Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 7 (2025) 2405

Journal of the Nigerian Society of Physical Sciences

Discretization of the Caputo time-fractional advection-diffusion problems with certain wavelet basis function

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Abstract

Considering the new wavelet-based Galerkin finite element technique constructed with Iweobodo-Mamadu-Njoseh wavelet (IMNW) as the basis function in seeking the numerical solution of time-fractional advection-diffusion equations (TFADE), the TFADE must be simplified to enhance an application of a numerical technique. Thus in this research work, we considered an implementation of the time and space discretization of the TFADE with the use of IMNW basis function. In order to successfully achieve our result, our methodology inculcated the Caputo fractional derivatives, time fractional advection-diffusion equations (TFADE), Wavelet, IMNW, and Galerkin finite element method. After a successful implementation of the time discretization, an implicit form of TFADE was obtained, followed by the implementation of the space discretization which generated the variational formulation of the equation for easy implementation of the scheme. The illustrated numerical solution from using the new technique provided a resulting numerical evidence which aligns with the exact solution.

DOI:10.46481/jnsps.2025.2405

Keywords: Caputo fractional derivatives, Time-fractional advection-diffusion equations (TFADE), Iweobodo-Mamadu-Njoseh wavelet (IMNW), Time discretization, Space discretization

Article History :
Received: 01 October 2024
Received in revised form: 06 November 2024
Accepted for publication: 29 March 2025
Available online: 07 May 2025

Communicated by: Pankaj Thakur

1. Introduction

Kumar and Pandit [1] described a differential equation as an equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variables. Its foundation. Differential equations are very significant in wide varieties of real life situations today, according to Iweobodo *et al*. [2], many problems involving chemical reactions, wave propagation, heat flow, stock market predictions, etc; are modeled with differential equations.

Fractional differential equation are differential equations possessing fractional or arbitrary order. Gupta and Saha [3] stated that fractional derivatives are important because they provide relevant tools used for describing the memory and hereditary properties of different processes and materials. Its use in science and engineering has caused it to become more popular and visible today.

Many fractional differential equations are not easy to solve with analytic methods, therefore researchers consider some numerical techniques. Eg Issa *et al*. [4], Basim *et al*. [5]. In the same vein, Shiralashetti *et al*. [6] stated that the impossibility to obtain the exact solutions of some differential equations has

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necessitated either discretization of differential equations leading to numerical solutions, or their qualitative study which is concerned with the deduction of important properties of the solutions without actually solving them.

The wavelet-based method of solutions to time fractional differential equations is among the recently developed methods, and a few researchers have applied it in seeking approximate solutions to time fractional differential equations.

Studies have shown that some wavelets emanated from orthogonal polynomials, hence, Iweobodo *et al*. [7] developed a new wavelet from Mamadu-Njoseh polynomial (MNP), and this new wavelet has been peacefully applied in seeking numerical solutions of one dimensional differential equations. Although, Iweobodo *et al*. [8] iterated some steps in applying the new wavelet in solving time fractional differential equations, however, the technique has not been directly applied in solving any fractional derivative. Hence in this research work, a time and space discretizaion process of the Caputo fractional advection-diffusion equation with respect to IMNW will be considered for the ease of providing its numerical solution.

2. Materials and methods

2.1. Caputo fractional calculus

Different types of fractional differential equations and their descriptions exist in literature, namely Riemann-Liouville, Caputo, Riesz, and Grunwald-Letnikov (Fadugba) [9]. However, our interest in this work is on the Caputo fractional differential equations.

The Caputo fractional derivative was defined by Podlubny [10] as:

$$ {}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+t}} d\tau \quad n-1 < \alpha \leqslant n, n \in N. \quad (1) $$

In Ref. [3] some properties of the Caputo fractional derivatives were stated as:

$$ {}_0D_t^{\alpha} t^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}, \quad 0 < \alpha < \beta+1, \beta > -1. \quad (2) $$

and

$$ J^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^k(0^+) \frac{t^k}{k!}, \quad n-1 < \alpha \leqslant n, n \in N. \quad (3) $$

Also, $D^{\alpha} C = 0$, where C is a constant.

Still in Ref. [3], it is believed that the Caputo fractional derivative must satisfy the following linearity condition

$$ D^{\alpha}(\gamma f(t) + \delta g(t)) = \gamma D^{\alpha} f(t) + \delta D^{\alpha} g(t), $$

with γ and δ as constants. Also, the following Leibnitz's rule holds

$$ D^{\alpha}(g(t)f(t)) = \sum_{k=0}^{n-1} g^k(t) D^{\alpha-k} f(t), \quad (4) $$

where $f(t)$ is a continuous function in $[0, t]$, and $g(t)$ is a continuous differentiable function (for sufficient order) in $[0, t]$ and and ${}^cD_t^{\alpha}$ denotes the Caputo time-fractional differential equation defined as follows:

$$ {}^cD_t^{\alpha} u(\zeta, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\Lambda)^{\alpha} \frac{\partial u(\zeta, \Lambda)}{\partial \Lambda} d\Lambda, \quad t \geqslant 0, \quad (5) $$

where $\Gamma(.)$ is the Euler Gamma function defined as:

$$ \Gamma(\alpha+1) = \int_0^{\infty} s^{\alpha} \exp^{-s} ds. \quad (6) $$

2.2. Time fractional advection-diffusion equations

These are fractional derivatives which arise as a result of putting together the fractional diffusion equations and the fractional advection equations.

The fractional diffusion equation is usually of the form:

$$ \frac{\partial^{\alpha} W}{\partial t} = k \frac{\partial^2 W}{\partial \zeta^2}, $$

also, the fractional advection equation is usually of the form

$$ \frac{\partial^{\alpha} W}{\partial t} = v \frac{\partial W}{\partial \zeta}. $$

A combination of the two equations above gives:

$$ \frac{\partial^{\alpha} W}{\partial t} = k \frac{\partial^2 W}{\partial \zeta^2} + v \frac{\partial W}{\partial t} \quad (7) $$

Where W is the dependent variable, t and ζ are the independent variable, α is the fractional order, k is the diffusion coefficient, and v is the advection coefficient. Equation (7) above is called the time fractional advection-diffusion equation.

2.3. Wavelet

This is a collection of functions formulated from the dilation and translation of one single-function called "mother wavelet". When the dilation parameter a and the translation parameter b change continuously, the wavelet is represented mathematically as:

$$ \psi_{a,b}(\zeta) = |a|^{-\frac{1}{2}} \psi\left(\frac{\zeta-b}{a}\right), \forall a, b \in \mathfrak{R}, a \neq 0. \quad (8) $$

When the two parameters a and b are discrete with $a = a_0^{-k}$ and $b = nb_0 a_0^{-k}$, $a_0 > 1, b_0 > 0$, then the wavelet is represented as

$$ \psi_{k,n}(\zeta) = |a|^{-\frac{1}{2}} \psi\left(a_0^k \zeta - nb_0\right), \forall a, b \in \mathfrak{R}, a \neq 0, \quad (9) $$

where $\psi_{k,n}(\zeta)$ forms a wavelet basis for $L_2(\mathfrak{R})$. To be particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(\zeta)$ forms an orthonormal basis.

$$ \psi_{j,k}(\zeta) = 2^{\frac{j}{2}} \psi(2^j \zeta - k). \quad (10) $$

Saeed and Rehman [11] iterated that the set $\psi_{j,k}(\zeta)$ generates an orthogonal basis of $L_2(\mathfrak{R})$, meaning that;

$$ < \psi_{j,k}(\zeta), \psi_{l,m}(\zeta) >= \delta_{jl} \delta_{km}. \quad (11) $$

2.4. Iweobodo-Mamadu-Njoseh Wavelet (IMNW)

In Ref. [7] a new wavelet was developed from MNP, this orthonormal wavelet which was described by Rayal *et al* [12] as Mamadu-Mjoseh wavelet, was further described by the developers in Ref. [8] as IMNW. In a more recent time, Rayal *et al*. [13] has considered the use this wavelet with approximation approach and collocation nodes in obtaining the solutions of a fractional pollution model. This wavelet is defined as:

$$\gamma_{n,m}(x) = \begin{cases} 2^{\frac{k}{2}} \left(\overline{MN}\right)_m (2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leqslant x \leqslant \frac{n}{2^{k-1}} \\ 0, & \text{Otherwise} \end{cases} \quad (12)$$

where

$$(\overline{MN})_m = \sqrt{\frac{2}{\pi}} MN_m, \quad (13)$$

where $m = 0, 1, \cdots, M-1$, $n = 1, 2, \cdots, 2^{k-1}$, k is any positive integer, and MN_m are the Mamadu-Njoseh polynomials of degree m. See Mamadu and Njoseh [14].

2.5. Galerkin finite element method

Incorporating the Galerkin technique with the finite element method has eradicated the constraint of finding a variational formulation for many mathematical problems (Moshen) [15].

The Galerkin method is a type of weighted residual technique. It is usually known with the finite element method (FEM) due to its exhibition of the weight function and trial solution, these factors assist in making its implementation easy to achieve. It represents the assumed solution $u^*(\zeta)$ (which satisfies the differential equation under consideration) as the sum of a number of assumed trial functions possessing coefficients which are yet to be known.

$$u^*(\zeta) = \sum_{i=1}^{n} c_i \Psi_i(\zeta) = c_1 \Psi_1(\zeta) + c_2 \Psi_2(\zeta) + \cdots + c_n \Psi_n(\zeta), \quad (14)$$

where c_i denote unknown coefficients and $\Psi_i(\zeta)$ are the trial functions.

The assumed solution equation (14) above is substituted into the original equation to obtain the residual equation as:

$$R(\zeta) = D(u^*(\zeta), \zeta). \quad (15)$$

The unknown coefficients are obtained by multiplying the trial function with the residual equation, then performing the inner product operation on the result (obtaining the weak formulation) and setting it to zero. This will amount to:

$$\int_a^b \Phi_i(\zeta) R(\zeta) d\zeta = 0, \quad i = 0, 1, \cdots, n. \quad (16)$$

We then substitute back the obtained coefficients into the reformulated equation bearing the trial function and its derivatives to obtain the approximate solution.

3. Discretization of the Caputo TFADE with IMNW

3.1. Preliminaries

The Caputo time-fractional advection diffusion equation (1) can be converted into the Riemann-Louville fractional derivatives in the form

$${}_0^R D_t^\alpha [u - u_0](x,t) - u_{xx}(x,t) + u_x(x,t) = g(x,t). \quad (17)$$

By definition, we know that:

$$\begin{aligned} {}_0^R D_t^\alpha (u_0) &= \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u_0 ds \\ &= \frac{u_0}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\frac{1}{1-\alpha} t^{1-\alpha} \right) \\ &= \frac{u_0}{\Gamma(1-\alpha)} t^{-\alpha}. \end{aligned}$$

Hence,

$${}_0^R D_t^\alpha u(x,t) = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-s)^{-\alpha-1} u(s) ds. \quad (18)$$

. Assuming that $[0, T]$ is a partition such that $0 < t_0 < t_1 < \cdots < t_n = T$. Applying $nt_j = j$, for $j = 1, 2, \cdots, n$, then equation (17) can be approximated in time step as:

$${}_0^R D_t^\alpha u(x, t_j) = \frac{1}{\Gamma(-\alpha)} \int_0^{t_j} (t_j - s)^{\alpha+1} u(s) ds,$$

Assuming that $t_j = t_j w + s$, we have:

$$\begin{aligned} {}_0^R D_t^\alpha u(x, t_j) &= \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \int_0^1 \frac{u(t_j - t_j w) - u(0)}{w^{\alpha+1)}} dw \\ &= \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \int_0^1 F(w) w^{-\alpha-1} dw. \end{aligned} \quad (19)$$

Here, $F(w) = u(t_j - t_j w) = u(0)$.
By the quadrature formula as captured in Cialet [16], we replace the integral sign with summation for $t_j = \frac{n}{j}$, so that for every j, we have:

$${}_0^R D_t^\alpha u(x, t_j) = \frac{1}{\Gamma(-\alpha)} \left[\sum_{r=0}^{j} \alpha_{rj} u(t_j - t_r) + G_j(g) \right],$$

here, $\|G_j(g)\| \leqslant k_j^{\alpha-2} \sup_{0 \leqslant t \leqslant T} \|u''(t_j - t_r)\|$. Therefore,

$$\begin{aligned} {}_0^R D_t^\alpha u(x, t_j) &= \frac{\Delta t_j^{-\alpha}}{\Gamma(2-\alpha)} \sum_{r=0}^{j} (-\alpha)(1-\alpha) j^{-\alpha} \alpha_{rj} u(t_j - t_r) \\ &+ \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} G_j(g) \\ &= \Delta t^{-\alpha} \sum_{r=0}^{j} \frac{(-\alpha)(1-\alpha) j^{-\alpha} \alpha_{rj}}{\Gamma(2-\alpha)} u(t_j - t_r) + \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} G_j(g) \\ &= \Delta t^{-\alpha} \sum_{r=0}^{j} w_{rj} u(t_j - t_r) + \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} G_j(g). \end{aligned}$$

Here, $\Gamma(2-\alpha)w_{rj} = (-\alpha+\alpha^2)j^{D-\alpha}\alpha_{rj}$, and w_{rj} and α_{rj} satisfy (Ref. [16] and Mamadu *et al* [17]) with definition:

$$w_{rj} = \frac{1}{\Gamma(2-\alpha)}\begin{cases}1, & r=0\\ -2r^{1-\alpha}+(r-1)^{1-\alpha}+(r+1)^{1-\alpha}, & r=1,2,\cdots,j-1\\ -(r-1)r^{-\alpha}+(r-1)^{1-\alpha}-r^{1-\alpha}, & r=j\end{cases}$$

$$\alpha_{rj} = \frac{1}{\Gamma(2-\alpha)}\begin{cases}-1, & r=0\\ -2r^{1-\alpha}-(r-1)^{1-\alpha}-(r+1)^{1-\alpha}, & r=1,2,\cdots,j\\ (r-1)r^{-\alpha}-(r-1)^{1-\alpha}+r^{1-\alpha}, & r=j\end{cases} \tag{20}$$

3.2. Time Discretization

Studying the finite difference method (Chen *et al*) [18] at the point $t=t_j$ we will obtain:

$${}_0^RD_t^\alpha\left[u(x,t)-u_0(x,t)\right]|_{t=t_j} = au_{xx}(x,t_j)-vu_x(x,t_j)+g(x,t_j). \tag{21}$$

But,

$${}_0^RD_t^\alpha[u(x,t)-u_0(x,t)]_{t=t_j} = \Delta t^{-\alpha}\sum_{r=0}^{j}w_{rj}\left[u(t_j-t_r)-u(0)\right]+\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}G_j(g). \tag{22}$$

Therefore,

$$\Delta t^{-\alpha}\sum_{r=0}^{j}w_{rj}\left[u(t_j-t_r)-u(0)\right]+\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}G_j(g) = au_{xx}(x,t_j)-vu_x(x,t_j)+g(x,t_j).$$

Denoting $u_j=u(x,t_j)$ gives:

$$\Delta t^{-\alpha}\sum_{r=0}^{j}w_{rj}\left[u_{j-r}-u(0)\right] = au_{xx}(x,t_j)-vu_x(x,t_j)+G(x,t_j),$$

Let $r=0$, we obtain:

$$\Delta t^{-\alpha}\left(w_{0j}u_j-w_{0j}u_0\right)+\Delta t^{-\alpha}\sum_{r=1}^{j}w_{rj}u_{j-r}+ \Delta t^{-\alpha}\sum_{r=0}^{j}w_{rj}u_0 = au_{xx}(x,t_j)-vu_x(x,t_j)+G(x,t_j),$$

$$\Rightarrow \Delta t^{-\alpha}(u_j-u_0)w_{0j}+\Delta t^{-\alpha}\sum_{r=1}^{j}w_{rj}u_{j-r}+ \Delta t^{-\alpha}\sum_{r=0}^{j}w_{rj}u_0 = au_{xx}(x,t_j)-vu_x(x,t_j)+g(x,t_j),$$

From Ref. [8], we saw that:

$$\sum_{r=0}^{j}w_{rj} = \frac{-\alpha(1-\alpha)j^{-\alpha}}{(2-\alpha)}\left(\frac{-1}{\alpha}\right) = j^{-\alpha}h_\alpha,$$

and

$$h_\alpha = \frac{1}{\Gamma(1-\alpha)}.$$

Therefore, the implicit formula for equation (17) is:

$$\Delta t^{-\alpha}(u_j-u_0)w_{0j}+\Delta t^{-\alpha}\sum_{r=1}^{j}w_{rj}u_{j-r}+\Delta t^{-\alpha}j^{-\alpha}h_\alpha u_0 = au_{xx}(x,t_j)-vu_x(x,t_j)+g(x,t_j). \tag{23}$$

3.2.1. Discretization in Time with IMNW-based Galerkin Technique

Let the approximate solution of (1) be defined as:

$$u_j(x,t_j) = \sum_{n=1}^{2^{k-1}}\sum_{r=0}^{j=M-1}c_{n,r}\gamma_{n,r}, \tag{24}$$

where $c_{n,r}$ are the coefficients of the new wavelets, and $\gamma_{n,r}$ is a basis function of the new wavelet.

Applying equation (24) on (23), we have:

$$\Delta t^{-\alpha}w_{0,j}\sum_{n=1}^{2^{k-1}}\sum_{r=0}^{j}c_{n,r}\gamma_{n,r}+ \Delta t^{-\alpha}\sum_{n=1}^{2^{k-1}}\sum_{r=0}^{j-1}w_{rj}(c_{n,r}\gamma_{n,r}-u_0)+\Delta t^{-\alpha}w_{0,j}u_0 = a\left(\sum_{n=1}^{2^{k-1}}\sum_{r=0}^{j}c_{n,r}\gamma_{n,r}\right)_{xx}-v\left(\sum_{n=1}^{2^{k-1}}\sum_{r=0}^{j}c_{n,r}\gamma_{n,r}\right)_{x} + g(x,t). \tag{25}$$

Equation (25) is rewritten to achieve a residual equation $R(x,t)$, from there, we can easily solve for the unknown coefficients $c_{n,r}$ using mathematics software with the parameters α, a, v, $w_{r,j}$ given. Then the approximate solution is obtained by substituting back the obtained $c_{n,r}$ together with the given parameters into equation (24).

3.3. Space Discretization of TFADE with IMNW

Let V_v be a linear piecewise finite element space, and $[0,T]$ be the space partitioning of $[a,b]$ defined as:

$$0 = x_0 < x_1 < x_2 < \cdots x_n = T.$$

Assuming that:
$V_v = \{S_v(x) : S_v(x) \text{ is continuous and linear in } [0,T]\}$.
We obtain the variational formulation of the fractional Advection-Diffusion equation (1) by computing $u(t)\in H_0^1(a,b)$ such that:

$$\left\langle {}_0^RD_t^\alpha[u(x,t)-U_0], S(x)\right\rangle = \langle U_{xx}, S(x)\rangle - \langle U_x, S(x)\rangle + \langle g(x,t), S(x)\rangle,\ S(x)\in H_0^1. \tag{26}$$

Because we are considering the finite element technique, we want to compute $U_v(t) \in V_v$, $\ni$.

$$\left\langle {}_0^R D_t^\alpha [u(x,t) - U_0], Q \right\rangle = \left\langle \frac{\partial^2 u}{\partial x^2}, Q \right\rangle - \left\langle \frac{\partial u}{\partial x}, Q \right\rangle + \langle g(x,t), Q \rangle, \quad Q \in V_v. \tag{27}$$

Let $B_v = -\Delta_h : V_v \to V_v$ satisfies

$$(B_v U_v) = \left\langle \frac{\partial^2 u}{\partial x^2}, Q \right\rangle - \left\langle \frac{\partial u}{\partial x}, Q \right\rangle, \quad Q \in V_v. \tag{28}$$

Assuming $G_v : G \to V_v$ defines an L_2 operator as:

$$\langle G_v s, Q \rangle = \langle s, Q \rangle, \ \forall Q \in V_v, \ s \in L_2.$$

Thus, we can rewrite equation (26) as:

$$\left\langle {}_0^R D_t^\alpha [u(x,t) - U_0], S(x) \right\rangle = B_v U_v + G_v g, \ t > 0. \tag{29}$$

But,

$${}_0^R D_t^\alpha [u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u(s-x)}{(t-s)^\alpha} ds, \ \alpha \in (0,1). \tag{30}$$

Therefore, using the quadrature formula on (29), we get:

$$\Delta t^{-\alpha} \sum_{r=0}^{j} w_{rj} \left\langle u(t_j - t_r) - U_0, Q \right\rangle = \left\langle \frac{\partial^2 u_j}{\partial x^2}, Q \right\rangle - \left\langle \frac{\partial u_j}{\partial x}, Q \right\rangle + \left\langle g_j, Q \right\rangle, \ j = 0(1)n, \ \forall Q \in V_v, \tag{31}$$

where

$$w_{rj} = \frac{1}{\Gamma(2-\alpha)} \begin{cases} 1, & r = 0 \\ -2r^{1-\alpha} + (r-1)^{1-\alpha} + (r+1)^{1-\alpha}, & r = 1, 2, \cdots, j-1 \\ -(r-1)r^{-\alpha} + (r-1)^{1-\alpha} - r^{1-\alpha}, & r = j \end{cases}$$

Let,

$$U_j \equiv U_v = u(x,t) = \sum_{n=1}^{2^{k-1}} \sum_{r=0}^{j=M-1} c_{n,r} \gamma_{n,r}$$

be the approximate solution of $U_v(t_j)$, with $\gamma_{n,r}$ being the basis function in the finite space V_v and $c_{n,r}$ being the coefficient of the new wavelet, $n = 1$, $r = 0(1)M-1$. Therefore we have

$$\left\langle \Delta t^{-\alpha} \sum_{r=0}^{j} w_{rj} [u(t_j - t_r) - U_0], Q \right\rangle = \left\langle \frac{\partial^2 u_j}{\partial x^2}, Q \right\rangle - \left\langle \frac{\partial u_j}{\partial x}, Q \right\rangle + \left\langle g_j, Q \right\rangle, \ j = 0(1)n, \ \forall Q \in V_v. \tag{32}$$

Since $U_j = u(t_j) = \sum_{n=1}^{2^{k-1}} \sum_{r=0}^{j=M-1} c_{n,r}\gamma_{n,r}$, therefore, it becomes:

$$\begin{aligned} &\langle \Delta t^{-\alpha} w_{0,j} \sum_{n=1}^{2^{k-1}} \sum_{r=0}^{j} c_{n,r}\gamma_{n,r} \\ &+ \Delta t^{-\alpha} \sum_{n=1}^{2^{k-1}} \sum_{r=0}^{j-1} w_{rj}(c_{n,r}\gamma_{n,r} - u_0) + \Delta t^{-\alpha} w_{0,j} u_0, Q \rangle \\ &= \left\langle a \left[\sum_{n=1}^{2^{k-1}} \sum_{r=0}^{j} c_{n,r}\gamma_{n,r} \right]_{xx}, Q \right\rangle - \left\langle \beta \left[\sum_{n=1}^{2^{k-1}} \sum_{r=0}^{j} c_{n,r}\gamma_{n,r} \right]_{x}, Q \right\rangle + \left\langle g_j, Q \right\rangle \\ &\equiv \int c_{n,r}\gamma_{n,r} Q dx \\ &j = 0(1)n, \ \forall Q \in V_v. \end{aligned} \tag{33}$$

4. Numerical Illustration

Consider the Caputo TFADE (Doley *et al*) [19],

$$ {}^c D_t^\alpha u(x,t) = a \frac{\partial^2 u(x,t)}{\partial x^2} - v \frac{\partial u(x,t)}{\partial x} + f(x,t) \tag{34}$$

$x \in [0,1], \quad t \geqslant 0, \quad 0 < \alpha < 1,$

with the initial condition $u(x,0) = x - x^2$, and boundary conditions $u(0,t) = u(1,t) = 0$. Where:

$$f(x,t) = \frac{(2x - 2x^2)t^{2-\alpha}}{\Gamma(3-\alpha)} + v(1-2x)(t^2+1) + 2(t^2+1).$$

The exact equation is $u(x,t) = (x - x^2)(t^2 + 1)$.

Solution

Let

$$u(x,t) = \sum_{n=1}^{2^{k-1}} \sum_{r=0}^{j=M-1} c_{n,r}\gamma_{n,r}.$$

Assuming $k = 1$, and $j = 2$

$$\gamma_{1,0} = \frac{2}{\sqrt{\pi}} \left(x - x^2 \right)$$

$$\gamma_{1,1} = \frac{2}{\sqrt{\pi}} (2x-1) \left(x - x^2 \right)$$

$$\gamma_{1,2} = \frac{2}{\sqrt{\pi}} (\frac{20x^2}{3} - \frac{20x}{3} + 1)(x - x^2).$$

$\Rightarrow$

$$u(x,t) = c_{1,0} \frac{2}{\sqrt{\pi}} (x - x^2) + c_{1,1} \frac{2}{\sqrt{\pi}} (2x-1)(x-x^2) + c_{1,2} \frac{2}{\sqrt{\pi}} (\frac{20x^2}{3} - \frac{20x}{3} + 1)(x - x^2)$$

$$u'(x,t) = c_{1,0} \frac{2}{\sqrt{\pi}} (-2x) + c_{1,1} \frac{2}{\sqrt{\pi}} (-6x^2 + 6x - 1) + c_{1,2} \frac{2}{\sqrt{\pi}} (\frac{-80x^3}{3} + \frac{120x^2}{3} - 46x)$$

$$u''(x,t) = c_{1,0} \frac{2}{\sqrt{\pi}} (-2) + c_{1,1} \frac{2}{\sqrt{\pi}} (-12x + 6) + c_{1,2} \frac{2}{\sqrt{\pi}} (\frac{-240x^2}{3} + \frac{240x}{3} - 46).$$

$${}^c D_t^\alpha u(x,t) = \Delta t^{-\alpha} w_{0,j} \sum_{n=1}^{2^{k-1}} \sum_{r=0}^{j} c_{n,r}\gamma_{n,r} + \Delta t^{-\alpha} \sum_{n=1}^{2^{k-1}} \sum_{r=0}^{j-1} w_{rj}(c_{n,r}\gamma_{n,r} - u_0) + \Delta t^{-\alpha} w_{0,j} u_0.$$

Substituting into the original equation, we have:

$$\Delta t^{-\alpha} w_{0,j}[c_{1,0}\frac{2}{\sqrt{\pi}}(x-x^2)+c_{1,1}\frac{2}{\sqrt{\pi}}(2x-1)(x-x^2)$$
$$+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{20x^2}{3}-\frac{20x}{3}+1)(x-x^2)]$$
$$+\Delta t^{-\alpha}\sum_{r=0}^{j} w_{rj}[c_{1,0}\frac{2}{\sqrt{\pi}}(x-x^2)+c_{1,1}\frac{2}{\sqrt{\pi}}(2x-1)(x-x^2)$$
$$+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{20x^2}{3}-\frac{20x}{3}+1)(x-x^2)-u(0)]+\Delta t^{-\alpha} w_{0,j}u_0$$
$$=a[c_{1,0}\frac{2}{\sqrt{\pi}}(-2)+c_{1,1}\frac{2}{\sqrt{\pi}}(-12x+6)+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{-240x^2}{3}$$
$$+\frac{240x}{3}-46)]-v[c_{1,0}\frac{2}{\sqrt{\pi}}(-2x)+c_{1,1}\frac{2}{\sqrt{\pi}}(-6x^2+6x-1)$$
$$+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{-80x^3}{3}+\frac{120x^2}{3}-46x)]+f(x,t).$$

Our residual equation is therefore given as:

$$R(x,t)=\Delta t^{-\alpha} w_{0,j}\Big[c_{1,0}\frac{2}{\sqrt{\pi}}(x-x^2)$$
$$+c_{1,1}\frac{2}{\sqrt{\pi}}(2x-1)(x-x^2)$$
$$+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{20x^2}{3}-\frac{20x}{3}+1)(x-x^2)\Big]$$
$$+\Delta t^{-\alpha}\sum_{r=0}^{j} w_{rj}\Big[c_{1,0}\frac{2}{\sqrt{\pi}}(x-x^2)$$
$$+c_{1,1}\frac{2}{\sqrt{\pi}}(2x-1)(x-x^2)$$
$$+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{20x^2}{3}-\frac{20x}{3}+1)(x-x^2)-u(0)\Big]$$
$$+\Delta t^{-\alpha} w_{0,j}u_0-a\Big[c_{1,0}\frac{2}{\sqrt{\pi}}(-2)$$
$$+c_{1,1}\frac{2}{\sqrt{\pi}}(-12x+6)$$
$$+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{-240x^2}{3}+\frac{240x}{3}-46)\Big]$$
$$+v\Big[c_{1,0}\frac{2}{\sqrt{\pi}}(-2x)+c_{1,1}\frac{2}{\sqrt{\pi}}(-6x^2+6x-1)$$
$$+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{-80x^3}{3}+\frac{120x^2}{3}-46x)\Big]-f(x,t),$$

discretizing in space produces a system of equations,

$$\int_0^1 \gamma_{1,0}(x)R(x)dx=0,$$
$$\int_0^1 \gamma_{1,1}(x)R(x)dx=0, \tag{35}$$
$$\int_0^1 \gamma_{1,2}(x)R(x)dx=0, m=0,1,2,\cdots,$$

which is represented as:

$$\int_0^1 \frac{2}{\sqrt{\pi}}[\Delta t^{-\alpha} w_{0,j}[c_{1,0}\frac{2}{\sqrt{\pi}}(x-x^2)$$
$$+c_{1,1}\frac{2}{\sqrt{\pi}}(2x-1)(x-x^2)+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{20x^2}{3}$$
$$-\frac{20x}{3}+1)(x-x^2)]+\Delta t^{-\alpha}\sum_{r=0}^{j} w_{rj}[c_{1,0}\frac{2}{\sqrt{\pi}}(x-x^2)$$
$$+c_{1,1}\frac{2}{\sqrt{\pi}}(2x-1)(x-x^2)$$
$$+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{20x^2}{3}-\frac{20x}{3}+1)(x-x^2)-u(0)]\Delta t^{-\alpha} w_{0,j}u_0$$
$$-a[c_{1,0}\frac{2}{\sqrt{\pi}}(-2)+c_{1,1}\frac{2}{\sqrt{\pi}}(-12x+6)$$
$$+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{-240x^2}{3}+\frac{240x}{3}-46)]+v[c_{1,0}\frac{2}{\sqrt{\pi}}(-2x)$$
$$+c_{1,1}\frac{2}{\sqrt{\pi}}(-6x^2+6x-1)+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{-80x^3}{3}\frac{120x^2}{3}$$
$$-46x)]-f(x,t)]dx=0,$$
$$\int_0^1 \frac{2}{\sqrt{\pi}}(2x-1)[\Delta t^{-\alpha} w_{0,j}[c_{1,0}\frac{2}{\sqrt{\pi}}(x-x^2)$$
$$+c_{1,1}\frac{2}{\sqrt{\pi}}(2x-1)(x-x^2)+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{20x^2}{3}$$
$$-\frac{20x}{3}+1)(x-x^2)]+\Delta t^{-\alpha}\sum_{r=0}^{j} w_{rj}[c_{1,0}\frac{2}{\sqrt{\pi}}(x-x^2)$$
$$+c_{1,1}\frac{2}{\sqrt{\pi}}(2x-1)(x-x^2)$$
$$+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{20x^2}{3}-\frac{20x}{3}+1)(x-x^2)-u(0)]\Delta t^{-\alpha} w_{0,j}u_0$$
$$-a[c_{1,0}\frac{2}{\sqrt{\pi}}(-2)+c_{1,1}\frac{2}{\sqrt{\pi}}(-12x+6)$$
$$+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{-240x^2}{3}+\frac{240x}{3}-46)]+v[c_{1,0}\frac{2}{\sqrt{\pi}}(-2x)$$
$$+c_{1,1}\frac{2}{\sqrt{\pi}}(-6x^2+6x-1)+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{-80x^3}{3}\frac{120x^2}{3}$$
$$-46x)]-f(x,t)]dx=0,$$
$$\int_0^1 \frac{2}{\sqrt{\pi}}(\frac{20x^2}{3}-\frac{20x}{3}+1)[\Delta t^{-\alpha} w_{0,j}[c_{1,0}\frac{2}{\sqrt{\pi}}(x-x^2)$$
$$+c_{1,1}\frac{2}{\sqrt{\pi}}(2x-1)(x-x^2)+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{20x^2}{3}$$
$$-\frac{20x}{3}+1)(x-x^2)]+\Delta t^{-\alpha}\sum_{r=0}^{j} w_{rj}[c_{1,0}\frac{2}{\sqrt{\pi}}(x-x^2)$$
$$+c_{1,1}\frac{2}{\sqrt{\pi}}(2x-1)(x-x^2)$$
$$+c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{20x^2}{3}-\frac{20x}{3}+1)(x-x^2)-u(0)]\Delta t^{-\alpha} w_{0,j}u_0$$

$$- a[c_{1,0}\frac{2}{\sqrt{\pi}}(-2) + c_{1,1}\frac{2}{\sqrt{\pi}}(-12x+6)$$
$$+ c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{-240x^2}{3} + \frac{240x}{3} - 46)] + v[c_{1,0}\frac{2}{\sqrt{\pi}}(-2x)$$
$$+ c_{1,1}\frac{2}{\sqrt{\pi}}(-6x^2+6x-1) + c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{-80x^3}{3}\frac{120x^2}{3}$$
$$- 46x)] - f(x,t)]dx = 0.$$

Solving the equations with Maple 18 software for $j = 2, t = 0.5$, $\Delta t = 1$.
At $\alpha = 0.5$,

$$W_{0,2} = 0.752252778,$$
$$W_{1,2} = -0.44065949,$$
$$W_{2,2} = -0.843516341$$

we obtain the coefficients $c_{1,0} = 1.107922464$, $c_{1,1} = 0.0002885768462$, and $c_{1,2} = -0.004797036447$ which are substituted into the assumed solution to get the approximate solution as

$$1.102643738(\frac{2}{\sqrt{\pi}})x(1-x)+$$
$$0.0008873787527(\frac{2}{\sqrt{\pi}})(2x-1)x(1-x)-$$
$$\frac{0.0006114146302}{3}(\frac{2}{\sqrt{\pi}}(20x^2-20x+3)x(1-x).$$

At $\alpha = 0.75$

$$W_{0,2} = 0.882610121,$$
$$W_{1,2} = -0.715614006,$$
$$W_{2,2} = -0.691799233$$

, solving with Maple 18 gives the coefficients
$c_{1,0} = 1.098277505$, $c_{1,1} = -0.001637407048$, and $c_{1,2} = 0.001130632789$
which are substituted into the assumed solution to get the approximate solution as

$$1.098277505(\frac{2}{\sqrt{\pi}})x(1-x)-$$
$$0.001637407048(\frac{2}{\sqrt{\pi}})(2x-1)x(1-x)+$$
$$\frac{0.001130632789}{3}(\frac{2}{\sqrt{\pi}}(20x^2-20x+3)x(1-x).$$

5. Conclusion

Being mindful of the importance of the numerical approach to many mathematical problems today, and the difficulty in approaching fractional derivatives analytically, we have successfully applied the IMNW as a basis function in discretizing the Caputo time fractional Advection-diffusion equations both in time and space. An implicit form of the equation has been obtained, making it easy for the application of numerical methods.

Data availability

We do not have any research data outside the submitted manuscript file.

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