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Homomorphic and restricted homomorphic products of soft graphs

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Abstract

Molodtsov pioneered the notion of soft set theory, presenting it as a mathematical tool for dealing with uncertainty. Numerous researchers have subsequently developed models leveraging this theory to tackle challenges in decision-making and medical diagnosis. Soft set theory emerges as a flexible framework adept at handling uncertain and imprecise information, a domain where classical set theory often struggles. Expanding on the soft set concept, researchers have introduced the idea of a soft graph. This innovative concept allows for the creation of diverse representations of graph-based relations by incorporating parameterisation. In this work, we present and investigate some of the features of the homomorphic and restricted homomorphic products of soft graphs. This paper establishes the structural properties of these products, ensuring that they are well-defined and maintain the essential characteristics of soft graphs. Additionally, we derive combinatorial identities related to the counts of vertices and edges, as well as the degree sums, offering deeper insights into the composition and behaviour of these graph products.

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1. Introduction

Soft set theory [1] is a mathematical framework designed to manage uncertainty and imprecision, which are often challenging for classical set theory. Unlike traditional approaches such as probability theory and fuzzy set theory, soft set theory offers the advantage of parameterisation, allowing it to better handle vague and ambiguous information. Building on this foundation, the concept of soft graphs emerged as an extension of soft set theory. Soft graphs incorporate parameterization into graph theory, enabling flexible and dynamic modelling of relationships that can accommodate a wide range of applications. This innovative approach allows for the creation of various adaptable representations, making soft graphs a powerful tool for addressing complex problems in diverse fields such as decision-making, medical diagnosis, and beyond.

In 1999, Molodtsov [1] introduced the notion of soft set theory. Molodtsov has successfully applied the principles of soft set theory across various fields in Mathematics. In real-world scenarios, soft set theory proves to be more advantageous than

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other established theories like probability theory or fuzzy set theory, given the limitations inherent in the latter. For instance, fuzzy set theory lacks the tools for parameterization. Renowned authors such as Maji *et al.* [2, 3], Muhammed [4] and Saleh *et al.* [5, 6] have applied soft set theory in diverse decision-making scenarios. Building on the foundation of soft set theory, researchers such as Thumbakara and George [7, 8] introduced the concept of soft graphs. Soft graphs extend the traditional graph model to incorporate uncertainty, enabling the representation and analysis of uncertain relationships between entities. Later modifications by Akram and Nawas [9–12] introduced variations such as fuzzy soft graphs, which further expanded the applicability of soft graphs. Akram and Zafar [13, 14] studied soft trees and fuzzy soft trees. Advancements in the field of soft graphs have been significant. Researchers like Thenge, Jain, and Reddy [15–17] have contributed to the development of soft graphs, particularly focusing on parameterization, which is essential for practical applications. George, Thumbakara, and Jose have further expanded the domain by introducing concepts such as soft hypergraphs [18], soft digraphs [19, 20], and soft disemigraphs [21], and thoroughly investigating their properties and applications.

The study of soft graphs has also led to the exploration of graph product operations. Product operations [22] allow the combination of two graphs to create a new graph with specific properties. Additionally, researchers like Baghernejad and Borzooei [23] have demonstrated the utility of soft graphs and soft multigraphs in managing complex systems such as urban traffic flows. Further contributions to the field include the introduction of novel concepts such as Eulerian and Hamiltonian soft graphs [24, 25], graph isomorphism [26], and various product operations on soft graphs [27, 28] and soft digraphs [29–34]. Additionally, researchers introduced soft semigraphs [35–39] and soft directed hypergraphs [40], applying principles from soft sets to these structures and defining operations and properties associated with them.

The study of soft graphs represents a significant advancement in graph theory, enabling the representation and analysis of uncertain relationships in complex systems. The application of soft set theory to graphs opens up new possibilities for solving practical problems in diverse fields. In this work, we introduce and study some of the features of homomorphic product and restricted homomorphic product of soft graphs.

2. Preliminaries

In this section, we lay the foundation for comprehending soft sets and soft graphs. Also, we provide a brief overview of topics including the part and degree associated with soft graphs.

Definition 2.1. Ref. [1] "Let $\mathfrak{R}$ be a set of parameters and U be an initial universe set. Then a pair $(\zeta, \mathfrak{R})$ is called a *soft set* (over U) if and only ζ is a mapping of $\mathfrak{R}$ into the power set of U. That is, $\zeta : \mathfrak{R} \to \mathcal{P}(U)$."

Definition 2.2. Ref. [9, 10] "Let $\mathfrak{R}$ be any nonempty set and $\Theta^* = (\varpi, \tau)$ be a simple graph with vertex set ϖ and edge set τ. Let R represent any relationship between elements of $\mathfrak{R}$ and ϖ. Define ζ from $\mathfrak{R}$ to $\mathcal{P}(\varpi)$ by $\zeta(\varepsilon) = \{y \in \varpi : \varepsilon R y\}$. Also, define ψ from $\mathfrak{R}$ to $\mathcal{P}(\tau)$ by $\psi(\varepsilon) = \{v_i v_j \in \tau : \{v_i, v_j\} \subseteq \zeta(\varepsilon)\}$. Then $(\zeta, \mathfrak{R})$ and $(\psi, \mathfrak{R})$ are soft sets over ϖ and τ respectively. If the 4-tuple $\Theta = (\Theta^*, \zeta, \psi, \mathfrak{R})$ meets the criteria listed below, it is referred to be a soft graph.

1. $\Theta^* = (\varpi, \tau)$ is a simple graph,
2. $\mathfrak{R} \neq \phi$ is the set of parameters,
3. $(\zeta, \mathfrak{R})$ is a soft set over ϖ,
4. $(\psi, \mathfrak{R})$ is a soft set over τ,
5. $(\zeta(\varepsilon), \psi(\varepsilon))$ is a subgraph of Θ^* for all $\varepsilon \in \mathfrak{R}$.

The soft graph Θ is also denoted by $\{H(\varepsilon) : \varepsilon \in \mathfrak{R}\}$ if $H(\varepsilon)=(\zeta(\varepsilon), \psi(\varepsilon))$. Then $H(\varepsilon)$ is called the *part* or *part*of Θ corresponding to the parameter ε in $\mathfrak{R}$. Let t be any vertex of the part $H(\varepsilon)$ of Θ for some $\varepsilon \in \mathfrak{R}$. Then the degree of the vertex t in that part $H(\varepsilon)$ is called *part degree* of the vertex t in $H(\varepsilon)$ and is denoted by *deg* $v[H(\varepsilon)]$. "

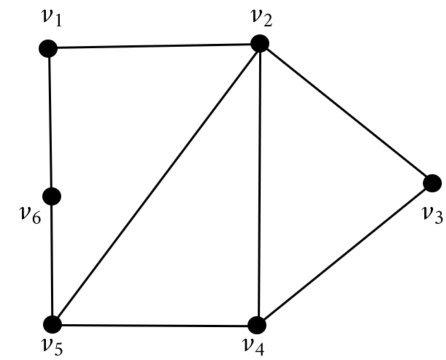

Figure 1: Graph $\Theta_1^* = (\varpi_1, \tau_1)$.

3. Homomorphic product of soft graphs

In this section, we define and explore the homomorphic product of soft graphs.

Definition 3.1. The *homomorphic product* of two soft graphs Θ_1 and Θ_2, which is denoted by $\Theta_1 \ltimes \Theta_2$ is defined as $\Theta_1 \ltimes \Theta_2 = \{H_1(\varepsilon) \ltimes H_2(\tau) : (\varepsilon, \tau) \in \mathfrak{R}_1 \times \mathfrak{R}_2\}$. Here $H_1(\varepsilon) \ltimes H_2(\tau)$ denotes the homomorphic product of the parts $H_1(\varepsilon)$ of Θ_1 and $H_2(\tau)$ of Θ_2 which is defined as follows: $H_1(\varepsilon) \ltimes H_2(\tau)$ is a graph having set of vertices $\varpi(H_1(\varepsilon) \ltimes H_2(\tau)) = \zeta_1(\varepsilon) \times \zeta_2(\tau)$ and set of edges $\tau(H_1(\varepsilon) \ltimes H_2(\tau))$, where there is a edge between the vertices (t_1, t_1') and (t_2, t_2') in $H_1(\varepsilon) \ltimes H_2(\tau)$ if and only if

1. $t_1 = t_2$ or
2. t_1 and t_2 are adjacent in $H_1(\varepsilon)$ and t_1' and t_2' are not adjacent in $H_2(\tau)$.

Example 1. Let $\Theta_1^* = (\varpi_1, \tau_1)$ be the graph depicted in Figure 1.
Let $\mathfrak{R}_1 = \{v_3, v_6\} \subseteq \varpi_1$ be a set of parameters. Define ζ_1 from $\mathfrak{R}_1$ to $\mathcal{P}(\varpi_1)$ by $\zeta_1(\varepsilon) = \{u \in \varpi_1 \mid d(u, \varepsilon) \leq 1\}, \forall \varepsilon \in \mathfrak{R}_1$. That is, $\zeta_1(v_3) = \{v_2, v_3, v_4\}$ and $\zeta_1(v_6) = \{v_1, v_5, v_6\}$. Here $(\zeta_1, \mathfrak{R}_1)$

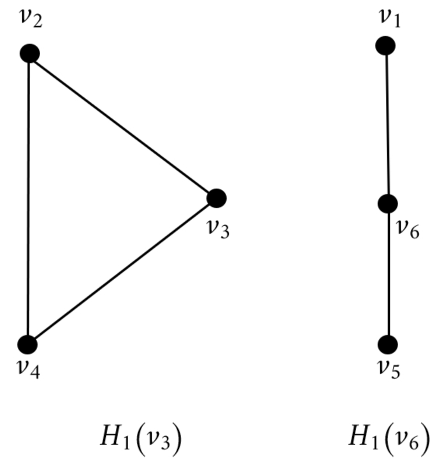

Figure 2: Soft Graph $\Theta_1 = \{H_1(v_3), H_1(v_6)\}$.

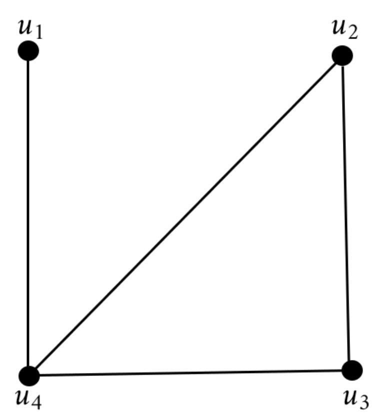

Figure 3: Graph $\Theta_2^* = (\varpi_2, \tau_2)$.

is a soft set over ϖ_1. Also, define ψ_1 from $\Re_1$ to $\mathcal{P}(\tau_1)$ by $\psi_1(\varepsilon) = \{v_i v_j \in \tau_1 \mid \{v_i, v_j\} \subseteq \zeta_1(\varepsilon)\}, \forall \varepsilon \in \Re_1$. That is, $\psi_1(v_3) = \{v_2v_3, v_2v_4, v_3v_4\}$ and $\psi_1(v_6) = \{v_1v_6, v_5v_6\}$. Here, $(\psi_1, \Re_1)$ is a soft set over τ_1. Then $H_1(v_3) = (\zeta_1(v_3), \psi_1(v_3))$ and $H_1(v_6) = (\zeta_1(v_6), \psi_1(v_6))$ are subgraphs of Θ_1^*. Therefore $\Theta_1 = \{H_1(v_3), H_1(v_6)\}$ is a soft graph of Θ_1^* and is depicted in Figure 2.
Let $\Theta_2^* = (\varpi_2, \tau_2)$ be the graph depicted in Figure 3.
Let $\Re_2 = \{u_1\} \subseteq \varpi_2$. Define $\zeta_2 : \Re_2 \to \mathcal{P}(\varpi_2)$ by $\zeta_2(\varepsilon) = \{u \in \varpi_2 \mid d(u, \varepsilon) \leq 1\}, \forall \varepsilon \in \Re_2$. That is, $\zeta_2(u_1) = \{u_1, u_4\}$. Here, $(\zeta_2, \Re_2)$ is a soft set over ϖ_2. Also, define $\psi_2 : \Re_2 \to \mathcal{P}(\tau_2)$ by $\psi_2(\varepsilon) = \{v_i v_j \in \tau_2 \mid \{v_i, v_j\} \subseteq \zeta_2(\varepsilon)\}, \forall \varepsilon \in \Re_2$. That is, $\psi_2(u_1) = \{u_1u_4\}$. Here, $(\psi_2, \Re_2)$ is a soft set over τ_2. Then, $H_2(u_1) = (\zeta_2(u_1), \psi_2(u_1))$ is a subgraph of Θ_2^*. Therefore, $\Theta_2 = \{H_2(u_1)\}$ is a soft graph of Θ_2^* and is depicted in Figure 4.
Then $\Theta_1 \ltimes \Theta_2 = \{H_1(v_3) \ltimes H_2(u_1), H_1(v_6) \ltimes H_2(u_1)\}$ is depicted in Figure 5.

Theorem 3.1. The homomorphic product $\Theta_1 \ltimes \Theta_2$ is also a soft graph of $\Theta_1^* \ltimes \Theta_2^*$.

Proof. $\Theta_1 \ltimes \Theta_2$ is defined as $\Theta_1 \ltimes \Theta_2 = \{H_1(\varepsilon) \ltimes H_2(\tau) : (\varepsilon, \tau) \in \Re_1 \times \Re_2\}$. Here $H_1(\varepsilon) \ltimes H_2(\tau)$ denotes the homomorphic product of the parts $H_1(\varepsilon)$ of Θ_1 and $H_2(\tau)$ of Θ_2 which is defined as follows: $H_1(\varepsilon) \ltimes H_2(\tau)$ is a graph having set of vertices $\varpi(H_1(\varepsilon) \ltimes H_2(\tau)) = \zeta_1(\varepsilon) \times \zeta_2(\tau)$ and set of edges $\tau(H_1(\varepsilon) \ltimes H_2(\tau))$, where there is a edge between the vertices (t_1, t_1') and (t_2, t_2') in $H_1(\varepsilon) \ltimes H_2(\tau)$ if and only if

1. $t_1 = t_2$ or
2. t_1 and t_2 are adjacent in $H_1(\varepsilon)$ and t_1' and t_2' are not adja-

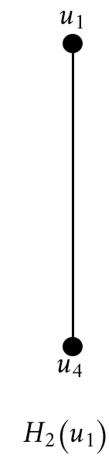

Figure 4: Soft Graph $\Theta_2 = \{H_2(u_1)\}$.

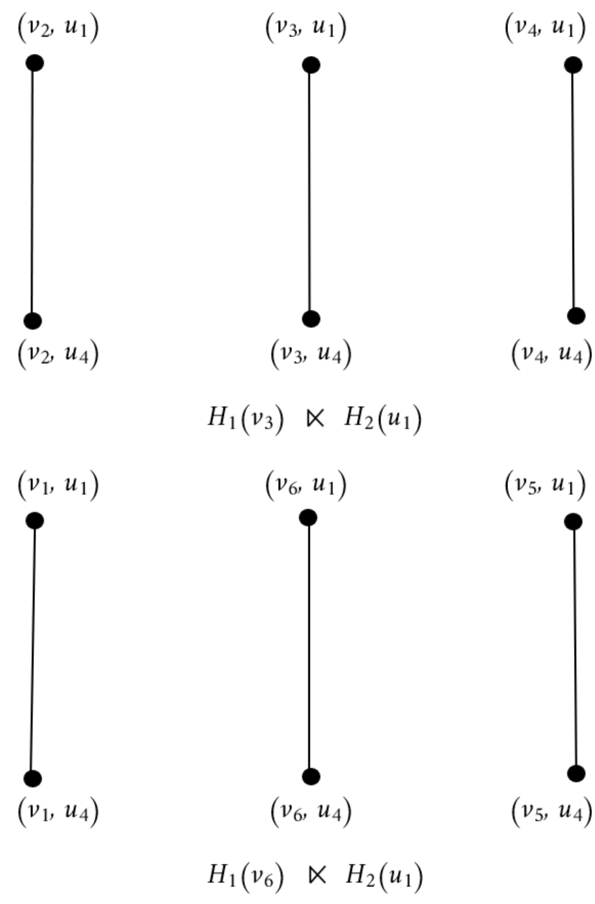

Figure 5: $\Theta = \Theta_1 \ltimes \Theta_2 = \{H_1(v_3) \ltimes H_2(u_1), H_1(v_6) \ltimes H_2(u_1)\}$.

cent in $H_2(\tau)$.

$\Theta_1^* \ltimes \Theta_2^*$ is a graph having set of vertices $\varpi(\Theta_1^* \ltimes \Theta_2^*) = \varpi_1 \times \varpi_2$ and set of edges $\tau(\Theta_1^* \ltimes \Theta_2^*)$ where two vertices (t_1, t_1') and (t_2, t_2') in $\Theta_1^* \ltimes \Theta_2^*$ are adjacent if and only if

1. $t_1 = t_2$ or
2. t_1 and t_2 are adjacent in Θ_1^* and t_1' and t_2' are not adjacent in Θ_2^*.

Let $\mathfrak{R}_{\Theta_1 \ltimes \Theta_2} = \mathfrak{R}_1 \times \mathfrak{R}_2$. Define $\zeta_{\Theta_1 \ltimes \Theta_2}$ from $\mathfrak{R}_{\Theta_1 \ltimes \Theta_2}$ to $\mathcal{P}[\varpi(\Theta_1^* \ltimes \Theta_2^*)]$ by $\zeta_{\Theta_1 \ltimes \Theta_2}(\varepsilon, \tau) = \zeta_1(\varepsilon) \times \zeta_2(\tau), \forall(\varepsilon, \tau) \in \mathfrak{R}_1 \times \mathfrak{R}_2$. Then $(\zeta_{\Theta_1 \ltimes \Theta_2}, \mathfrak{R}_{\Theta_1 \ltimes \Theta_2})$ is a soft set over $\varpi(\Theta_1^* \ltimes \Theta_2^*)$. Also, define $\psi_{\Theta_1 \ltimes \Theta_2}$ from $\mathfrak{R}_{\Theta_1 \ltimes \Theta_2}$ to $\mathcal{P}[\tau(\Theta_1^* \ltimes \Theta_2^*)]$ by $\psi_{\Theta_1 \ltimes \Theta_2}(\varepsilon, \tau) = \{(u, v)(y, z) \in \tau(\Theta_1^* \ltimes \Theta_2^*) \mid \{(u, v), (y, z)\} \in \zeta_{\Theta_1 \ltimes \Theta_2}(\varepsilon, \tau)\}, \forall(\varepsilon, \tau) \in \mathfrak{R}_1 \times \mathfrak{R}_2$. Then $(\psi_{\Theta_1 \ltimes \Theta_2}, \mathfrak{R}_{\Theta_1 \ltimes \Theta_2})$ is a soft set over $\tau(\Theta_1^* \ltimes \Theta_2^*)$. If we represent $(\zeta_{\Theta_1 \ltimes \Theta_2}(\varepsilon, \tau), \psi_{\Theta_1 \ltimes \Theta_2}(\varepsilon, \tau))$ by $H_{\Theta_1 \ltimes \Theta_2}(\varepsilon, \tau)$, then $H_{\Theta_1 \ltimes \Theta_2}(\varepsilon, \tau)$ is a subgraph of $\Theta_1^* \ltimes \Theta_2^*, \forall(\varepsilon, \tau) \in \mathfrak{R}_1 \times \mathfrak{R}_2$, since $\zeta_1(\varepsilon) \times \zeta_2(\tau) \subseteq \varpi_1 \times \varpi_2$ and any edge in $\psi_{\Theta_1 \ltimes \Theta_2}(\varepsilon, \tau)$ is also a edge in $\tau(\Theta_1^* \ltimes \Theta_2^*)$. Then $\Theta_1 \ltimes \Theta_2$ can be represented by the 4-tuple $(\Theta_1^* \ltimes \Theta_2^*, \zeta_{\Theta_1 \ltimes \Theta_2}, \psi_{\Theta_1 \ltimes \Theta_2}, \mathfrak{R}_{\Theta_1 \ltimes \Theta_2})$ and also by $\{H_{\Theta_1 \ltimes \Theta_2}(\varepsilon, \tau) : (\varepsilon, \tau) \in \mathfrak{R}_1 \times \mathfrak{R}_2\}$ and $\Theta_1 \ltimes \Theta_2$ is a soft graph of $\Theta_1^* \ltimes \Theta_2^*$. □

Theorem 3.2. The homomorphic product $\Theta_1 \ltimes \Theta_2$ contains $\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2} |\zeta_1(\varepsilon_i)||\zeta_2(\varepsilon_j)|$ vertices and $\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2}(|\zeta_1(\varepsilon_i)|\binom{|\zeta_2(\varepsilon_j)|}{2} + |\psi_1(\varepsilon_i)| \left[|\zeta_2(\varepsilon_j)|(|\zeta_2(\varepsilon_j)| - 1) - 2|\psi_2(\varepsilon_j)|\right])$ edges.

Proof. By definition, $\Theta_1 \ltimes \Theta_2 = \{H_1(\varepsilon) \ltimes H_2(\tau) : (\varepsilon, \tau) \in \mathfrak{R}_1 \times \mathfrak{R}_2\}$. The parameter set of $\Theta_1 \ltimes \Theta_2$ is $\mathfrak{R}_1 \times \mathfrak{R}_2$. Consider the part $H_1(\varepsilon_i) \ltimes H_2(\varepsilon_j)$ of $\Theta_1 \ltimes \Theta_2$ corresponding to the parameter $(\varepsilon_i, \varepsilon_j) \in \mathfrak{R}_1 \times \mathfrak{R}_2$. The vertex set of $H_1(\varepsilon_i) \ltimes H_2(\varepsilon_j)$ is $\zeta_1(\varepsilon_i) \times \zeta_2(\varepsilon_j)$ which contains $|\zeta_1(\varepsilon_i)||\zeta_2(\varepsilon_j)|$ elements. This is the case for all parts of $\Theta_1 \ltimes \Theta_2$. Therefore total count of vertices in $\Theta_1 \ltimes \Theta_2$ is $\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2} |\zeta_1(\varepsilon_i)||\zeta_2(\varepsilon_j)|$. We know, there is a edge between the vertices (t_p, t_q) and (t_m, t_n) in $H_1(\varepsilon_i) \ltimes H_2(\varepsilon_j)$ if and only if

1. $t_p = t_m$ or
2. t_p and t_m are adjacent in $H_1(\varepsilon_i)$ and t_q and t_n are not adjacent in $H_2(\varepsilon_j)$.

Each edge in $H_1(\varepsilon_i) \ltimes H_2(\varepsilon_j)$ is generated by one of two distinct conditions, and these conditions cannot occur simultaneously. To determine the total number of edges in $H_1(\varepsilon_i) \ltimes H_2(\varepsilon_j)$, we sum the number of edges produced by each condition. Let t be any vertex in $H_1(\varepsilon_i)$. The part $H_2(\varepsilon_j)$ contains $|\zeta_2(\varepsilon_j)|$ vertices. We can select 2 different vertices t' and t'' from $H_2(\varepsilon_j)$ in $\binom{|\zeta_2(\varepsilon_j)|}{2}$ different ways. Corresponding to each choice we get a edge joining the vertices (t, t') and (t, t'') in $H_1(\varepsilon_i) \ltimes H_2(\varepsilon_j)$. Like t, there are totally $|\zeta_1(\varepsilon_i)|$ vertices in $H_1(\varepsilon_i)$. Hence the first condition of adjacency gives $|\zeta_1(\varepsilon_i)|\binom{|\zeta_2(\varepsilon_j)|}{2}$ edges in $H_1(\varepsilon_i) \ltimes H_2(\varepsilon_j)$. We can choose two different vertices t_p and t_m in $H_1(\varepsilon_i)$ such that t_p and t_m are adjacent in $H_1(\varepsilon_i)$ in $|\psi_1(\varepsilon_i)|$ different ways. Similarly we can select 2 different vertices t_q and t_n in $H_2(\varepsilon_j)$ in such a way that t_q and t_n are not adjacent in $H_2(\varepsilon_j)$ in $\left(\frac{|\zeta_2(\varepsilon_j)|(|\zeta_2(\varepsilon_j)|-1)}{2} - |\psi_2(\varepsilon_j)|\right)$ different ways. Let t_p and t_m be two vertices in $H_1(\varepsilon_i)$ such that t_p and t_m are adjacent in $H_1(\varepsilon_i)$ and let t_q and t_n be two vertices in $H_2(\varepsilon_j)$ such that t_q and t_n are not adjacent in $H_2(\varepsilon_j)$. Thus we have 2 edges in $H_1(\varepsilon_i) \ltimes H_2(\varepsilon_j)$ in such a way that one edge joins the vertices (t_p, t_q) and (t_m, t_n) and the other joins (t_p, t_n) and (t_m, t_q). So, the second adjacency criteria gives $2|\psi_1(\varepsilon_i)|\left(\frac{|\zeta_2(\varepsilon_j)|(|\zeta_2(\varepsilon_j)|-1)}{2} - |\psi_2(\varepsilon_j)|\right)$ edges in $H_1(\varepsilon_i) \ltimes H_2(\varepsilon_j)$. Hence the total count of edges in $H_1(\varepsilon_i) \ltimes H_2(\varepsilon_j)$ is $|\zeta_1(\varepsilon_i)|\binom{|\zeta_2(\varepsilon_j)|}{2} + 2|\psi_1(\varepsilon_i)|\left(\frac{|\zeta_2(\varepsilon_j)|(|\zeta_2(\varepsilon_j)|-1)}{2} - |\psi_2(\varepsilon_j)|\right)$. This is the case for all parts of $\Theta_1 \ltimes \Theta_2$. Therefore total count of edges in $\Theta_1 \ltimes \Theta_2$ is

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2}(|\zeta_1(\varepsilon_i)|\binom{|\zeta_2(\varepsilon_j)|}{2}$$

$$+2|\psi_1(\varepsilon_i)|[\frac{|\zeta_2(\varepsilon_j)|(|\zeta_2(\varepsilon_j)| - 1)}{2} - |\psi_2(\varepsilon_j)|]) =$$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2}(|\zeta_1(\varepsilon_i)|\binom{|\zeta_2(\varepsilon_j)|}{2}$$

$$+|\psi_1(\varepsilon_i)|[[|\zeta_2(\varepsilon_j)|(|\zeta_2(\varepsilon_j)| - 1) - 2|\psi_2(\varepsilon_j)|]]).$$

□

Example 2. Consider the graphs given in Example 1. Here we have, total count of vertices in $\Theta_1 \ltimes \Theta_2 = 12$ and

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2} |\zeta_1(\varepsilon_i)||\zeta_2(\varepsilon_j)| = (3.2) + (3.2) = 12.$$

That is, total count of vertices in $\Theta_1 \ltimes \Theta_2 =$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2} |\zeta_1(\varepsilon_i)||\zeta_2(\varepsilon_j)|.$$

Also total count of edges in $\Theta_1 \ltimes \Theta_2 = 6$ and

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2}(|\zeta_1(\varepsilon_i)|\binom{|\zeta_2(\varepsilon_j)|}{2}$$

$$+|\psi_1(\varepsilon_i)|\left[|\zeta_2(\varepsilon_j)|(|\zeta_2(\varepsilon_j)| - 1) - 2|\psi_2(\varepsilon_j)|\right]) =$$

$$= (3.1 + 3.(2.1 - 2.1)) + (3.1 + 2.(2.1 - 2.1)) = 6.$$

That is, total count of edges in $\Theta_1 \ltimes \Theta_2$=

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2}(|\zeta_1(\varepsilon_i)|\binom{|\zeta_2(\varepsilon_j)|}{2}$$

$$+|\psi_1(\varepsilon_i)|[[|\zeta_2(\varepsilon_j)|(|\zeta_2(\varepsilon_j)| - 1) - 2|\psi_2(\varepsilon_j)|]]).$$

Theorem 3.3. Let $\Theta_1^* = (\varpi_1, \tau_1)$ and $\Theta_2^* = (\varpi_2, \tau_2)$ be two graphs and $\Theta_1 = (\Theta_1^*, \zeta_1, \psi_1, \mathfrak{R}_1)$ and $\Theta_2 = (\Theta_2^*, \zeta_2, \psi_2, \mathfrak{R}_2)$ be two soft graphs of Θ_1^* and Θ_2^* respectively. Then

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2} \sum_{(u,v)\in\zeta_{\Theta_1 \ltimes \Theta_2}(\varepsilon_i,\varepsilon_j)} deg(u, v)[H_{\Theta_1 \ltimes \Theta_2}(\varepsilon_i, \varepsilon_j)] =$$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2}(2|\zeta_1(\varepsilon_i)|\binom{|\zeta_2(\varepsilon_j)|}{2} + 2|\psi_1(\varepsilon_i)|[[|\zeta_2(\varepsilon_j)|(|\zeta_2(\varepsilon_j)| - 1)$$

$$-2|\psi_2(\varepsilon_j)|]]).$$

Proof. Consider any part $H_{\Theta_1 \ltimes \Theta_2}(\varepsilon_i, \varepsilon_j)$ of $\Theta_1 \ltimes \Theta_2$ which is given by $H_1(\varepsilon_i) \ltimes H_2(\varepsilon_j)$. By Theorem 3.2, we have number of edges in $H_1(\varepsilon_i) \ltimes H_2(\varepsilon_j)$ is

$$|\zeta_1(\varepsilon_i)|\binom{|\zeta_2(\varepsilon_j)|}{2} + |\psi_1(\varepsilon_i)|\left[|\zeta_2(\varepsilon_j)|(|\zeta_2(\varepsilon_j)| - 1) - 2|\psi_2(\varepsilon_j)|\right]$$

Then, we get

$$\sum_{(u,v)\in\zeta_{\Theta_1\ltimes\Theta_2}(\varepsilon_i,\varepsilon_j)} deg(u,v)[H_{\Theta_1\ltimes\Theta_2}(\varepsilon_i,\varepsilon_j)] =$$

$$2|\zeta_1(\varepsilon_i)|\binom{|\zeta_2(\varepsilon_j)|}{2} + 2|\psi_1(\varepsilon_i)|\left[|\zeta_2(\varepsilon_j)|(|\zeta_2(\varepsilon_j)| - 1) - 2|\psi_2(\varepsilon_j)|\right].$$

This is the case for all parts $H_{\Theta_1 \ltimes \Theta_2}(\varepsilon_i, \varepsilon_j)$ of $\Theta_1 \ltimes \Theta_2$. Hence,

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2} \sum_{(u,v)\in\zeta_{\Theta_1\ltimes\Theta_2}(\varepsilon_i,\varepsilon_j)} deg(u,v)[H_{\Theta_1\ltimes\Theta_2}(\varepsilon_i,\varepsilon_j)] =$$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2} (2|\zeta_1(\varepsilon_i)|\binom{|\zeta_2(\varepsilon_j)|}{2} + 2|\psi_1(\varepsilon_i)|[|\zeta_2(\varepsilon_j)|(|\zeta_2(\varepsilon_j)| - 1) - 2|\psi_2(\varepsilon_j)|]).$$

□

Example 3. Consider the graphs given in Example 1. Here we have,

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2} \sum_{(u,v)\in\zeta_{\Theta_1\ltimes\Theta_2}(\varepsilon_i,\varepsilon_j)} deg(u,v)[H_{\Theta_1\ltimes\Theta_2}(\varepsilon_i,\varepsilon_j)] =$$

$$(1+1+1+1+1+1) + (1+1+1+1+1+1) = 12$$

and

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2} (2|\zeta_1(\varepsilon_i)|\binom{|\zeta_2(\varepsilon_j)|}{2} + 2|\psi_1(\varepsilon_i)|[|\zeta_2(\varepsilon_j)|(|\zeta_2(\varepsilon_j)| - 1) - 2|\psi_2(\varepsilon_j)|]).$$

$$= (2.3.1 + 2.3.(2.1 - 2.1)) + (2.3.1 + 2.2.(2.1 - 2.1)) = 12.$$

That is,

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2} \sum_{(u,v)\in\zeta_{\Theta_1\ltimes\Theta_2}(\varepsilon_i,\varepsilon_j)} deg(u,v)[H_{\Theta_1\ltimes\Theta_2}(\varepsilon_i,\varepsilon_j)] =$$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2} (2|\zeta_1(\varepsilon_i)|\binom{|\zeta_2(\varepsilon_j)|}{2} + 2|\psi_1(\varepsilon_i)|[|\zeta_2(\varepsilon_j)|(|\zeta_2(\varepsilon_j)| - 1) - 2|\psi_2(\varepsilon_j)|]).$$

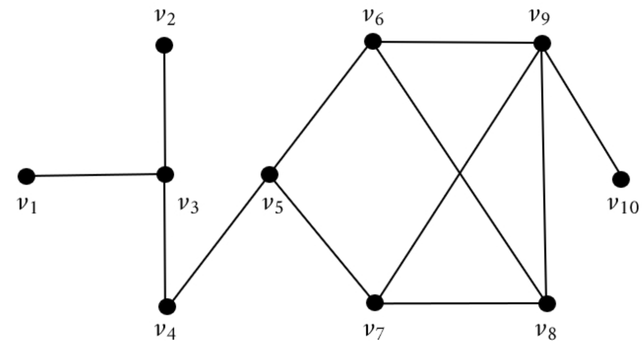

Figure 6: Graph $\Theta^* = (\varpi, \tau)$.

4. Restricted homomorphic product of soft graphs

This section delves into the concept of the restricted homomorphic product of soft graphs.

Definition 4.1. Let $\Theta^* = (\varpi, \tau)$ be a graph and $\Theta_1 = (\Theta^*, \zeta_1, \psi_1, \mathfrak{R}_1) = \{H_1(\varepsilon) : \varepsilon \in \mathfrak{R}_1\}$ and $\Theta_2 = (\Theta^*, \zeta_2, \psi_2, \mathfrak{R}_2) = \{H_2(\varepsilon) : \varepsilon \in \mathfrak{R}_2\}$ be two soft graphs of Θ^* such that $\mathfrak{R}_1 \cap \mathfrak{R}_2 \neq \phi$. Then the *restricted homomorphic product* of Θ_1 and Θ_2, which is denoted by $\Theta_1 \divideontimes \Theta_2$ is defined as $\Theta_1 \divideontimes \Theta_2 = \{H_1(\varepsilon) \ltimes H_2(\varepsilon) : \varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2\}$. Here $H_1(\varepsilon) \ltimes H_2(\varepsilon)$ denotes the homomorphic product of the parts $H_1(\varepsilon)$ of Θ_1 and $H_2(\varepsilon)$ of Θ_2 which is defined as follows: $H_1(\varepsilon) \ltimes H_2(\varepsilon)$ is a graph having set of vertices $\varpi(H_1(\varepsilon) \ltimes H_2(\varepsilon)) = \zeta_1(\varepsilon) \times \zeta_2(\varepsilon)$ and set of edges $\tau(H_1(\varepsilon) \ltimes H_2(\varepsilon))$, where there is a edge between the vertices (t_1, t_1') and (t_2, t_2') in $H_1(\varepsilon) \ltimes H_2(\varepsilon)$ if and only if

1. $t_1 = t_2$ or
2. t_1 and t_2 are adjacent in $H_1(\varepsilon)$ and t_1' and t_2' are not adjacent in $H_2(\varepsilon)$.

Example 4. Let $\Theta^* = (\varpi, \tau)$ be the graph depicted in Figure 6.

Let $\mathfrak{R}_1 = \{v_3, v_{10}\} \subseteq \varpi$ be a set of parameters . Define ζ_1 from $\mathfrak{R}_1$ to $\mathcal{P}(\varpi)$ by $\zeta_1(\varepsilon) = \{u \in \varpi \mid d(u, \varepsilon) \leq 2\}, \forall \varepsilon \in \mathfrak{R}_1$. That is, $\zeta_1(v_3) = \{v_1, v_2, v_3, v_4, v_5\}$ and $\zeta_1(v_{10}) = \{v_6, v_7, v_8, v_9, v_{10}\}$. Here $(\zeta_1, \mathfrak{R}_1)$ is a soft set over ϖ. Also, define ψ_1 from $\mathfrak{R}_1$ to $\mathcal{P}(\tau)$ by $\psi_1(\varepsilon) = \{v_iv_j \in \tau \mid \{v_i, v_j\} \subseteq \zeta_1(\varepsilon)\}$, $\forall \varepsilon \in \mathfrak{R}_1$. That is, $\psi_1(v_3) = \{v_1v_3, v_2v_3, v_3v_4, v_4v_5\}$ and $\psi_1(v_{10}) = \{v_6v_8, v_6v_9, v_7v_8, v_7v_9, v_8v_9, v_9v_{10}\}$. Here, $(\psi_1, \mathfrak{R}_1)$ is a soft set over τ. Then $H_1(v_3) = (\zeta_1(v_3), \psi_1(v_3))$ and $H_1(v_{10}) = (\zeta_1(v_{10}), \psi_1(v_{10}))$ are subgraphs of Θ^*. Therefore $\Theta_1 = \{H_1(v_3), H_1(v_{10})\}$ is a soft graph of Θ^* and is depicted in Figure 7.
Consider another parameter set $\mathfrak{R}_2 = \{v_3, v_7\} \subseteq \varpi$. Define $\zeta_2 : \mathfrak{R}_2 \to \mathcal{P}(\varpi)$ by $\zeta_2(\varepsilon) = \{u \in \varpi \mid d(u, \varepsilon) \leq 1\}, \forall \varepsilon \in \mathfrak{R}_2$. That is, $\zeta_2(v_3) = \{v_1, v_2, v_3, v_4\}$ and $\zeta_2(v_7) = \{v_5, v_7, v_8, v_9\}$. Here, $(\zeta_2, \mathfrak{R}_2)$ is a soft set over ϖ. Also, define $\psi_2 : \mathfrak{R}_2 \to \mathcal{P}(\tau)$ by $\psi_2(\varepsilon) = \{v_iv_j \in \tau \mid \{v_i, v_j\} \subseteq \zeta_2(\varepsilon)\}, \forall \varepsilon \in \mathfrak{R}_2$. That is, $\psi_2(v_3) = \{v_1v_3, v_2v_3, v_3v_4\}$ and $\psi_2(v_7) = \{v_5v_7, v_7v_8, v_7v_9\}$. Here, $(\psi_2, \mathfrak{R}_2)$ is a soft set over τ. Then, $H_2(v_3) = (\zeta_2(v_3), \psi_2(v_3))$ and $H_2(v_7) = (\zeta_2(v_7), \psi_2(v_7))$ are subgraphs of Θ^*. Therefore, $\Theta_2 = \{H_2(v_3), H_2(v_7)\}$ is a soft graph of Θ^* and is depicted in Figure 8.
Then the restricted homomorphic product $\Theta_1 \divideontimes \Theta_2$ is given

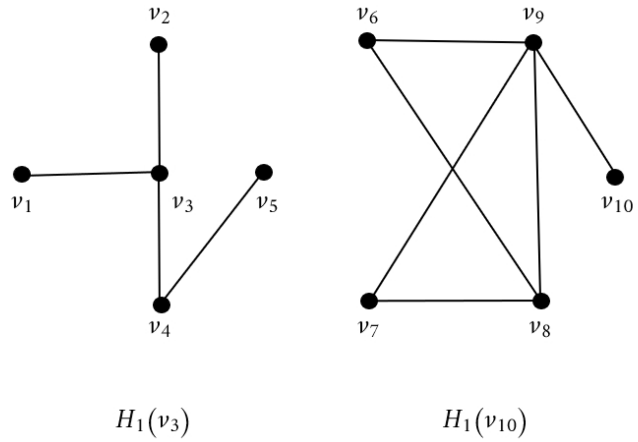

Figure 7: Soft Graph $\varTheta_1 = \{H_1(v_3), H_1(v_{10})\}$.

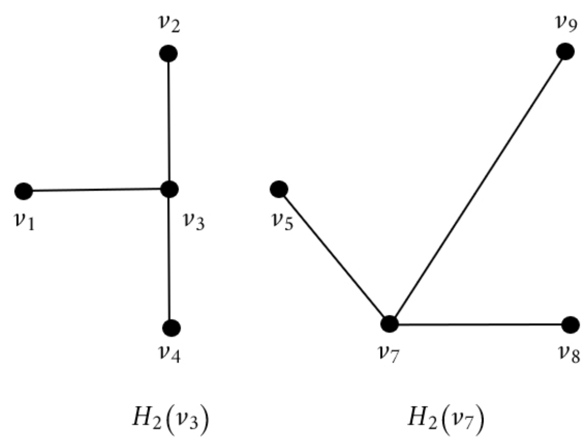

Figure 8: Soft Graph $\varTheta_2 = \{H_2(v_3), H_2(v_7)\}$.

by $\varTheta = \varTheta_1 \circledast \varTheta_2 = \{H_1(v_3) \ltimes H_2(v_3)\}$ and is depicted in Figure 9.

Theorem 4.1. Let $\varTheta^* = (\varpi, \tau)$ be a graph and $\varTheta_1 = (\varTheta^*, \zeta_1, \psi_1, \mathfrak{R}_1) = \{H_1(\varepsilon) : \varepsilon \in \mathfrak{R}_1\}$ and $\varTheta_2 = (\varTheta^*, \zeta_2, \psi_2, \mathfrak{R}_2) = \{H_2(\varepsilon) : \varepsilon \in \mathfrak{R}_2\}$ be two soft graphs of $\varTheta^*$ such that $\mathfrak{R}_1 \cap \mathfrak{R}_2 \neq \phi$. Then $\varTheta_1 \circledast \varTheta_2$ is a soft graph of $\varTheta^* \ltimes \varTheta^*$.

Proof. $\varTheta_1 \circledast \varTheta_2$ is defined as $\varTheta_1 \circledast \varTheta_2 = \{H_1(\varepsilon) \ltimes H_2(\varepsilon) : \varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2\}$. Here $H_1(\varepsilon) \ltimes H_2(\varepsilon)$ denotes the homomorphic product of the parts $H_1(\varepsilon)$ of $\varTheta_1$ and $H_2(\varepsilon)$ of $\varTheta_2$ which is defined as follows: $H_1(\varepsilon) \ltimes H_2(\varepsilon)$ is a graph having set of vertices $\varpi(H_1(\varepsilon) \ltimes H_2(\varepsilon)) = \zeta_1(\varepsilon) \times \zeta_2(\varepsilon)$ and set of edges $\tau(H_1(\varepsilon) \ltimes H_2(\varepsilon))$, where there is a edge between the vertices (t_1, t_1') and (t_2, t_2') in $H_1(\varepsilon) \ltimes H_2(\varepsilon)$ if and only if

1. $t_1 = t_2$ or
2. t_1 and t_2 are adjacent in $H_1(\varepsilon)$ and t_1' and t_2' are not adjacent in $H_2(\varepsilon)$.

$\varTheta^* \ltimes \varTheta^*$ is a graph having set of vertices $\varpi(\varTheta^* \ltimes \varTheta^*) = \varpi \times \varpi$ and set of edges $\tau(\varTheta^* \ltimes \varTheta^*)$, where there is a edge between the two vertices (t_1, t_1') and (t_2, t_2') in $\varTheta^* \ltimes \varTheta^*$ if and only if

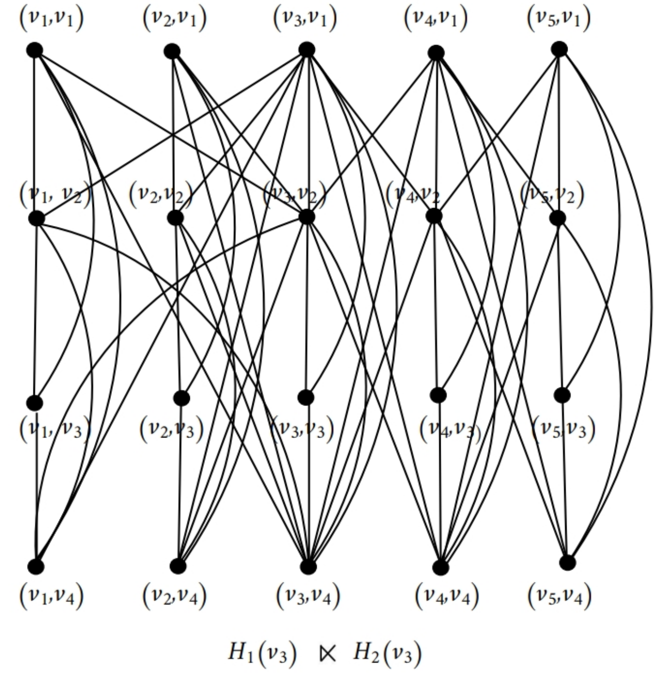

Figure 9: $\varTheta = \varTheta_1 \circledast \varTheta_2 = \{H_1(v_3) \ltimes H_2(v_3)\}$.

1. $t_1 = t_2$ or
2. t_1 and t_2 are adjacent in $\varTheta^*$ and t_1' and t_2' are not adjacent in $\varTheta^*$.

Let $\mathfrak{R}_{\varTheta_1 \circledast \varTheta_2} = \mathfrak{R}_1 \cap \mathfrak{R}_2$ be the set of parameters. Define $\zeta_{\varTheta_1 \circledast \varTheta_2}$ from $\mathfrak{R}_{\varTheta_1 \circledast \varTheta_2}$ to $\mathcal{P}[\varpi(\varTheta^* \ltimes \varTheta^*)]$ by $\zeta_{\varTheta_1 \circledast \varTheta_2}(\varepsilon) = \zeta_1(\varepsilon) \times \zeta_2(\varepsilon), \forall \varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2$. Then $(\zeta_{\varTheta_1 \circledast \varTheta_2}, \mathfrak{R}_{\varTheta_1 \circledast \varTheta_2})$ is a soft set over $\varpi(\varTheta^* \ltimes \varTheta^*)$. Also, define $\psi_{\varTheta_1 \circledast \varTheta_2}$ from $\mathfrak{R}_{\varTheta_1 \circledast \varTheta_2}$ to $\mathcal{P}[\tau(\varTheta^* \ltimes \varTheta^*)]$ by $\psi_{\varTheta_1 \circledast \varTheta_2}(\varepsilon) = \{(u, v)(y, z) \in \tau(\varTheta^* \ltimes \varTheta^*) \mid \{(u, v), (y, z)\} \in \zeta_{\varTheta_1 \circledast \varTheta_2}(\varepsilon)\}, \forall \varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2$. Then $(\psi_{\varTheta_1 \circledast \varTheta_2}, \mathfrak{R}_{\varTheta_1 \circledast \varTheta_2})$ is a soft set over $\tau(\varTheta^* \ltimes \varTheta^*)$. Also if we denote $(\zeta_{\varTheta_1 \circledast \varTheta_2}(\varepsilon), \psi_{\varTheta_1 \circledast \varTheta_2}(\varepsilon))$ by $H_{\varTheta_1 \circledast \varTheta_2}(\varepsilon)$, then $H_{\varTheta_1 \circledast \varTheta_2}(\varepsilon)$ is a subgraph of $\varTheta^* \ltimes \varTheta^*, \forall \varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2$, since $\zeta_1(\varepsilon) \times \zeta_2(\varepsilon) \subseteq \varpi \times \varpi$ and any edge in $\psi_{\varTheta_1 \circledast \varTheta_2}(\varepsilon)$ is also a edge in $\tau(\varTheta^* \ltimes \varTheta^*)$. Then $\varTheta_1 \circledast \varTheta_2$ can be represented by the 4-tuple $(\varTheta^* \ltimes \varTheta^*, \zeta_{\varTheta_1 \circledast \varTheta_2}, \psi_{\varTheta_1 \circledast \varTheta_2}, \mathfrak{R}_{\varTheta_1 \circledast \varTheta_2})$ and also by $\{H_{\varTheta_1 \circledast \varTheta_2}(\varepsilon) : \varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2\}$ and $\varTheta_1 \circledast \varTheta_2$ is a soft graph of $\varTheta^* \ltimes \varTheta^*$. □

Theorem 4.2. The restricted homomorphic product $\varTheta_1 \circledast \varTheta_2$ contains $\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} |\zeta_1(\varepsilon)||\zeta_2(\varepsilon)|$ vertices and $\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} (|\zeta_1(\varepsilon)|\binom{|\zeta_2(\varepsilon)|}{2} + |\psi_1(\varepsilon)|[|\zeta_2(\varepsilon)|(|\zeta_2(\varepsilon)| - 1) - 2|\psi_2(\varepsilon)|])$ edges.

Proof. By definition, $\varTheta_1 \circledast \varTheta_2 = \{H_1(\varepsilon) \ltimes H_2(\varepsilon) : \varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2\}$. The parameter set of $\varTheta_1 \circledast \varTheta_2$ is $\mathfrak{R}_1 \cap \mathfrak{R}_2$. Consider the part $H_1(\varepsilon) \ltimes H_2(\varepsilon)$ of $\varTheta_1 \circledast \varTheta_2$ corresponding to the parameter $\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2$. The vertex set of $H_1(\varepsilon) \ltimes H_2(\varepsilon)$ is $\zeta_1(\varepsilon) \times \zeta_2(\varepsilon)$ which contains $|\zeta_1(\varepsilon)||\zeta_2(\varepsilon)|$ elements. This is the case for all parts of $\varTheta_1 \circledast \varTheta_2$. Therefore total count of vertices in $\varTheta_1 \circledast \varTheta_2$ is $\sum_{\varepsilon \in \mathfrak{R}_1 \cap \mathfrak{R}_2} |\zeta_1(\varepsilon)||\zeta_2(\varepsilon)|$. Also, there is a edge between the two vertices (t_p, t_q) and (t_m, t_n) in $H_1(\varepsilon) \ltimes H_2(\varepsilon)$ if and only if

1. $t_p = t_m$ or

2. t_p and t_m are adjacent in $H_1(\varepsilon)$ and t_q and t_n are not adjacent in $H_2(\varepsilon)$.

Each edge in $H_1(\varepsilon) \ltimes H_2(\varepsilon)$ is generated by one of two distinct conditions, and these conditions cannot occur simultaneously. To determine the total number of edges in $H_1(\varepsilon) \ltimes H_2(\varepsilon)$, we sum the number of edges produced by each condition. Let t be any vertex in $H_1(\varepsilon)$. The part $H_2(\varepsilon)$ contains $|\zeta_2(\varepsilon)|$ vertices. We can select 2 different vertices t' and t'' from $H_2(\varepsilon)$ in $\binom{|\zeta_2(\varepsilon)|}{2}$ different ways. Corresponding to each choice we get a edge joining the vertices (t, t') and (t, t'') in $H_1(\varepsilon) \ltimes H_2(\varepsilon)$. Like t, there are $|\zeta_1(\varepsilon)|$ vertices in $H_1(\varepsilon)$. So, the first adjacency criteria gives $|\zeta_1(\varepsilon)|\binom{|\zeta_2(\varepsilon)|}{2}$ edges in $H_1(\varepsilon) \ltimes H_2(\varepsilon)$. We can choose two different vertices t_p and t_m in $H_1(\varepsilon)$ such that t_p and t_m are adjacent in $H_1(\varepsilon)$ in $|\psi_1(\varepsilon)|$ different ways. Similarly we can select 2 different vertices t_q and t_n in $H_2(\varepsilon)$ such that t_q and t_n are not adjacent in $H_2(\varepsilon)$ in $\left(\frac{|\zeta_2(\varepsilon)|(|\zeta_2(\varepsilon)|-1)}{2} - |\psi_2(\varepsilon)|\right)$ different ways. Let t_p and t_m be two vertices in $H_1(\varepsilon)$ such that t_p and t_m are adjacent in $H_1(\varepsilon)$ and let t_q and t_n be two vertices in $H_2(\varepsilon)$ such that t_q and t_n are not adjacent in $H_2(\varepsilon)$. Thus we get 2 edges in $H_1(\varepsilon) \ltimes H_2(\varepsilon)$ in such a way that one edge joins the vertices (t_p, t_q) and (t_m, t_n) and the other joins (t_p, t_n) and (t_m, t_q). So, the second adjacency criteria gives $2|\psi_1(\varepsilon)|\left(\frac{|\zeta_2(\varepsilon)|(|\zeta_2(\varepsilon)|-1)}{2} - |\psi_2(\varepsilon)|\right)$ edges in $H_1(\varepsilon) \ltimes H_2(\varepsilon)$. Hence the total count of edges in $H_1(\varepsilon) \ltimes H_2(\varepsilon)$ is $|\zeta_1(\varepsilon)|\binom{|\zeta_2(\varepsilon)|}{2} + 2|\psi_1(\varepsilon)|\left(\frac{|\zeta_2(\varepsilon)|(|\zeta_2(\varepsilon)|-1)}{2} - |\psi_2(\varepsilon)|\right)$. This is the case for all parts of $\Theta_1 \circledast \Theta_2$. Therefore total count of edges in $\Theta_1 \circledast \Theta_2$ is

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2}\left(|\zeta_1(\varepsilon)|\binom{|\zeta_2(\varepsilon)|}{2} + 2|\psi_1(\varepsilon)|\left[\frac{|\zeta_2(\varepsilon)|(|\zeta_2(\varepsilon)|-1)}{2} - |\psi_2(\varepsilon)|\right]\right)$$

$$= \sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2}\left(|\zeta_1(\varepsilon)|\binom{|\zeta_2(\varepsilon)|}{2} + |\psi_1(\varepsilon)|\left[|\zeta_2(\varepsilon)|(|\zeta_2(\varepsilon)|-1) - 2|\psi_2(\varepsilon)|\right]\right).$$

□

Example 5. Consider the graph given in Example 4. Here we have, total count of vertices in $\Theta_1 \circledast \Theta_2 = 20$ and

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2}|\zeta_1(\varepsilon)||\zeta_2(\varepsilon)| = 5.4 = 20.$$

That is, total count of vertices in $\Theta_1 \circledast \Theta_2 =$

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2}|\zeta_1(\varepsilon)||\zeta_2(\varepsilon)|.$$

Also total count of edges in $\Theta_1 \circledast \Theta_2 = 54$ and

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2}(|\zeta_1(\varepsilon)|\binom{|\zeta_2(\varepsilon)|}{2} + |\psi_1(\varepsilon)|[|\zeta_2(\varepsilon)|(|\zeta_2(\varepsilon)|-1)$$

$$-2|\psi_2(\varepsilon)|]) =$$

$$= 5.6 + 4.(4.3 - 2.3) = 54.$$

That is, total count of edges in $\Theta_1 \circledast \Theta_2$=

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2}(|\zeta_1(\varepsilon)|\binom{|\zeta_2(\varepsilon)|}{2} + |\psi_1(\varepsilon)|[|\zeta_2(\varepsilon)|(|\zeta_2(\varepsilon)|-1)$$

$$-2|\psi_2(\varepsilon)|]).$$

Theorem 4.3. Let $\Theta^* = (\varpi, \tau)$ be a graph and $\Theta_1 = (\Theta^*, \zeta_1, \psi_1, \mathfrak{R}_1)$ and $\Theta_2 = (\Theta^*, \zeta_2, \psi_2, \mathfrak{R}_2)$ be two soft graphs of Θ^* such that $\mathfrak{R}_1 \cap \mathfrak{R}_2 \neq \phi$. Then

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2}\ \sum_{(u,v)\in\zeta_{\Theta_1\circledast\Theta_2}(\varepsilon)} deg(u,v)[H_{\Theta_1\circledast\Theta_2}(\varepsilon)] =$$

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2}\left(2|\zeta_1(\varepsilon)|\binom{|\zeta_2(\varepsilon)|}{2} + 2|\psi_1(\varepsilon)|\left[|\zeta_2(\varepsilon)|(|\zeta_2(\varepsilon)|-1) - 2|\psi_2(\varepsilon)|\right]\right).$$

Proof. Consider any part $H_{\Theta_1\circledast\Theta_2}(\varepsilon)$ of $\Theta_1 \circledast \Theta_2$ which is given by $H_1(\varepsilon) \ltimes H_2(\varepsilon)$. By Theorem 4.2, we have number of edges in $H_1(\varepsilon) \ltimes H_2(\varepsilon)$ is

$$|\zeta_1(\varepsilon)|\binom{|\zeta_2(\varepsilon)|}{2} + |\psi_1(\varepsilon)|\left[|\zeta_2(\varepsilon)|(|\zeta_2(\varepsilon)|-1) - 2|\psi_2(\varepsilon)|\right]$$

Hence,

$$\sum_{(u,v)\in\zeta_{\Theta_1\circledast\Theta_2}(\varepsilon)} deg(u,v)[H_{\Theta_1\circledast\Theta_2}(\varepsilon)] =$$

$$2|\zeta_1(\varepsilon)|\binom{|\zeta_2(\varepsilon)|}{2} + 2|\psi_1(\varepsilon)|\left[|\zeta_2(\varepsilon)|(|\zeta_2(\varepsilon)|-1) - 2|\psi_2(\varepsilon)|\right].$$

This is the case for all parts $H_{\Theta_1\circledast\Theta_2}(\varepsilon)$ of $\Theta_1 \circledast \Theta_2$. Hence,

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2}\ \sum_{(u,v)\in\zeta_{\Theta_1\circledast\Theta_2}(\varepsilon)} deg(u,v)[H_{\Theta_1\circledast\Theta_2}(\varepsilon)] =$$

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2}\left(2|\zeta_1(\varepsilon)|\binom{|\zeta_2(\varepsilon)|}{2} + 2|\psi_1(\varepsilon)|\left[|\zeta_2(\varepsilon)|(|\zeta_2(\varepsilon)|-1) - 2|\psi_2(\varepsilon)|\right]\right).$$

□

Example 6. Consider the graphs given in Example 4. Here we have,

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2}\ \sum_{(u,v)\in\zeta_{\Theta_1\circledast\Theta_2}(\varepsilon)} deg(u,v)[H_{\Theta_1\circledast\Theta_2}(\varepsilon)] =$$

5+5+9+7+5+5+5+9+7+5+3+3+3+3+3+5+5+9+7+5 = 108

and

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2}\left(2|\zeta_1(\varepsilon)|\binom{|\zeta_2(\varepsilon)|}{2} + 2|\psi_1(\varepsilon)|\left[|\zeta_2(\varepsilon)|(|\zeta_2(\varepsilon)|-1) - 2|\psi_2(\varepsilon)|\right]\right).$$

$$= 2.5.6 + 2.4.(4.3 - 2.3) = 108.$$

That is,

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\mathfrak{R}_1\times\mathfrak{R}_2}\ \sum_{(u,v)\in\zeta_{\Theta_1\circledast\Theta_2}(\varepsilon)} deg(u,v)[H_{\Theta_1\circledast\Theta_2}(\varepsilon)] =$$

$$\sum_{\varepsilon\in\mathfrak{R}_1\cap\mathfrak{R}_2}\left(2|\zeta_1(\varepsilon)|\binom{|\zeta_2(\varepsilon)|}{2} + 2|\psi_1(\varepsilon)|\left[|\zeta_2(\varepsilon)|(|\zeta_2(\varepsilon)|-1) - 2|\psi_2(\varepsilon)|\right]\right).$$

5. Conclusion

Soft graphs emerged from the integration of soft sets into graph theory, offering a novel approach to representing complex relationships. Through parameterization, soft graphs can generate multiple representations of such relationships, contributing to their versatility. The field of soft graph theory is witnessing rapid progress due to the effective utilization of parameterization techniques. We introduced and explored the features of homomorphic and restricted homomorphic products in this study.

Data availability

We do not have any research data outside the submitted manuscript file.

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