	0.51799273	0.51803987	**0.51804109**	0.51804094
10	0.57838762	0.59758366	0.59861944	0.59865801	**0.59865902**	0.59865896
20	0.65785759	0.67693971	0.6778449	**0.67787452**	0.67787378	0.67787378
30	0.70376054	0.72271135	0.7235455	**0.7235707**	0.72357006	0.72357017
40	0.73607827	0.75492349	0.75571109	**0.75573358**	0.75573302	0.75573318
50	0.76101276	0.77977338	0.78052725	**0.78054787**	0.7805474	0.78054749
60	0.78130581	0.79999719	0.80072507	**0.8007443**	0.80074389	0.80074389
70	0.7984113	0.8170448	0.81775179	**0.81776994**	0.81776953	0.81776954
80	0.81319283	0.83177707	0.83246675	**0.83248403**	0.83248368	0.83248369
90	0.8262052	0.84474687	0.84542188	**0.84543843**	0.8454381	0.84543812
100	0.83782589	0.85633031	0.85699265	**0.85700858**	0.85700827	0.8570083
(c) For $\sigma = 0.3$						

where s, n, m, and u are the parameters and l_i, $i = 1, 2, \ldots, m+1$ are the decision variables.

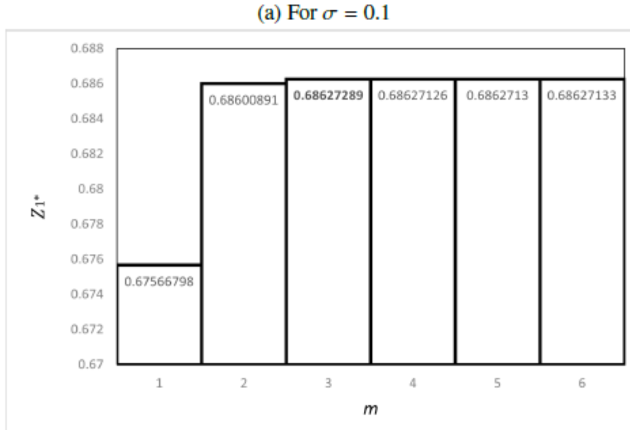
To compare the results of the above NLP (7) with NLP (6), we solved it using software R. For the relevant numerical analysis, we choose $n \in \{2, 5, 10, 20, \ldots, 100\}$, $u \in \{0, 0.1, \ldots, 0.8\}$, $m \in \{1, 2, \ldots, 6\}$ and $s = 1$. As the standard normal distribution is symmetric, we compared the results with case (a) (i.e., close to symmetry) of the NLP (7).

Table 3 and Table 4 refer to the NLP (7) where the optimal number of bid levels and the corresponding auctioneer's maximum expected revenue per unit of time for the fixed number of participants are shown respectively. The results are quite similar to the results in the study of Li *et al.* [3] when the valuations of the bidders follow the truncated normal distribution.

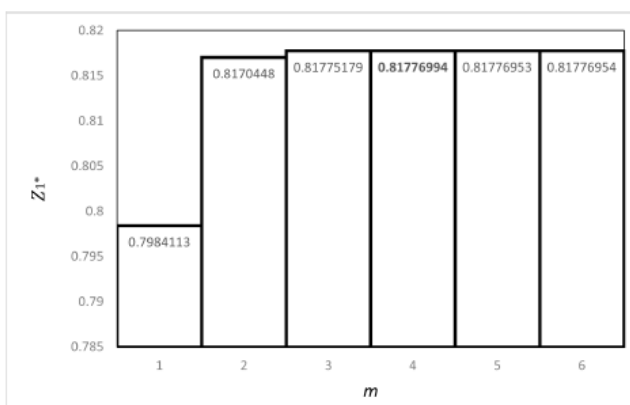
In our research, Table 3 is the case for $u = 0$, and it can be seen that with the increase in the number of bidders n, the number of bid level m to maximize the auctioneer's expected revenue per unit of time also increases until it reaches $m = 5$ and then stays constant. Also, with the increase in the number of bidders n, the revenue per unit of time increases for each value of m. It shows that only five bid levels are required to maximize the auctioneer's expected revenue per unit of time



(a) For $\sigma = 0.1$



(b) For $\sigma = 0.2$



(c) For $\sigma = 0.3$

Figure 3. Auctioneer's maximum expected revenue per unit of time vs. number of bid levels when $u = 0$ $n = 70$ and $s = 1$.

when the valuations of the bidders follow the standard normal distribution and the salvage value u is taken as zero. If we compare these results with the results in Table 1 (a), we see that in most of the cases, the maximum number of bid levels required to maximize the auctioneer's expected revenue per unit of time is three which is a better representation of the discrete Dutch auctions in the real-world scenario [3, 19, 20].

Table 4 shows that for each $u \in \{0, 0.1, \ldots, 0.8\}$, with the increase in the number of participants n in the auction, the maximum expected revenue per unit of time increases. Also, with the increase in the salvage value u, the optimal number of bid levels m decreases and reaches $m = 1$ when $u > 0.6$ for each value of n. In only a few cases, the optimal number of bid levels m^* reaches 5, and in all other cases $m^* \leq 4$. These trends here are quite similar to those in Table 2 except for the cases where the maximum expected revenue per unit of time is unpredictable as in real-world auctions.

Table 2. Auctioneer's maximum expected revenue per unit of time for a fixed number of bidders whose valuations follow lognormal distribution vs. number of bidders and salvage value with $s = 1$, $\mu = 0$ and $\sigma \in \{0.1, 0.2, 0.3\}$.

n	$u = 0$		$u = 0.1$		$u = 0.2$		$u = 0.3$		$u = 0.4$		$u = 0.5$		$u = 0.6$		$u = 0.7$		$u = 0.8$	
	m^*	Z^*_{1,m^*}	m^*	Z^*_{1,m^*}	m^*	Z^*_{1,m^*}	m^*	Z^*_{1,m^*}	m^*	Z^*_{1,m^*}	m^*	Z^*_{1,m^*}	m^*	Z^*_{1,m^*}	m^*	Z^*_{1,m^*}	m^*	Z^*_{1,m^*}
2	4	0.44997542	4	0.44997547	4	0.44997554	3	0.44997632	2	0.45007428	1	0.45432627	4	0.49176881	1	0.46550849	1	0.47504228
5	3	0.49176554	3	0.49176629	3	0.49176716	3	0.49176816	2	0.49177693	1	0.49343213	4	0.51889613	1	0.50034713	1	0.50595054
10	3	0.51889514	3	0.51889549	4	0.51889688	3	0.51889636	3	0.51889691	1	0.51953525	4	0.54297793	1	0.52460021	1	0.52854731
20	3	0.54297772	3	0.54297791	3	0.54297811	3	0.54297835	3	0.54297862	1	0.54303244	4	0.55590954	1	0.54690246	1	0.54981683
30	3	0.55590941	3	0.55590954	3	0.55590968	3	0.55590984	3	0.55591003	2	0.5559142	4	0.56464356	1	0.55910809	1	0.56159836
40	3	0.56464349	3	0.5646436	3	0.56464371	3	0.56464384	4	0.56464411	2	0.56464537	4	0.57119124	1	0.56742565	1	0.56967138
50	3	0.57119118	3	0.57119127	3	0.57119136	3	0.57119147	3	0.57119159	2	0.57119138	4	0.57640501	1	0.5736952	1	0.575777
60	4	0.57640466	1	0.05000000	5	0.57640476	6	0.57640483	5	0.57640489	4	0.57640547	4	0.58072361	1	0.57870643	1	0.58066848
70	3	0.58072358	1	0.05000000	1	0.10000000	4	0.58072344	5	0.58072349	4	0.58072402	4	0.58440158	1	0.58286895	1	0.58473848
80	6	0.58440131	1	0.05000000	1	0.10000000	6	0.58440145	6	0.58440149	4	0.58440197	4	0.58759943	1	0.58642183	1	0.58821703
90	6	0.58759917	1	0.05000000	1	0.10000000	6	0.5875993	6	0.58759935	4	0.58759978	4	0.5904246	1	0.58951642	1	0.59125015
100	3	0.59042458	1	0.05000000	1	0.10000000	6	0.59042448	5	0.59042452	4	0.59042492	2	0.59043432	1	0.59225442	1	0.59393617
(a) For $\sigma = 0.1$																		
2	5	0.42495318	4	0.42495327	3	0.42496654	2	0.42536314	1	0.57497469	1	0.43964938	4	0.50076864	1	0.4635029	1	0.48100463
5	4	0.50076599	4	0.50076623	4	0.50076651	3	0.50077199	1	0.57929075	1	0.50640113	4	0.55377068	1	0.52089756	1	0.53134511
10	4	0.55376978	4	0.55376987	4	0.55376998	4	0.5537701	1	0.58296774	1	0.55570454	4	0.60341959	1	0.5660992	1	0.57335127
20	5	0.60341907	4	0.60341928	4	0.60341932	4	0.60341937	3	0.60341979	2	0.60349381	4	0.63110875	1	0.61091963	1	0.6161728
30	6	0.6311062	4	0.63110856	4	0.63110859	4	0.63110862	4	0.63110866	2	0.63113393	4	0.65022085	1	0.63664977	1	0.64109097
40	6	0.65021807	4	0.65022071	4	0.65022068	4	0.65022076	4	0.65022078	2	0.65022251	4	0.66476663	1	0.6546427	1	0.65861991
50	3	0.66476108	5	0.66476649	5	0.66476646	5	0.66476618	4	0.66476658	4	0.6647666	4	0.67648299	1	0.66844357	1	0.67211186
60	3	0.67647794	1	0.05000000	1	0.10000000	6	0.67648285	6	0.67648286	3	0.6764832	4	0.68627757	1	0.67961879	1	0.6830626
70	3	0.68627289	1	0.05000000	1	0.10000000	6	0.68627394	5	0.68627492	3	0.68627761	4	0.69468347	1	0.68899727	1	0.69226842
80	3	0.69467909	1	0.05000000	1	0.10000000	6	0.69467968	6	0.69468195	3	0.69468339	4	0.70203988	1	0.69707019	1	0.70020315
90	3	0.70203589	1	0.05000000	1	0.10000000	6	0.70203641	6	0.70203718	3	0.70203986	4	0.70857542	1	0.70415231	1	0.70717136
100	3	0.70857246	1	0.05000000	1	0.10000000	6	0.70857288	5	0.70857352	3	0.70857616	2	0.70859621	1	0.71045728	1	0.71338027
(b) For $\sigma = 0.2$																		
2	4	0.41005332	3	0.41008587	2	0.41064083	1	0.41439497	1	0.68184957	1	0.43798637	4	0.51805543	1	0.4720754	1	0.49444572
5	5	0.51804109	4	0.51804118	3	0.51805322	2	0.51833421	1	0.69018593	1	0.52891161	4	0.59866318	1	0.55005334	1	0.56397092
10	5	0.59865902	5	0.59865904	4	0.59865915	3	0.59866839	1	0.69748646	1	0.60201777	4	0.67787658	1	0.61718268	1	0.62694127
20	4	0.67787452	4	0.67787476	4	0.67787503	4	0.67787533	1	0.70397652	2	0.67810932	4	0.72357197	1	0.68778504	1	0.69487103
30	4	0.7235707	4	0.72357085	4	0.72357102	4	0.72357121	3	0.72357205	2	0.72364041	4	0.75573451	1	0.72991327	1	0.7359053
40	4	0.75573358	4	0.75573369	4	0.75573382	4	0.75573396	4	0.75573412	3	0.7557393	4	0.7805486	1	0.76001314	1	0.76537921
50	4	0.78054787	4	0.78054796	4	0.78054806	5	0.78054758	4	0.7805483	4	0.78054844	4	0.80074492	1	0.78344215	1	0.78839141
60	4	0.8007443	1	0.05000000	1	0.10000000	4	0.80074456	6	0.80074412	3	0.80074597	4	0.81777047	1	0.80262506	1	0.80727153
70	4	0.81776994	4	0.81777001	1	0.10000000	4	0.81777016	6	0.8177698	3	0.81777056	4	0.8324845	1	0.81886666	1	0.82328032
80	4	0.83248403	6	0.83248372	1	0.10000000	4	0.83248423	5	0.83248386	6	0.8324839	4	0.84543885	1	0.83295004	1	0.83717743
90	4	0.84543843	1	0.05000000	1	0.10000000	4	0.84543861	6	0.84543827	5	0.84543831	4	0.85700897	2	0.84559956	1	0.84945606
100	4	0.85700858	1	0.05000000	1	0.10000000	4	0.85700875	5	0.85700841	6	0.85700847	6	0.8570085	2	0.85714354	1	0.86045459
(c) For $\sigma = 0.3$																		

Table 3. Auctioneer's maximum expected revenue per unit of time for a fixed number of bidders whose valuations follow standard normal distribution vs. number of bidders and number of bid levels with $u = 0$, $s = 1$.

n	$Z^*_{2,m=1}$	$Z^*_{2,m=2}$	$Z^*_{2,m=3}$	$Z^*_{2,m=4}$	$Z^*_{2,m=5}$	$Z^*_{2,m=6}$
2	**0.15159223**	0.15053723	0.13881274	0.12756683	0.11827636	0.11090719
5	0.28936203	**0.29757087**	0.28844512	0.28060803	0.27597318	0.27295334
10	0.42394118	**0.44273236**	0.43987166	0.43747813	0.43659812	0.43634733
20	0.57016821	0.59691713	**0.59775745**	0.59738084	0.59727475	0.59725696
30	0.65642471	0.6858042	**0.68746598**	0.6874113	0.6873847	0.68738094
40	0.71689002	0.74735379	0.74924787	**0.74928088**	0.74927258	0.74927136
50	0.7631264	0.79407212	0.79602389	**0.79608667**	0.79608439	0.79608391
60	0.80039396	0.83154351	0.83349029	0.83356364	**0.83356375**	0.83356312
70	0.83151396	0.86272481	0.86464218	0.86471862	**0.86471977**	0.86471926
80	0.85816929	0.88936348	0.89124211	0.89131846	**0.89132007**	0.89131961
90	0.88144177	0.91257497	0.91441203	0.91448688	**0.9144887**	0.9144883
100	0.90206662	0.9331132	0.93490879	0.93498153	**0.93498341**	0.9349831

2.3. Model with a random number of bidders and lognormal valuations

In real-world scenarios, the number of bidders in the auction is not fixed but it varies very often. So, in this section, we modeled the discrete Dutch auction with the number of bidders as a random variable. In this case, the arrival of the bidders is taken as Poisson's distributed having a non-negative mean λ, i.e., $\lambda \geq 0$ [16, 35, 36]. So, the probability of having n number of bidders in the auction is given by

$$P(n) = \frac{\lambda^n e^{-\lambda}}{n!}, \quad n = 0, 1, \ldots. \tag{8}$$

Hence, the probability of the item to be sold at the bid level

Table 4. Auctioneer's maximum expected revenue per unit of time for a fixed number of bidders whose valuations follow standard normal distribution vs. number of bidders and salvage value with $s = 1$.

n	$u = 0$		$u = 0.1$		$u = 0.2$		$u = 0.3$		$u = 0.4$		$u = 0.5$		$u = 0.6$		$u = 0.7$		$u = 0.8$	
	m^*	Z^*_{2,m^*}	m^*	Z^*_{2,m^*}	m^*	Z^*_{2,m^*}	m^*	Z^*_{2,m^*}	m^*	Z^*_{2,m^*}	m^*	Z^*_{2,m^*}	m^*	Z^*_{2,m^*}	m^*	Z^*_{2,m^*}	m^*	Z^*_{2,m^*}
2	1	0.15159223	1	0.18376061	1	0.21732315	1	0.25226957	1	0.28857758	1	0.32621282	1	0.36512937	1	0.40527056	1	0.44657026
5	2	0.29757087	1	0.31050312	1	0.33310504	1	0.3572504	1	0.38301359	1	0.41045725	1	0.4396287	1	0.47055662	1	0.50324838
10	2	0.44273236	2	0.44679466	1	0.45323952	1	0.46969119	1	0.48750015	1	0.50678852	1	0.52767893	1	0.55029008	1	0.57473123
20	3	0.59775745	2	0.59868508	2	0.60065192	2	0.60285364	1	0.61205945	1	0.62489097	1	0.63890139	1	0.65423027	1	0.67102801
30	3	0.68746598	3	0.68761172	2	0.68815336	2	0.68951981	2	0.69104563	1	0.69958334	1	0.71059621	1	0.72265519	1	0.73589826
40	4	0.74928088	3	0.74933719	3	0.74943665	2	0.75007116	2	0.75117182	1	0.75362183	1	0.76294106	1	0.77313272	1	0.78431807
50	4	0.79608667	4	0.79609143	3	0.79615534	3	0.79623213	2	0.79709524	2	0.79805489	1	0.80392392	1	0.81289731	1	0.82273199
60	5	0.83356375	4	0.83356686	3	0.83358946	3	0.83364685	2	0.83406197	2	0.83485297	1	0.83747451	1	0.84558371	1	0.85445751
70	5	0.86471977	4	0.86472097	4	0.86472356	3	0.86476638	2	0.86489501	2	0.8655708	2	0.86631889	1	0.87326309	1	0.88141424
80	5	0.89132007	4	0.89132026	4	0.89132224	3	0.89134444	3	0.89138549	2	0.89187114	2	0.89252443	1	0.8972227	1	0.90480791
90	5	0.9144887	6	0.91448836	4	0.91448989	3	0.91449873	3	0.91453327	2	0.91482472	2	0.91540641	1	0.91831507	1	0.92544318
100	5	0.93498341	4	0.93498271	4	0.93498399	4	0.93498541	3	0.93501355	2	0.9351597	2	0.93568541	1	0.93713286	1	0.9438828

l_i, $i = 1, 2, \ldots, m$ consistent with equation (2) is

$$P(l_i) = \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \left[F(l_{i+1})^n - F(l_i)^n\right], \qquad = e^{-\lambda}(e^{\lambda F(l_{i+1})} - e^{\lambda F(l_i)}). \tag{9}$$

If no bidder comes to the auction, obviously the object will go unsold, and in that case, the probability of such an occurrence is;

$$P(n = 0) = \frac{\lambda^0 e^{-\lambda}}{0!}, \qquad = e^{-\lambda}. \tag{10}$$

Thus the expected auction duration D_3 in this case will be equal to the sum of the expected time when no bidder arrives, the expected time for which the item goes unsold, and the expected time to sell the item. Taking (9) and (10) into consideration, we have;

$$\begin{aligned} D_3 &= sE(m), \\ &= s\left\{\sum_{i=1}^{m}(m+2-i)P(l_i) + (m+1)\left(1 - P(n=0) - \sum_{i=1}^{m} P(l_i)\right) + P(n=0)\right\}, \\ &= s\left\{\sum_{i=1}^{m}(m+2-i)(e^{-\lambda}(e^{\lambda F(l_{i+1})} - e^{\lambda F(l_i)})) + (m+1)\left(1 - (e^{-\lambda}) - \sum_{i=1}^{m}(e^{-\lambda}(e^{\lambda F(l_{i+1})} - e^{\lambda F(l_i)}))\right) + (e^{-\lambda})\right\}, \\ D_3 &= s\left\{e^{-\lambda} + \left(1 - e^{\lambda} + e^{-\lambda+\lambda F(l_1)} - e^{-\lambda+\lambda F(l_{m+1})}\right)(1+m) + \sum_{i=1}^{m} e^{-\lambda}\left(e^{\lambda F(l_{i+1})} - e^{\lambda F(l_i)}\right)(2-i+m)\right\}, \\ &= e^{-\lambda} s\left\{-m + \left(e^{\lambda} + e^{\lambda F(l_1)} - e^{\lambda F(l_{m+1})}\right)(1+m) + \sum_{i=1}^{m}\left(e^{\lambda F(l_{i+1})} - e^{\lambda F(l_i)}\right)(2-i+m)\right\}. \end{aligned} \tag{11}$$

Now, the auctioneer's expected revenue per unit of time, Z_3 can be written as;

$$\begin{aligned} Z_3 &= \frac{\sum_{i=1}^{m} l_i P(l_i) + u\left(1 - P(n=0) - \sum_{i=1}^{m} P(l_i)\right)}{D_3}, \\ Z_3 &= \frac{\left[\sum_{i=1}^{m} l_i\left(e^{-\lambda}(e^{\lambda F(l_{i+1})} - e^{\lambda F(l_i)})\right) + u\left(1 - e^{-\lambda} - \sum_{i=1}^{m}\left(e^{-\lambda}(e^{\lambda F(l_{i+1})} - e^{\lambda F(l_i)})\right)\right)\right]}{\left[e^{-\lambda} s\left\{-m + \left(e^{\lambda} + e^{\lambda F(l_1)} - e^{\lambda F(l_{m+1})}\right)(1+m) + \sum_{i=1}^{m}\left(e^{\lambda F(l_{i+1})} - e^{\lambda F(l_i)}\right)(2-i+m)\right\}\right]}, \\ &= \frac{\left[\left(-1 + e^{\lambda} + e^{\lambda F(l_1)} - e^{\lambda F(l_{m+1})}\right)u + \sum_{i=1}^{m} l_i\left(e^{\lambda F(l_{i+1})} - e^{\lambda F(l_i)}\right)\right]}{\left[s\left\{-m + \left(e^{\lambda} + e^{\lambda F(l_1)} - e^{\lambda F(l_{m+1})}\right)(m+1) + \sum_{i=1}^{m}\left(e^{\lambda F(l_{i+1})} - e^{\lambda F(l_i)}\right)(2-i+m)\right\}\right]}. \end{aligned} \tag{12}$$

Hence, our required model can be formulated as an NLP given below, where $l_1, l_2, \ldots, l_m$ are decision variables and m, λ, s and u are the parameters.

Maximize

$$Z_3 = \frac{\left[\left(-1 + e^{\lambda} + e^{\lambda F(l_1)} - e^{\lambda F(l_{m+1})}\right)u + \sum_{i=1}^{m} l_i\left(e^{\lambda F(l_{i+1})} - e^{\lambda F(l_i)}\right)\right]}{\left[s\left\{-m + \left(e^{\lambda} + e^{\lambda F(l_1)} - e^{\lambda F(l_{m+1})}\right)(m+1) + \sum_{i=1}^{m}\left(e^{\lambda F(l_{i+1})} - e^{\lambda F(l_i)}\right)(2-i+m)\right\}\right]},$$

subject to the constraints;

$$\begin{aligned} l_{i+1} &\geq l_i, \quad i = 1, 2, \ldots, m \quad , \\ l_1 &\geq u. \end{aligned} \tag{13}$$

If the bidders' valuations are drawn from the lognormal distribution, it follows that $f(l_i) = \frac{1}{l_i \sigma \sqrt{2\pi}} \exp\left(-\frac{(ln l_i - \mu)^2}{2\sigma^2}\right)$ and $F(l_i) = \frac{1}{2}\,\text{erfc}\left(-\frac{ln l_i - \mu}{\sigma\sqrt{2}}\right)$ where erfc is the complementary error function defined by $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt$.

Hence the modified NLP we get is as under;
Maximize

$$Z_3 = \frac{\left[\left(-1 + e^{\lambda} + e^{\lambda\left[\frac{1}{2}\,\mathrm{erfc}\left(-\frac{ln l_1 - \mu}{\sigma\sqrt{2}}\right)\right]} - e^{\lambda\left[\frac{1}{2}\,\mathrm{erfc}\left(-\frac{ln l_{m+1} - \mu}{\sigma\sqrt{2}}\right)\right]}\right)u + \sum_{i=1}^{m} l_i\left(e^{\lambda\left[\frac{1}{2}\,\mathrm{erfc}\left(-\frac{ln l_{i+1} - \mu}{\sigma\sqrt{2}}\right)\right]} - e^{\lambda\left[\frac{1}{2}\,\mathrm{erfc}\left(-\frac{ln l_i - \mu}{\sigma\sqrt{2}}\right)\right]}\right)\right]}{\left[s\left\{-m + \left(e^{\lambda} + e^{\lambda\left[\frac{1}{2}\,\mathrm{erfc}\left(-\frac{ln l_1 - \mu}{\sigma\sqrt{2}}\right)\right]} - e^{\lambda\left[\frac{1}{2}\,\mathrm{erfc}\left(-\frac{ln l_{m+1} - \mu}{\sigma\sqrt{2}}\right)\right]}\right)(m+1) + \sum_{i=1}^{m}\left(e^{\lambda\left[\frac{1}{2}\,\mathrm{erfc}\left(-\frac{ln l_{i+1} - \mu}{\sigma\sqrt{2}}\right)\right]} - e^{\lambda\left[\frac{1}{2}\,\mathrm{erfc}\left(-\frac{ln l_i - \mu}{\sigma\sqrt{2}}\right)\right]}\right)(2 - i + m)\right\}\right]},$$

subject to;

$$\begin{aligned} l_{i+1} &\geq l_i, \quad i = 1, 2, \ldots, m \quad, \\ l_1 &\geq u. \end{aligned} \tag{14}$$

Numerical analysis

The NLP (14) for different combinations of $n \in \{2, 5, 10, 20, \ldots, 100\}$, $u \in \{0, 0.1, \ldots, 0.8\}$, $m \in \{1, 2, \ldots, 6\}$ and $s = 1$ is set up and solved using the software R to understand how various combinations of the parameters can effect the auction outcomes. To start the discussion, in Table 5, we represented a special case of the auctioneer's maximum expected revenue per unit of time when the valuations of the bidders follow the lognormal distribution and the salvage value u is set as zero. Table 5(a), Table 5(b), and Table 5(c) are respectively the cases when $\sigma = 0.1$, $\sigma = 0.2$, and $\sigma = 0.3$ while $\mu = 0$ in each case. The highest value of Z_3^* among all the bid levels corresponding to each value of λ where $m \in \{1, 2, \ldots, 6\}$ is boldfaced. Again, we see that in most of the cases, Z_3^* increases with m and reaches the highest value at $m = 3$ or 4 with only a few exceptions when the revenue per unit of time is maximum for $m = 5$ or 6 which is slightly higher in value than that of $m = 3$ or 4. It verifies the fact that in most cases, only four bid levels are required to maximize the auctioneer's expected revenue per unit of time when the number of bidders in discrete Dutch auction follows the Poisson distribution [3], and sometimes it is tricky to predict.

Furthermore, in Table 6, a summary of the optimal number of bid levels along with the auctioneer's maximum expected revenue per unit of time Z_{3,m^*}^* for each of $u \in \{0, 0.1, \ldots, 0.8\}$ is given. Almost similar observations and interpretations can be made here as in Section 2.2 and the Table 6 shows that in most cases the optimal number of bid levels is 3 or 4 except in a few cases when it could be 5 or 6 with a very small incriment. Moreover, with the increase in the value of u, the optimal number of bid levels reduces to 1, *i.e.*, as few as one bid level is enough to maximize the auctioneer's expected revenue per unit of time regardless of the size λ of the bidding population. It can also be seen that Z_{3,m^*}^* is a non-decreasing function of u except for only a few exceptions when $Z_{3,m^*}^* = 0.1$ or 0.05.

Lastly, a comparison between the results for $Z_{1,m}^*$ in Section 2.1 with fixed number of bidders and $Z_{3,m}^*$ in Section 2.3 with random number of bidders shows that $Z_{1,m}^* > Z_{3,m}^*$ vis-a-vis in all cases except a few. The cause of these exceptions, the unpredictability of the revenue, is already discussed in Section 2.1. It makes intuitive and logical sense because the complete information about the bidding population's size goes in the auctioneer's favor and he/she can design more effective auctions to get better payoffs in such cases.

2.4. Model with random number of bidders and standard normal valuations

In this section, we took the valuation of the bidders following the standard normal distribution $N(\mu, \sigma^2)$ where $\mu = 0$ and $\sigma = 1$ with the number of bidders as a Poisson random variable. The modified discrete Dutch auction is formulated below.

If the valuations of the bidders follow the standard normal distribution, then the PDF and CDF are respectively $f(l_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}l_i^2}$ and $F(l_i) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{l_i}{\sqrt{2}}\right)\right]$, where erf is the error function given by $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$. Hence, the NLP (14) in this case becomes;

Maximize

$$Z_4 = \frac{\left[\left(-1 + e^{\lambda} + e^{\lambda\left[\frac{1}{2}\left(1+\mathrm{erf}\left(\frac{l_1}{\sqrt{2}}\right)\right)\right]} - e^{\lambda\left[\frac{1}{2}\left(1+\mathrm{erf}\left(\frac{l_{m+1}}{\sqrt{2}}\right)\right)\right]}\right)u + \sum_{i=1}^{m} l_i\left(e^{\lambda\left[\frac{1}{2}\left(1+\mathrm{erf}\left(\frac{l_{i+1}}{\sqrt{2}}\right)\right)\right]} - e^{\lambda\left[\frac{1}{2}\left(1+\mathrm{erf}\left(\frac{l_i}{\sqrt{2}}\right)\right)\right]}\right)\right]}{\left[s\left\{-m + (m+1)\left(e^{\lambda} + e^{\lambda\left[\frac{1}{2}\left(1+\mathrm{erf}\left(\frac{l_1}{\sqrt{2}}\right)\right)\right]} - e^{\lambda\left[\frac{1}{2}\left(1+\mathrm{erf}\left(\frac{l_{m+1}}{\sqrt{2}}\right)\right)\right]}\right) + \sum_{i=1}^{m}(2 - i + m)\left(e^{\lambda\left[\frac{1}{2}\left(1+\mathrm{erf}\left(\frac{l_{i+1}}{\sqrt{2}}\right)\right)\right]} - e^{\lambda\left[\frac{1}{2}\left(1+\mathrm{erf}\left(\frac{l_i}{\sqrt{2}}\right)\right)\right]}\right)\right\}\right]},$$

subject to;

$$\begin{aligned} l_{i+1} &\geq l_i, \quad i = 1, 2, \ldots, m \quad, \\ l_1 &\geq u. \end{aligned} \tag{15}$$

The NLP (15) for different combinations of $n \in \{2, 5, 10, 20, \ldots, 100\}$, $u \in \{0, 0.1, \ldots, 0.8\}$, $m \in \{1, 2, \ldots, 6\}$ and $s = 1$ is set up and solved using the software R to understand the effect of different combinations of the parameters on the auction outcomes. In Table 7, we have shown a special case of the auctioneer's maximum expected revenue per unit of time when the valuations of the bidders follow the standard normal distribution and the salvage value u is set as zero. The highest value of Z_4^* among all the bid levels corresponding to each value of λ where $m \in \{1, 2, \ldots, 6\}$ is boldfaced. Here, we see that with the increase in the value of m, Z_4^* keeps on increasing until reaches its maximum value. When λ is small, the optimal bid level is $m = 2$ or 3 but when the number of bidders increases, it turns $m = 4$ or 5 and doesn't go beyond $m = 5$. It verifies the fact that only five bid levels are required to maximize the auctioneer's expected revenue per unit of time when the number of bidders in discrete Dutch auction follows the Poisson distribution and the valuations of the bidders follow the standard normal distribution.

Furthermore, in Table 8, a summary of the optimal number of bid levels along with the auctioneer's maximum expected revenue per unit of time Z_{4,m^*}^* for each of $u \in \{0, 0.1, \ldots, 0.8\}$ is given. Almost similar observations and interpretations can

Table 5. Auctioneer's maximum expected revenue per unit of time for a variable number of bidders whose valuations follow lognormal distribution vs. number of bidders and number of bid levels with $u = 0$, $s = 1$, $\mu = 0$, and $\sigma \in \{0.1, 0.2, 0.3\}$.

λ	$Z^*_{3,m=1}$	$Z^*_{3,m=2}$	$Z^*_{3,m=3}$	$Z^*_{3,m=4}$	$Z^*_{3,m=5}$	$Z^*_{3,m=6}$
2	0.39518572	0.40652595	0.40710181	**0.40712359**	0.40712311	0.40712318
5	0.46478831	0.4740611	0.47452285	**0.47454006**	0.47453928	0.47453928
10	0.50411712	0.51123585	0.51147535	**0.51148453**	0.51148399	0.51148398
20	0.5339513	0.53965087	**0.53977422**	0.53977246	0.53977255	0.53977247
30	0.5486898	0.5538073	**0.55389837**	0.5538974	0.55389741	0.55389747
40	0.55832568	0.56311211	**0.56318798**	0.56318731	0.56318731	0.56318735
50	0.56542214	0.56998816	**0.570055**	0.57005447	0.57005447	0.5700545
60	0.5710092	0.57541464	**0.57547542**	0.57547498	0.575475	0.57547502
70	0.57560047	0.57988194	**0.57993831**	0.57993793	0.57993794	0.57993794
80	0.57948776	0.58366959	**0.58372258**	0.58372227	0.58372225	0.58372225
90	0.58285225	0.58695159	0.58697908	0.58700159	0.58700159	**0.5870016**
100	0.58581386	0.58984332	0.58986913	0.58989113	0.58986921	**0.58989114**
(a) For $\sigma = 0.1$						
2	0.36424917	0.38101932	0.38228251	0.38234358	**0.38234486**	0.38234459
5	0.46077752	0.47530803	0.47618152	**0.47621043**	0.47620965	0.4762092
10	0.52779777	0.54136422	0.54203169	0.54205666	**0.5420569**	0.54205654
20	0.58517499	0.5975154	0.59799615	**0.59801103**	0.59801039	0.59801047
30	0.61551777	0.62719526	0.62759219	**0.62760241**	0.62760214	0.62760215
40	0.63600699	0.64727104	0.64762202	**0.64763007**	0.64762666	0.64762661
50	0.65141038	0.66238544	0.66270686	0.66271029	**0.66271337**	0.66270983
60	0.66371917	0.67447715	0.67477762	**0.67478341**	0.67477987	0.67477614
70	0.67395137	0.68453816	**0.68482282**	0.68482127	0.68482106	0.68482114
80	0.68269606	0.69314307	**0.69341526**	0.69341367	0.69341367	0.69341384
90	0.69032415	0.70065402	**0.70091606**	0.70091425	0.7009145	0.70091462
100	0.697084	0.70731369	**0.70756724**	0.70756514	0.70756584	0.70756555
(b) For $\sigma = 0.2$						
2	0.34776507	0.36729663	0.36874431	**0.36875009**	0.36873919	0.36873807
5	0.46805136	0.48669614	**0.48761929**	0.48755673	0.48753914	0.48753764
10	0.56198961	0.5818029	0.58294301	**0.5829707**	0.58296602	0.58296513
20	0.64938874	0.66916139	0.67019258	0.67023198	**0.67023306**	0.67023289
30	0.69802012	0.71752711	0.71845188	**0.71848371**	0.71848306	0.71848303
40	0.73172332	0.75102599	0.75188248	**0.75190966**	0.75190882	0.75190888
50	0.75749805	0.77664611	0.77745505	**0.77747922**	0.77747856	0.77747856
60	0.77835591	0.79738328	0.79815682	**0.79817886**	0.79817827	0.79817832
70	0.79586757	0.81479781	0.81554368	**0.81556413**	0.81556359	0.81556366
80	0.81095549	0.82980556	0.83052904	**0.83054825**	0.83054778	0.83054779
90	0.82420734	0.8429899	0.84369474	**0.84371296**	0.84371254	0.84371262
100	0.83602047	0.85474521	0.85543424	**0.85545163**	0.85545126	0.85545126
(c) For $\sigma = 0.3$						

be made here as in Section 2.2 and the Table 8 shows that in most cases the optimal number of bid levels is 2 or 3 except in a few cases when it could be 4 or 5. Moreover, with the increase in the value of u the optimal number of bid levels reduces to 1 regardless of the size λ of the bidding population. It can also be seen that Z^*_{4,m^*} is a non-decreasing function of u.

Lastly, a comparison between the results for $Z^*_{2,m}$ in Section 2.2 with fixed number of bidders and $Z^*_{4,m}$ in Section 2.4 with a random number of bidders shows that $Z^*_{2,m} > Z^*_{4,m}$ vis-a-vis in all cases, making intuitive sense that the complete information about the participants in the auction goes in the auctioneer's favor [3].

3. Results and discussions

In this manuscript, the discrete Dutch auction is modeled as four different NLPs all aimed at maximizing the auctioneer's expected revenue per unit of time. The scenarios considered include the situations where the valuations of the bidders follow the lognormal distribution, *i.e.*, some of the bidders hold significantly higher valuations for the item to be auctioned off compared to the majority. The analysis is conducted for both fixed and random numbers of bidders. Additionally, scenarios involving bidders with valuations following the standard normal distribution are explored under similar conditions. Our major

Table 6. Auctioneer's maximum expected revenue per unit of time for a variable number of bidders whose valuations follow lognormal distribution vs. number of bidders and salvage value with $s = 1$, $\mu = 0$ and $\sigma \in \{0.1, 0.2, 0.3\}$.

λ	$u = 0$		$u = 0.1$		$u = 0.2$		$u = 0.3$		$u = 0.4$		$u = 0.5$		$u = 0.6$		$u = 0.7$		$u = 0.8$	
	m^*	Z^*_{3,m^*}	m^*	Z^*_{3,m^*}	m^*	Z^*_{3,m^*}	m^*	Z^*_{3,m^*}	m^*	Z^*_{3,m^*}	m^*	Z^*_{3,m^*}	m^*	Z^*_{3,m^*}	m^*	Z^*_{3,m^*}	m^*	Z^*_{3,m^*}
2	4	0.40712359	4	0.40712388	4	0.40712423	3	0.40713087	1	0.40830981	1	0.41330317	1	0.41955622	1	0.42769432	1	0.43882281
5	4	0.47454006	4	0.47454035	4	0.47454069	2	0.47458965	1	0.4756777	2	0.47512548	1	0.48421253	1	0.49010214	1	0.49794628
10	4	0.51148453	4	0.51148469	4	0.51148489	3	0.51148716	2	0.51153899	1	0.5131475	1	0.51604363	1	0.51970985	1	0.52462126
20	3	0.53977422	3	0.53977474	3	0.53977533	3	0.53977599	3	0.53977676	1	0.54005839	1	0.54201605	1	0.54448438	1	0.54776614
30	3	0.55389837	3	0.55389862	3	0.55389889	3	0.55389921	3	0.55389957	2	0.55390941	1	0.55546389	1	0.55751387	1	0.56021609
40	3	0.56318798	3	0.56318814	3	0.56318832	3	0.56318853	3	0.56318877	2	0.56319284	1	0.56441478	1	0.56624025	1	0.56863058
50	3	0.570055	3	0.57005512	3	0.57005526	3	0.57005542	3	0.5700556	2	0.57005688	1	0.57107228	1	0.57275333	1	0.57494308
60	3	0.57547542	3	0.57547552	3	0.57547563	3	0.57547576	4	0.57547603	3	0.57547608	2	0.57549439	2	0.57551761	2	0.57554774
70	3	0.57993831	3	0.57993839	3	0.57993849	3	0.5799386	3	0.57993872	5	0.57993831	2	0.57995438	1	0.58220306	1	0.58414252
80	3	0.58372258	1	0.05000000	5	0.58372235	6	0.58372242	6	0.5837225	3	0.58372308	2	0.58373655	1	0.58584152	1	0.5876957
90	6	0.5870016	1	0.05000000	1	0.10000000	6	0.58700173	5	0.5870018	3	0.58700233	2	0.58701425	1	0.58900239	1	0.59078693
100	4	0.58989113	1	0.05000000	1	0.10000000	4	0.58989129	6	0.58989132	3	0.58989182	2	0.58990251	1	0.59179322	1	0.59351947
(a) For $\sigma = 0.1$																		
2	5	0.38234486	4	0.38234797	2	0.38254083	2	0.38356215	1	0.39185621	1	0.40189615	1	0.41394219	1	0.42860039	1	0.44664196
5	4	0.47621043	3	0.4762363	2	0.4764648	2	0.47717709	1	0.48223714	1	0.48966179	1	0.49842145	1	0.50897093	1	0.52196753
10	5	0.5420569	4	0.54205834	3	0.54207164	2	0.5421341	2	0.54246672	1	0.54660389	1	0.55230594	1	0.55918042	1	0.56768421
20	4	0.59801103	4	0.59801119	4	0.59801138	4	0.59801158	3	0.59801645	1	0.59839008	1	0.6023325	1	0.60704287	1	0.61281108
30	4	0.62760241	4	0.62760247	4	0.62760254	4	0.62760262	3	0.62760316	2	0.62766308	1	0.63003655	1	0.63394821	1	0.63869396
40	4	0.64763007	5	0.64762983	4	0.64763015	4	0.64763019	4	0.64763024	2	0.6476493	1	0.64908938	1	0.65256506	1	0.65675252
50	5	0.66271337	4	0.66271369	4	0.66271371	4	0.66271374	4	0.66271378	3	0.6627149	1	0.66356024	1	0.66675339	1	0.67057974
60	4	0.67478341	5	0.67478145	6	0.67478354	4	0.67478367	4	0.67478369	3	0.6747843	2	0.6748528	2	0.67495169	2	0.67507109
70	3	0.68482282	6	0.68482609	3	0.68482476	5	0.68482818	4	0.68482825	5	0.6848282	2	0.68488126	1	0.68776342	1	0.69113777
80	3	0.69341526	1	0.05000000	1	0.10000000	6	0.6934202	5	0.69342021	3	0.6934204	2	0.69346149	1	0.69598224	1	0.69920246
90	3	0.70091606	1	0.05000000	1	0.10000000	4	0.70091691	6	0.70092059	3	0.70092071	2	0.70095295	1	0.70317903	1	0.70627338
100	3	0.70756724	1	0.05000000	1	0.10000000	6	0.70756775	6	0.70756903	3	0.70757151	2	0.70759677	1	0.70957654	1	0.71256556
(b) For $\sigma = 0.2$																		
2	4	0.36875009	3	0.36895959	2	0.37069186	1	0.37665353	1	0.38907022	1	0.40334394	1	0.41982229	1	0.43888425	1	0.4609063
5	3	0.48761929	2	0.48778494	2	0.48902075	1	0.49053164	1	0.49988643	1	0.51051002	1	0.52268646	1	0.5367691	1	0.55318486
10	4	0.5829707	3	0.58299395	3	0.58305137	2	0.58346617	2	0.5841772	1	0.59074425	1	0.59894521	1	0.6084371	1	0.61954841
20	5	0.67023306	5	0.6702331	4	0.67023353	3	0.67024087	2	0.67032961	2	0.6707224	1	0.67583018	1	0.68244517	1	0.69012156
30	4	0.71848371	4	0.71848403	4	0.71848437	4	0.71848477	3	0.71849164	2	0.71865529	1	0.72052489	1	0.72604039	1	0.73238922
40	4	0.75190966	4	0.75190985	4	0.75191007	4	0.75191031	3	0.75191117	2	0.75195031	2	0.7522072	1	0.7569596	1	0.76257489
50	4	0.77747922	4	0.77747936	4	0.77747951	4	0.77747968	4	0.77747987	3	0.77748539	2	0.77766975	1	0.78091375	1	0.78605177
60	4	0.79817886	4	0.79817897	4	0.79817909	4	0.79817922	4	0.79817937	3	0.79818237	2	0.79830141	1	0.80046303	2	0.79878033
70	4	0.81556413	5	0.81556366	4	0.81556432	6	0.81556384	4	0.81556456	4	0.81556469	2	0.81563957	1	0.81697534	1	0.82151338
80	4	0.83054825	6	0.83054785	1	0.10000000	4	0.83054851	6	0.8305481	3	0.83054897	2	0.83058904	1	0.83126725	1	0.83560013
90	4	0.84371296	6	0.84371262	1	0.10000000	4	0.84371319	6	0.8437128	3	0.84371289	2	0.84372715	2	0.84390102	1	0.8480303
100	4	0.85545163	3	0.85543703	1	0.10000000	4	0.85545183	5	0.85545148	6	0.85545148	4	0.8554521	2	0.85560859	1	0.85915283
(c) For $\sigma = 0.3$																		

Table 7. Auctioneer's maximum expected revenue per unit of time for a variable number of bidders whose valuations follow standard normal distribution vs. number of bidders and number of bid levels with $u = 0$, $s = 1$.

λ	$Z^*_{4,m=1}$	$Z^*_{4,m=2}$	$Z^*_{4,m=3}$	$Z^*_{4,m=4}$	$Z^*_{4,m=5}$	$Z^*_{4,m=6}$
2	0.14887966	**0.14990085**	0.13907163	0.12813552	0.11884314	0.1112753
5	0.27474548	**0.27921953**	0.26689333	0.25561712	0.24751556	0.24152551
10	0.41056476	**0.4261292**	0.42060115	0.41607403	0.41384138	0.41270035
20	0.56090046	0.58636238	**0.58640135**	0.58561455	0.58535963	0.58529909
30	0.64946585	0.67830358	**0.67968006**	0.6795058	0.67944645	0.67943602
40	0.71133661	0.74157787	**0.74336061**	0.74334873	0.74332977	0.74332677
50	0.75851025	0.7893893	0.79130087	**0.7913446**	0.79133801	0.79133689
60	0.79644567	0.82761163	0.82955106	**0.82961609**	0.82961421	0.82961373
70	0.82806518	0.85933927	0.86126561	0.86133872	**0.86133889**	0.86133824
80	0.85510805	0.88639271	0.8882887	0.88836421	**0.88836534**	0.88836472
90	0.87868989	0.90992953	0.91178825	0.91186358	**0.91186516**	0.91186467
100	0.89956738	0.93072961	0.93254894	0.93262286	**0.93262464**	0.93262425

findings encapsulate several noteworthy insights.

The emotional attachment effect, modeled using lognormal distribution, results in higher bids and increased auctioneer revenue. This aligns with the study of Adam *et al.* [26], which investigates the impact of incidental arousal, arousal unrelated to the auction environment, on bidding behavior. Through a series of experiments, they demonstrate that such arousal can significantly increase bidding in real monetary stakes auctions. In the first experiment, physiological measurements confirmed that induced arousal influenced bidding patterns. The second experiment illustrated that even non-competitive forms of arousal, such as cognitive dissonance, can lead to elevated bids. The third experiment revealed that bidders often remain unaware of how incidental arousal affects their decisions, complicating efforts to counteract its influence. These findings indicate that external factors, such as music and ambient conditions, can in-

Table 8. Auctioneer's maximum expected revenue per unit of time for a variable number of bidders whose valuations follow standard normal distribution vs. number of bidders and salvage value with $s = 1$.

λ	$u = 0$		$u = 0.1$		$u = 0.2$		$u = 0.3$		$u = 0.4$		$u = 0.5$		$u = 0.6$		$u = 0.7$		$u = 0.8$	
	m^*	Z^*_{4,m^*}	m^*	Z^*_{4,m^*}	m^*	Z^*_{4,m^*}	m^*	Z^*_{4,m^*}	m^*	Z^*_{4,m^*}	m^*	Z^*_{4,m^*}	m^*	Z^*_{4,m^*}	m^*	Z^*_{4,m^*}	m^*	Z^*_{4,m^*}
2	2	0.14990085	1	0.1785749	1	0.20944124	1	0.24148605	1	0.27470755	1	0.30909401	1	0.34462332	1	0.38126294	1	0.41897015
5	2	0.27921953	1	0.29774366	1	0.32205815	1	0.34775932	1	0.37491201	1	0.40357268	1	0.43378655	1	0.46558461	1	0.49898092
10	2	0.4261292	2	0.43124075	1	0.44250953	1	0.4602335	1	0.47926546	1	0.49971524	1	0.52169319	1	0.54530651	1	0.57065457
20	3	0.58640135	2	0.58847266	2	0.59080634	2	0.59340141	1	0.60544457	1	0.61894793	1	0.63362404	1	0.6496058	1	0.66703594
30	3	0.67968006	3	0.67987314	2	0.68099303	2	0.68255024	2	0.6842827	1	0.6947192	1	0.70617532	1	0.71868053	1	0.73236904
40	3	0.74336061	3	0.74347258	3	0.74359714	2	0.744608	2	0.74583062	1	0.74954403	1	0.75918394	1	0.76970279	1	0.78122001
50	4	0.7913446	3	0.79137631	3	0.79145959	2	0.79175502	2	0.79270085	2	0.79374849	1	0.80066921	1	0.80989561	1	0.81998936
60	4	0.82961609	4	0.82962036	3	0.82966768	3	0.82973514	2	0.83033685	2	0.83119072	1	0.83460745	1	0.84292012	1	0.85200352
70	5	0.86133889	4	0.86134172	3	0.86135655	3	0.86140879	2	0.8616659	2	0.86238908	1	0.8632423	1	0.87087095	1	0.87919643
80	5	0.88836534	4	0.88836646	4	0.88836893	3	0.88840486	3	0.88845145	2	0.88906064	2	0.88975441	1	0.89505247	1	0.90278582
90	5	0.91186516	4	0.91186533	4	0.91186725	3	0.91188546	3	0.91192419	2	0.91230916	2	0.91292352	1	0.91632929	1	0.9235855
100	5	0.93262464	5	0.9326247	4	0.93262581	3	0.93263223	3	0.93266522	2	0.93288389	2	0.93343661	1	0.93530267	1	0.94216494

advertently amplify arousal and contribute to what is known as "auction fever." The study underscores the importance of recognizing emotional influences in economic decision-making and offers practical insights for both auction organizers and participants. Moreover, several studies support the finding that emotional attachment can lead to increased revenue [27–29]. Our results further confirm that higher bidding and increased auctioneer revenue align with these findings, highlighting the critical role of emotional attachment in enhancing bidding behavior.

To optimize the auctioneer's expected revenue per unit of time, it is observed that a small number of bid levels generally suffices. Specifically, in situations where the salvage value is relatively substantial, a single bid level proves to be adequate. This strategic choice ensures that the discrete Dutch auction attains a swift conclusion, minimizing its time duration.

A notable observation emerges regarding the predictability of auctioneer revenue per unit of time when bidder valuations follow a lognormal distribution. The inherent unpredictability is rationalized by the practical reality that precise predictions are challenging until the conclusion of the auction.

Furthermore, an increase in the bidding population consistently results in higher auctioneer revenue per unit of time, irrespective of the salvage value. This finding underscores the inherent competitiveness in auctions, where a greater number of bidders fosters heightened competition and, consequently, increased revenue per unit of time.

Accurate knowledge about the number of bidders proves advantageous for the auctioneer, enabling the design of a more effective auction format tailored to the bidding population size. Consequently, the revenue per unit of time in scenarios with a fixed number of bidders surpasses that in cases where the number of bidders follows a Poisson distribution.

Lastly, it is observed that the auctioneer's maximum expected revenue per unit of time exhibits a non-decreasing trend with the salvage value, underscoring the pivotal role of this parameter in shaping auction outcomes.

4. Conclusion

Our investigation into modified discrete Dutch auctions has illuminated key insights into the optimization of auctioneer revenue dynamics. The formulated NLP models, addressing scenarios where bidder valuations adhere to both lognormal and standard normal distributions, have provided valuable observations.

Our findings highlight the efficacy of employing a strategic choice of bid levels to shorten auction durations while maximizing the auctioneer's expected revenue per unit of time, particularly in cases where salvage values are substantial. Notably, the inherent unpredictability in predicting auctioneer revenue per unit of time is underscored when bidder valuations follow a lognormal distribution, reflecting the dynamic and unpredictable nature of real-world auction scenarios. The study underscores the positive correlation between an increased bidding population and augmented auctioneer revenue per unit of time, emphasizing the competitive dynamics inherent in auctions. Accurate knowledge about the number of bidders emerges as a significant advantage for the auctioneer, allowing for the design of more effective auction formats tailored to the bidding population size. Crucially, our analysis reveals a non-decreasing trend in the auctioneer's maximum expected revenue per unit of time with the salvage value. This underscores the pivotal role played by the salvage value in influencing auction outcomes, demonstrating its significance as a determining factor. In essence, our research contributes valuable insights into the strategic design of discrete Dutch auctions, offering practical considerations for auctioneers dealing with perishable products or services and navigating dynamic auction environments with varying bidder valuations. These findings contribute to the broader understanding of auction dynamics and provide a foundation for future research in auction theory and design.

While we assert the originality and substantive contribution of our study to the domain of auction design, it is crucial to acknowledge certain limitations inherent in our work. Notably, our inability to provide closed-form formulas for computing optimal solutions in the four models discussed is a noteworthy constraint, despite our dedicated efforts to tackle the inherent complexity of these problems. In contemplating future avenues for research, several promising directions come to the fore. First and foremost, the assumption in our models regarding the rationality of all bidders, bidding based on their true valuations to maximize their chances of winning without overbidding, merits further exploration. Analyzing scenarios where bidders may acquire additional information during the auction or strategically wait for a lower asking price could yield valuable insights. Secondly, we express a keen interest in delving into the impact of bidders' risk attitudes, whether they are risk-averse, risk-neutral, or risk-seeking, on auction outcomes. The incorporation of risk-utility functions, such as the Arrow-Pratt coefficient of absolute risk aversion and the Arrow-Pratt–De Finetti coefficient of relative risk aversion [37], holds the potential for a significant understanding. Furthermore, while our existing models treat clock speed as a predetermined parameter, a fruitful avenue for future research involves investigating the influence of clock speed on the auctioneer's overall revenue. This entails exploring scenarios where clock speed is not fixed and its adjustment affects the decision-making dynamics of bidders. Additionally, our focus on maximizing auctioneer revenue without expanding auction house capacity prompts consideration of the effects of such expansion. This entails assessing factors like building or renting more auction rooms, extending operating hours, or acquiring additional facilities. Lastly, our study could be extended by introducing costs associated with setting each bid level in a discrete Dutch auction. This would involve accounting for various expenses incurred, such as rent for auction rooms and administrative charges, with the overarching objective of maximizing the auctioneer's expected net revenue per unit of time. While similar concepts have been explored in the context of English [35, 38] and Dutch auctions [22], our approach may offer unique insights and merits further exploration in subsequent research endeavors.

Data Availability

We do not have any research data outside the manuscript file.

References

[1] A. Cheema, D. Chakravarti & A. R. Sinha, "Bidding behavior in descending and ascending auctions", Marketing Science **31** (2012) 779. https://doi.org/10.1287/mksc.1120.0730.

[2] M. T. Adam, A. Eidels, E. Lux & T. Teubner, "Bidding behavior in Dutch auctions: Insights from a structured literature review", International Journal of Electronic Commerce **21** (2017) 363. https://doi.org/10.1080/10864415.2016.1319222.

[3] Z. Li, J. Yue & C. C. Kuo "Design of discrete dutch auctions with consideration of time", European Journal of Operational Research **265** (2018) 1159. https://doi.org/10.1016/j.ejor.2017.08.043.

[4] P. Miettinen, "Information acquisition during a dutch auction", Journal of Economic Theory **148** (2013) 1213. https://doi.org/10.1016/j.jet.2012.09.018.

[5] R. P. McAfee & J. McMillan, "Auctions with a stochastic number of bidders", Journal of economic theory **43** (1987) 1 https://doi.org/10.1016/0022-0531(87)90113-X.

[6] M. A. Robinson & R. Robinson, "Dutch-auction IPOs:institutional development and underpricing performance", Journal of Economics and Finance **36** (2012) 521. https://doi.org/10.1007/s12197-010-9166-3.

[7] V. Gretschko, A. Rasch & A. Wambach, "On the strictly descending multi-unit auction", Journal of Mathematical Economics **50** (2014) 79. https://doi.org/10.1016/j.jmateco.2013.10.009.

[8] C. Brown & K. Davis, "Taxes, tenders and the design of australian off-market share repurchases", Accounting & Finance **52** (2012) 109. https://doi.org/10.1111/j.1467-629X.2011.00445.x.

[9] M. Bichler, V. Grimm, S. Kretschmer & P. Sutterer, "Market design for renewable energy auctions: An analysis of alternative auction formats", Energy Economics **92** (2020) 104904. https://doi.org/10.1016/j.eneco.2020.104904.

[10] G. Rubio-Domingo & P. Linares, "The future investment costs of offshore wind: An estimation based on auction results", Renewable and Sustainable Energy Reviews **148** (2021) 111324. https://doi.org/10.1016/j.rser.2021.111324.

[11] H. Qiu, K. Zhu, N. C. Luong, C. Yi, D. Niyato & D. I. Kim, "Applications of auction and mechanism design in edge computing: A survey", IEEE Transactions on Cognitive Communications and Networking **8** (2022) 1034. https://doi.org/10.1109/TCCN.2022.3147196.

[12] Z. Shi, C. de Laat, P. Grosso & Z. Zhao, "Integration of blockchain and auction models: A survey, some applications, and challenges", IEEE Communications Surveys & Tutorials **25** (2022) 497. https://doi.org/10.1109/COMST.2022.3222403.

[13] H. Qiu & T. Li, "Auction method to prevent bid-rigging strategies in mobile blockchain edge computing resource allocation", Future Generation Computer Systems, **128** (2022) 1. https://doi.org/10.1016/j.future.2021.09.031.

[14] N. Sharghivand, F. Derakhshan & N. Siasi, "A comprehensive survey on auction mechanism design for cloud/edge resource management and pricing", IEEE Access **9** (2021) 126502. https://doi.org/10.1109/ACCESS.2021.3110914.

[15] Z. Li & C. C. Kuo, "Revenue-maximizing dutch auctions with discrete bid levels", European Journal of Operational Research **215** (2011) 721. https://doi.org/10.1016/j.ejor.2011.05.039.

[16] Z. Li and C. C. Kuo, "Design of discrete dutch auctions with an uncertain number of bidders", Annals of Operations Research **211** (2013) 255. https://doi.org/10.1007/s10479-013-1331-6.

[17] M. H. Rothkopf & R. M. Harstad, "On the role of discrete bid levels in oral auctions", European Journal of Operational Research **74** (1994) 572. https://doi.org/10.1016/0377-2217(94)90232-1.

[18] M. Truong, A. Gupta, W. Ketter & E. V. Heck, "Buyers' strategic behavior in b2b multichannel auction markets: When an online posted price channel is incorporated into a dutch auction system", Information Systems Research **33** (2022) 1344. https://doi.org/10.1287/isre.2022.1180.

[19] A. Kambil & E. Van Heck, "Reengineering the dutch flower auctions:A framework for analyzing exchange organizations", Information Systems Research, **9** (1998) 1. https://doi.org/10.1287/isre.9.1.1.

[20] M. Gallegati, G. Giulioni, A. Kirman & A. Palestrini, "What's that got to do with the price of fish? buyers behavior on the ancona fish market", Journal of Economic Behavior & Organization, **80** (2011) 20. https://doi.org/10.1016/j.jebo.2011.01.011.

[21] J. Yu, *Discrete approximation of continuous allocation mechanisms*, Ph.D. dissertation, Deaprtment of Humanities and Social Sciences, California Institute of Technology, California, USA, 1999. https://doi.org/10.7907/8SF1-5M95.

[22] W. H. Yuen, C. W. Sung & W. S. Wong, "Optimal price decremental strategy for dutch auctions", Communications in Information and Systems **2** (2022) 411. https://doi.org/10.4310/CIS.2002.v2.n4.a6.

[23] R. Cassady, *Auctions and auctioneering*, University of California Press, California, USA, 1967, pp. 92-94. https://doi.org/10.1525/9780520322257.

[24] T. Nedelec, C. Calauzenes, V. Perchet & N. El Karoui, "Revenuemaximizing auctions: a bidder's standpoint", Operations Research, **70** (2022) 2767. https://doi.org/10.1287/opre.2022.2316.

[25] T. Chipty, S. R. Cosslett & L. F. Dunn, "A race against the clock: Auctioneer strategies and selling mechanisms in live outcry auctions", Available at SSRN 2566616 **1** (2015) 1. https://doi.org/10.2139/ssrn.2566616.

[26] M. T. Adam, G. Ku & E. Lux, "Auction fever: The unrecognized effects of incidental arousal", Journal of Experimental Social Psychology, **80** (2019) 52. https://doi.org/10.1016/j.jesp.2018.07.009.

[27] M. T. Adam, J. Krämer, C. Jähnig, S. Seifert, & C. Weinhardt, "Understanding auction fever: A framework for emotional bidding", Electronic Markets, **21** (2011) 197. https://doi.org/10.1007/s12525-011-0068-9.

[28] M. T. P. Adam & J. Krämer, "Evaluating the emotional bidding framework: new evidence from a decade of neurophysiology", Electronic Markets, **32** (2022) 1529. https://doi.org/10.1007/s12525-022-00555-x.

[29] G. Ku, D. Malhotra & J. K. Murnighan, "Towards a competitive arousal model of decision-making: A study of auction fever in live and internet

auctions", Organizational Behavior and Human decision processes, **96** (2005) 89. https://doi.org/10.1016/j.obhdp.2004.10.001.
[30] M. Isaac, S. Pevnitskaya & K. S. Schnier, "Individual behavior and bidding heterogeneity in sealed bid auctions where the number of bidders is unknown", Economic Inquiry, **50** (2012) 516. https://doi.org/10.1111/j.1465-7295.2011.00393.x.
[31] D. Levin & E. Ozdenoren, "Auctions with uncertain numbers of bidders", Journal of Economic Theory, **118** (2004) 229. https://doi.org/10.1016/j.jet.2003.11.007.
[32] W. L. Winston, M. Venkataramanan & J. B. Goldberg, *Introduction to mathematical programming*, Duxbury Press, Belmont, CA, 2004. https://dl.acm.org/doi/abs/10.5555/1202147.
[33] E. Katok & A. M. Kwasnica, "Time is money: The effect of clock speed on seller's revenue in dutch auctions", Experimental Economics **11** (2008) 344. https://doi.org/10.1007/s10683-007-9169-x.
[34] E. Guerci, A. Kirman & S. Moulet, "Learning to bid in sequential dutch auctions", Journal of Economic Dynamics and Control **48** (2014) 374. https://doi.org/10.1016/j.jedc.2014.09.029.
[35] E. David, A. Rogers, N. R. Jennings, J. Schiff, S. Kraus & M. H. Rothkopf, "Optimal design of english auctions with discrete bid levels", ACM Transactions on Internet Technology **7** (2007) 12. https://doi.org/10.1145/1239971.1239976.
[36] J. Gallien & S. Gupta, "Temporary and permanent buyout prices in online auctions", Management Science, **53** (2007) 814. https://doi.org/10.1287/mnsc.1060.0650.
[37] M. Gentry, T. Li & J. Lu, "Existence of monotone equilibrium in first price auctions with private risk aversion and private initial wealth", Games and Economic Behavior **94** (2015) 214. https://doi.org/10.1016/j.geb.2015.07.003.
[38] U. Bergmann & A. Konovalov, "Auction design and order of sale with budget-constrained bidders", Experimental Economics **27** (2024) 36. https://doi.org/10.1007/s10683-023-09812-y.