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Semi-analytical solution and numerical simulations of a coinfection model of Malaria and Zika virus disease

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Abstract

In this work, a model for the coinfection of malaria and zika virus disease is studied. The model incorporates various control measures against the spread of malaria and zika virus disease such as vaccination, treatment and biological control of mosquitoes using sterile insect technique. The existence and uniqueness of solutions to the model were first shown. Thereafter, the model is shown to be well-posed epidemiologically by showing that all solutions to the system are positive and bounded. Then, the solution of the model is obtained using the homotopy perturbation method which is a semi-analytical method. The solutions obtained are shown to be comparable with those obtained from Runge-Kutta method of order 4. Furthermore, the performance of the controls in comparison to each other when applied seperately and when combined were shown. The results showed that combining the three controls performed better than the rest. Hence, efforts should be made to incorporate controls that affect both humans and the vectors for effective control of malaria, zika virus disease and their coinfection.

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Keywords: Malaria, Zika virus, Approximate solution, Homotopy perturbation

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1. Introduction

The study of nonlinear differential problems has been of great importance in all areas of physical sciences and engineering [1]. Obtaining solutions to these nonlinear problems is important computationally but complicated and tedious using analytical or numerical approaches [2]. Several methods have been developed to find the exact, approximate, and numerical solution of nonlinear differential problems [3]. Many problems of applied sciences in real life rely mostly on numerical methods to obtain an approximate solution of the problems. These numerical methods have been developed to handle problems such as differential equations, partial differential equations, boundary value problems, integral equations, nonlinear equations, etc. Some of the numerical methods include Adomian Decomposition Method (ADM), Homotopy Pertubation Method (HPM), Homotopy Analysis Method (HAM), Variation Iteration Method (VIM), Taylor series method, etc. [4–6].

Ref. [7] used homotopy perturbation method, adomian decomposition method and homotopy analysis method to solve the Generalized Zakharov Equations and compare the results obtained from each solution showing their similarities and differeneces. Ref. [8] compared the suitability of Adomian decomposition method and homotopy perturbation method in

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solving some nonlinear differential equations. Their work showed that both performed very well and in the problems considered, the results obtained were similar. Ref. [9] combined homotopy perturbation method and Laplace transformation to solve some linear and nonlinear singular initial value problems of Lane-Emden type equations. They used several examples to show the accuracy of the methods. Ref. [10] employed homotopy perturbation method to find the approximate solutions of a Dengue fever model and also compared the results with the numerical simulation results obtained. Some of the works reviewed in literatures showed that the choice of an approximate method to use depends on the nature of the nonlinear problem involved. Therefore, we will adopt the HPM to solve the model given in Ref. [11] because of its advantage over some of the other traditional methods. Also, it will be more friendly to handle the complex nature of the model being analyzed in this work.

2. Mathematical model

The mathematical model for the coinfection has twenty-two (22) compartments consisting of the human and mosquito populations. The human population has sixteen (16) compartments which are Susceptible humans S_h, Vaccinated humans for malaria S_{hv}, Unvaccinated humans for malaria S_{hu}, Exposed humans to malaria E_{hm}, Exposed humans to zika E_{hz}, coinfected humans with both diseases E_{mz}, Infectious humans with malaria I_{hm}, symptomatic infectious humans with zika I_{hzS}, asymptomatic infectious humans with zika I_{hzA}, coinfectious humans with both diseases I_{mz}, Infectious humans undergoing treatment for malaria I_{hmT}, Infectious humans undergoing treatment for zika I_{hzT}, coinfectious humans undergoing treatment for both diseases I_{mzT}, Infectious humans not undergoing treatment for malaria I_{hmU}, coinfectious humans not undergoing treatment for both disease I_{mzU} and Recovered humans R_h. While the mosquito population has six (6) compartments which are Susceptible Anopheles mosquitoes S_{mv}, Exposed Anopheles mosquitoes E_{mv}, Infectious Anopheles Mosquitoes I_{mv}, Susceptible Aedes mosquitoes S_{zv}, Exposed Aedes mosquitoes E_{zv} and Infectious Aedes mosquitoes I_{zv}. The transmission dynamics of the system as well as the assumptions leading to the model formulation has been explained in details by Ref. [11]. Also, the description of parameters, review of related and significant literatures which serves as foundations for proposing the model, basic quantitative and qualitative analyses were also sufficiently provided in Ref. [11]. Hence, the focus in this work is to obtain a semi-analytical solution of the system and perform further simulations to gain more insight into the dynamics of the control measures proposed. Thus, the model for the system according to Ref. [11] is given to be;

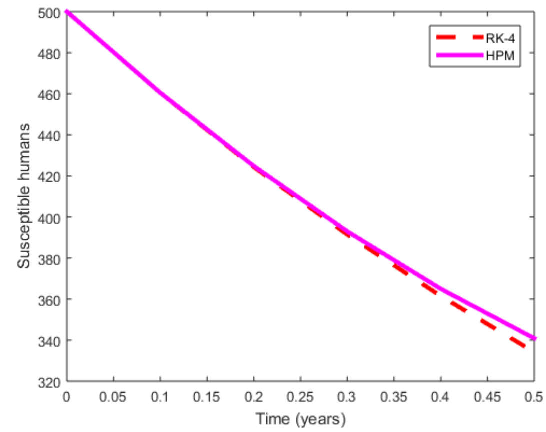

Figure 1. Susceptible humans.

$$
\begin{aligned}
\frac{dS_h}{dt} &= \Lambda_h + \theta R_h - (\rho_1 + \rho_2 + \tau_1) S_h \\
\frac{dS_{hu}}{dt} &= \rho_1 S_h - \alpha_1\beta_1 I_{mv} S_{hu} - \alpha_2\eta_1 I_{zv} S_{hu} - \tau_1 S_{hu} \\
\frac{dS_{hv}}{dt} &= \rho_2 S_h - \alpha_1\beta_2\varphi I_{mv} S_{hv} - \alpha_2\eta_1 I_{zv} S_{hv} - \tau_1 S_{hv} \\
\frac{dE_{hm}}{dt} &= \alpha_1\beta_1 I_{mv} S_{hu} + \alpha_1\beta_2\varphi I_{mv} S_{hv} + \phi_1 I_{hmU} + \phi_2 I_{hmzU} \\
&\quad - \alpha_2\eta_1 I_{zv} E_{hm} - (\delta_1 + \tau_1) E_{hm} \\
\frac{dE_{hz}}{dt} &= \alpha_2\eta_1 I_{zv}(S_{hu} + S_{hv}) - \alpha_1\beta_1 I_{mv} E_{hz} - ((\chi_1 + \chi_2)\delta_2 + \tau_1) E_{hz} \\
\frac{dE_{hmz}}{dt} &= \alpha_1\beta_1 I_{mv} E_{hz} + \alpha_2\eta_1 I_{zv} E_{hm} - (\delta_3 + \tau_1) E_{hmz} \\
\frac{dI_{hm}}{dt} &= \delta_1 E_{hm} - \alpha_2\eta_1 I_{zv} I_{hm} - (\tau_1 + \tau_2 + \varepsilon_1 + \varepsilon_2) I_{hm} \\
\frac{dI_{hmT}}{dt} &= \varepsilon_1 I_{hm} - (\gamma_1 + \tau_1 + \tau_2) I_{hmT} \\
\frac{dI_{hmU}}{dt} &= \varepsilon_2 I_{hm} - (\phi_1 + \tau_1 + \tau_2) I_{hmU} \\
\frac{dI_{hzS}}{dt} &= \delta_2\chi_1 E_{hz} - \alpha_1\beta_1 I_{mv} I_{hzS} - (\tau_1 + \tau_3 + \psi + \omega_1) I_{hzS} \\
\frac{dI_{hzA}}{dt} &= \delta_2\chi_2 E_{hz} - \alpha_1\beta_1 I_{mv} I_{hzA} - (\tau_1 + \tau_3 + \omega_3) I_{hzA} \\
\frac{dI_{hzT}}{dt} &= \psi I_{hzS} - (\tau_1 + \tau_3 + \omega_2) I_{hzT} \\
\frac{dI_{hmz}}{dt} &= \alpha_1\beta_1 I_{mv}(I_{hzS} + I_{hzA}) + \alpha_2\eta_1 I_{zv} I_{hm} + \delta_3 E_{hmz} \\
&\quad - (\tau_1 + \tau_4 + \sigma_1 + \sigma_2) I_{hmz} \\
\frac{dI_{hmzT}}{dt} &= \sigma_1 I_{hmz} - (\tau_1 + \tau_4 + \gamma_2) I_{hmzT} \\
\frac{dI_{hmzU}}{dt} &= \sigma_2 I_{hmz} - (\tau_1 + \tau_4 + \phi_2) I_{hmzU} \\
\frac{dR_h}{dt} &= \gamma_1 I_{hmT} + \gamma_2 I_{hmzT} + \omega_1 I_{hzS} + \omega_2 I_{hzT} + \omega_3 I_{hzA} - (\tau_1 + \theta) R_h \\
\frac{dS_{mv}}{dt} &= \Lambda_{mv} - \alpha_1(\beta_3 I_{hm} + \beta_4 I_{hmT} + \beta_5 I_{hmU} + \beta_6 I_{hmz} + \beta_7 I_{hmzT} \\
&\quad + \beta_8 I_{hmzU}) S_{mv} - (\kappa_1 M_{SIT} + \mu_m) S_{mv} \\
\frac{dE_{mv}}{dt} &= \alpha_1(\beta_3 I_{hm} + \beta_4 I_{hmT} + \beta_5 I_{hmU} + \beta_6 I_{hmz} + \beta_7 I_{hmzT} + \beta_8 I_{hmzU}) S_{mv} \\
&\quad - (\nu_1 + \mu_m) E_{mv} \\
\frac{dI_{mv}}{dt} &= \nu_1 E_{mv} - \mu_m I_{mv} \\
\frac{dS_{zv}}{dt} &= \Lambda_{zv} - \alpha_2(\eta_2 I_{hzS} + \eta_3 I_{hzA} + \eta_4 I_{hzT} + \eta_5 I_{hmz} + \eta_6 I_{hmzT} \\
&\quad + \eta_7 I_{hmzU}) S_{zv} - (\kappa_2 Z_{SIT} + \mu_z) S_{zv}
\end{aligned}
\qquad (1)
$$

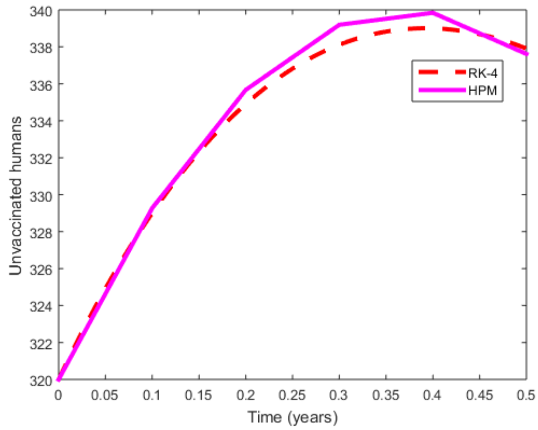

Figure 2. Unvaccinated humans.

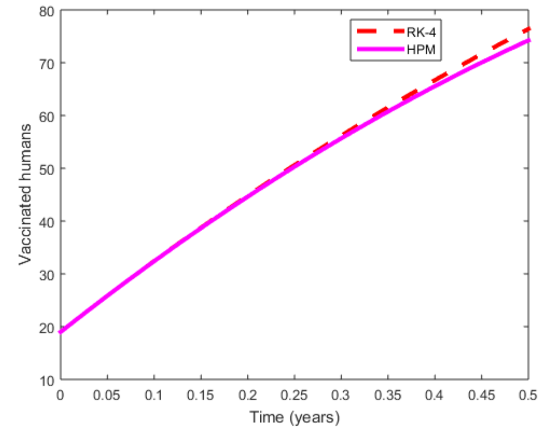

Figure 3. Vaccinated humans.

$$\frac{dE_{zv}}{dt} = \alpha_2(\eta_2 I_{hzS} + \eta_3 I_{hzA} + \eta_4 I_{hzT} + \eta_5 I_{hmz} + \eta_6 I_{hmzT} + \eta_7 I_{hmzU})S_{zv} - (\nu_2 + \mu_z)E_{zv}$$

$$\frac{dI_{zv}}{dt} = \nu_2 E_{zv} - \mu_z I_{zv},$$

where $S_h(0) = S_h^0, S_{hu}(0) = S_{hu}^0, S_{hv}(0) = S_{hv}^0, E_{hm}(0) = E_{hm}^0, E_{hz}(0) = E_{hz}^0, E_{mz}(0) = E_{mz}^0, I_{hm}(0) = I_{hm}^0, I_{mz}(0) = I_{mz}^0, I_{hmT}(0) = I_{hmT}^0, I_{hmU}(0) = I_{hmU}^0, I_{hzS}(0) = I_{hzS}^0, I_{hzA}(0) = I_{hzA}^0, I_{hzT}(0) = I_{hzT}^0, I_{mzT}(0) = I_{mzT}^0, I_{mzU}(0) = I_{mzU}^0, R_h(0) = R_h^0, S_{mv}(0) = S_{mv}^0, E_{mv}(0) = E_{mv}^0, I_{mv}(0) = I_{mv}^0, S_{zv}(0) = S_{zv}^0$, $E_{zv}(0) = E_{zv}^0$ and $I_{zv}(0) = I_{zv}^0$ are the initial conditions of the system with the total human and mosquito populations given by

$$N_h = S_h + S_{hu} + S_{hv} + E_{hm} + E_{hz} + E_{mz} + I_{hm} + I_{hmT} + I_{hmU} + I_{hzS} + I_{hzA} + I_{hzT} + I_{mz} + I_{mzT} + I_{mzU} + R_h$$

$$N_{mv} = S_{mv} + E_{mv} + I_{mv}$$

$$N_{zv} = S_{zv} + E_{zv} + I_{zv}.$$

3. Existence and uniqueness of solutions

To show the existence and uniqueness of the solutions to our models, we employ the Banach fixed point theorem which is a special kind of the Lipschitz continuity. Hence, we first state both concepts before adopting them.

Theorem 3.1 Banach fixed point theorem. Let Ω be a complete metric space and let $f : \Omega \to \Omega$ be a contraction, that is, $\exists$ $c \in (0, 1)$ such that for all $z_1, z_2 \in \Omega$, then

$$d(f(z_1), f(z_2)) \leq cd(z_1, z_2). \quad (2)$$

Then, f has a unique fixed point in Ω. That is, there exists a unique $z^* \in \Omega$ such that

$$f(z^*) = z^*. \quad (3)$$

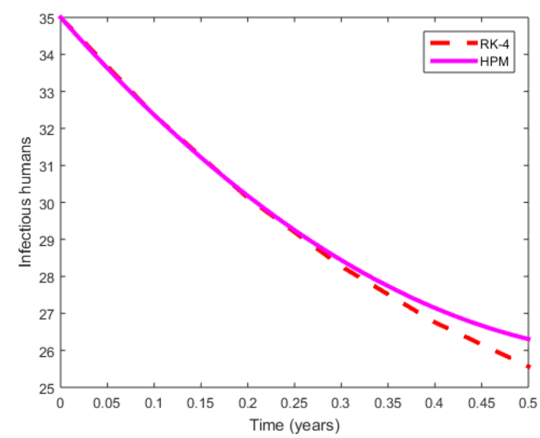

Figure 4. Infectious humans with malaria.

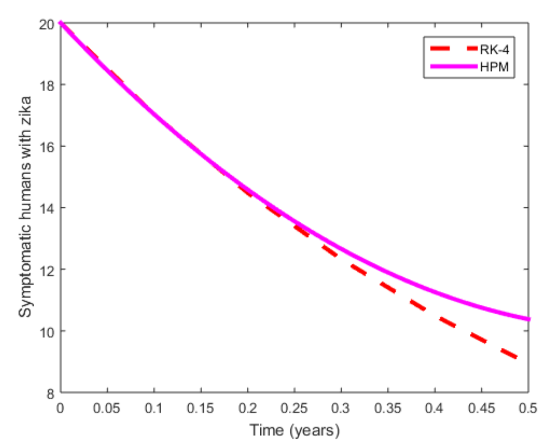

Figure 5. Symptomatic humans with zika.

Lemma 3.1 Lipschitz Continuity. A function $f(t, z)$ is said to be Lipschitz continuous in z if there exist a constant $K \geq 0$ such that

$$|f(t, z_1) - f(t, z_2)| \leq K|z_1 - z_2|. \quad (4)$$

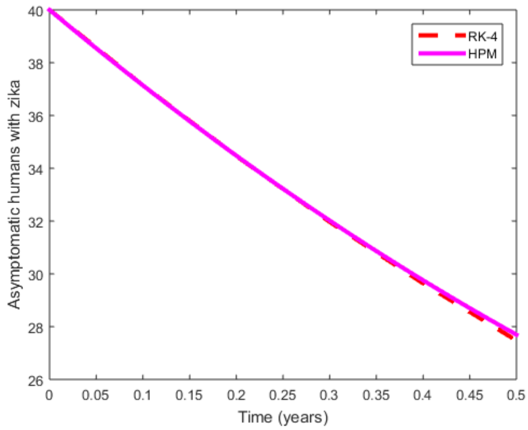

Figure 6. Asymptomatic humans with zika.

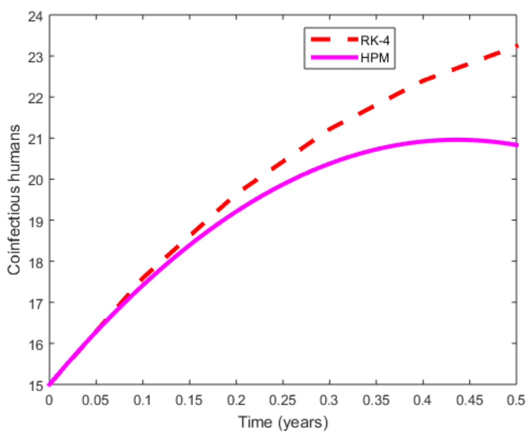

Figure 7. Coinfectious humans.

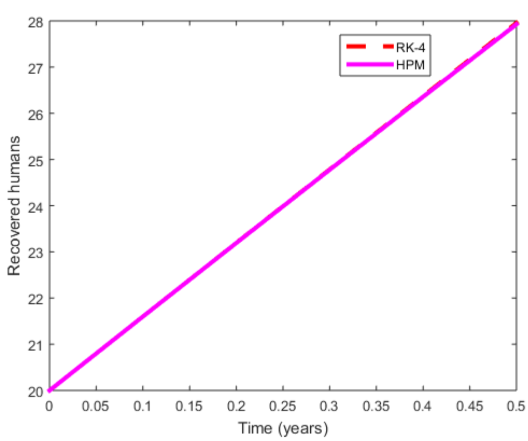

Figure 8. Recovered humans.

Here, K is called a Lipschitz constant and f is said to be K-Lipschitz $\forall z_1, z_2 \in \Omega$.

One of the properties of Lipschitz functions is that every Lipschitz function is absolutely continuous and therefore is differentiable almost everywhere. Also, a function that is differentiable everywhere is Lipschitz continuous if the first derivative is bounded. The existence and uniqueness can be established by either showing that each differential equations in the system is Lipschitz continuous and that the Lipschitz constant satisfy the Banach conditions for existence and uniqueness. In some works, existence and uniqueness was established by showing that the differential equations are continuously differentiable everywhere in $\mathcal{R}$ [12].

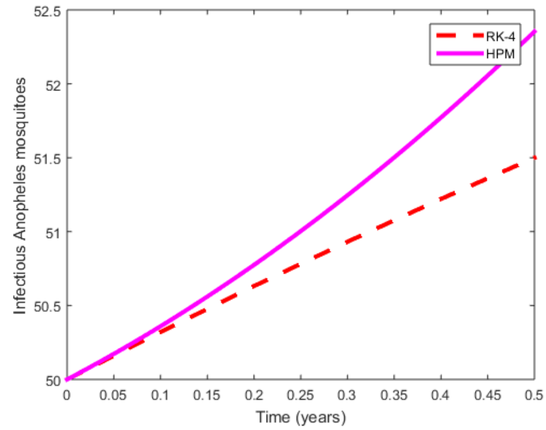

Figure 9. Infectious Anopheles mosquitoes.

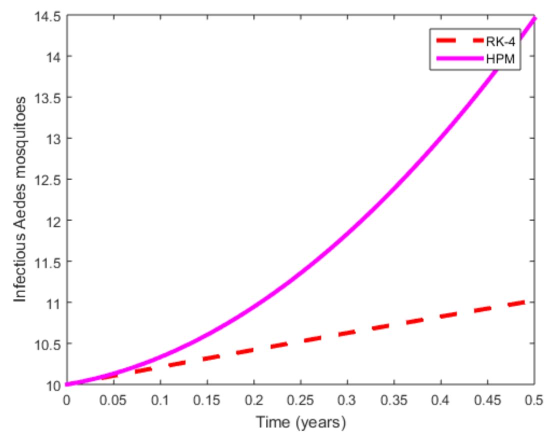

Figure 10. Infectious Aedes mosquitoes.

The co-infection model, (1) can be expressed as

$$Z'(t) = f(Z(t)), \quad Z(t_0) = Z_0, \tag{5}$$

where $Z(t) = (S_h, S_{hu}, S_{hv}, E_{hm}, E_{hz}, E_{hmz}, I_{hm}, I_{hmT}, I_{hmU}, I_{hzS}, I_{hzA}, I_{hzT}, I_{hmz}, I_{hmzT}, I_{hmzU}, R_h, S_{mv}, E_{mv}, I_{mv}, S_{zv}, E_{zv}, I_{zv})$
Let $||.||$ be the maximum norm in $\Omega \in \mathbb{R}^{22}$ taking to be the banach domain for continuous functions where

$$||Z(t)|| = \sum ||Z||_\infty.$$

Let $||S_h|| \leq k_1, ||S_{hu}|| \leq k_2, ||S_{hv}|| \leq k_3, ||E_{hm}|| \leq k_4, ||E_{hz}|| \leq k_5, ||E_{hmz}|| \leq k_6, ||I_{hm}|| \leq k_7, ||I_{hmT}|| \leq k_8, ||I_{hmU}|| \leq k_9, ||I_{hzS}|| \leq k_{10}, ||I_{hzA}|| \leq k_{11}, ||I_{hzT}|| \leq k_{12}, ||I_{hmz}|| \leq k_{13}, ||I_{hmzT}|| \leq k_{14}, ||I_{hmzU}|| \leq k_{15}, ||R_h|| \leq k_{16}, ||S_{mv}|| \leq k_{17}, ||E_{mv}|| \leq k_{18}, ||I_{mv}|| \leq k_{19}, ||S_{zv}|| \leq k_{20}, ||E_{zv}|| \leq k_{21}, ||I_{zv}|| \leq k_{22}, ||M_{SIT}|| \leq k_{23}, ||Z_{SIT}|| \leq k_{24}$ and $0 <$

$m_i < 1$ for $(i = 1, 2, 3, ..., 22)$.
From equation (1) and as was demonstrated in Ref. [13], we will have that for any S_{h1} and $S_{h2} \in \Omega$, then

$$\begin{aligned}||f(t, S_{h1}) - f(t, S_{h2})|| &= ||(\Lambda_h + \theta R_h - (\rho_1 + \rho_2 + \tau_1)S_{h1}) - (\Lambda_h + \theta R_h \\ &- (\rho_1 + \rho_2 + \tau_1)S_{h2})|| = ||(\rho_1 + \rho_2 + \tau_1)(S_{h1} - S_{h2})|| \\ &\leq (\rho_1 + \rho_2 + \tau_1)||S_{h1} - S_{h2}|| \leq m_1||S_{h1} - S_{h2}||.\end{aligned}$$

The Lipschitz continuity in S_h is established with m_1 as the Lipschitz constant. Similarly, we can establish the Lipschitz continuity in some of the other state variables as follows;

$$\begin{aligned}||f(t, S_{hu1}) - f(t, S_{hu2})|| &= ||(\rho_1 S_h - (\alpha_1\beta_1 I_{mv} + \alpha_2\eta_1 I_{zv} + \tau_1)S_{hu1}) \\ &- (\rho_1 S_h - (\alpha_1\beta_1 I_{mv} + \alpha_2\eta_1 I_{zv} + \tau_1)S_{hu2})|| \\ &= ||(\alpha_1\beta_1 I_{mv} + \alpha_2\eta_1 I_{zv} + \tau_1)(S_{hu1} - S_{hu2})|| \\ &\leq (\alpha_1\beta_1||I_{mv}|| + \alpha_2\eta_1||I_{zv}|| + \tau_1)||S_{hu1} - S_{hu2}|| \\ &\leq (\alpha_1\beta_1 k_{19} + \alpha_2\eta_1 k_{22} + \tau_1)||S_{hu1} - S_{hu2}|| \leq m_2||S_{hu1} - S_{hu2}||,\end{aligned}$$

$$\begin{aligned}||f(t, S_{hv1}) - f(t, S_{hv2})|| &= ||(\rho_2 S_h - (\alpha_1\beta_2\Phi I_{mv} + \alpha_2\eta_1 I_{zv} + \tau_1)S_{hv1}) \\ &- (\rho_2 S_h - (\alpha_1\beta_2\Phi I_{mv} + \alpha_2\eta_1 I_{zv} + \tau_1)S_{hv2})|| \\ &= ||(\alpha_1\beta_2\Phi I_{mv} + \alpha_2\eta_1 I_{zv} + \tau_1)(S_{hv1} - S_{hv2})|| \\ &\leq (\alpha_1\beta_2\Phi||I_{mv}|| + \alpha_2\eta_1||I_{zv}|| + \tau_1)||S_{hv1} - S_{hv2}|| \\ &\leq (\alpha_1\beta_2\Phi k_{19} + \alpha_2\eta_1 k_{22} + \tau_1)||S_{hv1} - S_{hv2}|| \leq m_3||S_{hv1} - S_{hv2}||,\end{aligned}$$

$$\begin{aligned}||f(t, E_{hm1}) - f(t, E_{hm2})|| &= ||(\alpha_1\beta_1 I_{mv} S_{hu} + \alpha_1\beta_2\Phi I_{mv} S_{hv} + \phi_1 I_{hmU} \\ &+ \phi_2 I_{mzU} - \alpha_2\eta_1 I_{zv} E_{hm1} - (\delta_1 + \tau_1)E_{hm1}) - (\alpha_1\beta_1 I_{mv} S_{hu} \\ &+ \alpha_1\beta_2\Phi I_{mv} S_{hv} + \phi_1 I_{hmU} + \phi_2 I_{hmzU} - \alpha_2\eta_1 I_{zv} E_{hm2} \\ &- (\delta_1 + \tau_1)E_{hm2})|| = ||(\alpha_2\eta_1 I_{zv} + \delta_1 + \tau_1)(E_{hm1} - E_{hm2})|| \\ &\leq (\alpha_2\eta_1||I_{zv}|| + \delta_1 + \tau_1)||E_{hm1} - E_{hm2}|| \\ &\leq (\alpha_2\eta_1 k_{22} + \delta_1 + \tau_1)||E_{hm1} - E_{hm2}|| \leq m_4||E_{hm1} - E_{hm2}||,\end{aligned}$$

$$\begin{aligned}||f(t, E_{hz1}) - f(t, E_{hz2})|| &= ||(\alpha_2\eta_1 I_{zv}(S_{hu} + S_{hv}) - \alpha_1\beta_1 I_{mv} E_{hz1} \\ &- ((\chi_1 + \chi_2)\delta_2 + \tau_1)E_{hz1}) - (\alpha_2\eta_1 I_{zv}(S_{hu} + S_{hv}) \\ &- \alpha_1\beta_1 I_{mv} E_{hz2} - ((\chi_1 + \chi_2)\delta_2 + \tau_1)E_{hz2})|| = ||(\alpha_1\beta_1 I_{mv} \\ &+ (\chi_1 + \chi_2)\delta_2 + \tau_1)(E_{hz1} - E_{hz2})|| \leq (\alpha_1\beta_1 k_{19} \\ &+ (\chi_1 + \chi_2)\delta_2 + \tau_1)||E_{hz1} - E_{hz2}|| \leq m_5||E_{hz1} - E_{hz2}||,\end{aligned}$$

$$\begin{aligned}||f(t, E_{hmz1}) - f(t, E_{hmz2})|| &= ||(\alpha_1\beta_1 I_{mv} E_{hz} + \alpha_2\eta_1 I_{zv} E_{hm} \\ &- (\delta_3 + \tau_1)E_{hmz1}) - (\alpha_1\beta_1 I_{mv} E_{hz} + \alpha_2\eta_1 I_{zv} E_{hm} \\ &- (\delta_3 + \tau_1)E_{hmz2})|| = ||(\delta_3 + \tau_1)(E_{hmz1} - E_{hmz2})||| \\ &\leq (\delta_3 + \tau_1)||E_{hmz1} - E_{hmz2}|| \leq m_6||E_{hz1} - E_{hz2}||.\end{aligned}$$

Following the same procedure, other Lipschitz conditions can be established as

$$\begin{aligned}||f(t, I_{hm1}) - f(t, I_{hm2})|| &\leq m_7||I_{hm1} - I_{hm2}|| \\ ||f(t, I_{hmT1}) - f(t, I_{hmT2})|| &\leq m_8||I_{hmT1} - I_{hmT2}|| \\ ||f(t, I_{hmU1}) - f(t, I_{hmU2})|| &\leq m_9||I_{hmU1} - I_{hmU2}|| \\ ||f(t, I_{hzS1}) - f(t, I_{hzS2})|| &\leq m_{10}||I_{hzS1} - I_{hzS2}||, \\ ||f(t, I_{hzA1}) - f(t, I_{hzA1})|| &\leq m_{11}||I_{hzA1} - I_{hzA2}|| \\ ||f(t, I_{hzT1}) - f(t, I_{hzT2})|| &\leq m_{12}||I_{hzT1} - I_{hzT2}|| \\ ||f(t, I_{hmz1}) - f(t, I_{hmz2})|| &\leq m_{13}||(I_{hmz1} - I_{hmz2})|| \\ ||f(t, I_{hmzT1}) - f(t, I_{hmzT2})|| &\leq m_{14}||I_{hmT1} - I_{hmT2}|| \\ ||f(t, I_{hmzU1}) - f(t, I_{hmzU2})|| &\leq m_{15}||I_{hmU1} - I_{hmU2}|| \\ ||f(t, R_{h1}) - f(t, R_{h2})|| &\leq m_{16}||R_{h1} - R_{h2}|| \\ ||f(t, S_{mv1}) - f(t, S_{mv2})|| &\leq m_{17}||S_{mv1} - S_{mv2}|| \\ ||f(t, E_{mv1}) - f(t, E_{mv2})|| &\leq m_{18}||E_{mv1} - E_{mv2}|| \\ ||f(t, I_{mv1}) - f(t, I_{mv2})|| &\leq m_{19}||I_{mv1} - I_{mv2}|| \\ ||f(t, S_{zv1}) - f(t, S_{zv2})|| &\leq m_{20}||S_{zv1} - S_{zv2}|| \\ ||f(t, E_{zv1}) - f(t, E_{zv2})|| &\leq m_{21}||E_{zv1} - E_{zv2}|| \\ ||f(t, I_{zv1}) - f(t, I_{zv2})|| &\leq m_{22}||I_{zv1} - I_{zv2}||,\end{aligned}$$

where $m_1 = \rho_1 + \rho_2 + \tau_1$, $m_2 = \alpha_1\beta_1 k_{19} + \alpha_2\eta_1 k_{22} + \tau_1$, $m_3 = \alpha_1\beta_2\Phi k_{19} + \alpha_2\eta_1 k_{22} + \tau_1$, $m_4 = \alpha_2\eta_1 k_{22} + \delta_1 + \tau_1$, $m_5 = \alpha_1\beta_1 k_{19} + (\chi_1 + \chi_2)\delta_2 + \tau_1$, $m_6 = \delta_3 + \tau_1$, $m_7 = \alpha_2\eta_1 k_{19} + \tau_1 + \tau_2 + \epsilon_1 + \epsilon_2$, $m_8 = \gamma_1 + \tau_1 + \tau_2$, $m_9 = \phi_1 + \tau_1 + \tau_2$, $m_{10} = \alpha_1\beta_1 k_{19} + \tau_1 + \tau_3 + \psi + \omega_1$, $m_{11} = \alpha_1\beta_1 k_{19} + \tau_1 + \tau_3 + \omega_3$, $m_{12} = \tau_1 + \tau_3 + \omega_2$, $m_{13} = \tau_1 + \tau_4 + \sigma_1 + \sigma_2$, $m_{14} = \gamma_2 + \tau_1 + \tau_4$, $m_{15} = \phi_2 + \tau_1 + \tau_2$, $m_{16} = \tau_1 + \theta$, $m_{18} = \nu_1 + \mu_m$, $m_{19} = \mu_m$, $m_{21} = \nu_2 + \mu_z$ $m_{17} = \alpha_1(\beta_3 k_7 + \beta_4 k_8 + \beta_5 k_9 + \beta_6 k_{13} + \beta_7 k_{14} + \beta_8 k_{15}) + \kappa_1 k_{23} + \mu_m$, $m_{20} = \alpha_2(\eta_2 k_{10} + \eta_3 k_{11} + \eta_4 k_{12} + \eta_5 k_{13} + \eta_6 k_{14} + \eta_7 k_{15}) + \kappa_2 k_{24} + \mu_z$, $m_{22} = \mu_z$ are the Lipschitz constants.

Thus, the Lipschitz continuity in the state variables have been established with the $m_i^{'}s$, $i = 1, 2, 3, ..., 22$ as the Lipschitz constants.

To establish that Banach fixed point theorem is satisfied, we need to show that the $m_i^{'}s$, $i = 1, 2, 3, ..., 22$ satisfy $0 < m_i < 1$. $m_1, m_6, m_8, m_9, m_{12}, m_{13}, m_{14}, m_{15}, m_{16}, m_{18}, m_{19}, m_{21}$, and $m_{22} < 1$ since the terms appearing there represents fractional outflow from each compartment which must be less than one. The other $m_i^{'}s$ contain some $k^{'}s$ which are the bounds of the infectious classes. To see the nature of these $k^{'}s$, we show that these infectious classes are bounded. For example,

$$\frac{dI_{zv}}{dt} = \nu_2 E_{zv} - \mu_z I_{zv} \implies \frac{dI_{zv}}{dt} + \mu_z I_{zv} = \nu_2 E_{zv}$$

Integrating the resulting differential equation by use of integrating factor method gives the result

$$I_{zv}(t) = \left[\nu_2 \int (E_{zv} e^{\mu_z t})dt + c\right] e^{-\mu_z t}. \quad (6)$$

Since E_{zv} is a function of time, t and the solution usually involves exponential terms, the result of (6) will always contain an exponential term. Hence, as $t \to \infty$, $I_{zv} \to 0$. Hence, $||I_{zt}|| \leq k_{22} = 0$ as $t \to \infty$.

Similarly,

$$I_{mv}(t) = \left[\nu_1 \int (E_{mv} e^{\mu_m t})dt + c\right] e^{-\mu_m t}, \quad (7)$$

and $I_{mv} \to 0$ as $t \to \infty$. All the infectious classes tend to zero as $t \to \infty$. This shows that each of the infectious classes are

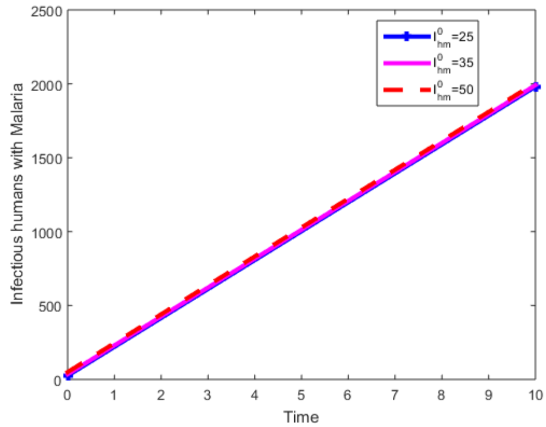

Figure 11. Infectious humans with malaria.

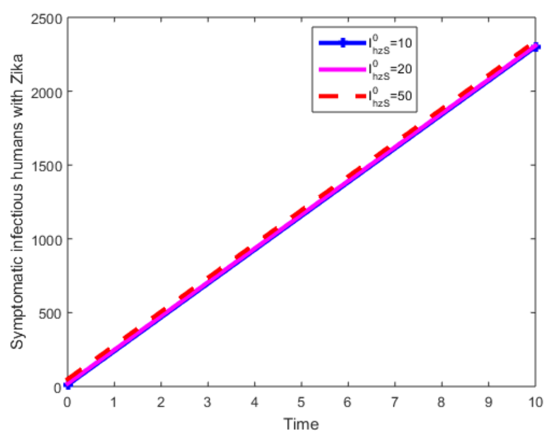

Figure 12. Symptomatic infectious humans with zika.

bounded likewise as I_{mv} and I_{zv}. Hence, we can represent them as $||I_{hm}|| \leq k_7 = 0, ||I_{hmT}|| \leq k_8 = 0, ||I_{hmU}|| \leq k_9 = 0, ||I_{hzS}|| \leq k_{10} = 0, ||I_{hzA}|| \leq k_{11} = 0, ||I_{hzT}|| \leq k_{12} = 0, ||I_{hmz}|| \leq k_{13} = 0, ||I_{hmzT}|| \leq k_{14} = 0, ||I_{hmzU}|| \leq k_{15} = 0, ||I_{mv}|| \leq k_{19} = 0, ||I_{zv}|| \leq k_{22} = 0.$
Substituting these values into the $m_i^{'}s$ gives
$m_2 = \tau_1 < 1, \quad m_3 = \tau_1 < 1, \quad m_4 = \delta_1 + \tau_1 < 1, \quad m_5 = (\chi_1 + \chi_2)\delta_2 + \tau_1 < 1, \quad m_7 = \tau_1 + \tau_2 + \epsilon_1 + \epsilon_2 < 1, \quad m_{10} = \tau_1 + \tau_3 + \psi + \omega_1 < 1, m_{11} = \tau_1 + \tau_3 + \omega_3 < 1, \quad m_{17} = \mu_m < 1, \quad m_{20} = \mu_z < 1.$
Hence, all the $m_i^{'}s$ have satisfied the condition that $0 < m_i < 1$ and by Banach fixed point theorem, the solutions to the system (1) exist and is unique.

3.1. Positivity of solutions and invariant region

Theorem 3.2. Let the initial data set for the model be $S^0_h, S^0_{hu}, S^0_{hv}, E^0_{hm}, E^0_{hz}, E^0_{mz}, I^0_{hm}, I^0_{mz}, I^0_{hmT}, I^0_{hmU}, I^0_{hzS}, I^0_{hzA}, I^0_{hzT}, I^0_{mzT}, I^0_{mzU}, R^0_h, S^0_{mv}, E^0_{mv}, I^0_{mv}, S^0_{zv}, E^0_{zv}$ which are all nonnegative at $t = 0$. Then, the solution $S_h(t), S_{hu}(t), S_{hv}(t), E_{hm}(t), E_{hz}(t), E_{mz}(t), I_{hm}(t), I_{mz}(t), I_{hmT}(t), I_{hmU}(t), I_{hzS}(t), I_{hzA}(t), I_{hzT}(t), I_{mzT}(t), I_{mzU}(t), R_h(t), S_{mv}(t), E_{mv}(t), I_{mv}(t), S_{zv}(t), E_{zv}(t), I_{zv}(t)$ of the system (1) given initial conditions, will remain positive for all $t > 0$.

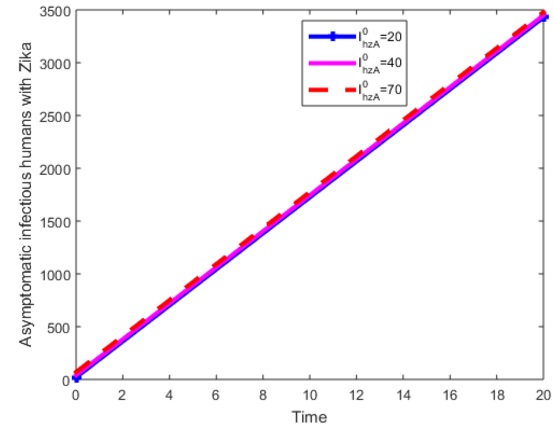

Figure 13. Asymptomatic infectious humans with zika.

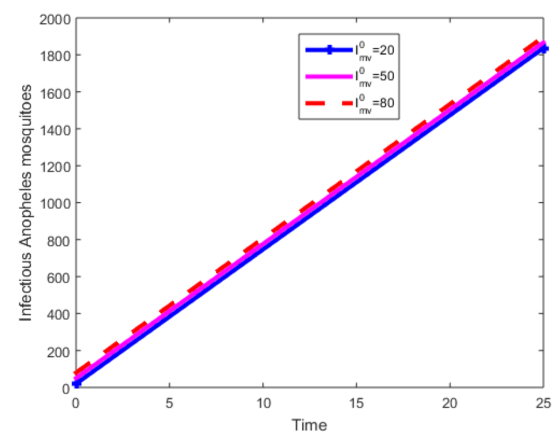

Figure 14. Infectious Anopheles mosquitoes.

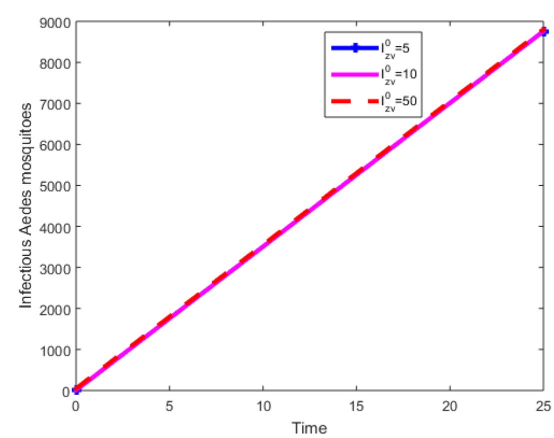

Figure 15. Infectious Aedes mosquitoes.

Proof. From (1), we have the following

$$\frac{dS_h}{dt} \geq -(\rho_1 + \rho_2 + \tau_1)S_h.$$
$$\frac{dS_{hu}}{dt} \geq -(\alpha_1\beta_1 I_{mv} + \alpha_2\eta_1 I_{zv} + \tau_1)S_{hu}.$$

$$\frac{dS_{hv}}{dt} \geq -(\alpha_1\beta_1\varphi I_{mv} + \alpha_2\eta_1 I_{zv} + \tau_1)S_{hv}. \quad (8)$$
$$\vdots$$
$$\frac{dI_{zv}}{dt} \geq -\mu_z I_{zv}.$$

Integrating the system (8) gives

$$\begin{aligned}
S_h(t) &\geq S_h^0 e^{-\int(\rho_1+\rho_2+\tau_1)dt} > 0,\\
S_{hu}(t) &\geq S_{hu}^0 e^{-\int(\alpha_1\beta_1 I_{mv}+\alpha_2\eta_1 I_{zv}+\tau_1)dt} > 0,\\
S_{hv}(t) &\geq S_{hv}^0 e^{-\int(\alpha_1\beta_1\varphi I_{mv}+\alpha_2\eta_1 I_{zv}+\tau_1)dt} > 0,\\
&\vdots\\
I_{zv}(t) &\geq I_{zv}^0 e^{-\int\mu_z dt} > 0.
\end{aligned}$$

Thus, the solution to the system remains positive $\forall t > 0$.

Furthermore, the total human and mosquito populations satisfy the differential equations

$$\begin{aligned}
\frac{dN_h}{dt} &= \Lambda_h - \tau_1 N_h - \tau_2(I_{hm} + I_{hmT} + I_{hmU})\\
&\quad - \tau_3(I_{hzS} + I_{hzA} + I_{hzT}) - \tau_4(I_{mz} + I_{mzT} + I_{mzU})\\
&\leq \Lambda_h - \tau_1 N_h.\\
\frac{dN_{mv}}{dt} &= \Lambda_{mv} - \mu_m I_{mv}. \quad (9)\\
\frac{dN_{zv}}{dt} &= \Lambda_{zv} - \mu_z I_{zv},
\end{aligned}$$

respectively. Integrating (9) and solving as $t \to \infty$ gives; $0 \leq N_{hm} \leq \frac{\Lambda_h}{\tau_1}$, $0 \leq N_{mv} \leq \frac{\Lambda_{mv}}{\mu_m}$ and $0 \leq N_{zv} \leq \frac{\Lambda_{zv}}{\mu_z}$ respectively. Hence, all the solutions of the system are positive and bounded in the region $\Omega = \Omega_1 \times \Omega_2 \times \Omega_3$ and proves that the region, Ω is positively invariant with respect to the flow generated by (9). Thus, the coinfection model (1) is biologically well-posed and defined since all the state variables remain non-negative for all $t > 0$ [14]. The basic mathematical analysis of the system (1) such as obtaining the coinfection-free equilibrium, coinfection reproduction number, stability and sensitivity analyses have been fully discussed in the previous work, see [11].

4. Homotopy perturbation method

The homotopy perturbation method (HPM) was introduced by Jihuan HE in 1998 according to Ref. [15]. HPM is a combination of the traditional perturbation method and homotopy and is based on finding the approximate solution of a nonlinear differential equation as an infinite series in the independent variable, say t. The method has been widely used in mathematical modelling of infectious diseases in finding approximate solutions to the associated models [15–17]. In applying HPM, consider s nonlinear differential equation of the form;

$$A(u) - f(t) = 0, t \in \Omega, \quad (10)$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0,$$

where A is the general nonlinear differential operator, B is the boundary operator, $f(t)$ is a known analytical function and Ω is the domain. The differential operator can be split into two part; the linear part, L and the nonlinear part, N such that Equation (10) can be rewritten as

$$L(u) + N(u) - f(t) = 0, t \in \Omega. \quad (11)$$

A homotopy, $v(t, p) : \Omega \times [0, 1] \to \mathbb{R}$ constructed by homotopy technique satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(t)] = 0, \quad (12)$$

where $p \in [0, 1]$ is the embedding parameter and u_0 is an initial approximate solution satisfying the boundary conditions. The homotopy equation can be rewritten as

$$H(v, p) = L(v) - (1 - p)L(u_0) + p[N(v) - f(t)] = 0. \quad (13)$$

From (13), we have that using the boundary points of $p \in [0, 1]$, then

$$H(v, 0) = L(u) - L(u_0) = 0$$
$$H(v, 1) = A(v) - f(t).$$

The solution to (13) can be represented as a Maclaurin's series in p as

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + ..., \quad (14)$$

such that as $p \to 1$, we get the approximate analytical solution to the system (1). The traditional homotopy perturbation method will be used to obtain the approximate solutions of the co-infection model.

To obtain the approximate solution for our co-infection model by HPM, we make use of two-term approximation for each class and write the solution to (1) in the form;

$$\begin{aligned}
&S_h(t) = S_h^0 + pS_h^1 + p^2 S_h^2, \quad S_{hu}(t) = S_{hu}^0 + pS_{hu}^1 + p^2 S_{hu}^2,\\
&S_{hv}(t) = S_{hv}^0 + pS_{hv}^1 + p^2 S_{hv}^2 \quad E_{hm}(t) = E_{hm}^0 + pE_{hm}^1 + p^2 E_{hm}^2,\\
&E_{hz}(t) = E_{hz}^0 + pE_{hz}^1 + p^2 E_{hz}^2, \quad E_{hmz}(t) = E_{hmz}^0 + pE_{hmz}^1 + p^2 E_{hmz}^2,\\
&I_{hm}(t) = I_{hm}^0 + pI_{hm}^1 + p^2 I_{hm}^2, \quad I_{hmT}(t) = I_{hmT}^0 + pI_{hmT}^1 + p^2 I_{hmT}^2,\\
&I_{hmU}(t) = I_{hmU}^0 + pI_{hmU}^1 + p^2 I_{hmU}^2, \quad I_{hzS}(t) = I_{hzS}^0 + pI_{hzS}^1 + p^2 I_{hzS}^2,\\
&I_{hzA}(t) = I_{hzA}^0 + pI_{hzA}^1 + p^2 I_{hzA}^2, \quad I_{hzT}(t) = I_{hzT}^0 + pI_{hzT}^1 + p^2 I_{hzT}^2,\\
&I_{hmz}(t) = I_{hmz}^0 + pI_{hmz}^1 + p^2 I_{hmz}^2, \quad I_{hmzT}(t) = I_{hmzT}^0 + pI_{hmzT}^1 + p^2 I_{hmzT}^2,\\
&I_{hmzU}(t) = I_{hmzU}^0 + pI_{hmzU}^1 + p^2 I_{hmzU}^2, \quad R_h(t) = R_h^0 + pR_h^1 + p^2 R_h^2,\\
&S_{mv}(t) = S_{mv}^0 + pS_{mv}^1 + p^2 S_{mv}^2, \quad E_{mv}(t) = E_{mv}^0 + pE_{mv}^1 + p^2 E_{mv}^2,\\
&I_{mv}(t) = I_{mv}^0 + pI_{mv}^1 + p^2 I_{mv}^2, \quad S_{zv}(t) = S_{zv}^0 + pS_{zv}^1 + p^2 S_{zv}^2,\\
&E_{zv}(t) = E_{zv}^0 + pE_{zv}^1 + p^2 ES_{zv}^2, \quad I_{zv}(t) = I_{zv}^0 + pI_{zv}^1 + p^2 I_{zv}^2.
\end{aligned}$$

and construct the homotopy for the system (1) as follows

$$\begin{aligned}
\frac{dS_h}{dt} &= p(\Lambda_h + \theta R_h - (\rho_1 + \rho_2 + \tau_1)S_h)\\
\frac{dS_{hu}}{dt} &= p(\rho_1 S_h - (\alpha_1\beta_1 I_{mv} + \alpha_2\eta_1 I_{zv} + \tau_1)S_{hu})\\
\frac{dS_{hv}}{dt} &= p(\rho_2 S_h - (\alpha_1\beta_2 I_{mv}\Phi + \alpha_2\eta_1 I_{zv} + \tau_1)S_{hv})\\
\frac{dE_{hm}}{dt} &= p(\alpha_1\beta_1 I_{mv}S_{hu} + \alpha_1\beta_2 I_{mv}\Phi S_{hv} + \phi_1 I_{hmU} + \phi_2 I_{mzU}
\end{aligned}$$

$$
\begin{aligned}
& -\alpha_2\eta_1 I_{zv}E_{hm} - (\delta_1+\tau_1)E_{hm}) \\
\frac{dE_{hz}}{dt} &= p(\alpha_2\eta_1 I_{zv}(S_{hu}+S_{hv}) - \alpha_1\beta_1 I_{mv}E_{hz} - ((\chi_1+\chi_2)\delta_2+\tau_1)E_{hz}) \\
\frac{dE_{hmz}}{dt} &= p(\alpha_1\beta_1 I_{mv}E_{hz} + \alpha_2\eta_1 I_{zv}E_{hm} - (\delta_3+\tau_1)E_{hmz}) \\
\frac{dI_{hm}}{dt} &= p(\delta_1 E_{hm} - \alpha_2\eta_1 I_{zv}I_{hm} - (\tau_1+\tau_2+\epsilon_1+\epsilon_2)I_{hm}) \\
\frac{dI_{hmT}}{dt} &= p(\epsilon_1 I_{hm} - (\gamma_1+\tau_1+\tau_2)I_{hmT}) \\
\frac{dI_{hmU}}{dt} &= p(\epsilon_2 I_{hm} - (\phi_1+\tau_1+\tau_2)I_{hmU}) \\
\frac{dI_{hzS}}{dt} &= p(\delta_2\chi_1 E_{hz} - \alpha_1\beta_1 I_{mv}I_{hzS} - (\tau_1+\tau_3+\psi+\omega_1)I_{hzS}) \\
\frac{dI_{hzA}}{dt} &= p(\delta_2\chi_2 E_{hz} - \alpha_1\beta_1 I_{mv}I_{hzA} - (\tau_1+\tau_3+\omega_3)I_{hzA}) \\
\frac{dI_{hzT}}{dt} &= p(\psi I_{hzS} - (\tau_1+\tau_3+\omega_2)I_{hzT}) \qquad (15) \\
\frac{dI_{hmz}}{dt} &= p(\alpha_1\beta_1 I_{mv}(I_{hzS}+I_{hzA}) + \alpha_2\eta_1 I_{zv}I_{hm} + \delta_3 E_{hmz} \\
& - (\tau_1+\tau_4+\sigma_1+\sigma_2)I_{hmz}). \\
\frac{dI_{hmzT}}{dt} &= p(\sigma_1 I_{hmz} - (\tau_1+\tau_4+\gamma_2)I_{hmzT}) \\
\frac{dI_{hmzU}}{dt} &= p(\sigma_2 I_{hmz} - (\tau_1+\tau_4+\phi_2)I_{hmzU}) \\
\frac{dR_h}{dt} &= p(\gamma_1 I_{hmT} + \gamma_2 I_{mzT} + \omega_1 I_{hzS} + \omega_2 I_{hzT} + \omega_3 I_{hzA} - (\tau_1+\theta)R_h) \\
\frac{dS_{mv}}{dt} &= p(\Lambda_{mv} - \alpha_1(\beta_3 I_{hm} + \beta_4 I_{hmT} + \beta_5 I_{hmU} + \beta_6 I_{hmz} + \beta_7 I_{hmzT} \\
& + \beta_8 I_{hmzU}) - (\kappa_1 M_{SIT} + \mu_m)S_{mv}) \\
\frac{dE_{mv}}{dt} &= p(\alpha_1(\beta_3 I_{hm} + \beta_4 I_{hmT} + \beta_5 I_{hmU} + \beta_6 I_{hmz} + \beta_7 I_{hmzT} \\
& + \beta_8 I_{hmzU})S_{mv} - (\nu_1+\mu_m)E_{mv}) \\
\frac{dI_{mv}}{dt} &= p(\nu_1 E_{mv} - \mu_m I_{mv}) \\
\frac{dS_{zv}}{dt} &= p(\Lambda_{zv} - \alpha_2(\eta_2 I_{hzS} + \eta_3 I_{hzA} + \eta_4 I_{hzT} + \eta_5 I_{hmz} + \eta_6 I_{hmzT} \\
& + \eta_7 I_{hmzU}) - (\kappa_2 Z_{SIT} + \mu_z)S_{zv}) \\
\frac{dE_{zv}}{dt} &= p(\lambda_z S_{zv} - (\nu_2+\mu_z)E_{zv}) \\
\frac{dI_{zv}}{dt} &= p(\nu_2 E_{zv} - \mu_z I_{zv}),
\end{aligned}
$$

where the coefficients p, p^1 and p^2 are to be determined. If we substitute the two terms approximate solutions of (1) into (15), neglecting terms that will yield powers more than order 2, since we are using two-term approximation, we will have the following by taking $a_1 = \rho_1+\rho_2+\tau_1, B_1 = \delta_1+\tau_1, B_2 = \tau_1+\tau_2+\epsilon_1+\epsilon_2, B_3 = \gamma_1+\tau_1+\tau_2, B_4 = \phi_1+\tau_1+\tau_2, B_5 = \delta_3+\tau_1, B_6 = \tau_1+\theta, B_7 = \nu_1+\mu_m, D_1 = (\chi_1+\chi_2)\delta_2+\tau_1, D_2 = \tau_1+\tau_3+\psi+\omega_1, D_3 = \tau_1+\tau_3+\omega_3, D_4 = \tau_1+\tau_3+\omega_2, D_5 = \tau_1+\tau_4+\sigma_1+\sigma_2, D_6 = \tau_1+\tau_4+\gamma_2, D_7 = \tau_1+\tau_4+\phi_2, D_8 = \nu_2+\mu_z$;

$$
\begin{aligned}
& \frac{dS_h^0}{dt} + p\frac{dS_h^1}{dt} + p^2\frac{dS_h^2}{dt} = p(\Lambda_h + \theta R_h^0 - a_1 S_h^0) + p^2(\theta R_h^1 - a_1 S_h^1) \\
& \frac{dS_{hu}^0}{dt} + p\frac{dS_{hu}^1}{dt} + p^2\frac{dS_{hu}^2}{dt} = p(\rho_1 S_h^0 - \alpha_1\beta_1 I_{mv}^0 S_{hu}^0 - \alpha_2\eta_1 I_{zv}^0 S_{hu}^0 \\
& - \tau_1 S_{hu}^0) + p^2(\rho_1 S_h^1 - \alpha_1\beta_1 I_{mv}^0 S_{hu}^1 - \alpha_1\beta_1 I_{mv}^1 S_{hu}^0 \\
& - \alpha_2\eta_1 I_{zv}^0 S_{hu}^1 - \alpha_2\eta_1 I_{zv}^1 S_{hu}^0 - \tau_1 S_{hu}^1) \\
& \frac{dS_{hv}^0}{dt} + p\frac{dS_{hv}^1}{dt} + p^2\frac{dS_{hv}^2}{dt} = p(\rho_1 S_h^0 - \alpha_1\beta_2\Phi I_{mv}^0 S_{hv}^0 - \alpha_2\eta_1 I_{zv}^0 S_{hv}^0 \\
& - \tau_1 S_{hv}^0) + p^2(\rho_1 S_h^1 - \alpha_1\beta_2\Phi I_{mv}^0 S_{hv}^1 - \alpha_1\beta_2\Phi I_{mv}^1 S_{hu}^0 \\
& - \alpha_2\eta_1 I_{zv}^0 S_{hv}^1 - \alpha_2\eta_1 I_{zv}^1 S_{hv}^0 - \tau_1 S_{hv}^1) \\
& \frac{dE_{hm}^0}{dt} + p\frac{dE_{hm}^1}{dt} + p^2\frac{dE_{hm}^2}{dt} = p(\alpha_1\beta_1 I_{mv}^0 S_{hu}^0 + \alpha_1\beta_2\Phi I_{mv}^0 S_{hv}^0 + \phi_1 I_{hmU}^0 \\
& + \phi_2 I_{hmzU}^0 - \alpha_2\eta_1 I_{zv}^0 E_{hm}^0 - B_1 E_{hm}^0) + p^2(\alpha_1\beta_1 I_{mv}^0 S_{hu}^1 \\
& + \alpha_1\beta_1 I_{mv}^1 S_{hu}^0 + \alpha_1\beta_2\Phi I_{mv}^0 S_{hv}^1 + \alpha_1\beta_2\Phi I_{mv}^1 S_{hv}^0 + \phi_1 I_{hmU}^1 \\
& + \phi_2 I_{hmzU}^1 - \alpha_2\eta_1 I_{zv}^0 E_{hm}^1 - \alpha_2\eta_1 I_{zv}^1 E_{hm}^0 - B_1 E_{hm}^1) \\
& \frac{dE_{hz}^0}{dt} + p\frac{dE_{hz}^1}{dt} + p^2\frac{dE_{hz}^2}{dt} = p(\alpha_2\eta_1 I_{zv}^0 S_{hu}^0 + \alpha_2\eta_1 I_{zv}^0 S_{hv}^0 - (\alpha_1\beta_1 I_{mv}^0 \\
& + D_1)E_{hz}^0) + p^2(\alpha_2\eta_1 I_{zv}^0 S_{hu}^1 + \alpha_2\eta_1 I_{zv}^1 S_{hu}^0 + \alpha_2\eta_1 I_{zv}^0 S_{hv}^1 \\
& + \alpha_2\eta_1 I_{zv}^1 S_{hv}^0 - \alpha_1\beta_1 I_{mv}^0 E_{hz}^1 - \alpha_1\beta_1 I_{mv}^1 E_{hz}^0 - D_1 E_{hz}^1). \\
& \frac{dE_{hmz}^0}{dt} + p\frac{dE_{hmz}^1}{dt} + p^2\frac{dE_{hmz}^2}{dt} = p(\alpha_1\beta_1 I_{mv}^0 E_{hz}^0 + \alpha_2\eta_1 I_{zv}^0 E_{hm}^0 \\
& - B_5 E_{hmz}^0) + p^2(\alpha_1\beta_1 I_{mv}^0 E_{hz}^1 + \alpha_1\beta_1 I_{mv}^1 E_{hz}^0 + \alpha_2\eta_1 I_{zv}^0 E_{hm}^1 \\
& + \alpha_2\eta_1 I_{zv}^1 E_{hm}^0 - B_5 E_{hmz}^1) \\
& \frac{dI_{hm}^0}{dt} + p\frac{dI_{hm}^1}{dt} + p^2\frac{dI_{hm}^2}{dt} = p(\delta_1 E_{hm}^0 - (\alpha_2\eta_1 I_{zv}^0 + B_2)I_{hm}^0) \\
& + p^2(\delta_1 E_{hm}^1 - \alpha_2\eta_1 I_{zv}^0 I_{hm}^1 - \alpha_2\eta_1 I_{zv}^1 I_{hm}^0 - B_2 I_{hm}^1) \\
& \frac{dI_{hmT}^0}{dt} + p\frac{dI_{hmT}^1}{dt} + p^2\frac{dI_{hmT}^2}{dt} = p(\epsilon_1 I_{hm}^0 - B_3 I_{hmT}^0) + p^2(\epsilon_1 I_{hm}^1 - B_3 I_{hmT}^1) \\
& \frac{dI_{hmU}^0}{dt} + p\frac{dI_{hmU}^1}{dt} + p^2\frac{dI_{hmU}^2}{dt} = p(\epsilon_2 I_{hm}^0 - B_4 I_{hmU}^0) + p^2(\epsilon_1 I_{hm}^1 - B_4 I_{hmU}^1) \\
& \frac{dI_{hzS}^0}{dt} + p\frac{dI_{hzS}^1}{dt} + p^2\frac{dI_{hzS}^2}{dt} = p(\delta_2\chi_1 E_{hz}^0 - D_2 I_{hzS}^0 - \alpha_1\beta_1 I_{mv}^0 I_{hzS}^0) \\
& + p^2(\delta_2\chi_1 E_{hz}^1 - D_2 I_{hzS}^1 - \alpha_1\beta_1 I_{mv}^0 I_{hzS}^1 - \alpha_1\beta_1 I_{mv}^1 I_{hzS}^0) \\
& \frac{dI_{hzA}^0}{dt} + p\frac{dI_{hzA}^1}{dt} + p^2\frac{dI_{hzA}^2}{dt} = p(\delta_2\chi_2 E_{hz}^0 - D_3 I_{hzA}^0 - \alpha_1\beta_1 I_{mv}^0 I_{hzA}^0) \\
& + p^2(\delta_2\chi_2 E_{hz}^1 - D_3 I_{hzA}^1 - \alpha_1\beta_1 I_{mv}^0 I_{hzA}^1 - \alpha_1\beta_1 I_{mv}^1 I_{hzA}^0) \\
& \frac{dI_{hzT}^0}{dt} + p\frac{dI_{hzT}^1}{dt} + p^2\frac{dI_{hzT}^2}{dt} = p(\psi I_{hzS}^0 - D_4 I_{hzT}^0) + p^2(\psi I_{hzS}^1 - D_4 I_{hzT}^1) \\
& \frac{dI_{hmz}^0}{dt} + p\frac{dI_{hmz}^1}{dt} + p^2\frac{dI_{hmz}^2}{dt} = p(\alpha_1\beta_1 I_{mv}^0 I_{hzS}^0 + \alpha_1\beta_1 I_{mv}^0 I_{hzA}^0 \\
& + \alpha_2\eta_1 I_{zv}^0 I_{hm}^0 + \delta_3 E_{hmz}^0 - D_5 I_{hmz}^0) + p^2(\alpha_1\beta_1 I_{mv}^0 I_{hzS}^1 + \alpha_1\beta_1 I_{mv}^1 I_{hzS}^0 \\
& + \alpha_1\beta_1 I_{mv}^0 I_{hzA}^1 + \alpha_1\beta_1 I_{mv}^1 I_{hzA}^0 + \alpha_1\beta_1 I_{mv}^0 I_{hzA}^1 + \alpha_1\beta_1 I_{mv}^1 I_{hzA}^0 \\
& + \delta_3 E_{hmz}^1 - D_5 I_{hmz}^1) \\
& \frac{dI_{hmzT}^0}{dt} + p\frac{dI_{hmzT}^1}{dt} + p^2\frac{dI_{hmzT}^2}{dt} = p(\sigma_1 I_{hmz}^0 - D_6 I_{hmzT}^0) \\
& + p^2(\sigma_1 I_{hmz}^1 - D_6 I_{hmzT}^1) \\
& \frac{dI_{hmzU}^0}{dt} + p\frac{dI_{hmzU}^1}{dt} + p^2\frac{dI_{hmzU}^2}{dt} = p(\sigma_2 I_{hmz}^0 - D_7 I_{hmzU}^0) \\
& + p^2(\sigma_2 I_{hmz}^1 - D_7 I_{hmzU}^1) \\
& \frac{dR_h^0}{dt} + p\frac{dR_h^1}{dt} + p^2\frac{dR_h^2}{dt} = p(\gamma_1 I_{hmT}^0 + \gamma_2 I_{hmzT}^0 + \omega_1 I_{hzS}^0 + \omega_2 I_{hzT}^0 \\
& + \omega_3 I_{hzA}^0 - B_6 R_h^0) + p^2(\gamma_1 I_{hmT}^1 + \gamma_2 I_{hmzT}^1 + \omega_1 I_{hzS}^1 + \omega_2 I_{hzT}^1 \\
& + \omega_3 I_{hzA}^1 - B_6 R_h^1) \\
& \frac{dS_{mv}^0}{dt} + p\frac{dS_{mv}^1}{dt} + p^2\frac{dS_{mv}^2}{dt} = p(\Lambda_{mv} - \alpha_1\beta_3 I_{hm}^0 S_{mv}^0 - \alpha_1\beta_4 I_{hmT}^0 S_{mv}^0 \\
& - \alpha_1\beta_5 I_{hmU}^0 S_{mv}^0 - \alpha_1\beta_6 I_{hmz}^0 S_{mv}^0 - \alpha_1\beta_7 I_{hmzT}^0 S_{mv}^0 - \alpha_1\beta_8 I_{hmzU}^0 S_{mv}^0 \\
& - (\kappa_1 M_{SIT} + \mu_m)S_{mv}^0) - p^2(\alpha_1\beta_3 I_{hm}^0 S_{mv}^1 + \alpha_1\beta_3 I_{hm}^1 S_{mv}^0
\end{aligned}
$$

$$
\begin{aligned}
&+ \alpha_1\beta_4 I^0_{hmT}S^1_{mv} + \alpha_1\beta_4 I^1_{hmT}S^0_{mv} + \alpha_1\beta_5 I^0_{hmU}S^1_{mv} + \alpha_1\beta_5 I^1_{hmU}S^0_{mv}\\
&+ \alpha_1\beta_6 I^0_{hmz}S^1_{mv} + \alpha_1\beta_6 I^1_{hmz}S^0_{mv} + \alpha_1\beta_7 I^0_{hmzT}S^1_{mv} + \alpha_1\beta_7 I^1_{hmzT}S^0_{mv}\\
&+ \alpha_1\beta_8 I^0_{hmzU}S^1_{mv} + \alpha_1\beta_8 I^1_{hmzU}S^0_{mv} - (\kappa_1 M_{SIT} + \mu_m)S^1_{mv})
\end{aligned}
$$

$$
\begin{aligned}
\frac{dE^0_{mv}}{dt} + p\frac{dE^1_{mv}}{dt} + p^2\frac{dE^2_{mv}}{dt} &= p(\alpha_1\beta_3 I^0_{hm}S^0_{mv} + \alpha_1\beta_4 I^0_{hmT}S^0_{mv}\\
&+ \alpha_1\beta_5 I^0_{hmU}S^0_{mv} + \alpha_1\beta_6 I^0_{hmz}S^0_{mv} + \alpha_1\beta_7 I^0_{hmzT}S^0_{mv}\\
&+ \alpha_1\beta_8 I^0_{hmzU}S^0_{mv} - B_7E^0_{mv}) + p^2(\alpha_1\beta_3 I^0_{hm}S^1_{mv} + \alpha_1\beta_3 I^1_{hm}S^0_{mv}.\\
&+ \alpha_1\beta_4 I^0_{hmT}S^1_{mv} + \alpha_1\beta_4 I^1_{hmT}S^0_{mv} + \alpha_1\beta_5 I^0_{hmU}S^1_{mv}\\
&+ \alpha_1\beta_5 I^1_{hmU}S^0_{mv} + \alpha_1\beta_6 I^0_{hmz}S^1_{mv} + \alpha_1\beta_6 I^1_{hmz}S^0_{mv}\\
&+ \alpha_1\beta_7 I^0_{hmzT}S^1_{mv} + \alpha_1\beta_7 I^1_{hmzT}S^0_{mv} + \alpha_1\beta_8 I^0_{hmzU}S^1_{mv}\\
&+ \alpha_1\beta_8 I^1_{hmzU}S^0_{mv} - B_7E^1_{mv})
\end{aligned}
$$

$$
\frac{dI^0_{mv}}{dt} + p\frac{dI^1_{mv}}{dt} + p^2\frac{dI^2_{mv}}{dt} = p(\nu_1E^0_{mv} - \mu_m I^0_{mv}) + p^2(\nu_1E^1_{mv} - \mu_m I^1_{mv})
$$

$$
\begin{aligned}
\frac{dS^0_{zv}}{dt} + p\frac{dS^1_{zv}}{dt} + p^2\frac{dS^2_{zv}}{dt} &= p(\Lambda_{zv} - \alpha_2\eta_2 I^0_{hzS}S^0_{zv} - \alpha_2\eta_3 I^0_{hzA}S^0_{zv}\\
&- \alpha_2\eta_4 I^0_{hzT}S^0_{zv} - \alpha_2\eta_5 I^0_{hmz}S^0_{zv} - \alpha_2\eta_6 I^0_{hmzT}S^0_{zv} - \alpha_2\eta_7 I^0_{hmzU}S^0_{zv}\\
&- (\kappa_2 Z_{SIT} + \mu_z)S^0_{zv}) - p^2(\alpha_2\eta_2 I^0_{hzS}S^1_{zv} + \alpha_2\eta_2 I^1_{hzS}S^0_{zv}\\
&+ \alpha_2\eta_3 I^0_{hzA}S^1_{zv} + \alpha_2\eta_3 I^1_{hzA}S^0_{zv} + \alpha_2\eta_4 I^0_{hzT}S^1_{zv} + \alpha_2\eta_4 I^1_{hzT}S^0_{zv}\\
&+ \alpha_2\eta_5 I^1_{hmz}S^0_{zv} + \alpha_2\eta_5 I^0_{hmz}S^1_{zv} + \alpha_2\eta_6 I^0_{hmzT}S^1_{zv} + \alpha_2\eta_6 I^1_{hmzT}S^0_{zv}\\
&+ \alpha_2\eta_7 I^0_{hmzU}S^1_{zv} + \alpha_2\eta_7 I^1_{hmzU}S^0_{zv} - (\kappa_2 Z_{SIT} + \mu_z)S^1_{zv})
\end{aligned}
$$

$$
\begin{aligned}
\frac{dE^0_{zv}}{dt} + p\frac{dE^1_{zv}}{dt} + p^2\frac{dE^2_{zv}}{dt} &= p(\alpha_2\eta_2 I^0_{hzS}S^0_{zv} + \alpha_2\eta_3 I^0_{hzA}S^0_{mv}\\
&+ \alpha_2\beta_4 I^0_{hzT}S^0_{zv} + \alpha_2\eta_5 I^0_{hmz}S^0_{zv} + \alpha_2\eta_6 I^0_{hmzT}S^0_{zv} + \alpha_2\eta_7 I^0_{hmzU}S^0_{zv}\\
&- D_8E^0_{zv}) + p^2(\alpha_2\eta_2 I^0_{hzS}S^1_{zv} + \alpha_2\eta_2 I^1_{hzS}S^0_{zv} + \alpha_2\eta_3 I^0_{hzA}S^1_{zv}\\
&+ \alpha_2\eta_3 I^1_{hzA}S^0_{zv} + \alpha_2\eta_4 I^0_{hzT}S^1_{zv} + \alpha_2\eta_4 I^1_{hzT}S^0_{zv} + \alpha_2\eta_5 I^0_{hmz}S^1_{zv}\\
&+ \alpha_2\eta_5 I^1_{hmz}S^0_{zv} + \alpha_2\eta_6 I^0_{hmzT}S^1_{zv} + \alpha_2\eta_6 I^1_{hmzT}S^0_{zv} + \alpha_2\eta_7 I^0_{hmzU}S^1_{zv}\\
&+ \alpha_2\eta_7 I^1_{hmzU}S^0_{zv} - D_8E^1_{zv})
\end{aligned}
$$

$$
\frac{dI^0_{zv}}{dt} + p\frac{dI^1_{zv}}{dt} + p^2\frac{dI^2_{zv}}{dt} = p(\nu_2E^0_{zv} - \mu_z I^0_{zv}) + p^2(\nu_2E^1_{zv} - \mu_z I^1_{zv}).
$$

By comparing coefficients on the right-hand side (RHS) and left-hand side (LHS), we have that for p^0

$$
\begin{aligned}
&\frac{dS^0_h}{dt} = 0,\quad \frac{dS^0_{hu}}{dt} = 0,\quad \frac{dS^0_{hv}}{dt} = 0,\quad \frac{dE^0_{hm}}{dt} = 0,\quad \frac{dE^0_{hz}}{dt} = 0,\\
&\frac{dE^0_{hmz}}{dt} = 0,\quad \frac{dI^0_{hm}}{dt} = 0,\quad \frac{dI^0_{hmT}}{dt} = 0,\quad \frac{dI^0_{hmU}}{dt} = 0,\quad \frac{dI^0_{hzS}}{dt} = 0,\\
&\frac{dI^0_{hzA}}{dt} = 0,\quad \frac{dI^0_{hzT}}{dt} = 0,\quad \frac{dI^0_{hmz}}{dt} = 0,\quad \frac{dI^0_{hmzT}}{dt} = 0,\quad \frac{dI^0_{hmzU}}{dt} = 0,\\
&\frac{dR^0_h}{dt} = 0,\quad \frac{dS^0_{mv}}{dt} = 0,\quad \frac{dE^0_{mv}}{dt} = 0,\quad \frac{dI^0_{mv}}{dt} = 0,\quad \frac{dS^0_{zv}}{dt} = 0,\\
&\frac{dE^0_{zv}}{dt} = 0,\quad \frac{dI^0_{zv}}{dt} = 0.
\end{aligned}
$$

Solving the resulting equations for p^0 by direct integration gives

$$
\begin{aligned}
&S_h(0) = S^0_h,\quad S_{hu}(0) = S^0_{hu},\quad S_{hv}(0) = S^0_{hv},\quad E_{hm}(0) = E^0_{hm}\\
&E_{hz}(0) = E^0_{hz},\quad E_{hmz}(0) = E^0_{hmz},\quad I_{hm}(0) = I^0_{hm} =,\quad I_{hmT}(0) = I^0_{hmT},\\
&I_{hmU}(0) = I^0_{hmU},\quad I_{hzS}(0) = I^0_{hzS},\quad I_{hzA}(0) = I^0_{hzA},\quad I_{hzT}(0) = I^0_{hzT},\\
&I_{hmz}(0) = I^0_{hmz},\ I_{hmzT}(0) = I^0_{hmzT},\quad I_{hmzU}(0) = I^0_{hmzU},\quad R_h(0) = R^0_h,\\
&S_{mv}(0) = S^0_{mv},\quad E_{mv}(0) = E^0_{mv},\quad I_{mv}(0) = I^0_{mv}, S_{zv}(0) = S^0_{zv},\\
&E_{zv}(0) = E^0_{zv},\quad I_{zv}(0) = I^0_{zv}.
\end{aligned}
$$

For p^1, we have

$$
\begin{aligned}
\frac{dS^1_h}{dt} &= \Lambda_h + \theta R^0_h - a_1S^0_h,\\
\frac{dS^1_{hu}}{dt} &= \rho_1S^0_h - \alpha_1\beta_1 I^0_{mv}S^0_{hu} - \alpha_2\eta_1 I^0_{zv}S^0_{hu} - \tau_1S^0_{hu},\\
\frac{dS^1_{hv}}{dt} &= \rho_2S^0_h - \alpha_1\beta_2\Phi I^0_{mv}S^0_{hv} - \alpha_2\eta_1 I^0_{zv}S^0_{hv} - \tau_1S^0_{hv},\\
\frac{dE^1_{hm}}{dt} &= \alpha_1\beta_1 I^0_{mv}S^0_{hu} + \alpha_1\beta_2\Phi I^0_{mv}S^0_{hv} + \phi_1 I^0_{hmU} + \phi_2 I^0_{hmzU},\\
&- \alpha_2\eta_1 I^0_{zv}E^0_{hm} - B_1E^0_{hm},\\
\frac{dE^1_{hz}}{dt} &= \alpha_2\eta_1 I^0_{zv}S^0_{hu} + \alpha_2\eta_1 I^0_{zv}S^0_{hv} - \alpha_1\beta_1 I^0_{mv}E^0_{hz} - D_1E^0_{hz},\\
\frac{dE^1_{hmz}}{dt} &= \alpha_1\beta_1 I^0_{mv}E^0_{hz} + \alpha_2\eta_1 I^0_{zv}E^0_{hm} - B_5E^0_{hmz},\\
\frac{dI^1_{hm}}{dt} &= \delta_1E^0_{hm} - (\alpha_2\eta_1 I^0_{zv} + B_2)I^0_{hm},\quad \frac{dI^1_{hmT}}{dt} = \epsilon_1 I^0_{hm} - B_3I^0_{hmT},\\
\frac{dI^1_{hmU}}{dt} &= \epsilon_2 I^0_{hm} - B_4I^0_{hmT},\quad \frac{dI^1_{hzS}}{dt} = \delta_2\chi_1E^0_{hz} - D_2I^0_{hzS} - \alpha_1\beta_1 I^0_{mv}I^0_{hzS},\\
\frac{dI^1_{hzA}}{dt} &= \delta_2\chi_2E^0_{hz} - D_3I^0_{hzA} - \alpha_1\beta_1 I^0_{mv}I^0_{hzA},\quad \frac{dI^1_{hzT}}{dt} = \psi I^0_{hzS} - D_4I^0_{hzT},\\
\frac{dI^1_{hmz}}{dt} &= \alpha_1\beta_1 I^0_{mv}I^0_{hzS} + \alpha_1\beta_1 I^0_{mv}I^0_{hzA} + \alpha_2\eta_1 I^0_{zv}I^0_{hm} + \delta_3E^0_{hmz},\\
&- D_5I^0_{hmz}\\
\frac{dI^1_{hmzT}}{dt} &= \sigma_1 I^0_{hmz} - D_6I^0_{hmzT},\quad \frac{dI^1_{hmzU}}{dt} = \sigma_2 I^0_{hmz} - D_7I^0_{hmzU},\\
\frac{dR^1_h}{dt} &= \gamma_1 I^0_{hmT} + \gamma_2 I^0_{hmzT} + \omega_1 I^0_{hzS} + \omega_2 I^0_{hzT} + \omega_3 I^0_{hzA} - (\tau_1 + \theta)R^0_h,\\
\frac{dS^1_{mv}}{dt} &= \Lambda_{mv} - \alpha_1\beta_3 I^0_{hm}S^0_{mv} - \alpha_1\beta_4 I^0_{hmT}S^0_{mv} - \alpha_1\beta_5 I^0_{hmU}S^0_{mv},\\
&+ \alpha_1\beta_6 I^0_{hmz}S^0_{mv} + \alpha_1\beta_7 I^0_{hmzT}S^0_{mv} + \alpha_1\beta_8 I^0_{hmzU}S^0_{mv}\\
&- (\kappa_1 M_{SIT} + \mu_m)S^0_{mv},\\
\frac{dE^1_{mv}}{dt} &= \alpha_1\beta_3 I^0_{hm}S^0_{mv} + \alpha_1\beta_4 I^0_{hmT}S^0_{mv} + \alpha_1\beta_5 I^0_{hmU}S^0_{mv},\\
&+ \alpha_1\beta_6 I^0_{hmz}S^0_{mv} + \alpha_1\beta_7 I^0_{hmzT}S^0_{mv} + \alpha_1\beta_8 I^0_{hmzU}S^0_{mv} - (\nu_1 + \mu_m)E^0_{mv},\\
\frac{dI^1_{mv}}{dt} &= \nu_1E^0_{mv} - \mu_m I^0_{mv},\\
\frac{dS^1_{zv}}{dt} &= \Lambda_{zv} - \alpha_2\eta_2 I^0_{hzS}S^0_{zv} - \alpha_2\eta_3 I^0_{hzA}S^0_{zv} - \alpha_2\eta_4 I^0_{hzT}S^0_{zv},\\
&- \alpha_2\eta_5 I^0_{hmz}S^0_{zv} - \alpha_2\eta_6 I^0_{hmzT}S^0_{zv} - \alpha_2\eta_7 I^0_{hmzU}S^0_{zv}\\
&- (\kappa_2 Z_{SIT} + \mu_z)S^0_{zv},\\
\frac{dE^1_{zv}}{dt} &= \alpha_2\eta_2 I^0_{hzS}S^0_{zv} + \alpha_2\eta_3 I^0_{hzA}S^0_{mv} + \alpha_2\beta_4 I^0_{hzT}S^0_{zv},\\
&+ \alpha_2\eta_5 I^0_{hmz}S^0_{zv} + \alpha_2\eta_6 I^0_{hmzT}S^0_{zv} + \alpha_2\eta_7 I^0_{hmzU}S^0_{zv} - (\nu_2 + \mu_z)E^0_{zv},\\
\frac{dI^1_{zv}}{dt} &= \nu_2E^0_{zv} - \mu_z I^0_{zv}.
\end{aligned}
$$

For p^2, we have

$$
\begin{aligned}
\frac{dS^2_h}{dt} &= \theta R^1_h - a_1S^1_h,\\
\frac{dS^2_{hu}}{dt} &= \rho_1S^1_h - \alpha_1\beta_1 I^0_{mv}S^1_{hu} - \alpha_1\beta_1 I^1_{mv}S^0_{hu} - \alpha_2\eta_1 I^0_{zv}S^1_{hu}\\
&- \alpha_2\eta_1 I^1_{zv}S^0_{hu} - \tau_1S^1_{hu},\\
\frac{dS^2_{hv}}{dt} &= \rho_2S^1_h - \alpha_1\beta_2\Phi I^0_{mv}S^1_{hv} - \alpha_1\beta_2\Phi I^1_{mv}S^0_{hu} - \alpha_2\eta_1 I^0_{zv}S^1_{hv}
\end{aligned}
$$

$$
\begin{aligned}
&- \alpha_2\eta_1 I^1_{zv} S^0_{hv} - \tau_1 S^1_{hv},\\
\frac{dE^2_{hm}}{dt} &= \alpha_1\beta_1 I^0_{mv} S^1_{hu} + \alpha_1\beta_1 I^1_{mv} S^0_{hu} + \alpha_1\beta_2\Phi I^0_{mv} S^1_{hv} + \alpha_1\beta_2\Phi I^1_{mv} S^0_{hv}\\
&+ \phi_1 I^1_{hmU} + \phi_2 I^1_{hmzU} - \alpha_2\eta_1 I^0_{zv} E^1_{hm} - \alpha_2\eta_1 I^1_{zv} E^0_{hm} - D_1 E^1_{hm},\\
\frac{dE^2_{hz}}{dt} &= \alpha_2\eta_1 I^0_{zv} S^1_{hu} + \alpha_2\eta_1 I^1_{zv} S^0_{hu} + \alpha_2\eta_1 I^0_{zv} S^1_{hv} + \alpha_2\eta_1 I^1_{zv} S^0_{hv}\\
&- \alpha_1\beta_1 I^0_{mv} E^1_{hz} - \alpha_1\beta_1 I^1_{mv} E^0_{hz} - D_1 E^1_{hz},\\
\frac{dE^2_{hmz}}{dt} &= \alpha_1\beta_1 (I^0_{mv} E^1_{hz} + I^1_{mv} E^0_{hz}) + \alpha_2\eta_1 (I^0_{zv} E^1_{hm} + I^1_{zv} E^0_{hm}) - B_5 E^1_{hmz},\\
\frac{dI^2_{hm}}{dt} &= \delta_1 E^1_{hm} - \alpha_2\eta_1 I^0_{zv} I^1_{hm} - \alpha_2\eta_1 I^1_{zv} I^0_{hm} - B_2 I^1_{hm},\\
\frac{dI^2_{hmT}}{dt} &= \epsilon_1 I^1_{hm} - B_3 I^1_{hmT}, \quad \frac{dI^2_{hmU}}{dt} = \epsilon_2 I^1_{hm} - B_4 I^1_{hmT},\\
\frac{dI^2_{hzS}}{dt} &= \delta_2\chi_1 E^1_{hz} - D_2 I^1_{hzS} - \alpha_1\beta_1 I^0_{mv} I^1_{hzS} - \alpha_1\beta_1 I^1_{mv} I^0_{hzS},\\
\frac{dI^2_{hzA}}{dt} &= \delta_2\chi_2 E^1_{hz} - D_3 I^1_{hzA} - \alpha_1\beta_1 I^0_{mv} I^1_{hzA} - \alpha_1\beta_1 I^1_{mv} I^0_{hzA},\\
\frac{dI^2_{hzT}}{dt} &= \psi I^1_{hzS} - D_4 I^1_{hzT},\\
\frac{dI^2_{hmz}}{dt} &= \alpha_1\beta_1 I^0_{mv} I^1_{hzS} + \alpha_1\beta_1 I^1_{mv} I^0_{hzS} + \alpha_1\beta_1 I^0_{mv} I^1_{hzA} + \alpha_1\beta_1 I^1_{mv} I^0_{hzA}\\
&+ \alpha_1\beta_1 I^0_{mv} I^1_{hzA} + \alpha_1\beta_1 I^1_{mv} I^0_{hzA} + \delta_3 E^1_{hmz} - D_5 I^1_{hmz},\\
\frac{dI^2_{hmzT}}{dt} &= \sigma_1 I^1_{hmz} - D_6 I^1_{hmzT}, \quad \frac{dI^2_{hmzU}}{dt} = \sigma_2 I^1_{hmz} - D_7 I^1_{hmzU},\\
\frac{dR^2_h}{dt} &= \gamma_1 I^1_{hmT} + \gamma_2 I^1_{hmzT} + \omega_1 I^1_{hzS} + \omega_2 I^1_{hzT} + \omega_3 I^1_{hzA} - B_6 R^1_h,\\
\frac{dS^2_{mv}}{dt} &= -\alpha_1\beta_3 I^0_{hm} S^1_{mv} - \alpha_1\beta_3 I^1_{hm} S^0_{mv} - \alpha_1\beta_4 I^0_{hmT} S^1_{mv}\\
&- \alpha_1\beta_4 I^1_{hmT} S^0_{mv} - \alpha_1\beta_5 I^0_{hmU} S^1_{mv} - \alpha_1\beta_5 I^1_{hmU} S^0_{mv}\\
&- \alpha_1\beta_6 I^0_{hmz} S^1_{mv} - \alpha_1\beta_6 I^1_{hmz} S^0_{mv} - \alpha_1\beta_7 I^0_{hmzT} S^1_{mv}\\
&- \alpha_1\beta_7 I^1_{hmzT} S^0_{mv} - \alpha_1\beta_8 I^0_{hmzU} S^1_{mv} - \alpha_1\beta_8 I^1_{hmzU} S^0_{mv}\\
&- (\kappa_1 M_{SIT} + \mu_m) S^1_{mv}\\
\frac{dE^2_{mv}}{dt} &= \alpha_1\beta_3 I^0_{hm} S^1_{mv} + \alpha_1\beta_3 I^1_{hm} S^0_{mv} + \alpha_1\beta_4 I^0_{hmT} S^1_{mv} + \alpha_1\beta_4 I^1_{hmT} S^0_{mv}\\
&+ \alpha_1\beta_5 I^0_{hmU} S^1_{mv} + \alpha_1\beta_5 I^1_{hmU} S^0_{mv} + \alpha_1\beta_6 I^0_{hmz} S^1_{mv} + \alpha_1\beta_6 I^1_{hmz} S^0_{mv}\\
&+ \alpha_1\beta_7 I^0_{hmzT} S^1_{mv} + \alpha_1\beta_7 I^1_{hmzT} S^0_{mv} + \alpha_1\beta_8 I^0_{hmzU} S^1_{mv}\\
&+ \alpha_1\beta_8 I^1_{hmzU} S^0_{mv} - (\nu_1 + \mu_m) E^1_{mv}\\
\frac{dI^2_{mv}}{dt} &= \nu_1 E^1_{mv} - \mu_m I^1_{mv}.\\
\frac{dS^2_{zv}}{dt} &= \alpha_2\eta_2 I^0_{hzS} S^1_{zv} + \alpha_2\eta_2 I^1_{hzS} S^0_{zv} + \alpha_2\eta_3 I^0_{hzA} S^1_{zv} + \alpha_2\eta_3 I^1_{hzA} S^0_{zv}\\
&+ \alpha_2\eta_4 I^0_{hzT} S^1_{zv} + \alpha_2\eta_4 I^1_{hzT} S^0_{zv} \alpha_2\eta_5 I^0_{hmz} S^1_{zv} + \alpha_2\eta_5 I^1_{hmz} S^0_{zv}\\
&+ \alpha_2\eta_6 I^0_{hmzT} S^1_{zv} + \alpha_2\eta_6 I^1_{hmzT} S^0_{zv} + \alpha_2\eta_7 I^0_{hmzU} S^1_{zv} + \alpha_2\eta_7 I^1_{hmzU} S^0_{zv}\\
&- (\kappa_2 Z_{SIT} + \mu_z) S^1_{zv}\\
\frac{dE^2_{zv}}{dt} &= \alpha_2\eta_2 I^0_{hzS} S^1_{zv} + \alpha_2\eta_2 I^1_{hzS} S^0_{zv} + \alpha_2\eta_3 I^0_{hzA} S^1_{zv} + \alpha_2\eta_3 I^1_{hzA} S^0_{zv}\\
&+ \alpha_2\eta_4 I^0_{hzT} S^1_{zv} + \alpha_2\eta_4 I^1_{hzT} S^0_{zv} + \alpha_2\eta_5 I^0_{hmz} S^1_{zv} + \alpha_2\eta_5 I^1_{hmz} S^0_{zv}\\
&+ \alpha_2\eta_6 I^0_{hmzT} S^1_{zv} + \alpha_2\eta_6 I^1_{hmzT} S^0_{zv} + \alpha_2\eta_7 I^0_{hmzU} S^1_{zv}\\
&+ \alpha_2\eta_7 I^1_{hmzU} S^0_{zv} - (\nu_2 + \mu_z) E^1_{zv}\\
\frac{dI^2_{zv}}{dt} &= \nu_2 E^1_{zv} - \mu_z I^1_{zv}.
\end{aligned}
$$

Making use of the parameter values in Table 1, and the assumed initial conditions $S_h = 500, S_{hu} = 320, S_{hv} = 19, E_{hm} = 48, E_{hz} = 30, E_{hmz} = 20, I_{hm} = 35, I_{hmU} = 8, I_{hmT} = 21, I_{hzS} = 20, I_{hzA} = 40, I_{hzT} = 15, I_{hmz} = 15, I_{hmzU} = 3, I_{hmzT} = 9, R_h = 20, S_{mv} = 500, E_{mv} = 60, I_{mv} = 50, S_{zv} = 500, E_{zv} = 25, I_{zv} = 10$, solving the various systems of ordinary differential equations generated for p, p^1 andp^2 to obtain the values of the state variables, the solutions to the system (1) by HPM allowing $p = 1$ now becomes

$$
\begin{aligned}
S_h(t) &= S^0_h + pS^1_h + p^2 S^2_h = 500 - 414.728t + 192.9739t^2\\
S_{hu}(t) &= S^0_{hu} + pS^1_{hu} + p^2 S^2_{hu} = 320 + 107.0992t - 143.7306t^2\\
S_{hv}(t) &= S^0_{hv} + pS^1_{hv} + p^2 S^2_{hv} = 19 + 139.9203t - 58.9404t^2\\
E_{hm}(t) &= E^0_{hm} + pE^1_{hm} + p^2 E^2_{hm} = 48 + 214.9582t + 34.6583t^2\\
E_{hz}(t) &= E^0_{hz} + pE^1_{hz} + p^2 E^2_{hz} = 30 - 23.5836t + 10.1205t^2\\
E_{hmz}(t) &= E^0_{hmz} + pE^1_{hmz} + p^2 E^2_{hmz} = 20 + 18.7764t + 8.9226t^2\\
I_{hm}(t) &= I^0_{hm} + pI^1_{hm} + p^2 I^2_{hm} = 35 - 28.5971t + 22.4152t^2\\
I_{hmT}(t) &= I^0_{hmT} + pI^1_{hmT} + p^2 I^2_{hmT} = 21 + 16.4424t - 10.9234t^2\\
I_{hmU}(t) &= I^0_{hmU} + pI^1_{hmU} + p^2 I^2_{hmU} = 8 + 9.8071t - 5.0718t^2\\
I_{hzS}(t) &= I^0_{hzS} + pI^1_{hzS} + p^2 I^2_{hzS} = 20 - 32.3023t + 26.1163t^2\\
I_{hzA}(t) &= I^0_{hzA} + pI^1_{hzA} + p^2 I^2_{hzA} = 40 - 29.6086t + 10.0020t^2\\
I_{hzT}(t) &= I^0_{hzT} + pI^1_{hzT} + p^2 I^2_{hzT} = 15 + 14.4944t - 14.9391t^2\\
I_{hmz}(t) &= I^0_{hmz} + pI^1_{hmz} + p^2 I^2_{hmz} = 15 + 27.3212t - 31.3151t^2\\
I_{hmzT}(t) &= I^0_{hmzT} + pI^1_{hmzT} + p^2 I^2_{hmzT} = 8 + 9.7948t + 9.2887t^2\\
I_{hmzU}(t) &= I^0_{hmzU} + pI^1_{hmzU} + p^2 I^2_{hmzU} = 3 + 2.3979t + 2.3382t^2\\
R_h(t) &= R^0_h + pR^1_h + p^2 R^2_h = 20 + 16.0347t - 0.3653t^2\\
S_{mv}(t) &= S^0_{mv} + pS^1_{mv} + p^2 S^2_{mv} = 500 - 62,494.68t + 3,911,918.167t^2\\
E_{mv}(t) &= E^0_{mv} + pE^1_{mv} + p^2 E^2_{mv} = 60 + 57.324t - 4,267.7745t^2\\
I_{mv}(t) &= I^0_{mv} + pI^1_{mv} + p^2 I^2_{mv} = 50 + 3.32t + 2.7739t^2\\
S_{zv}(t) &= S^0_{zv} + pS^1_{zv} + p^2 S^2_{zv} = 500 - 62,711.3t + 3,938,997.241t^2\\
E_{zv}(t) &= E^0_{zv} + pE^1_{zv} + p^2 E^2_{zv} = 25 + 279.61t - 314.2182t^2\\
I_{zv}(t) &= I^0_{zv} + pI^1_{zv} + p^2 I^2_{zv} = 10 + 1.944t + 13.9265t^2.
\end{aligned}
$$

5. Numerical simulation

5.1. Comparison of HPM and RK – 4 for the co-infection model

In this section, the numerical simulation of the semi-analytical solutions by HPM is presented and discussed. The results are shown in Figures 1 -10.

In Figures 1 and 2, we could see that the solutions obtained from homotopy perturbation method is similar to that obtained from Runge-Kutta method of order 4. We could see that both trajectories started and progressed alike but slightly seperated at a point.

The same scenario is seen in Figures 3 and 4 which compared the solutions of the vaccinated and infectious humans with malaria obtained by HPM to the ones obtained by RK–4. In Figures 5 and 7, the symptomatic humans with zika and coinfectious humans with both diseases did not appear much similar as the trajectories differ after few steps. However, the trajectories for the asymptomatic humans with zika and recovered humans obtained by HPM appeared much similar to the

Table 1. Parameters, values and sources.

Parameters	Values	Sources	Parameters	Values	Sources
Λ_h	50	Assumed	β_3	0.0044	[19, 20]
Λ_{mv}	100	Assumed	β_4	0.0022	Assumed
Λ_{zv}	100	Assumed	β_5	0.0044	[19, 20]
Φ	0.0125	[11]	β_6	0.0022	Assumed
θ	0.0146	[18]	β_7	0.0044	Assumed
ρ_1	0.65	[11]	β_8	0.0022	Assumed
ρ_2	0.28	[11]	ω_1	0.1429	[21]
γ_1	0.25	[18]	ω_2	0.1667	[22]
γ_2	0.111	[11]	ω_3	0.118	[23]
τ_3	0.0003	[11]	χ_1	0.31	Assumed
τ_4	0.0006454	[11]	χ_2	0.62	Assumed
ϕ_1	0.13	Assumed	α_1	0.4	[18]
α_2	0.1	Assumed	β_1	0.034	[24]
ϕ_2	0.1	Assumed	τ_1	0.00004	[17]
β_2	0.013	[25]	τ_2	0.00032338	[25]
η_1	0.0009	[22]	η_2	0.07	[22]
η_3	0.07	[22]	η_4	0.05	Assumed
η_5	0.03	Assuned	η_6	0.02	Assumed
η_7	0.03	Assumed	ψ	0.85	[15]
ϵ_1	0.62	[11]	ϵ_2	0.31	Assumed
σ_1	0.72	[11]	σ_2	0.18	[11]
κ_1	0.25	[11]	κ_2	0.25	[11]
ν_1	0.1	[11, 18]	δ_1	0.0833	[11]
ν_2	0.1	[12]	δ_2	0.125	[15]
M_{SIT}	500	Assumed	δ_3	0.0833	[11]
Z^*_{SIT}	500	Assumed	μ_m	0.0556	[22]
μ_z	0.0556	[22]			

solutions obtained by RK–4 as shown in Figures 6 and 8 respectively.

Figures 9 and 10 show the trajectories of the solutions by HPM against RK–4. We coulc see that though the direction is the same but they are not close. A further study will require obtaining solutions by other semi-analytic methods and comparing with these ones to ascertain which method is most suitable for the system studied. The performance of the HPM may not entirely conform to the RK-4 since only two term approximations were used.

5.2. *Convergence and stability of solutions*

The behaviour of the solutions obtained from homotopy perturbation method is further investigated by comparing the solutions with that obtained from Runge-Kutta method of order 4. The HPM is a semi-analytical method while RK-4 is a purely numerical method. Hence, it is a good reference function to help investigate the convergence and stability of the solutios by HPM since the exact solutions cannot be obtained. First, we observe that the solutions by HPM for each state variable is either monotonically increasing or decreasing just as the ones obtained by RK-4 shown in Table 2. This shows that the solutions are either increasing or decreasing to a point. The convergence of the solutions by RK-4 will suggest the convergence of the solutions by HPM since both are following similar numerical pattern.

To check the stability of the solutions obtained, we simply see if there is a perturbation effect on the solutions by varying the initial guess. If the solutions converges to the same point irrespective of the initial guess, then the solution is considered to be stable. The effect of changes in the initial values of some state variables are shown in Figures 11-15. The Figures showed that irrespective of the initial values of the state variables, the solutions obtained by Homotopy perturbation method will converge to the same point with time. This shows that the solutions are not affected by changes in the starting point of the solutions hence, suggesting that the solutions are stable and converge.

5.3. *Effects of the Controls in the System*

In this section, we investigate the effects of the various controls employed in the system on the populations studied. The effects of the individual controls as well as when they are combined are analyzed. The simulation is shown in Figures 16–57.

5.3.1. *Effects of treatment only*

The effect of employing treatment only is shown in Figures 16–21. In these Figures, the populations of the infectious humans with malaria only (Figure 16), symptomatic humans with zika virus disease (Figure 17) and coinfectious humans with both diseases (Figure 18) all reduced with treatment as compared to when there was no treatment.

Table 2. Comparison of Solutions for $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5$.

	S_h	S_{hu}	S_{hv}	E_{hm}	E_{hz}	E_{hmz}	I_{hm}	I_{hmT}	I_{hmU}	I_{hzS}	I_{hzA}
HPM	500.00	320.00	19.0000	48.0000	30.0000	20.0000	35.0000	21.0000	8.0000	20.0000	40.0000
RK-4	500.00	320.00	19.0000	48.0000	30.0000	20.0000	35.0000	21.0000	8.0000	20.0000	40.0000
HPM	460.4569	329.2726	32.4026	69.8424	27.7428	21.9669	32.3644	22.5350	8.9300	17.0309	37.1392
RK-4	460.3985	329.0055	32.4261	69.8106	27.7284	21.7991	32.3572	22.5404	8.9325	17.0176	37.1374
HPM	424.7734	335.6706	44.6264	92.3780	25.6881	24.1122	30.1772	23.8515	9.7585	14.5842	34.4784
RK-4	424.3162	334.9033	44.7878	92.1023	25.6225	23.4482	30.1249	23.8937	9.7779	14.4813	34.4640
HPM	392.9493	339.1940	55.6715	115.6067	23.8357	26.4359	28.4382	24.9496	10.4857	12.6598	32.0176
RK-4	391.4405	338.1106	56.1796	114.6721	23.6714	24.9575	28.2696	25.0886	10.5495	12.3248	31.9691
HPM	364.9846	339.8428	65.5377	139.5286	22.1858	28.9381	27.1476	25.8292	11.1114	11.2577	29.7569
RK-4	361.4868	339.0015	66.6875	137.3428	21.8647	26.3366	26.7593	26.1511	11.2594	10.4917	29.6422
HPM	340.8794	337.6170	74.2251	164.1437	20.7383	31.6189	26.3053	26.4903	11.6356	10.3779	27.6962
RK-4	334.1955	337.9108	76.3899	159.9602	20.1926	27.5944	25.5634	27.1047	11.9185	8.9336	27.4734

	I_{hzT}	I_{hmz}	I_{hmzT}	I_{hmzU}	R_h	S_{mv}	E_{mv}	I_{mv}	S_{zv}	E_{zv}	I_{zv}
HPM	15.0000	15.0000	8.0000	3.0000	20.0000	500.00	60.0000	50.0000	500.00	25.0000	10.0000
RK-4	15.0000	15.0000	8.0000	3.0000	20.0000	500.00	60.0000	50.0000	500.00	25.0000	10.0000
HPM	16.3000	17.4190	10.0723	3.2632	21.5998	33369.7137	23.0547	50.3597	33618.8424	49.8188	10.3337
RK-4	16.3080	17.5947	9.0817	3.2627	21.5890	0.8006	59.5792	50.3210	0.7979	26.8634	10.2125
HPM	17.3013	19.2116	10.3305	3.5731	23.1923	14447.78	-99.2461	50.7750	14551.76	68.3532	10.9458
RK-4	17.3622	19.6407	10.3170	3.5643	23.1724	0.7988	58.6626	50.6315	0.7963	26.4905	10.4219
HPM	18.0038	20.3780	11.7744	3.9298	24.7775	33382.42	-306.9025	51.2457	33619.64	80.6034	11.8366
RK-4	18.2017	21.2185	11.6679	3.8951	24.7514	0.7986	57.7673	50.9313	0.7962	26.1218	10.6265
HPM	18.4075	20.9181	13.4041	4.3333	26.3554	60140.90	-599.9143	51.7718	60565.50	86.5691	13.0058
RK-4	18.8595	22.3974	13.1020	4.2473	26.3273	0.7987	56.8861	51.2206	0.7964	25.7573	10.8262
HPM	18.5124	20.8318	15.2196	4.7835	27.9260	94723.22	-978.2816	52.3534	95389.37	86.2505	14.4536
RK-4	19.3639	23.2380	14.5921	4.6138	27.9013	0.7991	56.0187	51.4995	0.7971	25.3969	11.0213

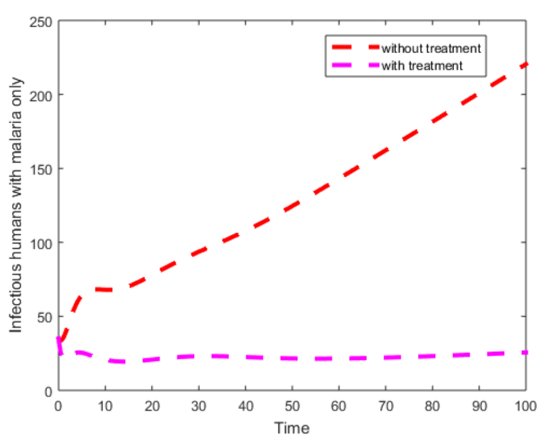

Figure 16. Infectious humans with malaria.

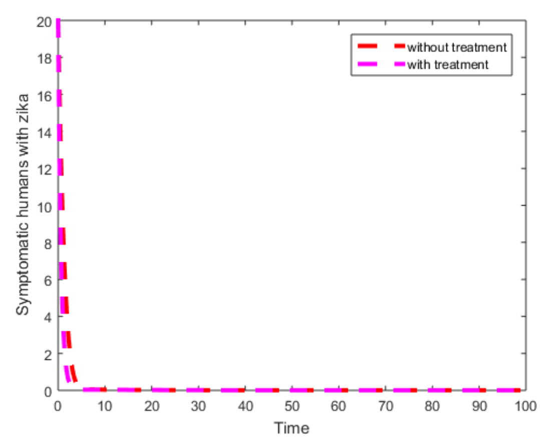

Figure 17. Symptomatic humans with zika.

This shows that employing treatment of infectious persons is critical in the control of infectious diseases. As the infectious classes are treated, the population of recovered humans increases as shown in Figure 19. The increase is because treatment increases the rate of recovery and reduces the time spent in the infectious class. In Figures 20 and 21, we also saw that treatment helped to reduce the number of infectious mosquitoes. As more people are treated, the rate of recovery increases thereby reducing the population of infectious humans that can infect the mosquitoes. In this case, the infectious human population was mostly affected by the measure than the infectious mosquito population because the measure was directly applied to human population.

5.3.2. Effects of vaccination only

The effect of vaccination is shown in Figures 22–27. In these Figures, vaccination was seen to reduce the population of susceptible humans as shown in Figure 22.

This is because vaccination protects more humans from malaria thus reducing the susceptible population of humans. The population of humans infectious with malaria and coinfectious humans also reduced as more people are protected from

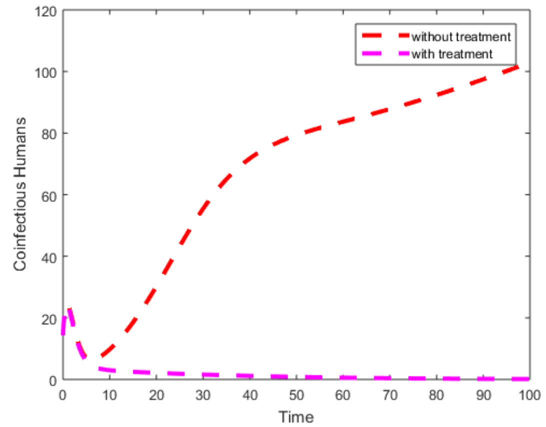

Figure 18. Coinfectious humans.

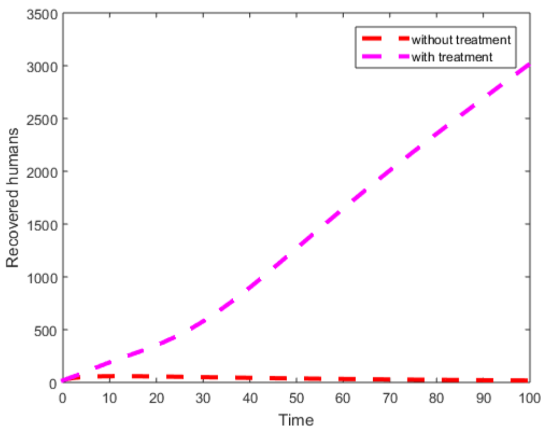

Figure 19. Recovered humans.

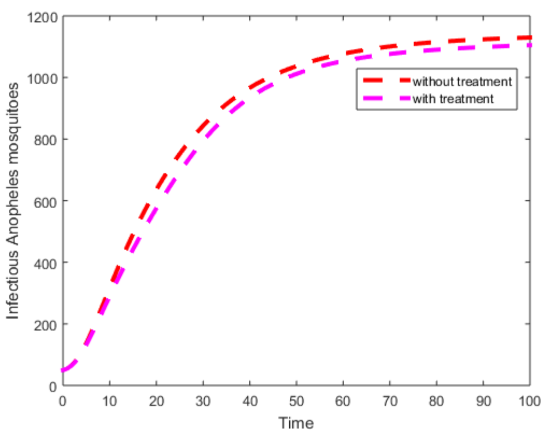

Figure 20. Infectious Anopheles mosquitoes.

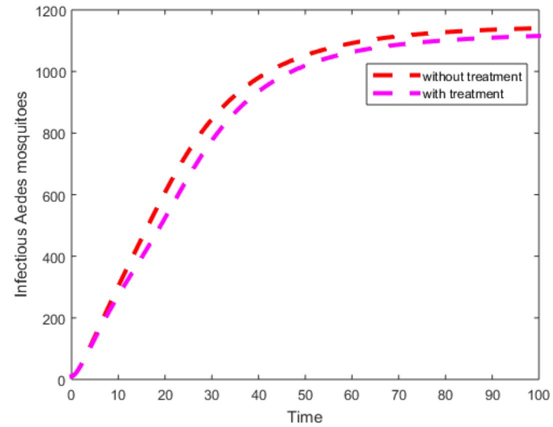

Figure 21. Infectious Aedes mosquitoes.

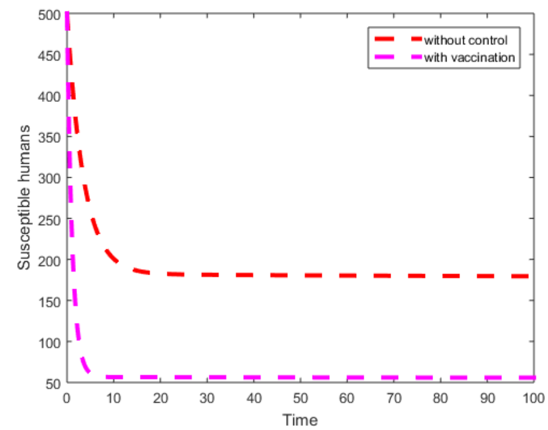

Figure 22. Susceptible humans.

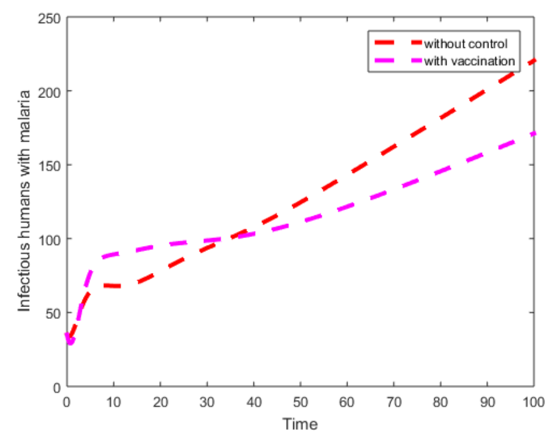

Figure 23. Infectious humans with malaria.

malaria by vaccination seen in Figures 23 and 25. However, this measure did not affect the infectious humans with zika virus disease (Figure 24) as the control measure is not targeted at them.

Also, the population of the infectious Anopheles mosquitoes were affected slightly as seen in Figure 26 while that of infectious Aedes mosquitoes (Figure 27) were not affected just like that of the symptomatic infectious humans with zika.

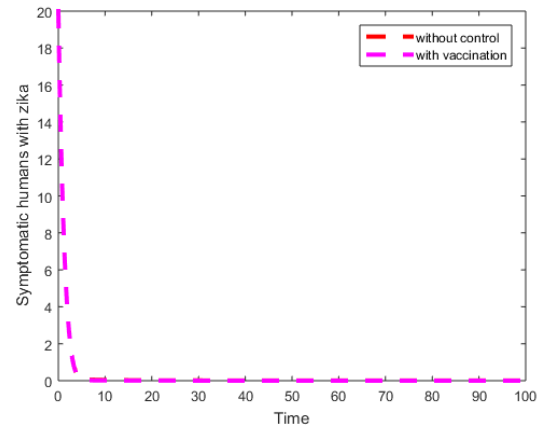

Figure 24. Symptomatic humans with zika.

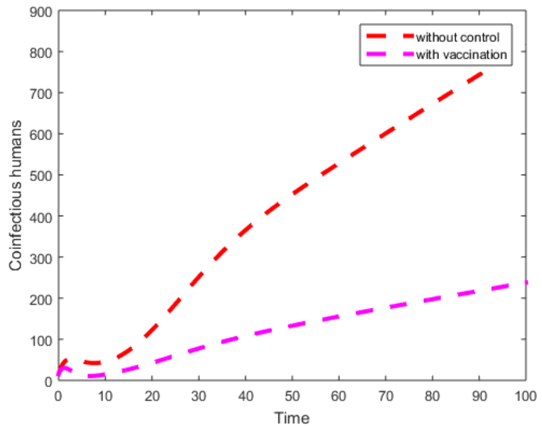

Figure 25. Coinfectious humans.

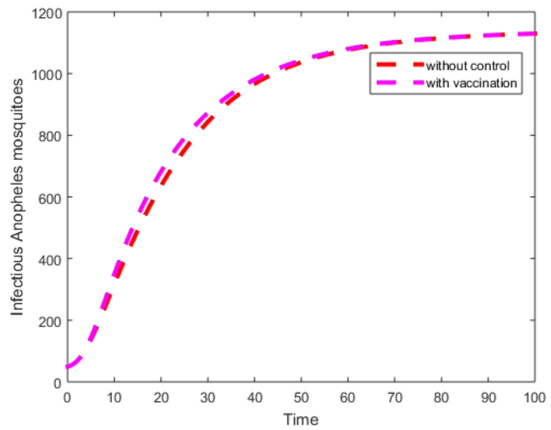

Figure 26. Infectious Anopheles mosquitoes.

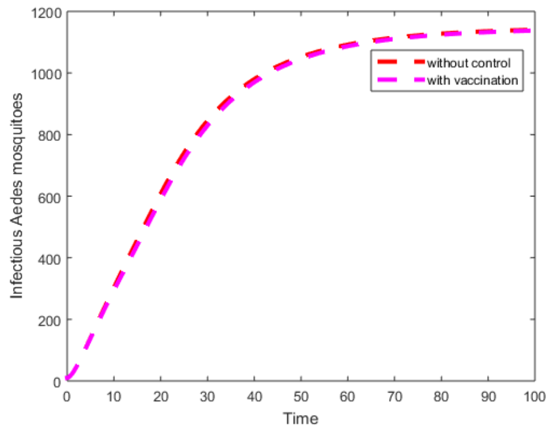

Figure 27. Infectious Aedes mosquitoes.

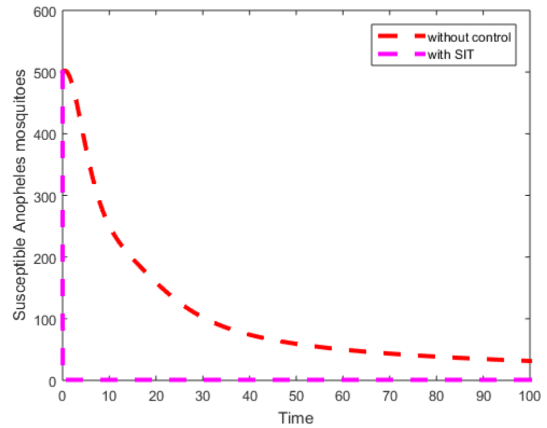

Figure 28. Susceptible Anopheles mosquitoes.

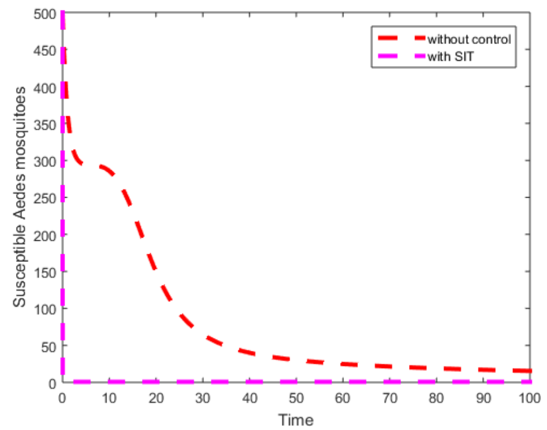

Figure 29. Susceptible Aedes mosquitoes.

5.3.3. *Effects of SIT only*

The effects of employing only sterile insect technique are shown in Figures 28–33. In these Figures, all the infectious classes were reduced under the application of SIT. Also, the susceptible mosquito populations were reduced. The application of SIT causes the female mosquitoes in the wild to lay eggs that do not hatch thus reducing the number of susceptible mosquitoes with time.

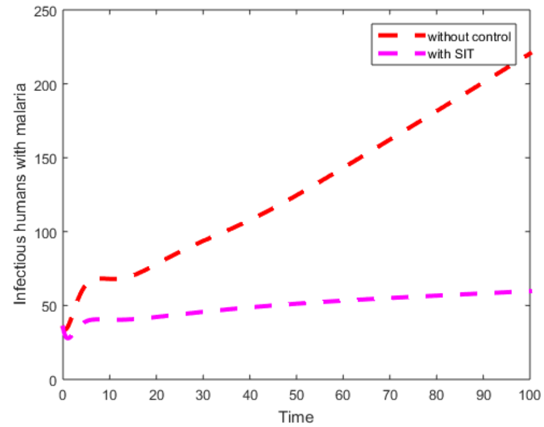

Figure 30. Infectious humans with malaria.

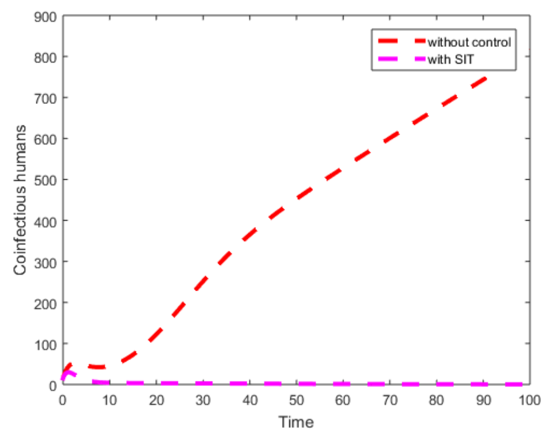

Figure 31. Coinfectious humans.

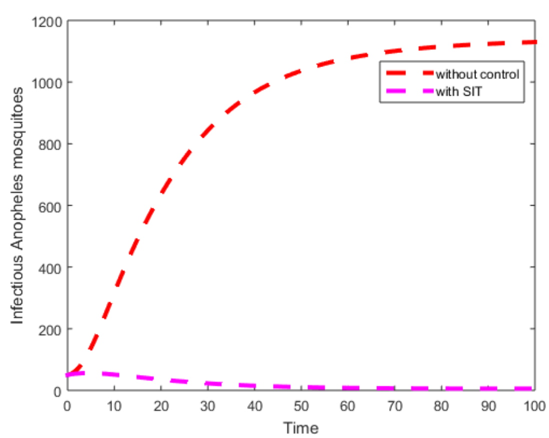

Figure 32. Infectious Anopheles mosquitoes.

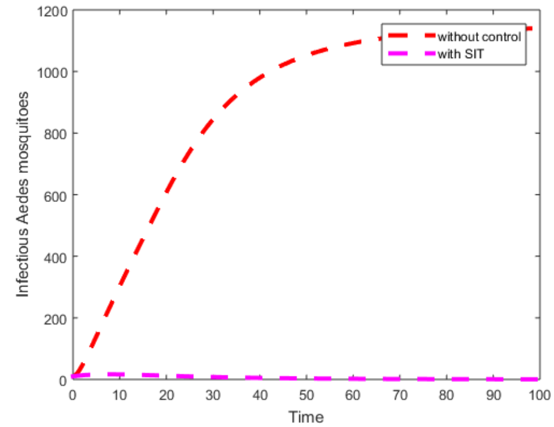

Figure 33. Infectious Aedes mosquitoes.

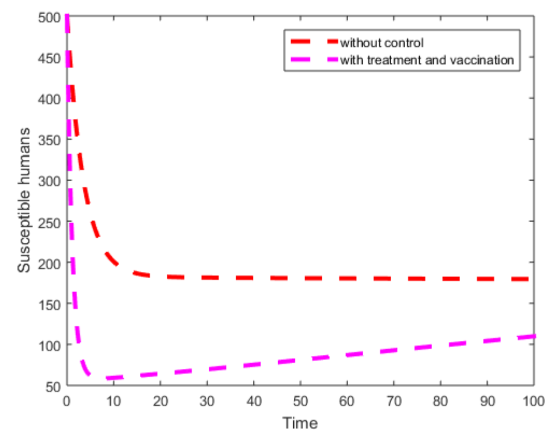

Figure 34. Susceptible humans.

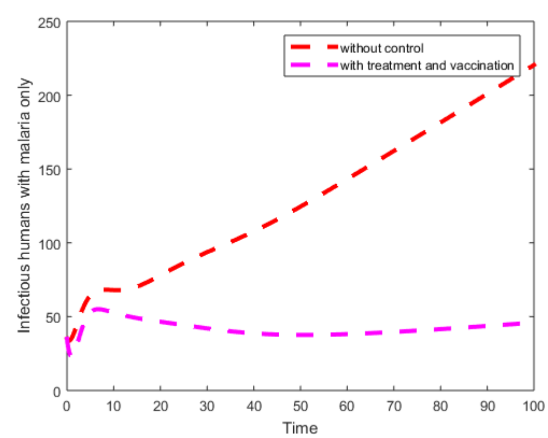

Figure 35. Infectious humans with malaria.

5.3.4. Effects of treatment and vaccination only

The effects of treatment and vaccination in the system are shown in Figures 34–39. In these Figures, vaccination and treatment were seen to reduce the population of susceptible humans as shown in Figure 34.

Also, the infectious and coinfectious human populations were reduced significantly but the mosquito population reduced slightly. The slight reduction in the mosquito population was because the control measures employed here were not directly on the mosquito population hence does not affect the population

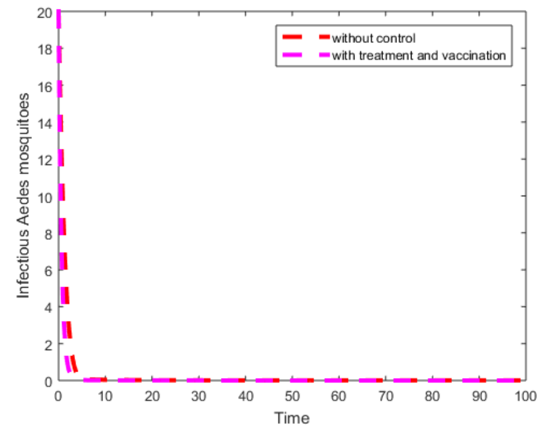

Figure 36. Symptomatic humans with zika.

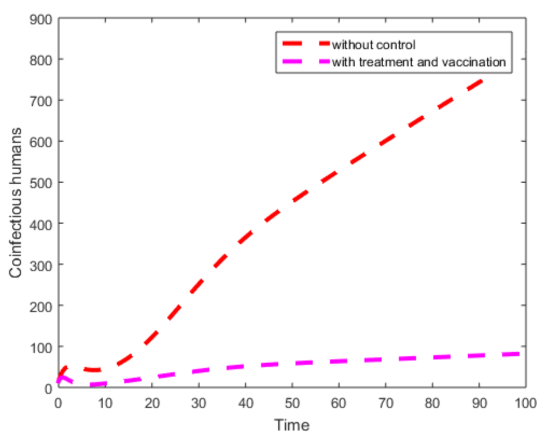

Figure 37. Coinfectious humans.

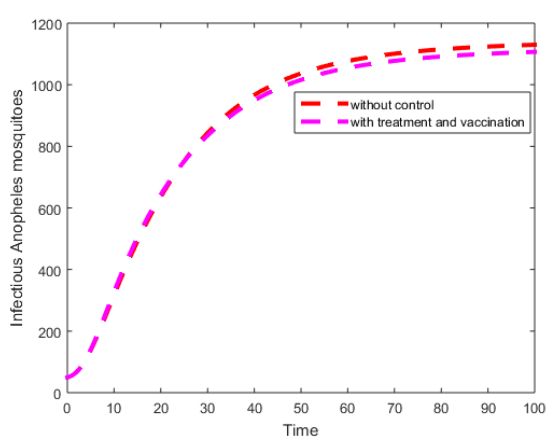

Figure 38. Infectious Anopheles mosquitoes.

of the susceptible mosquitoes.

5.3.5. *Effects of treatment and SIT only*

The effects of treatment and SIT are shown in Figures 40–45. Employing treatment and sterile insect technique as a con-

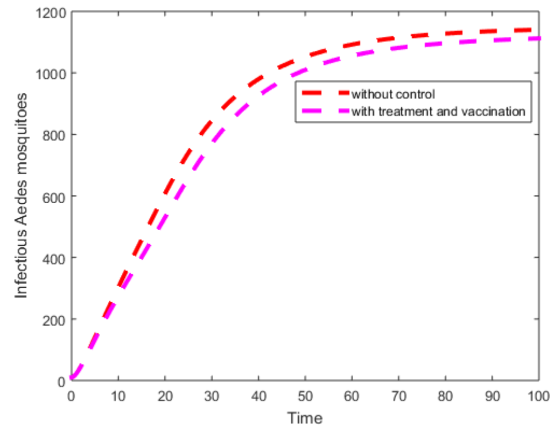

Figure 39. Infectious Aedes mosquitoes.

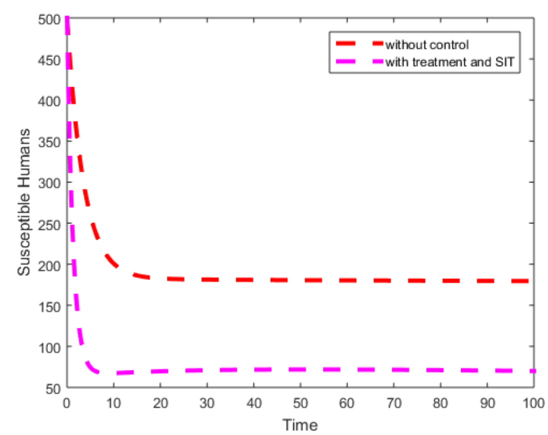

Figure 40. Susceptible humans.

trol measure reduce both the infectious and coinfectious population significantly as well as the susceptible mosquito populations. The reduction of the susceptible mosquitoes causes a consequential reduction in the infectious mosquito population thus reducing the infectious human population as the number of mosquitoes that can infect humans are reduced. This strategy produced a better result than applying any of the controls individually showing that the diseases are best fought by applying measures that affects both the human and mosquito populations simultaneously.

5.3.6. *Effects of vaccination and SIT only*

The effects of vaccination and SIT are shown in Figures 46–51. In these Figures, vaccination and SIT were seen to reduce the population of susceptible humans as shown in Figure 46. It also reduced the populations of the infectious and coinfectious humans greatly as seen in other cases. The population of the infectious mosquitoes were also reduced in this case showing that SIT and vaccination offers significant option in reducing the infectious human populations.

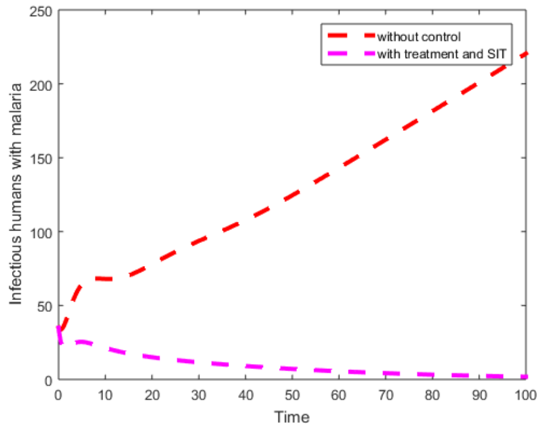

Figure 41. Infectious humans with malaria.

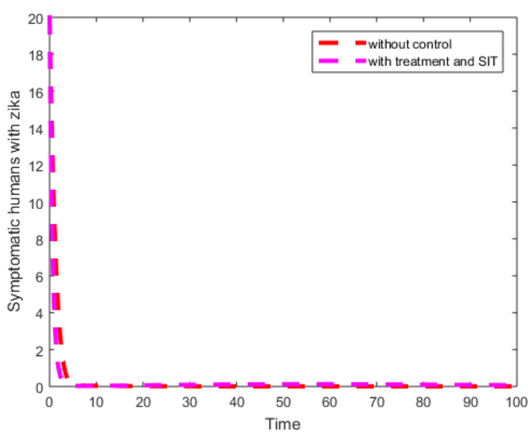

Figure 42. Symptomatic humans with zika.

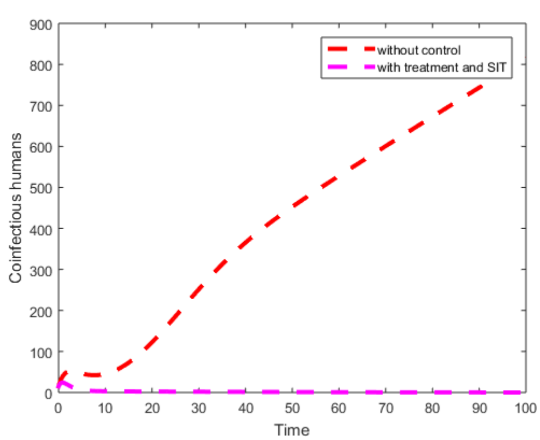

Figure 43. Coinfectious humans.

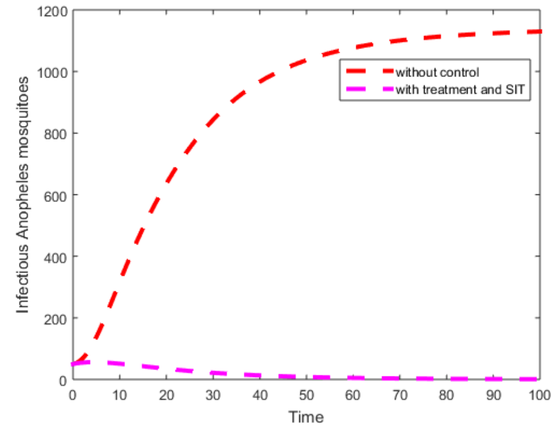

Figure 44. Infectious Anopheles mosquitoes.

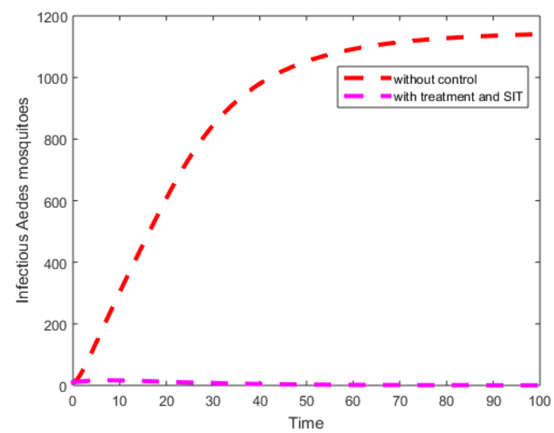

Figure 45. Infectious Aedes mosquitoes.

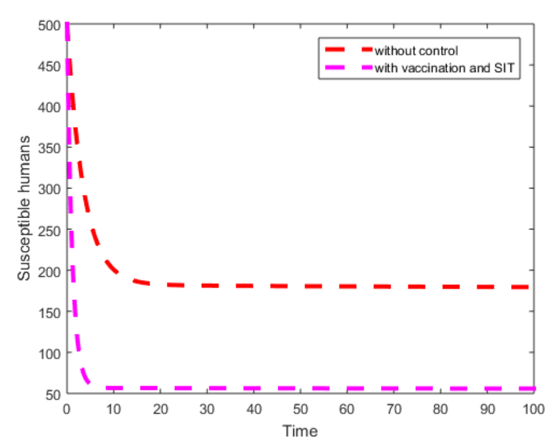

Figure 46. Susceptible humans.

5.3.7. Effects of treatment, SIT and vaccination only

Figures 52–57 show the effect of employing the three controls which are treatment, vaccination and use of SIT to control the vectors. The simulation also showed that the combination of these three control measures performed relatively better when compared to using only one or two control measures.

In this case, combining the three controls ensured that the three different populations had a measure employed to help reduce the spread of the diseases. This will ensure rapid and efficient control of the diseases. Hence, we suggest that the

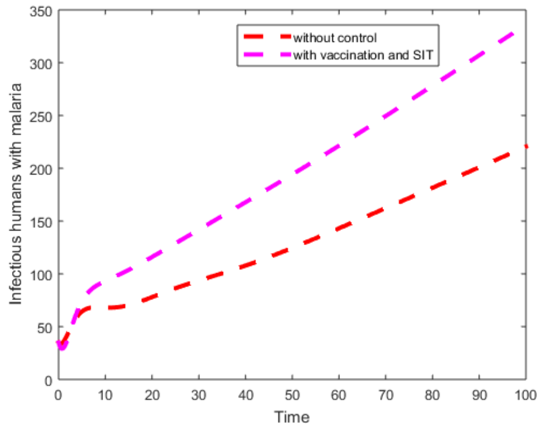

Figure 47. Infectious humans with malaria.

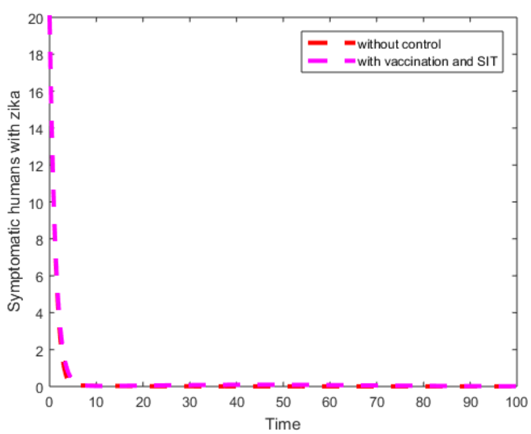

Figure 48. Symptomatic humans with zika.

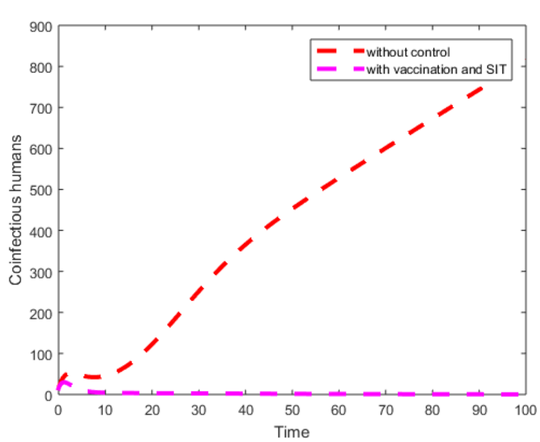

Figure 49. Coinfectious humans.

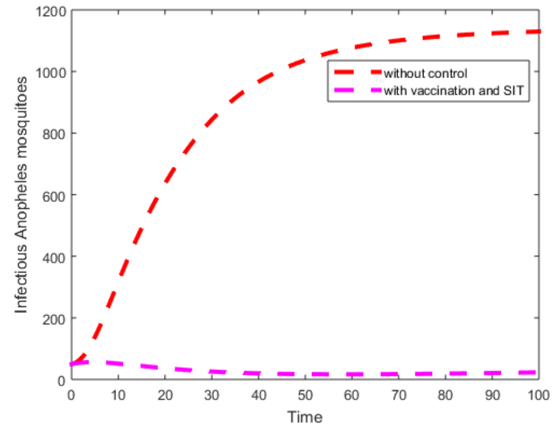

Figure 50. Infectious Anopheles mosquitoes.

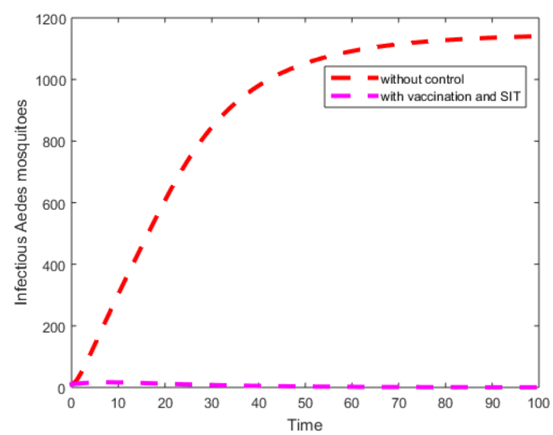

Figure 51. Infectious Aedes mosquitoes.

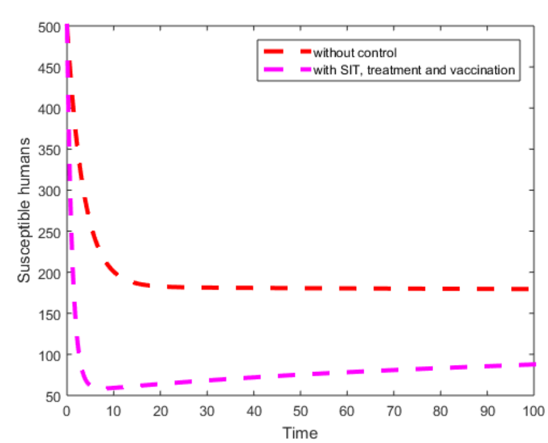

Figure 52. Susceptible humans.

three controls should be incorporated simultaneously in the fight against malaria and zika virus disease.

The analyses performed in this work has shown the importance of vaccination against infectious diseases such as malaria. It was shown that in the presence of vaccination in all cases where it was employed in the simulation, the population of infectious and coinfectious humans reduced drastically. The reduction is because the number of humans susceptible to the disease was reduced significantly by vaccination. This underlines the importance of vaccination and the need for health practition-

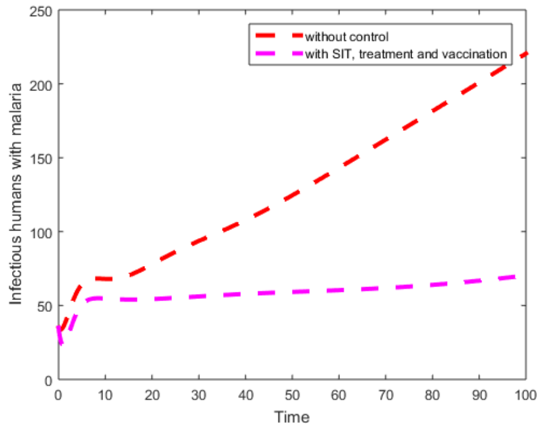

Figure 53. Infectious humans with malaria.

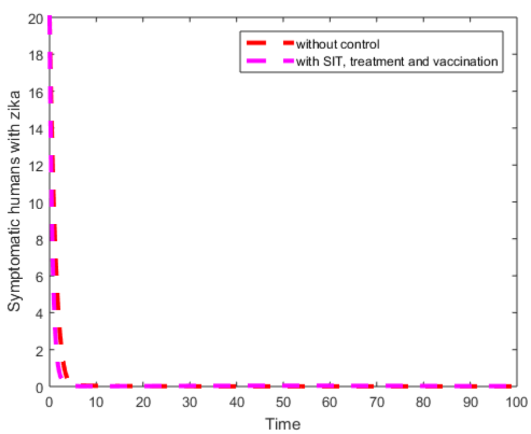

Figure 54. Symptomatic humans with zika.

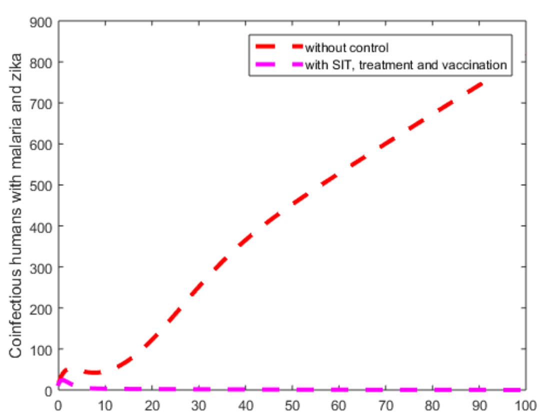

Figure 55. Coinfectious humans.

ers to keep encouraging people to embrace vaccination. Efforts should be made to ensure that every barrier against vaccination such as religious and ethnic sentiments as well as some conspiracy theories are overcome. Also, people should be educated that refusing to be vaccinated does not only put them at risk but

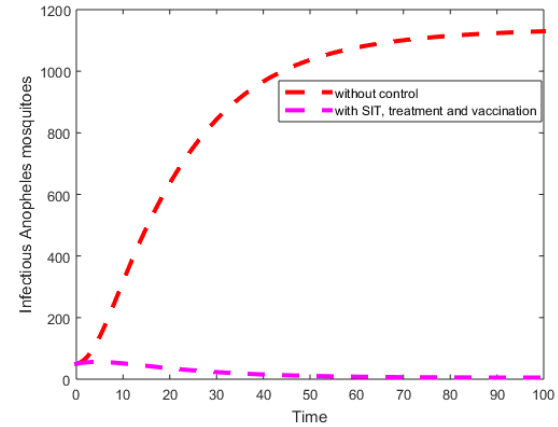

Figure 56. Infectious Anopheles mosquitoes.

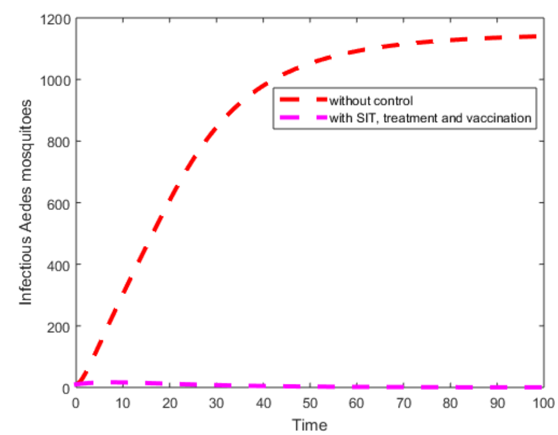

Figure 57. Infectious Aedes mosquitoes.

those around them thereby causing a global health burden. The importance of treatment was equally shown in this work. Treatment of infectious humans helped significantly to improve the rate of recovery and reduce the time infectious humans spent being sick. This ensures that there are few infectious humans who can infect the mosquitoes and renew the cycle of the spread of the diseases. Thus refusing to be treated when sick is not an option that should be entertained. Moreover, as proposed in Ref. [11], using sterile insect technique as a control measure against the continuous growth of mosquitoes was shown to be highly successful. The focus of the simulation done in Ref. [11] was to obtain what proportion of SIT mosquitoes that should be interacting with a given number of female mosquitoes in the wild for effective control. Here, the focus is to know how the various controls performed relative to one another and show the importance of incorporating them together rather than using each seperately.

6. Conclusion

In this paper, we looked at the coinfection model proposed in Ref. [11] and provided more insight into the study. The model incorporated vaccination and treatment as control measures against the spread of malaria, zika virus disease and their coinfection. It also proposes using sterile insect technique to control the population of the mosquitoes. The existence and uniqueness of solutions to the system were first established. Thereafter, the system was shown to be well-posed epidemiologically by showing that the solutions to the system are always positive and bounded. Then, the approximate solutions to the system were obtained using homotopy perturbation technique which is a semi-analytical method. Furthermore, numerical simulations were performed to show the effects of the various controlled adopted in the work. The performance of each control when adopted singly and when combined were all shown. The result showed that combining the three controls performed better than when they are adopted individually or combined one with each other. This shows that if malaria and zika virus will be effectively controlled, then efforts should not only be focused on humans nor mosquitoes seperately but simultaneously on both humans and mosquitoes. In future studies, the optimal control of the system and associated cost effectiveness analysis can be researched. Also, investigation into the dynamics of the enddemic equilibrium and bifurcation analysis can also be carried out. The performance of different incidence functions can also be employed and compared to the results obtained here. Furthermore, different controls such as physical and chemical procedures can also be incorporated into the system to see if better results will be obtained.

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Data availability

No additional research data were used beyond the content presented in the submitted manuscript.

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