Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 7 (2025) 2355

Journal of the Nigerian Society of Physical Sciences

Some properties of convolution and spherical analysis on the Euclidean motion group

U. E. Edeke[a,*], U. N. Bassey[b]

[a] Department of Mathematics, University of Calabar, Calabar, Nigeria
[b] Department of Mathematics, University of Ibadan, Ibadan, Nigeria

Abstract

The Euclidean motion group $G = SE(n)$ is a non-commutative and non-compact solvable Lie group which may also be referred to as an affine group of rigid motions on $\mathbb{R}^n$ realized as $G = \mathbb{R}^n \rtimes \mathrm{SO}(n)$. It is observed that this group is isomorphic to the collection of homogeneous transformation matrices. This article presents a novel method for calculating the Haar measure of G, and is demonstrated that it is the product of the Lebesque measure of $\mathbb{R}^2$ and the normalized measure of SO(2). A thorough description is given of the topology of the Schwartz space of G ($\mathcal{S}(G = SE(n))$) and how it is produced by a system of semi-norms. For $n = 2$, the convolution product of functions in $\mathcal{S}SE(2)$ realises the Radon transform of functions in $\mathcal{S}SE(2)$. It is demonstrated that this convolution is homogeneous, smooth and satisfies the moment condition. Further more, spherical analysis on the Gelfand pair $(\mathbb{R}^n \rtimes \mathrm{SO}(n), \mathrm{SO}(n))$ is presented, including an explicit determination of spherical function for G, when $n = 2$, by the method of separation of variables.

DOI:10.46481/jnsps.2025.2355

Keywords: Euclidean motion group, Radon transform, Convolution, Schwartz space, Gelfand pairs

Article History :
Received: 11 September 2024
Received in revised form: 08 December 2024
Accepted for publication: 11 January 2025
Available online: 06 February 2025

Communicated by: S. Fadugba

1. Introduction

Radon transform takes a function defined on $\mathbb{R}^n$ and transforms it into a function defined on a space of hyperplane of $\mathbb{R}^n$ [1, 2]. The use of the Radon transform is evident in tomography, which is the process of generating an image from projection data linked to an object's Cross Sectional Scan (CST) [1, 3]. This paper aims at investigating the Radon transform over $SE(n)$ using harmonic analysis and the method of convolution. It is demonstrated that this convolution is smooth, homogenous and is in the Schwartz space of SE(2).

The study of representations of semi-simple Lie groups is where the concept of a Gelfand pairs was initially observed [4]. Since then, analysis on symmetric spaces has also used this concept (Gelfand pair). A prototype of Fourier analysis on $\mathbb{R}^n$ is spherical analysis on the pair (G, K) [5]. Numerous special functions introduced in analysis have been shown to have tight ties to the idea of linear representations of Lie groups. The spherical functions are prominent among these functions. Both the continuous features of Lie groups and the classical Laplace spherical harmonics are generalized by the theory of spherical functions. In the current theory of infinite dimensional linear representation of Lie groups, spherical functions are crucial. We address spherical functions on the Euclidean motion groups in this paper. Without providing any proof, Jean Dieudonne [6] mentioned that the spherical function for SE(2) is a Bessel function of order zero. The aim to provide a proof of this claim in this study, and our finding reveals that the spherical functions

*Corresponding author Tel. No.: +234-703-014-8511.
Email address: ueedeke@gmail.com (U. E. Edeke)

for the pair (G, K) are spherical Bessel functions.
This research is organized into four sections as follows. Section two is concerned with preliminaries on the Euclidean motion group. Under this section, the structure of the Euclidean motion and Haar measures are carefully presented. Radon transform on $\mathbb{R}^n$ and its properties are also discussed, and is extended to Euclidean motion group. Results concerning properties of Convolution on SE(2) are presented in this section. In section four preliminaries concerning spherical functions on locally compact group, one parameter subgroups and vector fields on SE(2) are considered. Results concerning spherical functions on G and the pair (G, K) are also presented in this section.

2. Preliminaries

2.1. The Euclidean motion group

The group $SE(n)$ is realized as the semi-direct product of $\mathbb{R}^n$ with $SO(n)$. That is $SE(n) = \mathbb{R}^n \rtimes SO(n)$. Any member of $SE(n)$ may be denoted as $g = (\bar{x}, \xi)$, where $\xi \in SO(n)$ and $\bar{x} \in \mathbb{R}^n$. For any $g_1 = (\bar{x_1}, \xi_1)$ and $g_2 = (\bar{x_2}, \xi_2) \in SE(n)$, multiplication on SE(n) may be defined as [7]:

$$g_1 g_2 = (\bar{x_1} + \xi_1 \bar{x_2}, \xi_1 \xi_2),$$

and the inverse is defined as

$$g^{-1} = (-\xi^t \bar{x}, \xi^t).$$

Here ξ^t denotes a transpose. Alternatively, $SE(n)$ may also be identified as a matrix group whose arbitrary element may be identified as $(n+1) \times (n+1)$ matrix, given in the form

$$H(g) = \begin{pmatrix} \xi & \bar{x} \\ 0^t & 1 \end{pmatrix}, \tag{1}$$

where $\xi \in \mathrm{SO}(n)$ and $0^t = (0, 0..., 0)$. It is observed that $H(g_1)H(g_2) = H(g_1 g_2)$, $H(g^{-1}) = H^{-1}(g)$ and $g \mapsto H(g)$ is an isomorphism between $SE(n)$ and H(g). We state this result formally as a proposition below and present a proof.

2.1.1. Proposition

$g \mapsto H(g)$, being a map, is an isomorphism between $SE(n)$ and the set of transformation matrices, H(g), that are homogeneous.

Proof. We restrict the proof of this result to $n = 2$. Define the map $H : g \mapsto H(g)$ by $H(a, A) = \begin{pmatrix} \xi & \bar{x} \\ 0^t & 1 \end{pmatrix}$. We show that H is a bijective homomorphism which is also bi-continuous.

(i) *Homomorphism of H*. Let $g_1 = (x_1, \xi_1) \in SE(2)$ and $g_2 = (x_2, \xi_2) \in SE(2)$. We show that $H(g_1 g_2) = H(g_1)H(g_2)$. This follows because,

$$\begin{aligned} H(g_1 g_2) &= H[(x_1, \xi_1)(x_2, \xi_2)] = H(x_1 + \xi_1 x_2, \xi_1 \xi_2) \\ &= \begin{pmatrix} \xi_1 \xi_2 & x_1 + \xi_1 x_2 \\ 0^t & 1 \end{pmatrix} = \begin{pmatrix} \xi_1 & x_1 \\ 0^t & 1 \end{pmatrix} \begin{pmatrix} \xi_2 & x_2 \\ 0^t & 1 \end{pmatrix} \\ &= H(g_1)H(g_2). \end{aligned}$$

(ii) Injectivity of H. For $g_1 = (x_1, \xi_1)$ and $g_2 = (x_2, \xi_2)$, then $H(g_1) = H(g_2)$ implies that

$$\begin{pmatrix} \xi_1 & x_1 \\ 0^t & 1 \end{pmatrix} = \begin{pmatrix} \xi_2 & x_2 \\ 0^t & 1 \end{pmatrix}.$$

Therefore, $\xi_1 = \xi_2$ and $x_1 = x_2$, hence, $g_1 = g_2$. (iii) Surjectivity of H. For all $g \in SE(2)$, $\exists\, y \in GL(3, \mathbb{R})$ such that $H(g) = y$. Hence, H is surjective. (iv) Continuity of H and H^{-1}. Since every transformation map is continuous as well as its inverse, it therefore means that H and its inverse are continuous since H is a transformation map. □

We can now give the matrix representation of the element of $SE(2) \subset GL(3, \mathbb{R})$ by:

$$g((x_1, x_2), \phi) = \begin{pmatrix} \cos\phi & -\sin\phi & x_1 \\ \sin\phi & \cos\phi & x_2 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\phi \in [0, 2\pi]$, $(a_1, a_2) \in \mathbb{R}^2$ [8, 9] and in polar coordinate as

$$g(\bar{x}, \phi, \theta) = \begin{pmatrix} \cos\phi & -\sin\phi & x_1\cos\theta \\ \sin\phi & \cos\phi & x_2\sin\theta \\ 0 & 0 & 1 \end{pmatrix},$$

$\bar{x} = (x_1, x_2) \in \mathbb{R}^2$. $\phi, \theta \in [0, 2\pi]$. The group $SE(2)$ is a non - compact and non- commutative solvable Lie group [8]. It is also referred to as a group of rigid motions on $\mathbb{R}^n$ and plays a significant role in robotic, motion planning as well as dynamics [10]. $M(2)$ is a three (3) dimensional solvable Lie group with a connected component containing the identity. It is also called the Isometry group of $\mathbb{R}^2$, represented as $I(\mathbb{R})^2$.

In what follows, we focus on some aspects of harmonic analysis on $SE(2)$. In particular, we are concerned with some properties of convolution of functions on the group.

Before going on, it is pertinent to show explicitly how the Haar measure on the group $SE(2)$ may be computed. This is done in what follows.

2.2. Haar measure (Integration) on the Euclidean motion group

We begin this subsection by defining Radon measure on a space X that is locally compact and Hausdorff. The following definition is found in Refs. [11, 12]. Let $C_c(X)$ be the space of function on X that are continuous with compact support. A Radon measure on X is a function

$$\mu : C_c(X) \longrightarrow \mathbb{R},$$

such that for every $K \subseteq X$, which is compact, $\exists\, M_K$ such that

$$|\mu(f)| \leq M_K \parallel f \parallel_\infty, \ \forall f \in C_c(X).$$

Here and here after, $\parallel . \parallel_\infty$ is the uniform norm on $C_c(X)$. The action of μ on a function $f \in C_c(X)$ denoted by $\mu(f)$ is the integration of f and is defined by

$$\mu(f) = I(f) = \int_X f(x) d\mu(x).$$

A Radon measure μ on X is non- negative if for $f \in C_c(X)$ with $f(x) \geq 0$, we have $\mu(f) \geq 0$.

In order to describe the Haar measures on a locally compact group, we require the following preparations. Let $a \in G$ be fixed. The left and right actions of $a \in G$ are defined by:

$$L_a := x \mapsto ax : G \longrightarrow G,$$

and

$$R_a := x \mapsto xa : G \longrightarrow G,$$

respectively. Following these the left and right translations of the function

$$f : G \longrightarrow \mathbb{C} \text{ (} or\ \mathbb{R})$$

, are given by:

$${}_af(x) := (f \circ L_{a^{-1}})(x) = f(a^{-1}x),$$

and

$$f_a(x) := (f \circ R_a)(x) = f(xa),$$

respectively. μ is called left and right invariant respectively if $\mu({}_af) = \mu(f)$ and $\mu(f_a) = \mu(f)$. If $\mu({}_af) = \mu(f) \neq 0$, μ is called left Haar measure.

For a fixed element $a \in G$, recall that L_a is the left translation by a and R_a is the right translation by a. Given $g \in G$, we set $S(g) = |J(L_a)|$, where $J(L_a)|$ is the Jacobian matrix and $|.|$ is the absolute value. Similarly, we set $D(g) = |J(L_a)|$. Then the left and right Haar integrals of G are, respectively, computed by the formulas:

$$I_L(f) = \int_G \frac{f(g)}{S(g)} dg,$$

and

$$I_R(f) = \int_G \frac{f(g)}{D(g)} dg.$$

As given above, an element of $SE(2)$ may be written as:

$$g = \begin{pmatrix} \cos\phi & -\sin\phi & a_1 \\ \sin\phi & \cos\phi & a_2 \\ 0 & 0 & 1 \end{pmatrix}, \phi \in [0, 2\pi], (a_1, a_2) \in \mathbb{R}^2. \quad (2)$$

For $x \in SE(2)$, $L_x(g)$ is given as:

$$L_x(g) = \begin{pmatrix} \cos\phi & -\sin\phi & a_1 \\ \sin\phi & \cos\phi & a_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

$$= \begin{pmatrix} x_{11}\cos\phi - x_{21}\sin\phi + a_1x_{31} & x_{12}\cos\phi - x_{22}\sin\phi + a_1x_{32} & x_{13}\cos\phi - x_{23}\sin\phi + a_1x_{33} \\ x_{11}\sin\phi + x_{21}\cos\phi + a_2x_{31} & x_{12}\sin\phi + x_{22}\cos\phi + a_2x_32 & x_{13}\sin\phi + x_{23}\cos\phi + a_2x_{33} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}.$$

In order to compute the Jacobian of the left translation $L_x(g)$, we define the entries of matrix $L_x(g)$ by the functions $F_1, F_2, ..., F_9$ as follows:

$$F_1(x_{ij}) = x_{11}\cos\phi - x_{21}\sin\phi + a_1x_{31}$$

$$F_2(x_{ij}) = x_{12}\cos\phi - x_{22}\sin\phi + a_1x_{32}$$

$$F_3(x_{ij}) = x_{13}\cos\phi - x_{23}\sin\phi + a_1x_{33}$$

$$F_4(x_{ij}) = x_{11}\sin\phi + x_{21}\cos\phi + a_2x_{31}$$

$$F_5(x_{ij}) = x_{12}\sin\phi + x_{22}\cos\phi + a_2x_32$$

$$F_6(x_{ij}) = x_{13}\sin\phi + x_{23}\cos\phi + a_2x_{33}$$

$$F_7(x_{ij}) = x_{31},\ F_8(x_{ij}) = x_{32},\ F_9(x_{ij}) = x_{33}.$$

Therefore, the Jacobian is computed using the formula:

$$J(L_x) = \begin{pmatrix}
\frac{\partial F_1}{\partial x_{11}} & \frac{\partial F_1}{\partial x_{12}} & \frac{\partial F_1}{\partial x_{13}} & \frac{\partial F_1}{\partial x_{21}} & \frac{\partial F_1}{\partial x_{22}} & \frac{\partial F_1}{\partial x_{23}} & \frac{\partial F_1}{\partial x_{31}} & \frac{\partial F_1}{\partial x_{32}} & \frac{\partial F_1}{\partial x_{33}} \\
\frac{\partial F_2}{\partial x_{11}} & \frac{\partial F_2}{\partial x_{12}} & \frac{\partial F_2}{\partial x_{13}} & \frac{\partial F_2}{\partial x_{21}} & \frac{\partial F_2}{\partial x_{22}} & \frac{\partial F_2}{\partial x_{23}} & \frac{\partial F_2}{\partial x_{31}} & \frac{\partial F_2}{\partial x_{32}} & \frac{\partial F_2}{\partial x_{33}} \\
\frac{\partial F_3}{\partial x_{11}} & \frac{\partial F_3}{\partial x_{12}} & \frac{\partial F_3}{\partial x_{13}} & \frac{\partial F_3}{\partial x_{21}} & \frac{\partial F_3}{\partial x_{22}} & \frac{\partial F_3}{\partial x_{23}} & \frac{\partial F_3}{\partial x_{31}} & \frac{\partial F_3}{\partial x_{32}} & \frac{\partial F_3}{\partial x_{33}} \\
\frac{\partial F_4}{\partial x_{11}} & \frac{\partial F_4}{\partial x_{12}} & \frac{\partial F_4}{\partial x_{13}} & \frac{\partial F_4}{\partial x_{21}} & \frac{\partial F_4}{\partial x_{22}} & \frac{\partial F_4}{\partial x_{23}} & \frac{\partial F_4}{\partial x_{31}} & \frac{\partial F_4}{\partial x_{32}} & \frac{\partial F_4}{\partial x_{33}} \\
\frac{\partial F_5}{\partial x_{11}} & \frac{\partial F_5}{\partial x_{12}} & \frac{\partial F_5}{\partial x_{13}} & \frac{\partial F_5}{\partial x_{21}} & \frac{\partial F_5}{\partial x_{22}} & \frac{\partial F_5}{\partial x_{23}} & \frac{\partial F_5}{\partial x_{31}} & \frac{\partial F_5}{\partial x_{32}} & \frac{\partial F_5}{\partial x_{33}} \\
\frac{\partial F_6}{\partial x_{11}} & \frac{\partial F_6}{\partial x_{12}} & \frac{\partial F_6}{\partial x_{13}} & \frac{\partial F_6}{\partial x_{21}} & \frac{\partial F_6}{\partial x_{22}} & \frac{\partial F_6}{\partial x_{23}} & \frac{\partial F_6}{\partial x_{31}} & \frac{\partial F_6}{\partial x_{32}} & \frac{\partial F_6}{\partial x_{33}} \\
\frac{\partial F_7}{\partial x_{11}} & \frac{\partial F_7}{\partial x_{12}} & \frac{\partial F_7}{\partial x_{13}} & \frac{\partial F_7}{\partial x_{21}} & \frac{\partial F_7}{\partial x_{22}} & \frac{\partial F_7}{\partial x_{23}} & \frac{\partial F_7}{\partial x_{31}} & \frac{\partial F_7}{\partial x_{32}} & \frac{\partial F_7}{\partial x_{33}} \\
\frac{\partial F_8}{\partial x_{11}} & \frac{\partial F_8}{\partial x_{12}} & \frac{\partial F_8}{\partial x_{13}} & \frac{\partial F_8}{\partial x_{21}} & \frac{\partial F_8}{\partial x_{22}} & \frac{\partial F_8}{\partial x_{23}} & \frac{\partial F_8}{\partial x_{31}} & \frac{\partial F_8}{\partial x_{32}} & \frac{\partial F_8}{\partial x_{33}} \\
\frac{\partial F_9}{\partial x_{11}} & \frac{\partial F_9}{\partial x_{12}} & \frac{\partial F_9}{\partial x_{13}} & \frac{\partial F_9}{\partial x_{21}} & \frac{\partial F_9}{\partial x_{22}} & \frac{\partial F_9}{\partial x_{23}} & \frac{\partial F_9}{\partial x_{31}} & \frac{\partial F_9}{\partial x_{32}} & \frac{\partial F_9}{\partial x_{33}}
\end{pmatrix}$$

$$= \begin{pmatrix}
\cos\phi & 0 & 0 & -\sin\phi & 0 & 0 & a_1 & 0 & 0 \\
0 & \cos\phi & 0 & 0 & -\sin\phi & 0 & 0 & a_1 & 0 \\
0 & 0 & \cos\phi & 0 & 0 & -\sin\phi & 0 & 0 & a_1 \\
\sin\phi & 0 & 0 & \cos\phi & 0 & 0 & a_2 & 0 & 0 \\
0 & \sin\phi & 0 & 0 & \cos\phi & 0 & 0 & a_2 & 0 \\
0 & 0 & \sin\phi & 0 & 0 & \cos\phi & 0 & 0 & a_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

so that $S(x) = |J(L_a)| = 1$. The left Haar integral over $SE(2)$ is therefore given as:

$$I_L(f) = \int_{SE(2)} \frac{f(g)}{S(x)} dg = \int_{SE(2)} f(g)dg$$
$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_{\mathbb{R}^2} f(x_1, x_2)dx_1dx_2 = \int_{\mathbb{R}^2} f(x_1, x_2)dx_1dx_2.$$

Here $\frac{1}{2\pi}\int_0^{2\pi} d\phi$ is the normalised Haar integral of $SO(2)$ up to isomorphism with the circle group $\mathbb{T}$. A similar calculation shows that the right Haar integral on $SE(2)$ is the same as the left Haar integral. Thus the Lie group $SE(2)$ is unimodular. Henceforth, $dg = d\mu_G(g)$ denotes the Haar measure of $G = SE(2)$.

We close this subsection with the definition of the Schwartz space of $SE(2)$. Consider $SE(2)$ realized as $\mathbb{R} \rtimes \mathbb{T}$ where $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$. If we choose a system of coordinates (x, y, θ) on G with $x, y \in \mathbb{R}$ and $\theta \in \mathbb{T}$, then the Schwartz space of $SE(2)$ may be defined in the following way.

2.2.1. Definition (Ref. [10])

A complex-valued C^∞ function f on $G = SE(2)$ is called rapidly decreasing if for any $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}^3$ we have:

$$P_{N,\alpha}(f) = Sup_{\theta\in\mathbb{T},\xi\in\mathbb{R}^2} \mid (1 + ||\xi||^2)^N (D^\alpha f)(\xi, \theta) \mid < +\infty,$$

where

$$D^\alpha = \left(\frac{\partial}{\partial x}\right)^{\alpha_1}\left(\frac{\partial}{\partial y}\right)^{\alpha_2}\left(\frac{\partial}{\partial \theta}\right)^{\alpha_3},$$

$(\alpha = (\alpha_1, \alpha_2, \alpha_3); \xi = (x, y))$. The vector space of all rapidly decreasing functions on G is denoted by $\mathcal{S} = \mathcal{S}(G)$. Then $\mathcal{S}$

is a Frechet space in the topology given by the family of semi-norms $\{P_{N,\alpha} : N \in \mathbb{N}, \alpha \in \mathbb{N}^3\}$. The space $S'(G)$ of linear functionals on $S(G)$ that are continuous is called the space of tempered distribution on $G = SE(2)$.

3. Radon transform of functions on $\mathbb{R}^n$ and $SE(2)$

Let $X = (x_1, x_2, ..., x_n)$ be an arbitrary point in $\mathbb{R}^n$, $f(X) = f(x_1, x_2, ..., x_n)$ be a function on $\mathbb{R}^n$ and $dX = dx_1 dx_2 ... dx_n$ be a measure in $\mathbb{R}^n$. Let ξ be a unit vector that defines the hyperplane with the equation

$$P = \xi \cdot X = \xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n.$$

The Radon transform of f is defined as

$$\Re\{f(X)\} = \int_{\mathbb{R}^n} f(X)\delta(P - \xi \cdot X)dX.$$

We may also denote $\Re\{f(X)\}$ by $\check{f}(P, \xi)$. The properties of this transform on $\mathbb{R}^n$ as contained in Ref. [2] are:

1. Homogeneity property: This property states that, for arbitrary real number $s \neq 0$,

$$\check{f}(sP, s\xi) = | s |^{-1} \check{f}(P, \xi).$$

   If $s = -1$, $\check{f}(-P, -\xi) = \check{f}(P, \xi)$. This is the symmetric property.
2. Linearity property: Given f and g as functions and c_1, c_2 as constants, then

$$\Re\{c_1 f + c_2 g\} = c_1 \Re f + c_2 \Re g.$$

   This property allows this transform defined on $\mathbb{R}^n$ to be seen as a linear transform.
3. Shifting property: Considering $f(X - a)$, $(a \in \mathbb{R}^n)$, where $X - a$ are $(x_1 - a_1, x_2 - a_2, ..., x_n - a_n)$. Then

$$\begin{aligned} \Re f(X - a) &= \int_{\mathbb{R}^n} f(X - a)\delta(P - \xi \cdot X)dX \\ &= \int_{\mathbb{R}^n} f(Y)\delta(P - \xi.a - \xi.Y)dY. \end{aligned}$$

$dX = dx_1.dx_2...dx_n$ and $dY = dy_1.dy_2...dy_n$. If $a = -a$, we have:

$$\Re f(X + a) = \check{f}(P + \xi \cdot a, \xi).$$

We also give the definition of Radon transform of functions on $SE(2)$. Let f be a complex valued function on $SE(2)$. Then the Radon transform of f as may be found in Ref. [8] is defined as:

$$\check{f}(a \cdot e, A^T e) = \int_{\mathbb{R}^2} f(r)\delta(a \cdot e - A^T r)dr. \tag{3}$$

The definition of Radon transform on $SE(2)$ given in equation (3) above may be shown to be the convolution of two functions on $SE(2)$ as follows. Let

$$f^*(g) = f(g^{-1})$$

be a conjugate function. Then the convolution of f_1 and f_2^* on G is given by

$$(f_1 * f_2^*)(g) = \int_G f_1(h) f_2(g^{-1}h)\, dh = \int_G f_1(gh) f_2(h) dh.$$

When $G = SE(2)$, we write:

$$(f_1 * f_2^*)(a, A) = \int_{SO(2)} \int_{\mathbb{R}^2} f_1(a + Ar, AR) f_2(r, R)\, dr\, dR,$$

where dr is the Lebesque measure on $\mathbb{R}^2$ and dR is the Haar measure on $SO(2)$.

Letting $e_1 = (1, 0) \in \mathbb{R}^2$, $f_1(a, A) = \delta(a.e_1)$ and $f_2(r, R) = f(-r)$, we have:

$$(f_1 * f_2^*)(a, A) = \int_{SO(2)} \int_{\mathbb{R}^2} \delta(a + Ar) f_2(-r) dr dR.$$

Let dR be a measure on $SO(2)$ normalized so that the total measure equals 1. In other words, assume that $\int_{SO(2)} dR = 1$, then

$$(f_1 * f_2^*)(a, A) = \int_{\mathbb{R}^2} \delta(a + Ar) f(-r) dr.$$

By applying the transformation $r \to -r$, we get

$$\begin{aligned} (f_1 * f_2^*)(a, A) &= \int_{\mathbb{R}^2} \delta(a - Ar) f(r) dr = \int_{\mathbb{R}^2} f(r)\delta(a - Ar) dr \\ &= \int_{\mathbb{R}^2} f(r)\delta(a \cdot e_1 - A^T r) dr = \check{f}(a \cdot e_1, A^T e_1). \end{aligned}$$

We present the following result in the theorem below:

3.1. Theorem

Let $G = \mathbb{R}^2 \rtimes SO(2)$ and let $\Phi(a, A)$ be the convolution product of functions $f_1, f_2^* \in \mathcal{S}(G)$, with f_2 also a function in $\mathcal{S}(G)$. Then the following properties hold:

(i) $\Phi(a, A)$ satisfies the homogeneity condition

$$\Phi(\alpha a, \alpha A) = | \alpha |^{-1} \Phi(a, A), \ \alpha > 0.$$

(ii) $\Phi(a, A)$ is infinitely differentiable with respect to a.

(iii) $\Phi(a, A) \in \mathcal{S}(G)$

(iv) the integral

$$I = \int_{\mathbb{R}^2} ||\xi||^t \Phi(\xi, A) d\xi$$

contains a polynomial in t of degree $(n - 1)$ of the form

$$P_n(t) = (-1)^{n-1} ||\xi||^t \beta \Big(\sum_{k=1}^{n-1} (\ln ||\xi||)^k t^k \Big).$$

Proof

(i) We put

$$\Phi(a, A) = \int_{\mathbb{R}^2} f(r)\delta(a.e - rA^T) dr.$$

Then,

$$\Phi(\alpha a,\alpha A)=\int_{\mathbb{R}^2} f(r)\delta(\alpha a.e-\alpha A^T r)dr$$
$$=\int_{\mathbb{R}^2} f(r)\delta[\alpha(a.e-A^T r)]dr.$$

Using the properties below

$$\delta(ax)=a^{-1}\delta(x)=|a|^{-1}\delta(x), \tag{4}$$

we have:

$$\Phi(\alpha a,\alpha A)=\int_{\mathbb{R}^2} f(r)\alpha^{-1}\delta(a.e-A^T r)dr$$
$$=\frac{1}{\alpha}\int_{\mathbb{R}^2} f(r)\delta(a.e-A^T r)dr$$
$$=|\alpha|^{-1}\int_{\mathbb{R}^2} f(r)\delta(a.e-A^T r)dr=|\alpha|^{-1}\check{f}(a.e,A^T e).$$

Hence,

$$\Phi(\alpha a,\alpha A)=|\alpha|^{-1}\Phi(a,A).$$

(ii)

$$\Phi(a,A)=\int_{\mathbb{R}^2} f(r)\delta(a.e-A^T.er)dr.$$

Now

$$\frac{d}{da}\Phi(a,A)=\frac{d}{da}\int_{\mathbb{R}^2} f(r)\delta(a.e-A^T er)dr$$
$$=\int_{\mathbb{R}^2} f(r)dr\frac{d}{da}\delta(a.e-A^T er).$$

Our focus is on $\frac{d}{da}\delta(a.e-A^T er)$. $a=(a_1,a_2)$ and $r=(r_1,r_2)$, $A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ and $A^T=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$, then

$$a-A^T r=\begin{pmatrix}a_1\\a_2\end{pmatrix}-\begin{pmatrix}\cos\theta & \sin\theta\\ -sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}r_1\\r_2\end{pmatrix}$$
$$=\begin{pmatrix}a_1\\a_2\end{pmatrix}-\begin{pmatrix}\cos\theta r_1+\sin\theta r_2\\ -\sin\theta r_1+\cos\theta r_2\end{pmatrix}$$
$$=\begin{pmatrix}a_1-\cos\theta r_1-\sin\theta r_2\\ a_2+\sin\theta r_1-\cos\theta r_2\end{pmatrix}$$
$$=\begin{pmatrix}a_1-\cos\theta r_1-\sin\theta r_2, & a_2+\sin\theta r_1-\cos\theta r_2\end{pmatrix}\in\mathbb{R}^2.$$

Since for $X=(x_1,x_2,x_3,...,x_n)$, $\delta(X)=\prod_{j=1}^{n}\delta(x_j)$. It means that

$$\delta(a_1-cos\theta r_1-sin\theta r_2,a_2-sin\theta r_1+cos\theta r_2)$$
$$=\delta(a_1-cos\theta r_1-sin\theta r_2)\delta(a_2+sin\theta r_1-cos\theta r_2).$$

Therefore,

$$\frac{d}{da}\delta(a_1-cos\theta r_1-sin\theta r_2)\delta(a_2+sin\theta r_1-cos\theta r_2)$$
$$=H(a_1)\delta(a_2+sin\theta r_1-cos\theta r_2)$$
$$+H(a_2)\delta(a_1-cos\theta r_1-sin\theta r_2),$$

$$\frac{d^2}{da}\delta(a_1-cos\theta r_1-sin\theta r_2)\delta(a_2+sin\theta r_1-cos\theta r_2)$$
$$=H(a_1)H(a_2)+H(a_1)H(a_2)=2H(a_1)H(a_2),$$

$$\frac{d^3}{da}\delta(a_1-cos\theta r_1-sin\theta r_2)\delta(a_2+sin\theta r_1-cos\theta r_2)=0,$$

where H denotes the Heaviside's function. We can conclude that $f_1*f_2\in C^\infty(G)$.

(iii) Let $f_1,f_2\in S(G)$. We show that $f_1*f_2\in S(G)$. This means that we have to show that
(a) $f_1*f_2\in C^\infty(G)$
(b) $D^\alpha(f_1*f_2)(g)$ is bounded. (a) has been proved in property (ii) of the result above. Now to prove (b), let us give the definition of convolution of functions in $\mathcal{S}(G)$. Given $f_1,f_2\in\mathcal{S}(SE(2))$, the convolution of f_1 and f_2 is defined as

$$(f_1*f_2)(g)=\int_G f_1(h)f_2(h^{-1}g)dh.$$

Therefore,

$$|D^\alpha(f_1*f_2)(g)|=|\int_G D^\alpha f_1(h)f_2(h^{-1}g)dh|.$$

We note that $SE(2)$ is unimodular [12], this means

$$\int_G f(hg)dg=\int_G f(gh)dg=\int_G f(g^{-1})dg=\int_G f(g)dg.$$

So, by putting $g=hg$, we get:

$$|D^\alpha(f_1*f_2)(g)|=|\int_G D^\alpha f_1(h)f_2(h^{-1}(hg)dh|$$
$$=|\int_G D^\alpha f_1(h)f_2(g)dh|$$
$$\le\int_G|D^\alpha f_1(h)f_2(g)dh|.$$

We know that

$$D^\alpha(f_1*f_2)(g)=f_1*D^\alpha f_2(g).$$

But

$$f_1*D^\alpha f_2(h)=\int_G f_1(h)D^\alpha(h^{-1}g)dh.$$

Therefore

$$|D^\alpha(f_1*f_2)(g)|\le\int_G|f_1(h)||D^\alpha f_2(g)|dh$$
$$\le\int_G|f_1(h)|dh\frac{C}{|(1+||\xi||^2)^m|},$$

because $|D^\alpha f_2(g)|\le\frac{C_{\alpha,N}}{|(1+||\xi||^2)^m|}$. Since $||\xi||$ is a positive real constant, we may put $Q_N=|(1+||\xi||^2)^N|$, so that

$$|D^\alpha(f_1*f_2)(g)|\le\int_G|f_1(h)||D^\alpha f_2(g)|dh$$

$$\leq \int_G |f_1(h)|dh \frac{C}{|(1+||\xi||^2)^N|} \leq \frac{C}{Q_N}\int_G |f_1(h)|dh = \frac{C}{Q_N}||f_1||_{L^1(G)},$$

$$|(1+||\xi||^2)^N D^\alpha(f_1 * f_2)(\xi,\theta)| = |(1+||\xi||^2)^N||D^\alpha(f_1 * f_2)(\xi,\theta)| \leq C||f||_{L^1(G)} < +\infty.$$

On taking supremum, we have

$$P_{\alpha,N}(f_1 * f_2) = sup_{\theta\in\mathbb{T},\xi\in\mathbb{R}^2}|(1+||\xi||^2)^N D^\alpha(f_1 * f_2)(\xi,\theta)| < +\infty.$$

(iv) We prove that

$$I = \int_{\mathbb{R}^2} ||\xi||^t \Phi(\xi, A)d\xi$$

is a polynomial of degree t. To this end, we evaluate I using partial integration formula. To effect this, let $U = ||\xi||^t$ and let $dV = \Phi(\xi, A)$ so that $V = \int_{\mathbb{R}^2}\Phi(\xi, A)d\xi$. Therefore,

$$I = UV - \int_{\mathbb{R}^2} VdU.$$

Now

$$\frac{dU}{dt} = ||\xi||^t \ln||\xi|| \Rightarrow dU = ||\xi||^t \ln||\xi||dt.$$

Now,

$$I = ||\xi||^t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi - \int_{\mathbb{R}}\int_{\mathbb{R}^2}\Phi(\xi,A)d\xi||\xi||^t \ln||\xi||dt$$
$$= ||\xi||^t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi - \int_{\mathbb{R}}\ln||\xi||dt \int_{\mathbb{R}^2}||\xi||^t\Phi(\xi,A)d\xi$$

$$= ||\xi||^t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi - \ln||\xi||t \int_{\mathbb{R}^2}||\xi||^t\Phi(\xi,A)d\xi. \quad (5)$$

Integrating equation (5), we obtain

$$I = ||\xi||^t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi - \ln||\xi||t\Big(||\xi||^t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi - \ln||\xi||t \int_{\mathbb{R}^2}||\xi||^t\Phi(\xi,A)d\xi\Big)$$

$$= ||\xi||^t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi - ||\xi||^t\ln||\xi||t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi + (\ln||V||)^2t^2 \int_{\mathbb{R}^2}||\xi||^t\Phi(\xi,A)d\xi. \quad (6)$$

Integrating equation (6), we obtain further

$$I = ||\xi||^t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi - ||\xi||^t\ln||\xi||t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi + (\ln||\xi||)^2t^2 \times\Big(||\xi||^t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi - \ln||\xi||t \int_{\mathbb{R}^2}||\xi||^t\Phi(\xi,A)d\xi\Big)$$
$$= ||\xi||^t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi - ||\xi||^t\ln||\xi||t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi + (\ln||\xi||)^2t^2||\xi||^t \int_{\mathbb{R}^2}\Phi(\xi,A)da - (\ln||\xi||)^3t^3 \int_{\mathbb{R}^2}||\xi||^t\Phi(\xi,A)d\xi.$$

Integrating I further, we obtain:

$$I = ||\xi||^t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi - ||\xi||^t\ln||\xi||t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi + ||\xi||^t(\ln||\xi||)^2t^2||\xi||^t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi - (\ln||\xi||)^3t^3 \Big(||\xi||^t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi - \ln||\xi||t \int_{\mathbb{R}^2}||\xi||^t\Phi(\xi,A)d\xi\Big)$$
$$= ||\xi||^t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi - ||\xi||^t\ln||\xi||t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi + ||\xi||^t(\ln||\xi||)^2t^2||\xi||^t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi - ||\xi||^t(\ln||\xi||)^3t^3||\xi||^t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi + (\ln||\xi||)^4t^4\Big(||\xi||^t \int_{\mathbb{R}^2}\Phi(\xi,A)d\xi\Big).$$

We can conclude that the integral I contains a polynomial in t of degree $(n-1)$ of the form:

$$P_n(t) = (-1)^{n-1}||\xi||^t\beta\Big(\sum_{k=1}^{n-1}(\ln||\xi||)^k t^k\Big),$$

and a non terminating integral of the form:

$$(-1)^n\beta\ln(||\xi||)^n t^n||\xi||^t \int_{\mathbb{R}^2}||\xi||^t\Phi(\xi,A)d\xi. \qquad \square$$

The above result is an extension of the work of Gelfand and Gindikin [2] obtained for $\mathbb{R}^n$. We have extended the result to the motion group and obtained a unique polynomial in property (iv) which is used mainly in approximation theory for obtaining numerical solutions.

4. Explicit determination of spherical function for $SE(2)$

In this section, an explicit derivation of spherical function for $SE(n)$, when $n = 2$ is presented. Before then, some preliminaries about one-parameter subgroups, vector fields and spherical functions on G are required. We start with one parameter subgroups of $SE(2)$ in subsection 4.1 followed by vector fields in subsection 4.2, spherical functions are discussed in subsection 4.3.

4.1. One-parameter subgroup of SE(2)

We begin this section by identifying the Lie algebra of $SE(n)$ denoted by $se(n)$. This Lie algebra is the sub-algebra of $gl(n+1,\mathbb{R})$ which may be defined as:

$$se(n) = \Big\{X = \begin{pmatrix} P & Q \\ 0 & 0\end{pmatrix} : Q\in\mathbb{R}^n,\ P\in so(n)\Big\}.$$

$gl(n+1,\mathbb{R})$ is the Lie algebra of $GL(n+1,\mathbb{R})$, the general linear group. It consist of real matrices Z of dimension $n+1$ with the Lie bracket $[Z_1,Z_2] = Z_1Z_2 - Z_2Z_1$ for $Z_1, Z_2 \in gl(n+1,\mathbb{R})$ and $so(n)$, being the Lie algebra of $SO(n)$, is defined as $so(n) = \left\{P \in gl(n+1,\mathbb{R}) | P + P^t = 0\right\}$ [13]. $se(2)$ is a three dimensional Linear with the following as basis elements:

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

These basis elements obey these relations $[X_1, X_2] = 0$, $[X_2, X_3] = X_1$ and $[X_3, X_1] = X_2$ and are said to be commutative. They are shown below:

$$\begin{aligned}[X_1, X_2] &= X_1X_2 - X_2X_1 \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &- \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= 0.\end{aligned}$$

$$\begin{aligned}[X_2, X_3] &= X_2X_3 - X_3X_2 \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &- \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= X_1.\end{aligned}$$

$$\begin{aligned}[X_3, X_1] &= X_3X_1 - X_1X_3 \\ &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &- \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= X_2.\end{aligned}$$

It is now suitable to discuss the one-parameter subgroup of $SE(2)$. This is important for defining the left or right invariant differential operators on $SE(2)$. Before going on, the one-parameter subgroup of a (real) Lie group G in general is defined as follows [14].

4.1.1. Definition

Let G be a linear Lie group. A function $\zeta : \mathbb{R} \to G$ is a one-parameter subgroup of G if
(a) ζ is continuous
(b) $\zeta(0) = I$, I is the identity element of G
(c) $\zeta(t+s) = \zeta(t)\zeta(s)$, for all $s, t \in \mathbb{R}$.

There is an important theorem for calculating the one parameter subgroup of G which we state in theorem 4.1.2 below.

4.1.2. Theorem (Ref. [14])

Let $\zeta : \mathbb{R} \to GL(n,\mathbb{R})$ be a one-parameter subgroup. ζ is a C^∞ and $\zeta(t) = \exp(tB)$, with $B = \zeta'(0)$. In fact, ζ is seen to be even and real analytic.

Putting $t = 1$, then $\zeta(1) = \exp(B)$ and since the exponential map exp(.) is always defined from the Lie algebra of G to G itself, it stands to reason that $B \in gl(n,\mathbb{R})$. We can now calculate the one-parameter subgroups of $SE(2)$ using the formula specified in theorem 4.1.2. For X_1, we have

$$\begin{aligned}\zeta_1(t) &= \exp(tX_1) \\ &= I + tX_1 + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + ...\end{aligned}$$

Since

$$\begin{aligned}X_1^2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = X_1^n, \ \forall\, n > 2,\end{aligned}$$

we have

$$\zeta_1(t) = I + tX_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, the one-parameter subgroup of $SE(2)$ corresponding to X_2 is

$$\zeta_2(t) = \exp(tX_2) = I + tX_2 + \frac{t^2X_2^2}{2!} + \frac{t^3X_2^3}{3!} + ...$$

But

$$X_2^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = X_1^n, \; \forall \; n > 2.$$

Therefore,

$$\zeta_2(t) = I + tX_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally, $\zeta_3(t)$ is obtained as:

$$\zeta_3(t) = \exp(tX_3) = I + tX_3 + \frac{t^2X_3^2}{2!} + \frac{t^3X_3^3}{3!} + \dots \quad (7)$$

Now

$$X_3^2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_3^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_3^4 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_3^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_3^6 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $X_3^6 = X_3^2$, the exponential series equation (7) terminates at X_3^5. Therefore,

$$\begin{aligned} \zeta_3(t) &= I + tX_3 + \frac{t^2X_3^2}{2!} + \frac{t^3X_3^3}{3!} + \frac{t^4X_3^4}{4!} + \frac{t^5X_3^5}{5!} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + t\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &+ \frac{t^3}{3!}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &+ \frac{t^4}{4!}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{t^5}{5!}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \dots & -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \dots & 0 \\ t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \dots & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

4.2. Vector fields on SE(2)

An explicit invariant differential operator on SE(2) [15, 16], which is of interest in what follows, is the Laplace-Beltrami operator given by:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \psi^2}.$$

Before going on, we show that $(\mathbb{R}^n \rtimes SO(n), SO(n))$ is a Gelfand pair, where $G = \mathbb{R}^n \rtimes SO(n)$ is the general Euclidean motion group. For $g \in G$, $g = (\xi, k)$ with $\xi \in \mathbb{R}^n$, $k \in SO(2) =: K$. Product on G may be considered as an action of G on $\mathbb{R}^n$ given as:

$$g \cdot x = k \cdot x + \xi,$$

where $\xi, x \in \mathbb{R}^n$ and $k \in SO(n)$. Now, given $k_1, k_2 \in K$ and $\xi_1, \xi_2 \in \mathbb{R}^n$, one can define a product in G as $(k_1, \xi_1)(k_2, \xi_2) = (k_2 \cdot \xi_1 + \xi_2, k_1k_2)$. This shows that $(k_1, \xi_1) = (k_1, 0)(1, \xi_1)$ following the product defined above. Let us define a map ϕ as:

$$\phi : G \to G,$$

as

$$(k_1, \xi_1) \mapsto (k_1, -\xi_1) \; or \; \phi(k_1, \xi_1) = (k_1, -\xi_1).$$

Then ϕ is an automorphism of G that is involutive and continuous and $\phi(k_1, \xi_1) = [(k_1, 0)(1, \xi_1)] = (k_1, 0)(1, -\xi_1) = (k_1, 0)(k_1, \xi_1)^{-1}(k_1, 0)$. Therefore, $\phi(g) \in Kg^{-1}K$, $\forall g \in G$. This shows that (G, K) is a Gelfand pair. This claim is supported by Ref. [17]. Let us give an explicit derivation of spherical function for $G = \mathbb{R}^n \rtimes SO(n)$ as follows.

The Laplace-Beltrami operator on SE(2) [16] is given by:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \psi^2}. \quad (8)$$

Let the operator act on $\varphi = \varphi(r, \theta, \psi)$, then

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial \theta^2} + \frac{\partial^2\varphi}{\partial \psi^2}. \quad (9)$$

We have the following elleptic partial differential equation

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial \theta^2} + \frac{\partial^2\varphi}{\partial \psi^2} = 0. \quad (10)$$

Let us assume that the above equation has a solution of the form

$$\varphi(r, \theta, \psi) = R(r)F(\theta)\Psi(\psi). \quad (11)$$

Now,

$$\begin{cases} \dfrac{\partial\varphi}{\partial r} = F(\theta)\Psi(\psi)\dfrac{\partial R}{\partial r} \text{ and } \dfrac{\partial^2\varphi}{\partial r^2} & = F(\theta)\Psi(\psi)\dfrac{\partial^2 R}{\partial r^2} \\ \dfrac{\partial^2\varphi}{\partial \theta^2} & = R(r)\Psi(\psi)\dfrac{\partial^2 F}{\partial \theta^2} \\ \dfrac{\partial^2\varphi}{\partial \psi^2} & = R(r)F(\theta)\dfrac{\partial^2 \Psi}{\partial \psi^2}. \end{cases} \quad (12)$$

Substituting equation (12) into equation (10), we have:

$$F(\theta)\Psi(\psi)\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}F(\theta)\Psi(\psi)\frac{\partial R(r)}{\partial r} + \frac{1}{r^2}R(r)\Psi(\psi)\frac{\partial^2 F}{\partial \theta^2}$$

$$+ R(r)F(\theta)\frac{\partial^2\Psi}{\partial\psi^2} = 0. \quad (13)$$

Dividing equation (13) by $R(r)F(\theta)\Psi(\psi)$, we get:

$$\frac{1}{R(r)}\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{R(r)r}\frac{\partial R(r)}{\partial r} + \frac{1}{F(\theta)r^2}\frac{\partial^2 F(\theta)}{\partial\theta^2} + \frac{1}{\Psi(\psi)}\frac{\partial^2\Psi(\psi)}{\partial\psi^2} = 0.$$

$$\frac{1}{R(r)}\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{R(r)r}\frac{\partial R(r)}{\partial r} + \frac{1}{F(\theta)r^2}\frac{\partial^2 F(\theta)}{\partial\theta^2} = -\frac{1}{\Psi(\psi)}\frac{\partial^2\Psi(\psi)}{\partial\psi^2}. \quad (14)$$

The Left Hand Side of equation (14) depends only on (r, θ) while the Right Hand Side depends only on ψ. We can equate each side to a constant, say $-m^2$. Thus we get:

$$-\frac{1}{\Psi(\psi)}\frac{\partial^2\Psi}{\partial\Psi^2} = -m^2 \Rightarrow \frac{1}{\Psi(\psi)}\frac{\partial^2\Psi}{\partial\Psi^2} = m^2 \Rightarrow \frac{\partial^2\Psi}{\partial\Psi^2} = m^2\Psi(\psi), \quad (15)$$

and

$$\frac{1}{R(r)}\frac{\partial^2 R}{\partial r^2} + \frac{1}{R(r)r}\frac{\partial R}{\partial r} + \frac{1}{F(\theta)r^2}\frac{\partial^2 F}{\partial\theta^2} = -m^2. \quad (16)$$

Multiplying equation (16) by r^2, we get:

$$\frac{r^2}{R(r)}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R(r)}\frac{\partial R}{\partial r} + \frac{1}{F(\theta)}\frac{\partial^2 F}{\partial\theta^2} = -m^2 r^2$$

so that

$$\frac{r^2}{R(r)}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R(r)}\frac{\partial R}{\partial r} + m^2 r^2 = -\frac{1}{F(\theta)}\frac{\partial^2 F}{\partial\theta^2}. \quad (17)$$

Again equate both sides equation (17) to n^2, n being a fixed positive integer:

$$\frac{r^2}{R(r)}\frac{d^2 R(r)}{dr^2} + \frac{r}{R(r)}\frac{dR(r)}{dr} + m^2 r^2 = n^2 \quad (18)$$

$$-\frac{1}{F(\theta)}\frac{d^2 F(\theta)}{d\theta} = n^2. \quad (19)$$

We can now solve the ODE:

$$\frac{r^2}{R(r)}\frac{d^2 R(r)}{dr^2} + \frac{r}{R(r)}\frac{dR(r)}{dr} + m^2 r^2 - n^2 = 0. \quad (20)$$

Next, we transform this equation into Bessel equation. To do this, we let $mr = x$ so that $\frac{\partial x}{\partial r} = m$. Then,

$$\frac{dR(r)}{dr} = \frac{dR(r)}{dx}\frac{dx}{dr} = m\frac{dR(r)}{dx}.$$

$$\frac{d^2 R(r)}{dr^2} = \frac{d}{dr}\left(\frac{dR}{dr}\right) = \frac{d}{dr}\left(m\frac{dR(r)}{dx}\right) = m\frac{d^2 R(r)}{dx^2}\frac{dx}{dr} = m\frac{d^2 R(x)}{dx^2}m = m^2\frac{d^2 R(x)}{dx^2}.$$

Equation (20) becomes

$$m^2\frac{r^2}{R(r)}\frac{d^2 R(r)}{dx^2} + \frac{mr}{R(r)}\frac{R(r)}{dx} + (m^2 r^2 - n^2) = 0. \quad (21)$$

Multiply equation (21) by $R(r)$ to get

$$m^2 r^2\frac{{}^2 R(r)}{dx^2} + mr\frac{R(r)}{dx} + (m^2 r^2 - n^2)R(r) = 0,$$

where $mr = x$; $m^2 r^2 = x^2$, therefore,

$$x^2\frac{d^2 R(r)}{dx^2} + x\frac{R(r)}{dx} + (x^2 - n^2)R(r) = 0.$$

This may be re-written as:

$$\frac{d^2 R(r)}{dx^2} + \frac{1}{x}\frac{dR(r)}{dx} + (1 - \frac{n^2}{x^2})R(r) = 0. \quad (22)$$

The differential equation (22) is a Bessel differential equation and it has a solution of the form

$$J_\lambda(mr) = \Gamma(\frac{n}{2})\sum_{k=0}^{\infty}\frac{\lambda^k}{k!\Gamma(k + \frac{n}{2}}(\frac{mr}{2})^{2k} \quad (23)$$

$$= \Gamma(\frac{n}{2})(\frac{\sqrt{\lambda r}}{2})^{\frac{2-n}{2}} I_{\frac{n-2}{2}}(\sqrt{\lambda r}). \quad (24)$$

I_ν is the Bessel function of index ν. Different values of λ will give different solutions. In our own case, we are considering $SE(2)$, that is $n = 2$. Therefore, equation (24) can be further simplified to be

$$J_\lambda(mr) = \Gamma(1)\left(\frac{\sqrt{\lambda r}}{2}\right)^0 I_{\frac{2-2}{2}}\left(\sqrt{\lambda r}\right) \quad (25)$$

$$= I_0\left(\sqrt{\lambda r}\right). \quad \square \quad (26)$$

Expression equation (26) is the desired spherical function for $SE(2)$, generally referred to as the Bessel function of order zero.

A positive definite function [18, 19]

$$f : G \to \mathbb{C}$$

satisfies the following inequality

$$\sum_{i,j=1^m}\alpha_i\overline{\alpha_k}f(g_i^{-1}g_k) \geq 0, \quad (27)$$

for all subsets $\{g_1, ..., g_m\} \in G$ and all sequences $\{\alpha_1, ..., \alpha_m\} \in \mathbb{C}$. The integral analogue of the inequality equation (27) is given by

$$\int_G\int_G f(g_i^{-1}g_k)\varphi(g_i)\varphi(g_k)dg_i dg_k \geq 0, \quad (28)$$

where φ ranges over $L^1(G)$ or $C_c(G)$. If f is a continuous functions, equations (27) and (28) are equivalent.

A spherical function that also satisfies equation (27) is referred to as positive definite spherical function. Let $\widehat{(G, K)}$ stand for the set of spherical functions on G and let $\widehat{(G, K)}_+$ denotes the subset of $\widehat{(G, K)}$ that is positive definite. The set $\widehat{(G, K)}_+$

is isomorphic with $\mathbb{R}^+$. A measure π on $\widehat{(G,K)}$ such that for $f \in L^1(G,K)$ the plancherel theorem holds, that is

$$\int_{(K\backslash G/K)} |f(a)|^2 d(kak) = \int_{\widehat{(G,K)}} |\widehat{f}(\varphi)|^2 d\pi(\varphi),$$

is referred to as plancherel measure and its support is the full set $\widehat{(G,K)}$.

A Bochner measure μ on G can be decomposed with respect to π into the following

$$\mu = \mu_{ac} + \mu_d + \mu'_{sc}. \quad (29)$$

With respect to π, μ_{ac}, μ_d and μ_{sc} are known as absolutely continuous, discrete part of the singular part of μ and the continuous part of the singular part of μ respectively. There is a function $h_\mu \in L^1((G,K),\pi)$ such that for all measurable set E, we have

$$\mu_{ac}(E) = \int_E h_\mu d\pi.$$

A result concerning spherical functions on G, which are also positive definite , is presented as follows.

4.2.1. Theorem

Let $G = \mathbb{R}^n \rtimes SO(n)$ and let ϕ be a positive definite K-bi-invariant function on G that is continuous. Let $h_\mu \in L^1((G,K),\pi)$ be the representation of μ_{ac}. ϕ is strictly positive definite if h_μ is strictly positive for all φ in the support of π.

Proof Let us assume that there is a set of n elements $g_1, ..., g_n$ in G and $c_1, ..., c_n \in \mathbb{C}$ such that

$$\sum_{p,q=1}^{n} c_p \overline{c_q} \phi(g_p^{-1} g_q) = 0.$$

Without losing generality, let us assume that elements of G lie in distinct coset, that is $kg_pk \neq kg_qk$, for $p \neq q$. Letting $\alpha_\varphi(a^{-1}) = \overline{\alpha(a)}$ and the notation $g_q = a_q \tau_q$, we get

$$\int_{\widehat{(G,K)}} \int_K \left| \sum_{q=1}^{n} \overline{c_q} \alpha_\varphi(a_q \tau_q) \right|^2 d\tau d\mu(\varphi) = 0.$$

Decomposing μ, we can conclude that the following integral is equal to zero

$$\int_{(G,K)} \int_K \left| \sum_{q=1}^{n} \overline{c_q} \alpha_\varphi(a_q \tau_q) \right|^2 d\tau h_\mu(\varphi) d\pi(\varphi).$$

Since $h_\mu > 0$ on the support of π and the latter equal $\widehat{(G,K)}$, we have that

$$\sum_{q=1}^{n} \overline{c_q} \alpha(a_q) = 0,$$

for all α in $\widehat{\mathbb{R}^n}$. Evaluation of points or functionals on the dual group $\widehat{\mathbb{R}^n}$ are linearly independent as a result of the Gelfand-Raikov theorem, which leads to the contradiction $c_1 = ... = c_n = 0$. □

5. Conclusion

In this work, a unique way of computing the Haar measure of SE(2) is presented. It is shown in that the Radon transform of functions on SE(2) is obtained as the convolution of functions in the Schwartz space of SE(2). A major result concerning some properties of convolution of functions in SE(2) is proved. The explicit determination of spherical functions for SE(2) is presented. It is shown that this spherical function is the Bessel function of order zero, obtained by solving the Laplace-Beltrami operator of SE(2) radially by the method of separation of variable. One open problem encountered in this research is the extension of Paley-Wienner theorem on $\mathbb{R}^n$ to SE(2). It is our interest to use the theorem to solve this problem.

Data availability

We do not have any research data outside the submitted manuscript file.

References

[1] M. Andrew, *Analytic tomography*, Cambridge University Press, 2006. https://doi.org/10.1017/CBO9780511530012.

[2] S. R. Deans, *The radon transform and some of its applications*, John Wiley and Sons, New york, 1983. https://archive.org/details/radontransformso0000dean.

[3] J. Beatly, *The Radon transform and the mathematics of medical imaging*, Honors Theses, Coldby College, 2012. https://digitalcommons.colby.edu/honorstheses/646.

[4] N. Ja. Vilenkin & A. U. Klimyk, *Special functions and the theory of group representations*, Kluwer Academic Publishers, 1991. https://doi.org/10.1090/mmono/022.

[5] F. Astengo, B. Blasio & F. Ricci, *The Schwrtz correspondence for the complex motion group*, Journal of Functional Analysis **285** (2023) 110068. https://doi.org/10.1016/j.jfa.2023.110068.

[6] J. Dieudonne, "Gelfand pairs and spherical functions", International Journal Math. & Math. Sci. **2** (1979) 153. https://doi.org/10.1155/S0161171279000168.

[7] U. N. Bassey & U. E. Edeke, "Convolution equation and operators on the euclidean motion group", Journal of the Nigerian Society of Physical Sciences **6** (2024) 2029. https://doi.org/10.46481/jnsps.2024.2029.

[8] G. S. Chirikjian & A. B. Kyatkin, *Engineering applications of non commutative harmonic analysis*, CRC Press, New York, 2001. https://doi.org/10.1201/9780429289385.

[9] C. E. Yarman, B. Yazici, *A Wiener filtering approach over the Euclidean motion group for radon transform inversion*, Proceeding of SPIE 5032, The international Society for Optical Engineering, 2003. https://doi.org/10.1117/12.481372.

[10] M. Sugiura, *Unitary representation and harmonic analysis: an introduction*, North- Holland, New York, 1990. https://doi.org/10.1137/1019094.

[11] J. Dietsel & S. Angela, *The joys of Haar measure*, American Mathematical Society, Rhode-island, 2014. https://bookstore.ams.org/gsm-150.

[12] Bagchi, S.C.etal;*A first course on representation theory and linear Lie group*, Universities Press, 2000. http://m.universitiespress.com/details?id=9788173712845.

[13] M. Bhowmik & S. Sen, *An uncertainty principole of Paley and Wiener on Euclidean motion group*, J Fourier Anal Appl **23** (2017) 1445. https://doi.org/10.1007/s00041-016-9510-x.

[14] K. Kangni, *Lie groups and Lie algebra*, Universite Felix Houphouet Boigy d Abidjan, 2020. [Online]. https://www.cimpa.info/en/node/6769.

[15] F. D. Battesti & A. H. Dooley, *Solvability of differential operators II: semi simple Lie groups*, Proccedings of the Center for Mathematical Analysis, Australian National University, Canberra, Volume 14, 1986, pp. 85-94. http://maths-old.anu.edu.au/research.publications/proceedings/.

[16] R. L. Rubin, *Multipliers on the rigid motions of the plane and their relations to multipliers on direct products*, Proceedings of the American Mathematical Society, 1976. https://doi.org/10.2307/2042042.

[17] G. V. Dijk, *Introduction to harmonic analysis and generalized Gelfand pairs*, De Gruyter, Berlin, 2009. https://doi.org/10.1515/9783110220209.

[18] F. Astengo, B. Blasio & F. Ricci, *On the Schwartz correspondence for Gelfand pairs of polynomial growth*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. **32** (2021) 79. http://doi.org/10.4171/RLM/927.

[19] K. Issa, R. A. Bello & U. J. Abubakar, *Approximate analytical solution of fractional-order generalized integro-differential equations via fractional derivative of shifted Vieta-Lucas polynomial*, Journal of the Nigerian Society of Physical Sciences **6** (2024) 1821. https://doi.org/10.46481/jnsps.2024.1821.