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Expectation values and Fisher information theoretic measures of heavy flavoured mesons

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Abstract

The expectation values play important role in atomic physics and quantum mechanics. They are vital to obtaining the quantum information theoretic measures, Compton profile, electronic kinetic energy, Langevin-Pauli diamagnetic susceptibility, Dirac exchange energy and so on. In this work, however, we utilized the bound state solutions of the non-relativistic Schrödinger equation under the Cornell potential to obtain the expectation values using two analytical methods such as the integral approach and the Hellmann-Feynman theorem method. We applied the mean values to obtain the Fisher information theoretic measures for the Charmonium and Bottomonium mesons. Also, we found that the mean values and probability densities are sensitive to the meson masses and the principal quantum number for fixed orbital quantum states. The calculated average kinetic and potential energies were found to possess an oscillatory motion occasioned by the quark-anti-quark interactions.The Dirac exchange and the kinetic energies obey the lowest bound inequalities for 3D atomic systems. Also, the results obey the Fisher uncertainty product, Cramer-Rao and the Heisenberg uncertainty inequalities for 3D quantum systems.

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1. Introduction

In the quark model, mesons are bound states of quark-antiquark pairs which can be described by the non-relativistic Schrödinger equation for the heavy (quarkonia) and the heavy-light mesons [1]. Also, the properties of the light mesons have been investigated using the relativistic wave equations [1]. The quarks possess a quantum number of colours and interact via quantized fields which are described by the theory of quantum chromodynamics (QCD). The properties such as mass spectra,

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decay constants and thermodynamic functions have been investigated using QCD-inspired potential energy functions [1–5]. Potentials such as the Cornell potential and its extended forms have been extensively studied [6–13]. Gupta *et al.* [14] stated that the potentials have similar characteristics in the interval $[0.1, 1]$ Fermi (fm) but differ outside this range. Also, energy-dependent potentials have been utilized to study the mass spectrum, decay width, mean values of $\langle r\rangle_{nl}$, $\langle r^{-1}\rangle_{nl}$, and the root mean square radii $\sqrt{\langle r^2\rangle_{nl}}$ of the Charmonium ($c\bar{c}$) and Bottomonium ($b\bar{b}$) mesons [14–17].

Lombard *et al.* [18] was the first to use an energy-dependent potential function to study the bound states of $c\bar{c}$ and $b\bar{b}$ mesons. They concluded that the energy dependence of the potential saturates the mass spectra. Abushady and Inyang [19] investigated the mass spectra of heavy mesons with a trigonometry Rosen-Morse potential using the generalized fractional derivative. The potential applied in [19] was expanded to accommodate the confinement function and attractive Coulomb potential which together play a vital role in determining properties of heavy quarkonia. Equally, the meson masses were obtained using the analytical exact iteration method in [20]. The Dirac equation was solved under the generalized Cornell potential model where the authors applied the energy spectra to obtain the spin average masses of the heavy and heavy-light mesons [21].

The effects of external magnetic fields on the properties of mesons have been investigated. Coppola *et al.* [22] studied the masses of light pseudoscalar and vector mesons using the Nambu-Jona-Lasinio (NJL) model, finding that the magnetic field impacts both charged and neutral mesons differently. Yang *et al.* [23] extended the two-flavor NJL model to analyze the effects of a strong magnetic field on the mass spectra and decay properties of σ and π_0 mesons. They found that the external magnetic field has small effects on the π_0 meson at low temperatures but produces clear changes in the σ meson and quarks. Additionally, they found that a constant external magnetic field accelerates the decay rate, while the chemical potential reduces the effects.

The expectation values may be applied in analyzing the complexities of quantum systems [24–28]. They can also be applied in the description of basic measurable quantities of the system [29, 30]. For example, the expectation value $\langle r^2\rangle_{nl}$ can be used to access the Langevin-Pauli diamagnetic susceptibility and the magnetic screening factor [29]. Dehesa *et al.* [29] stated that the entropic moment $\langle\rho^\alpha\rangle$ has been used to describe the atomic Thomas-Fermi ($\alpha = 5/3$) and Dirac exchange ($\alpha = 4/3$) energies [31, 32]. The respective Compton profile, electron kinetic energy, and the Pauli relativistic correction to the kinetic energy can be obtained from the momentum power moments $\langle p^2\rangle$ and $\langle p^4\rangle$ [29, 33].

On the other hand, quantum information theory plays a vital role in data processing and quantum networking in modern technology. The information-theoretic measure such as the Shannon entropy under a point-like defect in spherically symmetric spacetime has been studied for the Charmonium and Bottomonium mesons [34]. The authors utilized the solutions of the Schrödinger equation with a Cornell potential and found that the point-like defect affects the Shannon entropy in both position and momentum spaces. Their results for the Shannon entropic sum satisfy the Bialynicky-Birula and Mycielski uncertainty relation. Recently, the relationships between the information content and the variance of a system have been investigated. Omugbe *et al.* [35] employed the wave function of the Schrödinger equation under a q-deformed hyperbolic potential to model the Shannon entropy and Fisher information, finding that the information contents in position and momentum spaces are sensitive to the potential parameters.

In this article, we present the bound states of the Charmonium and Bottomonium mesons using the Cornell potential (sum of Coulomb plus linear potentials). We will employ the solutions of the bound states to obtain the mean values such as $\langle p^2\rangle_{nl}$, $\langle T\rangle_{nl}$, $\langle V\rangle_{nl}$, $\langle r^2\rangle_{nl}$, $\langle r^{-1}\rangle_{nl}$, and $\langle r^{-2}\rangle_{nl}$ using two analytical methods: the integral approach and the Hellmann-Feynman theorem [36, 37]. The mean values will then be applied to obtain the local Fisher uncertainty, Heisenberg uncertainty, and other inequalities such as the kinetic energy and Dirac exchange energy, which have physical meaning in atomic and quantum physics.

The organization of the remaining parts of the article is as follows: In Section 2, we present the bound state solutions of the Schrödinger equation under the Cornell potential. In Section 3, we will present the analytical solution for obtaining the radial and momentum power moments ($\langle r^\alpha\rangle$, $\langle p^\alpha\rangle$) as well as the potential and kinetic mean energies ($\langle T\rangle$, $\langle V\rangle$). In Section 4, we will apply the mean values to study quantum Fisher information-theoretic measures and kinetic energy for the Charmonium and Bottomonium mesons. The discussion of results is presented in Section 5, and the conclusion is given in Section 6.

2. Bound state solutions of the Schrödinger equation with the Cornell potential

The time-independent Schrödinger equation for 3D quantum systems with reduced mass μ and the wave function $\psi_{nlm}(r,\theta,\phi)$ is given as:

$$\begin{aligned}&-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right.\\&\left.+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi_{nlm}(r,\theta,\phi)+V(r)\psi_{nlm}(r,\theta,\phi)\\&=E_{nl}\psi_{nlm}(r,\theta,\phi).\end{aligned}\tag{1}$$

Equation (1) contains both the radial part and angular parts (Spherical harmonics), which can be separated using the substitution given in (2). The notations n, l and m represent the respective principal quantum number which determines the size, the orbital quantum number which gives the shape and the magnetic quantum state which gives the orientation of quantum particles.

$$\psi_{nlm}(r,\theta,\phi)=\frac{R_{nl}(r)}{r}P_l^m(\cos\theta)\Phi(\phi),\tag{2}$$

$$\frac{d^2R_{nl}}{dr^2}+\frac{2\mu}{\hbar^2}\left(E_{nl}-V(r)-\frac{l(l+1)\hbar^2}{2\mu r^2}\right)R_{nl}(r)=0, \quad (3)$$

$$\frac{d^2\Phi}{d\phi^2}+m^2\Phi=0, \quad (4)$$

$$\frac{d}{dy}\left[(1-y^2)\frac{dP_l^m(y)}{dy}\right]+\left[l(l+1)-\frac{m^2}{1-y^2}\right]P_l^m(y)=0,\ y=\cos\theta, \quad (5)$$

Equation (5) is the associated Legendre equation. The respective solutions of Equation (4) and (5) are given by [38]:

$$\Phi(\phi)=e^{im\phi},\quad m=0,\pm1,\pm2,\ldots,\pm l, \quad (6)$$

$$P_l^m(y)=\frac{(-1)^m}{2^l l!}(1-y^2)^{m/2}\frac{d^{l-m}}{dy^{l-m}}(1-y^2)^l,\quad -l\le m\le l. \quad (7)$$

The product of Equations (6) and (7) constitutes the Spherical harmonics:

$$Y_l^m(\theta,\phi)=(-1)^m\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(\cos\theta)e^{im\phi}. \quad (8)$$

To solve the radial part, we use the Cornell potential [6–10] given by:

$$V(r)=\sigma r-\frac{\Lambda}{r},\quad \sigma,\Lambda>0, \quad (9)$$

where σ and Λ are constants. The linear term is the confinement component, while the Coulomb term arises from one-gluon exchange between the quark and its antiquark. The linear potential has been used to study the condensation properties of a Bose-Einstein ideal gas [39] and the energy levels of ultracold neutrons bouncing on a perfectly reflecting surface [40]. Substituting the potential in Equation (9) into the Schrödinger equation (3), we obtained:

$$\frac{d^2R_{nl}}{dr^2}+\frac{2\mu}{\hbar^2}\left(E_{nl}-\sigma r-\frac{\Lambda}{r}-\frac{l(l+1)\hbar^2}{2\mu r^2}\right)R_{nl}(r)=0. \quad (10)$$

The bound state solutions of Equation (10) are not exactly solvable but can be approximated using the Pekeris-type approximation ($r_0(\delta=1/r_0)$), which is assumed to be the characteristic radius of mesons [10]. The respective energy spectra and wave function have been obtained in closed form using the Nikiforov-Uvarov method [10]:

$$E_{nl}=\frac{3\sigma}{\delta}-\frac{\mu}{2\hbar^2}\left(\frac{3\sigma/\delta^2+\Lambda}{n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu\sigma}{\delta^3}}}\right)^2, \quad (11)$$

$$R_{nl}(s)=s^{W/(2\sqrt{T})}e^{-\sqrt{T}/s}N_{nl}\frac{d^n}{ds^n}\left[s^{2n-W/\sqrt{T}}e^{-2\sqrt{T}/s}\right],\quad s=\frac{1}{r}, \quad (12)$$

where, N_{nl} is the normalization constant, and the notations W and T are given by:

$$W=\frac{2\mu}{\hbar^2}\left(\frac{3\sigma}{\delta^2}+\Lambda\right),\quad T=\frac{2\mu}{\hbar^2}\left(\frac{3\sigma}{\delta}-E_{nl}\right).$$

The normalization constant of the radial wave function can be obtained using the condition for the probability of finding a quantum particle in space:

$$\int_0^\infty |R_{nl}(r)|^2dr=1. \quad (13)$$

Using Equations (11) and (12), we can describe many properties of the meson systems. In the next section, we present the analytical solution for the expectation values using two analytical methods. Later we will apply these mean values to study the Fisher information theoretic measures for Charmonium and Bottomonium mesons.

3. Calculation of expectation values using two equivalent methods

In this section, we applied the wave function to find the radial and momentum power moments. Equally, we will use the energy spectra equation with the Hellmann-Feynman theorem to access the mean values. The radial expectation values can be written as:

$$\langle r^\alpha\rangle_{nl}=\int_0^\infty r^\alpha R_{nl}^2(r)\,dr\int_0^\pi\int_0^{2\pi}Y_{lm}(\theta,\phi)Y^*_{l'm'}(\theta,\phi)\,d\Omega, \quad (14)$$

where is the differential solid angle given by $d\Omega=\sin(\theta)d\theta d\phi$. The spherical harmonics normalize to unity. The momentum expectation value $\langle p^2\rangle$ in the position representation is given by [29]:

$$\langle p^2\rangle_{nl}=\int_0^\infty\left(\frac{dR_{nl}(r)}{dr}\right)^2dr+l(l+1)\langle r^{-2}\rangle_{nl}. \quad (15)$$

Equivalently, the l-states expectation values for $\langle p^2\rangle$ can also be obtained from the equation:

$$\langle p^2\rangle_{nl}=-\int_0^\infty R_{nl}(r)\frac{d^2R^*_{nl}(r)}{dr^2}dr+l(l+1)\langle r^{-2}\rangle_{nl}. \quad (16)$$

For the respective positive and negative values of the radial integrals in Equations (14) and (15), the equation satisfies the momentum $\langle p^2\rangle_{nl}$ mean value inequality [29, 41]:

$$\langle p^2\rangle_{nl}\ge l(l+1)\langle r^{-2}\rangle_{nl}. \quad (17)$$

The expectation values for the kinetic and potential energies are given as:

$$\langle T\rangle_{nl}=\frac{1}{2\mu}\langle p^2\rangle_{nl}=\frac{1}{2\mu}\left(\int_0^\infty\left(\frac{dR_{nl}(r)}{dr}\right)^2dr+l(l+1)\langle r^{-2}\rangle_{nl}\right), \quad (18)$$

$$\langle V(r)\rangle_{nl}=\int_0^\infty V(r)R_{nl}^2(r)\,dr=\sigma\langle r\rangle_{nl}-\Lambda\langle r^{-1}\rangle_{nl}. \quad (19)$$

Using the above relations, the total energy can be written as:

$$E_{nl}=\langle T\rangle_{nl}+\langle V(r)\rangle_{nl}=\frac{1}{2\mu}\langle p^2\rangle_{nl}+\sigma\langle r\rangle_{nl}-\Lambda\langle r^{-1}\rangle_{nl}. \quad (20)$$

On the other hand, the Hellmann-Feynman theorem has been used to derive the mean values of quantum particles moving in Hamiltonian functions provided the energy spectra are known [42, 43]. The basic equations for obtaining the mean values can be written [36, 37] as:

$$\frac{\partial E_{nl}(q)}{\partial q} = \langle \psi_{nl}(q)|\frac{\partial H(q)}{\partial q}|\psi_{nl}(q)\rangle, \tag{21}$$

where H and q are the respectively Hamiltonian of the system and a parameter. The Hamiltonian with the Cornell potential is given as:

$$H = -\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \sigma r - \frac{\Lambda}{r} + \frac{l(l+1)\hbar^2}{2\mu r^2}. \tag{22}$$

If we let $q = l$, μ and substitute the energy and Hamiltonian in the respective Equations (11) and (22) into (21), the mean values of the respective radial power moment $\langle r^k\rangle_{nl}$, the momentum moment $\langle p^2\rangle_{nl}$, the potential energy $\langle V(r)\rangle_{nl}$ and kinetic energy $\langle T\rangle_{nl}$ are obtained analytically in closed form:

$$\langle r^{-2}\rangle_{nl} = \frac{2\mu}{2l+1}\frac{\partial E_{nl}}{\partial l}, \tag{23}$$

$$\langle p^2\rangle_{nl} = -2\mu^2\frac{\partial E_{nl}}{\partial \mu}, \tag{24}$$

$$\langle T\rangle_{nl} = -\mu\frac{\partial E_{nl}}{\partial \mu}, \tag{25}$$

$$\langle V(r)\rangle_{nl} = \sigma\frac{\partial E_{nl}}{\partial \sigma} + \Lambda\frac{\partial E_{nl}}{\partial \Lambda}. \tag{26}$$

$$\langle r^{-1}\rangle_{nl} = -\frac{\partial E_{nl}}{\partial \Lambda},, \tag{27}$$

$$\langle r\rangle_{nl} = \frac{\partial E_{nl}}{\partial \sigma}. \tag{28}$$

Using Equations (25) and (26), the total energy for the system can be written as

$$E_{nl} = -\mu\frac{\partial E_{nl}}{\partial \mu} + \sigma\frac{\partial E_{nl}}{\partial \sigma} + \Lambda\frac{\partial E_{nl}}{\partial \Lambda}. \tag{29}$$

3.1. Fisher information theoretic measures

The Fisher information measures tell us about the spread of the electron distribution which measures the localization and delocalization properties of quantum systems. It has been successfully applied in describing tumor cell growths, analyzing stock markets as well as the derivation of equations of motion [44–47]. When the lowest bound of the Fisher information product is obeyed it gives more information about the system. The Fisher's information can be written in terms of the position and momentum power moments via the conjecture [29, 41]:

$$I(\rho) = 4\langle p^2\rangle_{nl} - 2(2L+1)|m|\langle r^{-2}\rangle_{nl}, \tag{30}$$

$$I(\gamma) = 4\langle r^2\rangle_{nl} - 2(2L+1)|m|\langle p^{-2}\rangle_{nl}, \tag{31}$$

where, $L = l + \frac{(D-3)}{2}$ and m are the respective grand orbital quantum number and magnetic quantum state. If we set $m = 0$, the product of the Fisher information can be expressed as:

$$FP = I(\rho)I(\gamma) = 16\langle r^2\rangle_{nl}\langle p^2\rangle_{nl}. \tag{32}$$

The Heisenberg uncertainty product (HP) $\langle r^2\rangle\langle p^2\rangle$ was derived in [29, 41] as:

$$HP = \langle r^2\rangle_{nl}\langle p^2\rangle_{nl} \geq \left(l + \frac{3}{2}\right)^2. \tag{33}$$

In 3D, for the ground state ($l, m = 0$), the lowest bound for the HP can be expressed as:

$$\langle r^2\rangle_{nl}\langle p^2\rangle_{nl} \geq \frac{9}{4}. \tag{34}$$

Inserting Equation (34) into (32) gives the lowest bound for the Fisher's information product (FP):

$$I(\rho)I(\gamma) \geq 36. \tag{35}$$

4. Discussion of results

We have presented bound states solutions of the Schrödinger equation with a QCD-inspired potential function such as the Cornell potential. We utilized the wave function and energy equations to obtain the Charmonium and Bottomonium meson expectation values in position and momentum spaces by the integral and Hellmann-Feynman theorem methods. It is what stating that the meson masses play a dominant role in describing their properties. The masses of the Charm quark ($1.2 < m_c < 1.8$ GeV) and Bottom quark ($4.5 < m_b < 5.4$ GeV) have been experimentally determined by the Particle Data Group [48]. Presently, we applied the spectroscopic parameters $m_c = 1.209$ GeV, $\sigma = 0.2$ GeV^2, $\Lambda = 1.234$, $\delta = 0.232$ GeV for Charmonium, and $m_b = 4.823$ GeV, $\sigma = 0.2$ GeV^2, $\Lambda = 1.553$, $\delta = 0.381$ GeV for Bottomonium mesons as used in [10]. The notations σ, Λ, and δ are fitted free parameters.

In Figures 1(a-f), we plotted the s-wave wave functions $\psi_{n,0}$ (1s, 2s, and 3s) with their corresponding probability densities $\rho_{n,0}$. The wave functions and probability densities obey the boundary conditions $\psi_{n,0} = \rho_{n,0} = 0$ at $r = 0$ and $r = \infty$. The Bottomonium meson probability density and wave function peaks shift toward shorter distances and have higher amplitudes than those of the Charmonium meson. The peaks and troughs of the $c\bar{c}$ mesons occur at larger radial distances.

Also, in Figures 2 (a-d), we have investigated the variations of the mean kinetic; mean potential and total energies with the quantum numbers. For the Charmonium meson (see Figure 2 (a, b)), the mean kinetic energy increases to a peak at small quantum numbers before decreasing gradually as the quantum number increases. In contrast, the average value of the potential energy is bounded above the mean kinetic energy and increases steadily before converging to a constant value for large quantum numbers. In Figure 2 (c, d), this trend are reproduced for the Bottomonium meson except that for small quantum states, the mean kinetic energy dominate before decreasing as the quantum numbers become large. The properties of the quarkonia mean energies are analogous to the oscillatory motion of diatomic molecules where both the kinetic and potential energies possess inverse relationship [49] and may be ascribed to the oscillations between the quark and it anti-particle.

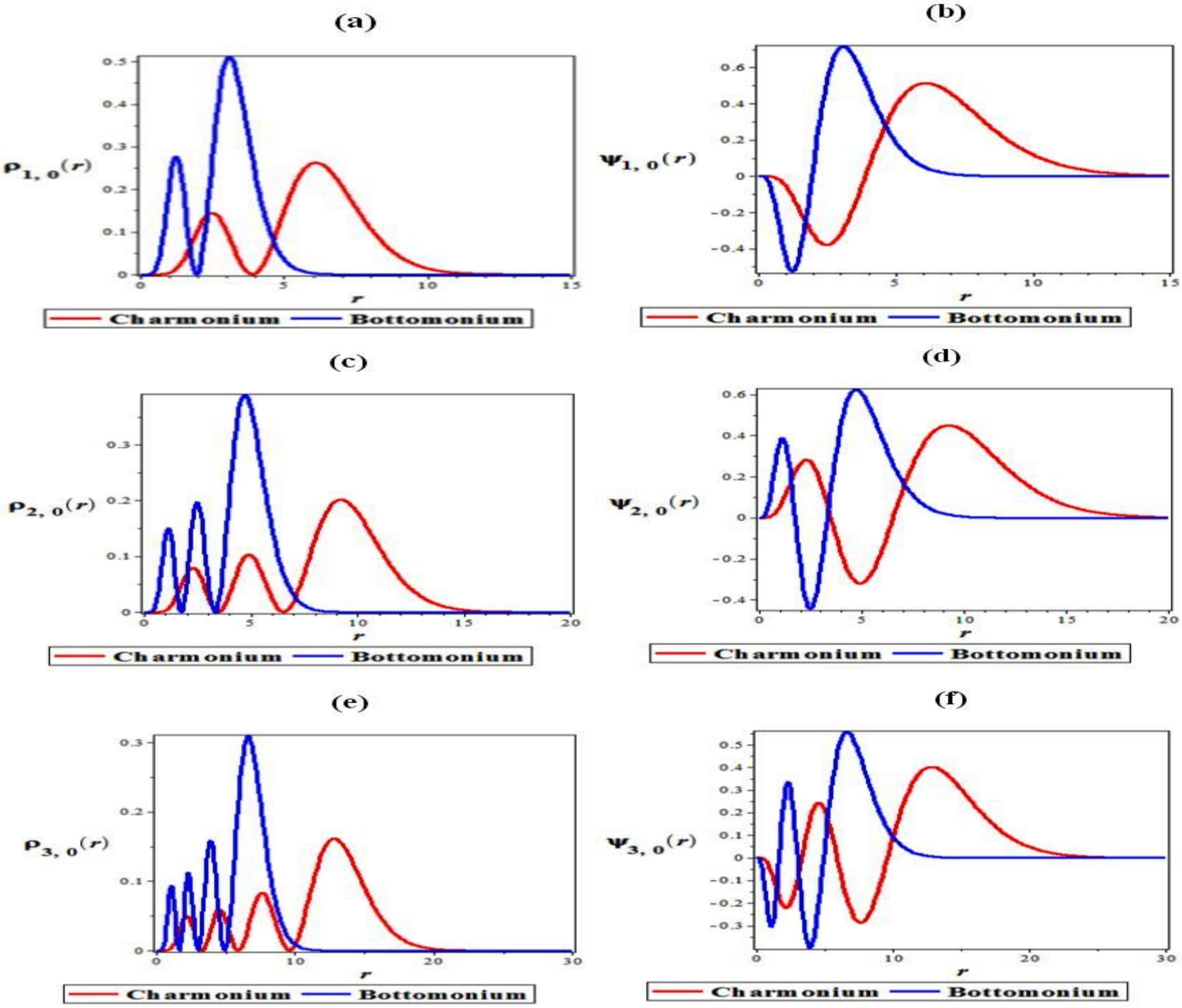

Figure 1: (a-f) S-wave probability density ($\rho_{n,0}$ for 1s, 2s, 3s) and wave function ($\psi_{n,0}$ for 1s, 2s, 3s) for $c\bar{c}$ and $b\bar{b}$ meson systems.

Table 1: Expectation values for different states of $\langle r^{-1}\rangle$ in GeV.

States		$c\bar{c}$		$b\bar{b}$	
n	l	(Equation 27)	(Equation 14)	(Equation 27)	(Equation 14)
1	0	0.2129317664	0.2129657725	0.4211587055	0.4210534901
2	0	0.1559039334	0.1559039340	0.3049185482	0.3049185465
3	0	0.1190577028	0.1190577024	0.2309160581	0.2309160567
1	1	0.1979467211	0.1979321156	0.3889726229	0.3889866102
2	1	0.1464428402	0.1464428410	0.2849176894	0.2849176905
3	1	0.1127070772	0.1127070765	0.2176493001	0.2176493012
1	2	0.1738703205	0.1738091127	0.3382233434	0.3381931549
2	2	0.1309204008	0.1309204017	0.2526377874	0.2526377869
3	2	0.1021214603	0.1021214596	0.1958596431	0.1958596414

Table 2: Expectation values for different states of $\langle r^{-2}\rangle$ in GeV^2

States		$c\bar{c}$		$b\bar{b}$	
n	l	(Equation 23)	(Equation 14)	(Equation 23)	(Equation 14)
1	0	0.06069637320	0.06070617130	0.2406313017	0.240577925
2	0	0.03802661470	0.03802661494	0.1482378857	0.148237884
3	0	0.02537688458	0.02537688456	0.0976932706	0.097693270
1	1	0.05182507172	0.05182227137	0.2024437404	0.202453649
2	1	0.03297764650	0.03297764662	0.1269126523	0.126912652
3	1	0.02226610357	0.02226610339	0.0847347812	0.084734781
1	2	0.03919081590	0.03917797273	0.1496488652	0.149634136
2	2	0.02560690629	0.02560690639	0.0966085411	0.096608541
3	2	0.01764093694	0.01764093681	0.0659455506	0.065945550

In Tables 1 and 2, the respective mean values of $\langle r^{-1}\rangle_{nl}$ GeV and $\langle r^{-2}\rangle_{nl}$ GeV^2 for $c\bar{c}$ and $b\bar{b}$ mesons decrease with increasing n for fixed l. The expectation values of $\langle r^{2}\rangle_{nl}$ GeV^{-2} for both $c\bar{c}$ and $b\bar{b}$ mesons, obtained in Table 3, increase as n increases for fixed l. Also, the quarkonia masses have a strong influence on the root mean square radii $\sqrt{\langle r^2\rangle_{nl}}$.

Our calculated values for $c\bar{c}$ and $b\bar{b}$ mesons increase with the principal quantum number n for fixed l, with $c\bar{c}$ having larger values. This trend is well reproduced compared to the results obtained in [14–17], where the authors utilized different QCD-inspired potential functions. Generally, the second-order momentum power moment obtained in Table 3 decreases with increasing quantum number n for fixed l.

The mean values in Table 3 were used to compute the Fisher

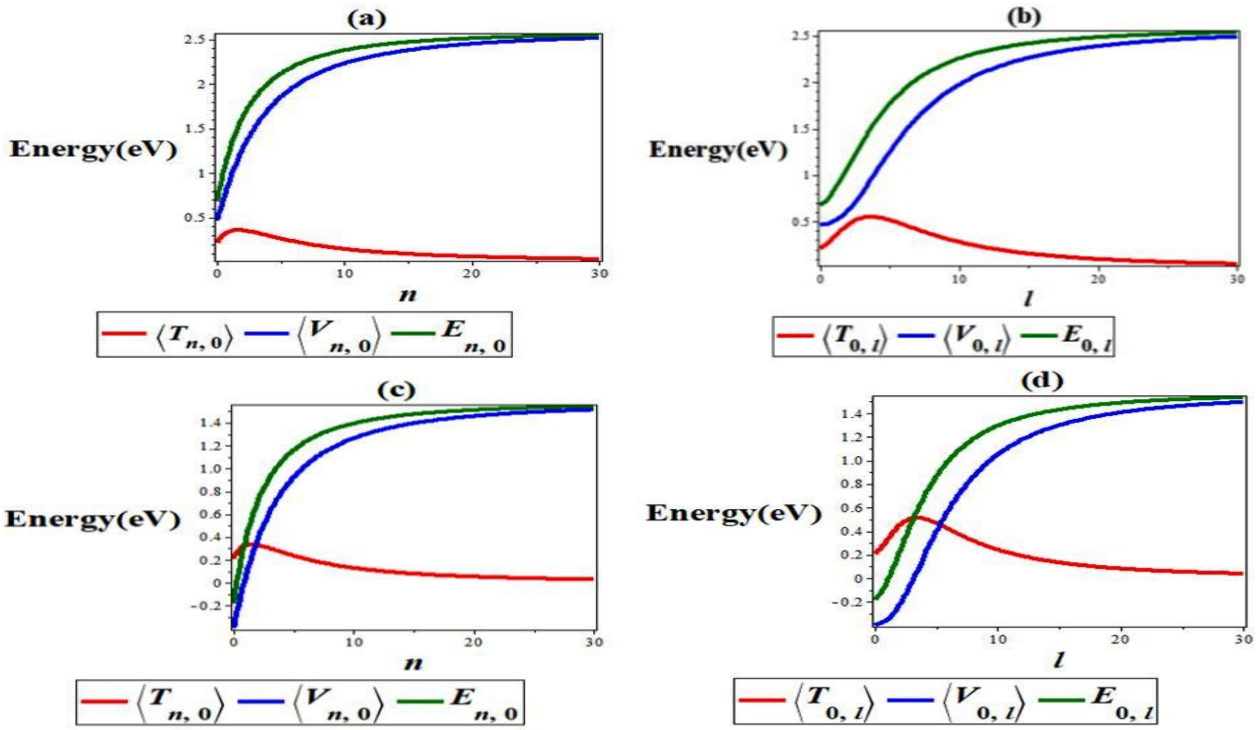

Figure 2: Expectation values for the kinetic, potential, and total energies for n and l quantum states. (a, b) Charmonium; (c, d) Bottomonium.

information theoretic measures in both momentum and position spaces. We found that a decrease in the position Fisher information is complemented by an increase in momentum Fisher information for both $c\bar{c}$ and $b\bar{b}$ mesons. In Tables 4 and 5, the Fisher's information product, the Cramer-Rao inequality ($I(\rho)\langle r^2\rangle \geq 4(l+\frac{3}{2})^2$), and the Heisenberg uncertainty product were found to be greater than their lowest bounds for both $c\bar{c}$ and $b\bar{b}$ mesons, indicating stability and a high probability of locating the heavy meson systems.

The momentum Fisher information for $c\bar{c}$ mesons is larger than for $b\bar{b}$ mesons due to the large mean square radii of the $c\bar{c}$ mesons, while the position space Fisher information for $b\bar{b}$ mesons is larger than for $c\bar{c}$ mesons, owing to their large momentum second-order power moment. Notably, the masses and square radii affect the Fisher information of the meson systems differently, but the local Fisher's product tends toward comparable values where we found the ratio $\frac{(I(\rho)I(\gamma))_{c\bar{c}}}{(I(\rho)I(\gamma))_{b\bar{b}}}$ to be approximately unity. This may be an indication that irrespective of the mesons' different masses, momentum, square radii and the spread of the radial probability density, their intrinsic properties may be the cause of the comparable values obtained for the Fisher information theoretic measures.. The results for the kinetic energy $\langle T\rangle_{nl} = \langle p^2\rangle_{nl}/2\mu$ are greater than the kinetic energy lowest bounds for 3D systems [29]:

$$\langle T\rangle_{nl} = \frac{\langle p^2\rangle_{nl}}{2\mu} \geq \frac{0.92867}{\langle r^2\rangle}, \tag{36}$$

$$\langle T\rangle_{nl} = \frac{\langle p^2\rangle_{nl}}{2\mu} \geq 0.34668\langle r^{-1}\rangle^2, \tag{37}$$

Table 3: Expectation values for different states of $\langle r^2\rangle$ (GeV^{-2}) and $\langle p^2\rangle$ (GeV^2)

States		$b\bar{b}$			$c\bar{c}$		
n	l	$\langle r^2\rangle$	$\langle p^2\rangle$ Equation (15)	$\langle p^2\rangle$ Equation (24)	$\langle r^2\rangle$	$\langle p^2\rangle$ Equation (15)	$\langle p^2\rangle$ Equation (24)
1	0	9.648017879	1.578326543	1.578326543	37.23427629	0.4183882178	0.4183882178
2	0	20.75153307	1.595807372	1.595807365	78.39165128	0.4305331327	0.4305331330
3	0	38.78059204	1.462590423	1.462590429	144.3356611	0.3997023059	0.3997023104
1	1	11.15023021	1.803021779	1.803001968	42.50594228	0.4780086065	0.4780142106
2	1	23.46047321	1.693477547	1.693477544	87.79353229	0.4574884046	0.4574884048
3	1	43.17132226	1.506678041	1.506678027	159.4098663	0.4124073084	0.4124073130
1	2	14.37933028	2.027805796	2.027894177	53.82201489	0.5424605909	0.5424605918
2	2	29.16379464	1.779370905	1.779370902	107.5397841	0.4840355449	0.4840355408
3	2	52.26461533	1.535588576	1.535588581	190.5988399	0.4227395444	0.4227395514

Table 4: Fisher information, Cramer-Rao inequality, and Heisenberg uncertainty product for Charmonium meson.

States		$c\bar{c}$				
n	l	$I(\rho)$	$I(\gamma)$	$FP \geq 16(l+\frac{3}{2})^2$	$C_R(\rho) \geq 4(l+\frac{3}{2})^2$	$HP \geq (l+\frac{3}{2})^2$
1	0	1.673552871	148.9371052	249.2541200	62.31352999	15.57838250
2	0	1.722132532	313.5666051	540.0032516	135.0008129	33.75020323
3	0	1.598809242	577.3426444	923.0607554	230.7651889	57.69129721
1	1	1.912056842	170.0237691	325.0951111	81.27377778	20.31844444
2	1	1.829953619	351.1741292	642.6323686	160.6580922	40.16452304
3	1	1.649629252	637.6394652	1051.868714	262.9671785	65.74179463
1	2	2.169842367	215.2880596	467.1411528	116.7852882	29.19632205
2	2	1.936142163	430.1591364	832.8492409	208.2123102	52.05307756
3	2	1.690958206	762.3953596	1289.178689	322.2946723	80.57366809

$$\langle T\rangle_{nl} = \frac{\langle p^2\rangle_{nl}}{2\mu} \geq \frac{\langle r^{-2}\rangle}{8}. \tag{38}$$

We obtained the Dirac exchange energy (E^{ex}_{nl}) for atomic systems in the plane wave approximation for both $c\bar{c}$ and $b\bar{b}$ meson systems. Our results satisfy the lowest bound [29]:

$$E^{\text{ex}}_{nl} = \int_0^\infty \left[R^2_{nl}(r)\right]^{4/3} dr \geq \frac{2.95424}{5^{4/3}\pi^{1/3}} \left(\frac{\langle r^{-1}\rangle^5}{\langle r^{-2}\rangle}\right)^{1/3}. \tag{39}$$

Table 5: Fisher information, Cramer-Rao inequality, and Heisenberg uncertainty product for Bottomonium meson.

States		$b\bar{b}$				
n	l	$I(\rho)$	$I(\gamma)$	$FP \geq 16(l+\frac{3}{2})^2$	$C_R(\rho) \geq 4(l+\frac{3}{2})^2$	$HP \geq (l+\frac{3}{2})^2$
1	0	6.313306172	38.59207152	243.6435633	60.91089082	15.22772271
2	0	6.38322946	83.00613227	529.8471889	132.4617972	33.11544931
3	0	5.850361716	155.1223682	907.5219639	226.880491	56.72012275
1	1	7.212007872	44.60092084	321.6621922	80.41554805	20.10388701
2	1	6.773910176	93.84189286	635.6765529	158.9191382	39.72978455
3	1	6.026712108	172.685289	1040.724522	260.1811306	65.04528265
1	2	8.111576708	57.51732112	466.5561623	116.6390406	29.15976016
2	2	7.117483608	116.6551786	830.2913213	207.5728303	51.89320757
3	2	6.142354304	209.0584613	1284.111140	321.0277860	80.25694649

5. Conclusions

We have presented the bound states solution of the Schrödinger equation with the Cornell potential function. The energy spectra and the normalized wave function were utilized to obtain the expectation values $\langle p^2\rangle_{nl}$, $\langle T\rangle_{nl}$, $\langle V\rangle_{nl}$, $\langle r^2\rangle_{nl}$, $\langle r^{-1}\rangle_{nl}$, and $\langle r^{-2}\rangle_{nl}$ using two analytical methods such as the integral approach and the Hellmann-Feynman theorem method. We applied the mean values to access the Fisher information measures for the Charmonium and Bottomonium mesons.

Our results obey the Fisher uncertainty product inequality, the position space Cramer-Rao inequality, as well as the Heisenberg uncertainty. The average energies were found to exhibit oscillatory motion due to the quark-antiquark interactions. The kinetic energy satisfies different kinetic energy inequality lowest bounds for 3D quantum systems. The obtained Dirac exchange energy obeys the lowest bounds inequality for atomic systems. The ratio $\frac{(I(\rho)I(\gamma))_{c\bar{c}}}{(I(\rho)I(\gamma))_{b\bar{b}}}$ tends to unity, indicating that the masses, momentum, square radii, and radial probability density spread have minimal effects on the Fisher information measures due to their relative values. These results may provide insights into understanding the intrinsic properties of heavy meson systems.

Data availability

The data used in this article were obtained from the analytical solutions derived within the study.

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