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Asymptotic stability analysis of a fractional epidemic model for Ebola virus disease in Caputo sense

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Abstract

In this work, a non-integer-order epidemic system modelling the nonlinear dynamics of Ebola virus disease is formulated in the sense of Caputo-fractional derivative. The existence and uniqueness of solution of the model is established. More importantly, the theoretical analysis carried out is aimed at establishing the local and global asymptotic stability properties of the disease-free steady state of the epidemic model using the fractional Routh-Hurwitz criterion and Lyapunov functional technique, respectively. It is proved that the steady state is locally and globally asymptotically stable at the value of the key epidemiological threshold quantity lower than unity. The result is numerically validated for different values of fractional order to show the asymptotic behavior of the disease dynamics. This result is significant for fighting and preventing Ebola epidemic in the population, since the Caputo derivative operator allows for effective description of the disease dynamics with memory, where the future evolution of the disease is governed by its prior history.

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1. Introduction

Ebola virus disease (EVD) is a dangerous hemorrhagic condition resulting into serious public health concern. The disease derives its name from Ebola River in the Democratic Republic of Congo where it was first discovered in 1976. Since its discovery, the disease has claimed more than 34000 cases and over 15000 deaths across some countries of the world with case fatality rates varying between 25% and 90%. Although, EVD is rare globally but it breaks out occasionally on the continent of Africa where the largest outbreak of the disease occurred between 2014–2016 [1]. The disease is caused by virus infection of genus *Ebolavirus* with four species affecting humans, namely *Zaire ebolavirus*, *Sudan ebolavirus*, *Taï Forest ebolavirus* and *Bundibugyo ebolavirus*. It is important to mention that these species and other two species such as *Reston ebolavirus* and *Bombali ebolavirus* also affect bats or nonhuman primates, including monkeys, gorillas and chimpanzees. Therefore, EVD is an animal-borne disease and it can spread to humans through direct contact with the blood, body fluids and tissues of animals [2]. In particular within human population, EVD can spread through direct contact with body fluids of infected person either living or dead. The signs and symptoms

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of the disease include some of fever, pains, fatigue, sore throat, loss of appetite, diarrhea, bleeding, red eyes, skin rash and hiccups. However, EVD is treatable, but it is deadly in the absence of good supportive clinical care [1, 2].

The study of EVD dynamics using mathematical modelling tools is vast in the literature (see, e.g., [3–10]). Takaidza *et al.* [4] formulated and analyzed the Ebola virus disease with nonlinear incidence rate and examined the optimal measures for controlling the disease using education and treatment strategies. In Ref. [7], a mathematical model which includes vaccination of individuals was developed to understand the population dynamics of EVD transmission which occurred in Liberia in 2014. And different cases of vaccination were discussed in order to study the effect of vaccination on the infected individuals over time using optimal control. Okyere *et al.* [8] applied optimal control theory to study existing mathematical models for EVD and showed that effective educational campaigns with vaccination of susceptible individuals and effective treatment of patients can help reduce the spread of the disease. Authors in Ref. [9] studied an SEIR type model to illustrate the mode of impact of EVD on the human population via domestic and wild animals. An attempt was made by Verma *et al.* [11] to incorporate fuzziness into EVD transmission models and characterized the spread trajectories of the disease outbreak by stability analysis.

While the aforementioned literature employed ordinary or integer-order differential equations (ODEs) to study the dynamics of the EVD transmission, as in other related studies on infectious disease modelling [12, 13], it is important to mention that fractional or non-integer-order differential equations (FDEs), which are generalizations of ODEs, can be used to model infectious diseases and other complex phenomena [14–20]. The use of FDEs in infectious disease modelling has been proven to provide superior data fitting results when compared to the use of ODEs. This is so because fractional order models have more degree of freedom for assessing disease dynamics than their classical order models counterparts [21, 22]. Few studies in the literature have considered the transmission dynamics of EVD using fractional derivatives. Raza *et al.* [23] studied fractional Ebola virus model with Caputo fractional derivative operator and employed Laplace Adomian decomposition method to derive numerical solutions for evaluating the effects of the fractional parameter on the EVD spread. Srivastava and Saad [24] investigated the numerical solutions of the fractal-fractional Ebola virus based on three different kernels, namely power law, exponential decay and the generalized Mittag-Leffler function. In Farman *et al.* [25], the authors presented and analysed a nonlinear time-fractional order system to model the outbreak of EVD in a community using Atangana-Baleanu Caputo derivative operator. The existence and uniqueness of solutions of the model were proved using fixed point theorem. In addition, simulations of the fractional-order system for the outbreaks of Ebola virus disease were performed using real data in Ref. [26]. An EVD model was studied by Ref. [27] using Caputo-Fabrizio arbitrary operator and β-conformable derivative operator to illustrate the disease dynamics and memory effects. In Ref. [28], an integer-order mathematical model of Ebola virus in bats was extended to a fractional order using Atangana-Baleanu operator, and stability of the solution for the model was presented.

Non-integer-order derivatives have attracted considerable attention due to their ability to model real-world system involving memory, wide range of interactions and hereditary properties which are ignored in classical derivatives [29, 30]. Since past information about the spread of any disease helps reduce the impact of such disease whenever it breaks out, it suffices to study the memory effect of Ebola virus disease as it spreads in the population in order to gain more insight into the dynamics of the EVD. The choice of the Caputo fractional derivative operator over other fractional derivatives in the literature stems from the singularity of its kernel capable of representing current history of infectious diseases more efficiently while still tracking the effect of the distant part history of the disease [31, 32]. In addition, Caputo fractional derivative operator obeys the fundamental law of fractional calculus where the derivative is the left-inverse of the integral operator unconditionally [33]. Therefore, this work is set out to investigate the effect of fractional order in terms of memory on the asymptotic dynamics of an EVD epidemic model with non-integer-order Caputo derivative operator.

The rest of the work is arranged as follows. Section 2 deals with the presentation of the preliminary concepts and formulation of the fractional EVD model. The existence and uniqueness results with the stability analysis of solution are presented in Section 3, while Section 4 discusses the numerical simulations. And the concluding remarks of the work are provided in Section 5.

2. Preliminary concepts and formulation

Understanding the following concepts, as presented in Refs. [9, 29, 34, 35], is prerequisite for the derivation of the fractional epidemic model for Ebola virus disease.

Definition 2.1. *Given a function $f : \mathbb{R}_+ \to \mathbb{R}$ and $\alpha \in \mathbb{R}_+$, the Riemann-Liouville fractional α-order integral operator of function $f(t)$, denoted by $I_t^\alpha f(t)$, is defined as:*

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \tag{1}$$

for $\alpha \in (0, 1)$, $t > 0$ and $\Gamma(\alpha)$ represents the gamma function of α defined by:

$$\Gamma(\alpha) = \int_0^\infty m^{\alpha-1} e^{-m} dm. \tag{2}$$

Definition 2.2. *Given a function $f : \mathbb{R}_+ \to \mathbb{R}$ and $\alpha \in \mathbb{R}_+$, the Caputo fractional α-order derivative of $f(t)$, denoted by ${}^cD_t^\alpha f(t)$, is defined as:*

$${}^cD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau, \tag{3}$$

for $0 < \alpha \leq 1$, where $D_t^\alpha = d^\alpha / dt^\alpha$.

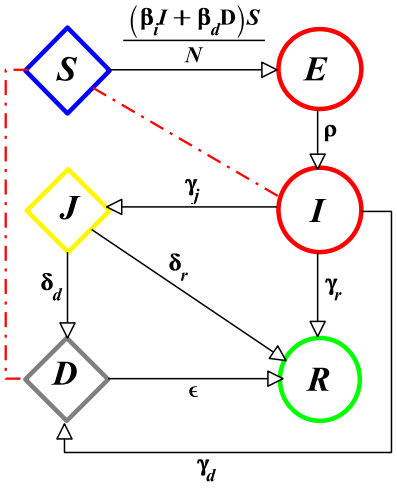


Figure 1: The Schematic flow of Ebola virus disease transmission with no vital dynamics.

Lemma 1. *Let $g : \mathbb{R}^n \to \mathbb{R}^n$, the Caputo fractional α-order system of the form*

$$ {}^cD_t^\alpha z(t) = g(z), \quad z(t_0) = z_0, \tag{4}$$

where $0 < \alpha \leq 1$ and $t_0 \in \mathbb{R}$ with $z_0 \in \mathbb{R}^n$, has a unique solution $z(t)$ on the interval $[t_0, +\infty)$ if function g satisfies the following properties:

p_1. $g(z)$ and $\frac{\partial g}{\partial z}$ are continuous $\forall\ z \in \mathbb{R}^n$

p_2. $\|g(z)\| \leq k_1 + k_2\|z\|\ \forall\ z \in \mathbb{R}^n$, where k_1 and k_2 are two non-negative constants.

Lemma 2. *Let $x(t) \in C[a,b]$ and ${}^cD_t^\alpha x(t) \in C[a,b]$ for $0 < \alpha \leq 1$, then*

$$ x(t) = x(a) + \frac{1}{\Gamma(\alpha)} {}^cD_t^\alpha x(\vartheta)(t-a)^\alpha, \quad a < \vartheta < t, \ \forall\ t \in (a,b]. \tag{5}$$

Moreover, if ${}^cD_t^\alpha x(t) \geq 0\ \forall\ t \in [a,b]$, then $x(t)$ is nondecreasing for each $t \in [a,b]$. And if ${}^cD_t^\alpha x(t) \leq 0\ \forall\ t \in [a,b]$, then $x(t)$ is non-increasing for each $t \in [a,b]$.

2.1. Non-integer-order EVD model

Here, the classical epidemic model for EVD studied by Adeyoyin [36] is generalized using the Caputo fractional derivative operator. See Figure 1 for the schematic diagram depicting the transmission dynamics of Ebola virus disease. Thus, the system of the fractional α-order differential equations is given by

Table 1: The EVD model parameters and description.

Parameter	Description
β_i	Transmission rate from infectious class
β_d	Transmission rate from dead class
ρ	Progression rate of the exposed class
γ_j	Rate at which infectious individuals are identified and isolated
γ_r	Rate at which infectious individuals recover due to treatment
γ_d	EVD-induced death rate for infectious class
δ_r	Rate at which isolated individuals recover due to treatment
δ_d	EVD-induced death rate for isolated class
ϵ	Burial rate for the dead class
α	Order of the fractional derivative

$$\begin{aligned}
{}^cD_t^\alpha S(t) &= -(\beta_i I + \beta_d D)\frac{S}{N}, \\
{}^cD_t^\alpha E(t) &= (\beta_i I + \beta_d D)\frac{S}{N} - \rho E, \\
{}^cD_t^\alpha I(t) &= \rho E - (\gamma_j + \gamma_r + \gamma_d) I, \\
{}^cD_t^\alpha J(t) &= \gamma_j I - (\delta_r + \delta_d) J, \\
{}^cD_t^\alpha D(t) &= \gamma_d I + \delta_d J - \epsilon D, \\
{}^cD_t^\alpha R(t) &= \gamma_r I + \delta_r J + \epsilon D,
\end{aligned} \tag{6}$$

with non-negative initial conditions,

$$\begin{aligned}
&S(0) = S_0, E(0) = E_0, I(0) = I_0, J(0) = J_0, D(0) = D_0, \\
&R(0) = R_0.
\end{aligned} \tag{7}$$

The Ebola virus disease model is formulated under the assumption of constant population, since the epidemic is usually short-lived. Hence, such vital dynamics as births and natural deaths are not considered [7, 36, 37]. The total constant population, N, is the sum of all the six classes of population. That is, $N = S + E + I + J + D + R$, where S is the susceptible class comprising of individuals who are vulnerable to EVD; E is the exposed class comprising individuals who are already infected but cannot transmit the disease; I represents infectious individuals who can transmit EVD; J is the isolated class comprising identified infectious individuals who cannot transmit the disease because they are kept in isolation unit and are prevented from mixing with the community; D denotes the highly infectious dead class comprising of individuals who died as a result of EVD and can transmit the disease until safe burial is done; R is the removed class comprising safely buried bodies and all treated individuals who are immune against re-infection. Parameters of the model are described in Table 1.

3. Analysis of the fractional EVD model

Since the total population is considered constant at the time of the EVD outbreak, then model equation (6) is normalized using $\bar{Y} = Y/N$, where $\bar{Y}$ is the population proportion given by $\bar{Y} = (\bar{S}, \bar{E}, \bar{I}, \bar{J}, \bar{D}, \bar{R})$ and $Y = (S, E, I, J, D, R)$, so that

$$ {}^{c}D_t^{\alpha}\bar{Y}(t) = \frac{1}{N}{}^{c}D_t^{\alpha}Y(t). $$

Consequently, the six-dimensional EVD model equation (6) is rescaled to the following four-dimensional system.

$$ \begin{aligned} {}^{c}D_t^{\alpha}E(t) &= (\beta_i I + \beta_d D) - (\beta_i I + \beta_d D)(E + I + J + D) \\ &\quad - \rho E, \\ {}^{c}D_t^{\alpha}I(t) &= \rho E - (\gamma_j + \gamma_r + \gamma_d)I, \\ {}^{c}D_t^{\alpha}J(t) &= \gamma_j I - (\delta_r + \delta_d)J, \\ {}^{c}D_t^{\alpha}D(t) &= \gamma_d I + \delta_d J - \epsilon D, \end{aligned} \tag{8} $$

Note that the last equation for the removed class R has been omitted since R is not required to obtain other state variables, and the value of R can be easily obtained from the values of I, J and D. Therefore, $S = 1 - (E + I + J + D)$ has been used in place of the state variable S. Further, it should be noted that the bars in $\bar{E}$, $\bar{I}$, $\bar{J}$, $\bar{D}$ have been dropped for convenience.

3.1. Existence of unique solution

Consider the biologically feasible region $\Phi = \{(E, I, J, D) \in \mathbb{R}_+^4 : 0 \le E + I + J + D \le 1\}$. It is important to show that a unique solution exists for the scaled system equation (8) in Φ. This is achieved through the following result.

Theorem 3.1. *For any non-negative initial conditions E_0, I_0, J_0, D_0, the fractional-order system equation (8) has a unique solution on $[0, +\infty)$ in Φ. Furthermore, the unique solution is non-negative for all $t \ge 0$.*

Proof. The proof of the existence of unique solution is based on Lemma 1. It is easy to see that the vector function of the model equation (8) satisfies the continuity property, p_1, of Lemma 1. Now, to satisfy the property p_2 of the Lemma 1, let

$$ Y(t) = \begin{pmatrix} E(t) \\ I(t) \\ J(t) \\ D(t) \end{pmatrix}, \tag{9} $$

so that model equation (8) can be written as

$$ {}^{c}D_t^{\alpha}Y(t) = C_1 Y(t) + (E + I + J + D)C_2 Y(t), \tag{10} $$

where C_1 and C_2 are, respectively, given by

$$ C_1 = \begin{pmatrix} -\rho & \beta_i & 0 & \beta_d \\ \rho & -(\gamma_j + \gamma_r + \gamma_d) & 0 & 0 \\ 0 & \gamma_j & -(\delta_r + \delta_d) & 0 \\ 0 & \gamma_d & \delta_d & -\epsilon \end{pmatrix} $$

and

$$ C_2 = \begin{pmatrix} 0 & -\beta_i & 0 & -\beta_d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. $$

It follows from equation (10) that

$$ \begin{aligned} \|{}^{c}D_t^{\alpha}Y(t)\| &= \|C_1 Y(t) + (E + I + J + D)C_2 Y(t)\| \\ &\le (\|C_1\| + \|C_2\|)\|Y\|, \text{ since } \|E + I + J + D\| \le 1. \end{aligned} $$

Hence, the property p_2 is satisfied, showing that a unique solution exists on $[0, +\infty)$ for the fractional-order EVD model equation (8). Furthermore, for non-negative parameters of the model, then

$$ \begin{aligned} {}^{c}D_t^{\alpha}E(t)|_{E=0} &= (\beta_i I + \beta_d D)(1 - (I + J + D)) \ge 0, \\ {}^{c}D_t^{\alpha}I(t)|_{I=0} &= \rho E \ge 0, \\ {}^{c}D_t^{\alpha}J(t)|_{J=0} &= \gamma_j I \ge 0, \\ {}^{c}D_t^{\alpha}D(t)|_{D=0} &= \gamma_d I + \delta_d J \ge 0. \end{aligned} $$

Therefore, it can be deduced from Lemma 2, known as the generalized mean-value theorem, that the unique solution of the system (8) is non-negative. This completes the proof. □

3.2. EVD-free equilibrium and basic reproduction number

In the absence of Ebola virus disease, the equilibrium point of the fractional EVD model equation (8) is given by $\mathcal{E}_0 = (0, 0, 0, 0)$, which corresponds to the equilibrium point $(1, 0, 0, 0, 0, 0)$ of the normalized full system where the entire population is susceptible before the disease outbreak.

To measure the potential spread of Ebola virus disease, an important epidemiological quantity called the basic reproduction number, usually denoted by $\mathcal{R}_0$, is required. This basic reproduction number can be defined as the average number of secondary cases produced by primary case in the course of infectiousness when the entire population is assumed susceptible. Using the notations in Ref. [38], matrices $\mathcal{F}$ and $\mathcal{V}$ representing the appearance of EVD and transfer of individuals within classes, respectively, are given by

$$ \mathcal{F} = \begin{pmatrix} (\beta_i I + \beta_d D)(1 - (E + I + J + D)) \\ 0 \\ 0 \\ 0 \end{pmatrix}, $$

$$ \mathcal{V} = \begin{pmatrix} \rho E \\ (\gamma_j + \gamma_r + \gamma_d)I - \rho E \\ (\delta_r + \delta_d)J - \gamma_j I \\ \epsilon D - (\gamma_d I + \delta_d J) \end{pmatrix}. $$

The Jacobian matrices of $\mathcal{F}$ and $\mathcal{V}$ computed at the EVD-free equilibrium $\mathcal{E}_0 = (0, 0, 0, 0)$ are, respectively, given by

$$ F = \begin{pmatrix} 0 & \beta_i & 0 & \beta_d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, V = \begin{pmatrix} \rho & 0 & 0 & 0 \\ -\rho & \gamma_j + \gamma_r + \gamma_d & 0 & 0 \\ 0 & -\gamma_j & \delta_r + \delta_d & 0 \\ 0 & -\gamma_d & -\delta_d & \epsilon \end{pmatrix}. $$

Hence, the dominant eigenvalue of the next generation matrix FV^{-1} is obtained as

$$\mathcal{R}_0 = \frac{\beta_i}{\gamma_j+\gamma_r+\gamma_d} + \frac{\beta_d(\gamma_j\delta_d + (\delta_r+\delta_d)\gamma_d)}{\epsilon(\gamma_j+\gamma_r+\gamma_d)(\delta_r+\delta_d)}. \tag{11}$$

3.3. Stability analysis

Local and global asymptotic stability of the EVD-free equilibrium point are investigated in the following results.

Theorem 3.2. *The EVD-free equilibrium point $\mathcal{E}_0$ of the fractional-order model equation (8) is locally asymptotically stable if $\mathcal{R}_0$ given in equation (11) is less than unity and unstable otherwise.*

Proof. Let $J_{\mathcal{E}_0}$ denote the Jacobian matrix of the system equation (8) evaluated at the EVD-free equilibrium. Then

$$J_{\mathcal{E}_0} = \begin{pmatrix} -\rho & \beta_i & 0 & \beta_d \\ \rho & -(\gamma_j+\gamma_r+\gamma_d) & 0 & 0 \\ 0 & \gamma_j & -(\delta_r+\delta_d) & 0 \\ 0 & \gamma_d & \delta_d & -\epsilon \end{pmatrix}. \tag{12}$$

The eigenvalues λ of $J_{\mathcal{E}_0}$ must satisfy the condition $|arg(\lambda)| > \alpha(\pi/2)$ [39, 40]. Let the characteristic polynomial of $J_{\mathcal{E}_0}$ be represented by $P(\lambda)$, so that

$$P(\lambda) = \lambda^4 + B_3\lambda^3 + B_2\lambda^2 + B_1\lambda + B_0(1-\mathcal{R}_0), \tag{13}$$

where

$$\begin{aligned} B_3 &= \epsilon+\delta_r+\delta_d+\gamma_j+\gamma_r+\gamma_d+\rho, \\ B_2 &= \epsilon(\delta_r+\delta_d)+(\gamma_j+\gamma_r+\gamma_d+\rho)(\epsilon+\delta_r+\delta_d) \\ &\quad +\rho(\gamma_j+\gamma_r+\gamma_d)-\rho\beta_i, \\ B_1 &= \epsilon(\delta_r+\delta_d)(\gamma_j+\gamma_r+\gamma_d+\rho) \\ &\quad +\rho((\gamma_j+\gamma_r+\gamma_d-\beta_i)(\epsilon+\delta_r+\delta_d)-\beta_d\gamma_d), \\ B_0 &= \rho\epsilon(\gamma_j+\gamma_r+\gamma_d)(\delta_r+\delta_d). \end{aligned}$$

And consider the discriminant, $\Delta(P)$, of the polynomial equation (13) given by

$$\Delta(P) = \begin{vmatrix} 1 & B_3 & B_2 & B_1 & B_0(1-\mathcal{R}_0) & 0 & 0 \\ 0 & 1 & B_3 & B_2 & B_1 & B_0(1-\mathcal{R}_0) & 0 \\ 0 & 0 & 1 & B_3 & B_2 & B_1 & B_0(1-\mathcal{R}_0) \\ 4 & 3B_3 & 2B_2 & B_1 & 0 & 0 & 0 \\ 0 & 4 & 3B_3 & 2B_2 & B_1 & 0 & 0 \\ 0 & 0 & 4 & 3B_3 & 2B_2 & B_1 & 0 \\ 0 & 0 & 0 & 4 & 3B_3 & 2B_2 & B_1 \end{vmatrix}.$$

$$\begin{aligned} \Delta(P) &= 16B_2^2B_0(1-\mathcal{R}_0) + 144B_2B_1^2B_0(1-\mathcal{R}_0) \\ &\quad + 144B_3^2B_2B_0^2(1-\mathcal{R}_0)^2 + 18B_3B_2B_1^3 + B_3^2B_2^2B_1^2 \\ &\quad + 18B_3^3B_2B_1B_0(1-\mathcal{R}_0) + 68B_3B_2^2B_1B_0(1-\mathcal{R}_0) \\ &\quad + 256B_0^3(1-\mathcal{R}_0)^3 - 64B_2^2B_0^2(1-\mathcal{R}_0)^2 - 4B_3^2B_2^3B_0(1-\mathcal{R}_0) \\ &\quad - 27B_3^4B_0^2(1-\mathcal{R}_0)^2 - 27B_1^4 - 4B_2^2B_1^3 - 4B_3^3B_1^3 \\ &\quad - 192B_3B_1B_0^2(1-\mathcal{R}_0)^2 - 64B_3^2B_1^2B_0(1-\mathcal{R}_0). \end{aligned}$$

By fractional order Routh-Hurwitz conditions [39, 40], all the roots of the polynomial equation $P(\lambda) = 0$ satisfy $|arg(\lambda)| > \alpha(\pi/2)$ with the following conditions:

i. If $\Delta(P) > 0$, $B_3 \geq 0$, $B_2 \geq 0$, $B_1 \geq 0$, $B_0(1-\mathcal{R}_0) > 0$, then $\mathcal{E}_0$ is locally asymptotically stable.
ii. If $\Delta(P) < 0$, $B_3 \geq 0$, $B_2 \geq 0$, $B_1 \geq 0$, $B_0(1-\mathcal{R}_0) > 0$, $\alpha < 1/2$, then $\mathcal{E}_0$ is locally asymptotically stable.
iii. If $\Delta(P) > 0$, $B_3 > 0$, $B_2 > 0$, $B_1 > 0$, $B_0(1-\mathcal{R}_0) > 0$, $B_3B_2B_1 - B_3^2B_0(1-\mathcal{R}_0) = B_1^2$ for all $\alpha \in (0,1)$, then $\mathcal{E}_0$ is locally asymptotically stable.

It follows that $B_0(1-\mathcal{R}_0) > 0$ is the necessary condition that all roots satisfy $|arg(\lambda)| > \alpha(\pi/2)$. Since $B_0 > 0$, then $B_0(1-\mathcal{R}_0) > 0$ is valid if $\mathcal{R}_0 < 0$. This implies that the EVD-free equilibrium point $\mathcal{E}_0$ is locally asymptotically stable if $\mathcal{R}_0 < 0$, and unstable if $\mathcal{R}_0 > 0$. Hence the proof. □

The following definition [29, 41] is useful for the next global stability result.

Definition 3.1. *Consider the Caputo fractional α-order system*

$${}^cD_t^\alpha z_i(t) = f(z_i(t)), \quad z_i(0) = z_0, \ \alpha \in (0,1], \tag{14}$$

with its solution $z_i(t)$, and f is a C^1-function $f : \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathcal{V}(z_i(t)) = \sum_{i=1}^n a_i\psi_i(z_i(t))$, $a_i > 0$, be a C^1-function defined on some domain in $\mathbb{R}_+^n$, then the Caputo fractional time derivative of $\mathcal{V}$ along the solution of equation (14) is given by

$${}^cD_t^\alpha \mathcal{V}(z_i(t)) = \sum_{i=1}^n a_i\, {}^cD_t^\alpha \psi_i(z_i(t)). \tag{15}$$

Theorem 3.3. *If $\mathcal{R}_0 \leq 1$, then the EVD-free equilibrium point $\mathcal{E}_0$ of the fractional-order model equation (8) is globally asymptotically stable in Φ.*

Proof. The proof is based on constructing a Lyapunov function $\mathcal{V} : \{E, I, J, D\} \in \mathbb{R}_+^4 \to \mathbb{R}$ defined by

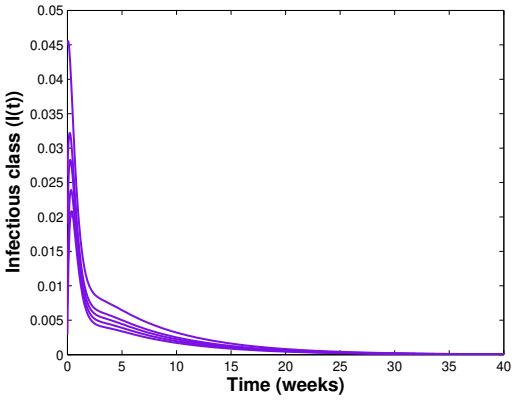
$$\begin{aligned} \mathcal{V} &= \left(\frac{\beta_i}{\gamma_j+\gamma_r+\gamma_d} + \beta_d M\right)E + \left(\frac{\beta_i}{\gamma_j+\gamma_r+\gamma_d} + \beta_d M\right)I \\ &\quad + \frac{\beta_d\delta_d}{\epsilon(\delta_r+\delta_d)}J + \frac{\beta_d}{\epsilon}D, \end{aligned} \tag{16}$$

where

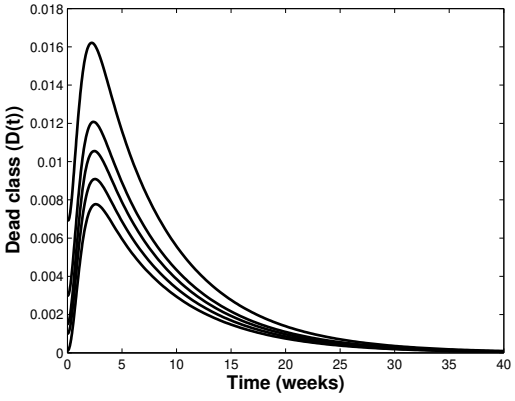
$$M = \frac{(\gamma_j\delta_d + (\delta_r+\delta_d)\gamma_d)}{\epsilon(\gamma_j+\gamma_r+\gamma_d)(\delta_r+\delta_d)}.$$

Using equation (15), which is analogous to the approach for the Lyapunov functions for the classical systems of ODEs (see, Refs. [42–45]), the Caputo fractional time derivative of $\mathcal{V}$ in equation (16) along the solution of the fractional-order EVD model equation (8) is given by:

$$\begin{aligned} {}^cD_t^\alpha \mathcal{V} &= \left(\frac{\beta_i}{\gamma_j+\gamma_r+\gamma_d} + \beta_d M\right){}^cD_t^\alpha E \\ &\quad + \left(\frac{\beta_i}{\gamma_j+\gamma_r+\gamma_d} + \beta_d M\right){}^cD_t^\alpha I + \frac{\beta_d\delta_d}{\epsilon(\delta_r+\delta_d)}{}^cD_t^\alpha J \\ &\quad + \frac{\beta_d}{\epsilon}{}^cD_t^\alpha D \\ &= \left(\frac{\beta_i}{\gamma_j+\gamma_r+\gamma_d} + \beta_d M\right)[(\beta_i I + \beta_d D) \end{aligned}$$



(a)



(b)

Figure 2: Global asymptotic behaviour of the fractional EVD model equation (8) at various initial data I_0 and D_0 when the fractional order $\alpha = 1$.

$$
\begin{aligned}
&- (\beta_i I + \beta_d D)(E + I + J + D) - \rho E] \\
&+ \left(\frac{\beta_i}{\gamma_j + \gamma_r + \gamma_d} + \beta_d M\right)[\rho E - (\gamma_j + \gamma_r + \gamma_d) I] \\
&+ \frac{\beta_d \delta_d}{\epsilon(\delta_r + \delta_d)}[\gamma_j I - (\delta_r + \delta_d) J] \\
&+ \frac{\beta_d}{\epsilon}[\gamma_d I + \delta_d J - \epsilon D].
\end{aligned}
$$

Since $0 \leq E + I + J + D \leq 1$ in Φ, then $(\beta_i I + \beta_d D)(1 - (E + I + J + D)) \leq (\beta_i I + \beta_d D)$. Using this inequality and simplifying, the Caputo fractional time derivative becomes:

$$
\begin{aligned}
{}^{c}D_t^{\alpha}\mathcal{V} &\leq \left(\frac{\beta_i}{\gamma_j + \gamma_r + \gamma_d} + \beta_d M\right)(\beta_i I + \beta_d D) \\
&\quad - \left(\frac{\beta_i}{\gamma_j + \gamma_r + \gamma_d} + \beta_d M\right)(\gamma_j + \gamma_r + \gamma_d) I \\
&\quad + \frac{\beta_d \delta_d \gamma_j}{\epsilon(\delta_r + \delta_d)} I + \frac{\beta_d \gamma_d}{\epsilon} I - \beta_d D \\
&= \left[\beta_i \left(\frac{\beta_i}{\gamma_j + \gamma_r + \gamma_d} + \beta_d M\right)\right. \\
&\quad \left. - \left(\frac{\beta_i}{\gamma_j + \gamma_r + \gamma_d} + \beta_d M\right)(\gamma_j + \gamma_r + \gamma_d) \right. \\
&\quad \left. + \frac{\beta_d \delta_d \gamma_j}{\epsilon(\delta_r + \delta_d)} + \frac{\beta_d \gamma_d}{\epsilon}\right] I \\
&\quad + \beta_d \left[\left(\frac{\beta_i}{\gamma_j + \gamma_r + \gamma_d} + \beta_d M\right) - 1\right] D \\
&= \beta_i(\mathcal{R}_0 - 1) I + \beta_d(\mathcal{R}_0 - 1) D.
\end{aligned}
$$

This shows that ${}^{c}D_t^{\alpha}\mathcal{V} \leq 0$ if $\mathcal{R}_0 \leq 1$, where ${}^{c}D_t^{\alpha}\mathcal{V} = 0$ if and only if $I = D = 0$. Therefore, the largest compact invariant set in $\{(E, I, J, D) \in \mathbb{R}_+^4 \mid {}^{c}D_t^{\alpha}\mathcal{V} = 0\}$ is the singleton $\{\mathcal{E}_0\}$. Employing LaSalle's invariance principle [46] establishes that the EVD-free equilibrium point $\mathcal{E}_0$ is globally asymptotically stable if $\mathcal{R}_0 \leq 1$. □

4. Simulations and discussion

For robustness of the theoretical analysis carried out on the fractional-order EVD model equation (8), it is important to validate the theoretical results at different α-order values. To do this, estimated values of the parameters used in Refs. [36, 47] are employed as follows: $\beta_i = 0.738$, $\beta_d = 1.025$, $\rho = 2.835$, $\gamma_d = 0.045$, $\gamma_r = 0.527$, $\gamma_j = 2.163$, $\delta_d = 0.3089$, $\delta_r = 0.7207$ and $\epsilon = 0.601$, together with the following initial conditions: $E_0 = 0.028$, $I_0 = 0.013$, $J_0 = 0$ and $D_0 = 0$. The fractional Euler's method derived in Ref. [48] is implemented using MATLAB to simulate the fractional EVD model equation (8). It should be noted that the parameter values used correspond to the EVD-free population where the basic reproduction number $\mathcal{R}_0 = 0.7026$. Hence, the global asymptotic behaviours of the fractional-order system around the EVD-free equilibrium point $\mathcal{E}_0$ at different initial sizes of both the infectious class, I_0, and dead class, D_0, with $\alpha = 1$ are graphically shown in Figure 2(a) and Figure 2(b), respectively. It is seen that solutions converge to the EVD-free equilibrium point asymptotically in agreement with Theorem 3.3. This confirms that Ebola virus disease can be eliminated when $\mathcal{R}_0 < 1$ notwithstanding the magnitude of the initial sizes of infectious individuals in the community. It is instructive to mention that the stability result established in Theorem 3.2 is dependent on the initial sizes of the infectious population.

Further, the effects of some values of fractional order α between 0 and 1 on the behaviours of the system starting at the fixed initial conditions are depicted in Figure 3. It can be observed that the numbers of the exposed, infectious, isolated and dead classes decrease asymptotically with time as the fractional order α decreases. As a result of this, convergence to the EVD-free equilibrium is rapid with decrease in the fractional order (i.e., $\alpha = 1$, $\alpha = 0.95$, $\alpha = 0.85$, $\alpha = 0.75$) for each of the trajectories of the system equation (8). This translates to the effect of memory on the disease dynamics, where increase in memory leads to reduction in the incidence of the disease in the population.

Therefore, the epidemiological insight gained from this study is that application of fractional calculus in modelling the

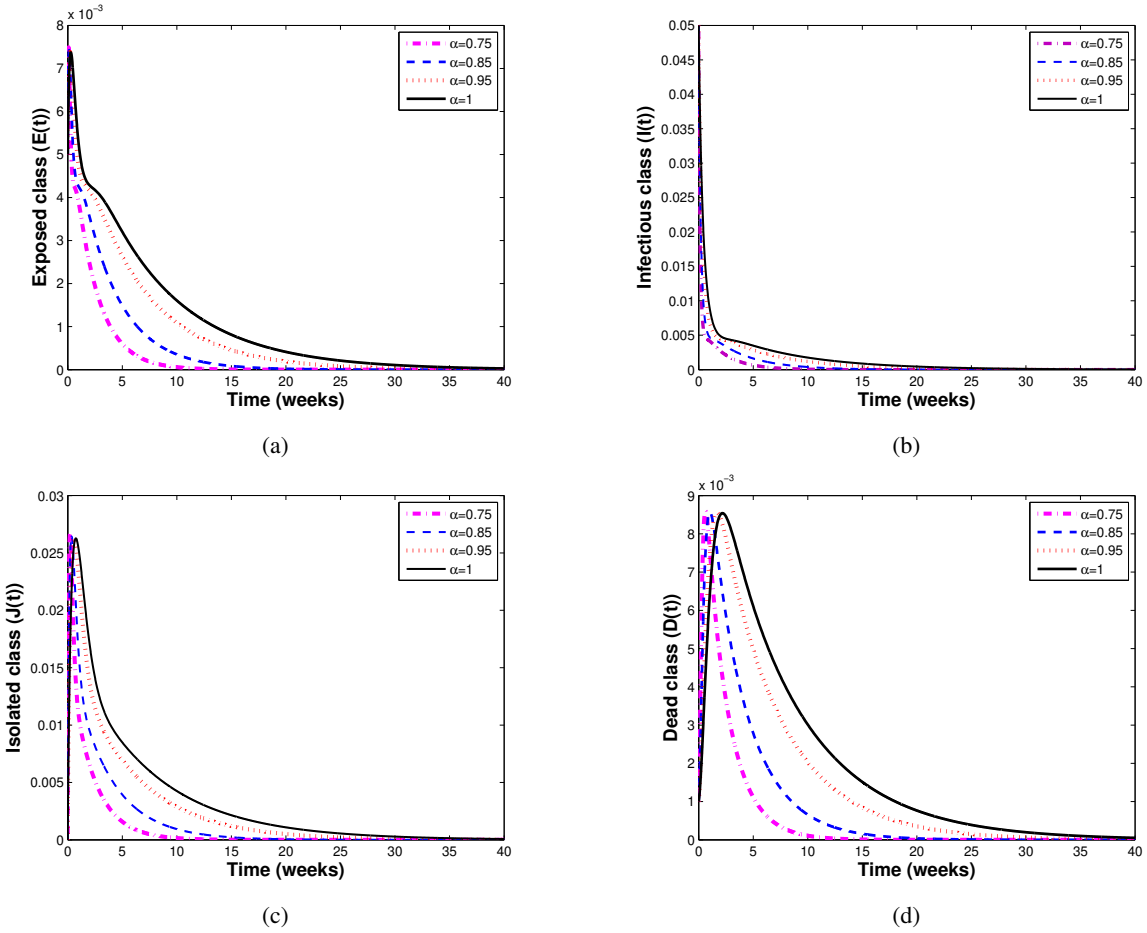


Figure 3: Convergence of the trajectories of the fractional EVD model equation (8) to the EVD-free equilibrium at different fractional orders $\alpha = (0.75,\ 0.85,\ 0.95,\ 1)$.

dynamics of the Ebola virus disease helps in describing the evolution of the disease more accurately than the classical model (i.e., when $\alpha = 1$), since the non-integer order present in fractional model (i.e., when $\alpha < 1$) can be used as a fit parameter for controlling the outbreak of the disease in the population.

5. Conclusion

The importance of fractional calculus in modelling phenomena cannot be overemphasized. This work has applied fractional calculus approach to model the spread of Ebola virus disease (EVD) in the population in order to underscore the importance of memory effect in infectious disease modelling. The epidemic model is governed by a system of fractional-order differential equations using the Caputo derivative operator, which is a generalization of an existing classical EVD epidemic model comprising of six epidemiological classes, namely the susceptible class, exposed class, infectious class, isolated class, dead class and the removed class. Under the constant population assumption, the six-dimensional system is rescaled and reduced to a four-dimensional fractional-order system. The existence of unique solution for the reduced fractional-order EVD model is proved using fractional calculus theory of continuity and boundedness with the generalized mean-value theorem.

Moreover, the basic reproduction number of the fractional-order EVD model is obtained and stability analysis is conducted to determine the asymptotic behaviour of the non-integer-order system around the EVD-free equilibrium point. It is shown, using the fractional Routh-Hurwitz criteria, that the EVD-free equilibrium point is locally asymptotically stable when the basic reproduction number is less than one. This shows that EVD can be eliminated from the community provided that the initial sizes of infectious individuals with dead bodies are small. However, to demonstrate that the EVD can be eliminated notwithstanding the magnitude of the initial sizes of the populations, it is established that EVD-free equilibrium point is globally asymptotically stable when the basic reproduction number is less than or equal to unity. The global stability result is achieved through a construction of suitable Lyapunov functional analogous to the classical approach. And numerical simulations of the fractional-order system are performed to corroborate the theoretical result of the global asymptotic dynamics. In addition, it is shown that the numbers of the exposed class, infectious class, isolated class and the dead bodies reduce asymptotically as the order of the fractional derivative decreases. Particu-

larly, rapid convergence of solution of the fractional-order system to the EVD-free equilibrium underscores the significance of the presence of memory effect in the fractional disease dynamics.

Data availability

We do not have any research data outside the submitted manuscript file.

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