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A New Multi-Step Method for Solving Delay Differential Equations using Lagrange Interpolation

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Abstract

This paper presents 2-step p-th order ($p = 2, 3, 4$) multi-step methods that are based on the combination of both polynomial and exponential functions for the solution of Delay Differential Equations (DDEs). Furthermore, the delay argument is approximated using the Lagrange interpolation. The local truncation errors and stability polynomials for each order are derived. The Local Grid Search Algorithm (LGSA) is used to determine the stability regions of the method. Moreover, applicability and suitability of the method have been demonstrated by some numerical examples of DDEs with constant delay, time dependent and state dependent delays. The numerical results are compared with the theoretical solution as well as the existing Rational Multi-step Method2 (RMM2).

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1. Introduction

Delay Differential Equations (DDEs) are differential equations in which the derivative of the unknown function depends not only at its present time but also at the previous times. In Ordinary Differential Equations (ODEs), a simple initial condition is given. But to specify DDEs, additional information is needed. Because the derivative depends on the solution at the previous times, an initial history function which gives information about the solution in the past needs to be specified. A general form of the first order DDE is

$$y'(t) = f(t, y(t), y(t-\tau)), \qquad t > t_0$$

$$y(t) = \vartheta(t), \quad t \le t_0 \tag{1}$$

where ϑ(t) is the initial function and τ is the delay term. The function ϑ(t) is also known as the 'history function', as it gives information about the solution in the past. If the delay term τ is a constant, then it is called constant delay. If it is function of time t, then it is called time dependent delay. If it is a function of time t and y(t), then it is called state dependent delay.

These equations arise in population dynamics, control systems, chemical kinetic, and in several areas of science and engineering [1, 2, 3]. Recently there has been a growing interest in obtaining the numerical solutions of DDEs. Rostann *et al.* [4] implemented Adomian decomposition method for the solution of system of DDEs. Two and three point one-step block method for solving DDEs was developed by [5]. Block method for solving Pantograph type functional DDEs was described by

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[6]. An exact/approximate solution of DDEs by using the combination of Laplace and the variational iteration method were obtained by [7]. RK method based on Harmonic Mean for solving DDEs with constant lags was proposed by [8]. Several numerical methods have been constructed for solving stiff DDEs, see [9, 10, 11]. Several multi-step techniques using varieties of interpolating polynomials and functions have been developed to solve ODEs such as [12, 13, 14, 15, 16, 17], just to mention a few.

In this paper, we present the 2-step p-th order ($p = 2, 3, 4$) multi-step method for solving DDEs. This method has been referred here as EPMM $(2, p)$, ($p = 2, 3, 4$). The organization of this paper is as follows: In Section Two, the derivation of EPMM $(2, p)$, ($p = 2, 3, 4$) is given. In Section Three, the stability analysis of EPMM $(2, p)$, (p = 2, 3, 4) has been presented. In Section Four, numerical illustrations of DDEs are provided. Moreover, the numerical results are compared with the existing Rational Multi-step Method2 (RMM2) to demonstrate the efficiency and suitability of the method.

2. Derivation of EPMM (2, *p*)

For 2-step p-th order EPMM, let us assume an approximation to the analytical solution $y(t_{n+2})$ of (1) given by

$$y_{n+2} = a_0 e^{2h} + 1 + \sum_{j=1}^{p} b_j h^j, \tag{2}$$

where a_0, b_j, $(j=1,2,\dots p)$ are parameters that may contain the approximation of $y(t_n)$ and higher derivatives of $y(t_n)$.
With EPMM in (2) we associate the difference operator L defined by

$$L[y(t);h]_{EPMM} = \left(y(t+2h) - (1 + \sum_{j=0}^{p} b_j h^j)\right) - a_0 e^{2h} \tag{3}$$

where y(t) is an arbitrary, continuous and differentiable function.
Expanding $y(t+2h)$ as Taylor series and collecting terms in (3),

$$L[y(t);h]_{EPMM} = C_0 h^0 + C_1 h^1 + \cdots + C_p h^p + C_{p+1} h^{p+1} + \dots \tag{4}$$

where $C_i, \quad i = 0, 1, \dots, p, p+1$ are the coefficients that need to be determined.
For 2-step second order EPMM, we take p = 2 and expand $y(t+2h)$ via Taylor series, (3) becomes

$$L[y(t);h]_{EPMM(2,2)} = -1 + y(t) + h\left(-b_1 + 2y'(t)\right)$$
$$+ h^2\left(2y''(t) - b_2\right) + h^3\left(\frac{4}{3}y'''(t)\right) + a_0 e^{2h} + O(h^4) \tag{5}$$

Using $e^{2h} \approx 1 + 2h + 2h^2$ in (2), we get

$$L[y(t);h]_{EPMM(2,2)} = -1 - a_0 + y(t) + h\left(-b_1 - 2a_0 + 2y'(t)\right)$$
$$+ h^2\left(2y''(t) - 2a_0 - b_2\right) + h^3\left(\frac{4}{3}y'''(t)\right) + a_0 e^{2h} + O(h^4) \tag{6}$$

Comparing (5) and (6), we have

$$\begin{cases} C_0 = -(1 + a_0) + y(t), \\ C_1 = -b_1 - 2a_0 + 2y'(t), \\ C_2 = 2y''(t) - 2a_0 - b_2, \\ C_3 = \frac{4}{3}y'''(t) \end{cases} \tag{7}$$

For second order EPMM, we put $C_0 = C_1 = C_2 = 0$ in (7) and get the following solutions:

$$a_0 = y(t) - 1, b_1 = 2(y'(t) - y(t) + 1), b_2 = 2(y''(t) - y(t) + 1) \tag{8}$$

If we write $y_n = y(t_n)$ and $y_n^{(m)} = y^{(m)}(t_n)$ for m = 1, 2,..., then (8) becomes

$$\begin{cases} A = y_n, \\ b_1 = 2\left(y_n' - y_n + 1\right), \\ b_2 = 2(y_n'' - y_n + 1) \end{cases} \tag{9}$$

Taking $p = 2$ and $e^{2h} \approx 1 + 2h + 2h^2$ in (2), we get

$$y_{n+2} = (a_0 + 1) + h(2a_0 + b_1) + h^2(2a_0 + b_2) \tag{10}$$

Substituting (9) into (10), we have

$$y_{n+2} = y_n + 2hy_n' + 2h^2 y_n'' \tag{11}$$

The local truncation error of EPMM $(2, 2)$ is given by,

$$LTE_{EPMM(2,2)} = h^3\left(\frac{4}{3}y_n'''\right) + O(h^4)$$

Taking p = 3 in (2) and on simplification, we get the formula for EPMM (2, 3)

$$y_{n+2} = y_n + 2hy_n' + 2h^2 y_n'' + \frac{4}{3}h^3 y_n''' \tag{12}$$

The local truncation error of EPMM (2, 3) is given by,

$$LTE_{EPMM(2,3)} = h^4\left(\frac{2}{3}y_n^{(4)}\right) + O(h^5)$$

Taking p = 4 in (2) and on simplification, we get the formula for EPMM (2, 4)

$$y_{n+2} = y(t) + 2hy'(t) + 2h^2 y''(t) + \frac{4}{3}h^3 y'''(t) + \frac{2}{3}h^4 y^{(4)}(t) \tag{13}$$

The local truncation error of EPMM (2, 4) is given by,

$$LTE_{EPMM(2,4)} = h^5\left(\frac{4}{15}y_n^{(5)}\right) + O(h^6)$$

3. Stability Analysis of EPMM

In this section, we derive the stability polynomials of EPMM $(2, p)$, (p = 2, 3, 4) and their corresponding stability regions were obtained.
We consider a commonly used linear test equation with a constant delay $\tau = mh$ where m is a positive integer,

$$y'(t) = \lambda y(t) + \mu y(t - \tau), t > t_0$$

$$y(t) = \phi(t), t \le t_0 \quad (14)$$

where $\lambda, \mu \in C$, $\tau > 0$ andΦ is continuous.
Using (11) in (14), we get

$$y_{n+2} = y_n + 2h(\lambda y_n + \mu y(t_n - \tau)) + 2h^2(\lambda y'_n + \mu y'(t_n - \tau)) \quad (15)$$

$$y(t_n - mh) = y(t_{n-m}) = \sum_{l=-r_1}^{s_1} L_l(c_i) y_{n-m+l}$$

with

$$L_l(c_i) = \prod_{j=-r_1}^{s_1} \frac{c_i - j_1}{l - j_1}, \qquad j_1 \ne l \text{ and } r_1, s_1 > 0$$

Taking

$$y(t_n - \tau) = \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l}$$

and

$$y'(t_n - \tau) = \lambda \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l} + \mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l} \quad (16)$$

Then (15) becomes

$$y_{n+2} = y_n + 2h\left(\lambda y_n + \mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l}\right)$$

$$+2h^2\begin{pmatrix} \lambda\left(\lambda y_n + \mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l}\right) \\ +\mu\lambda \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l} \\ +\mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l} \end{pmatrix}$$

$$y_{n+2} = y_n + 2\lambda h y_n + 2\lambda^2 h^2 y_n$$

$$+ \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l}\left(2\mu h + 4h^2\mu\lambda\right) + \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l}\left(2h^2\mu^2\right)$$

$$y_{n+2} = y_n\left(1 + 2\lambda h + 2(\lambda h)^2\right) + \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l}(\mu h(2 + 4\lambda h))$$

$$+ \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l}\left(2(\mu h)^2\right)$$

Let $\alpha = \lambda h$ and $\beta = \mu h$ then the above equation becomes

$$y_{n+2} = y_n\left(1 + 2\alpha + 2\alpha^2\right) + (\beta(2 + 4\alpha)) \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l}$$

$$+2\beta^2 \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l}$$

To obtain the stability polynomial, the delay term is approximated using three points Lagrange interpolation.
By putting $n - m + l = 0$ and $n - 2m + l = 0$ and by taking $l =$ -1, 0, 1, the stability polynomial will be in the standard form.

The recurrence is stable if the zeros of ζ_i of the stability polynomial

$$S(\alpha, \beta : \zeta) = \zeta^{n+2} - \left(1 + \alpha + 2\alpha^2\right)\zeta^n$$

$$-\beta(2 + 4\alpha)\left(L_{-1}(c) + L_0(c)\zeta + L_1(c)\zeta^2\right)$$

$$-2\beta^2\left(L_{-1}(c) + L_0(c)\zeta + L_1(c)\zeta^2\right)$$

satisfies the root condition$|\zeta_i| \le 1$. From this, the stability polynomial for the method GRMM (2, 2) with $\tau = 1$ is given as

$$S(\alpha, \beta : \zeta) = \zeta^{n+2} - \left(1 + \alpha + 2\alpha^2\right)\zeta^n - \left(2\beta + 2\beta^2 + 4\alpha\beta\right)$$

Similarly, by considering suitable number of points in Lagrange interpolation according to the order of the method, we can obtain the corresponding stability polynomials of EPMM (2, p). When $p = 3$, the stability polynomial for EPMM (2, 3) is given as

$$S(\alpha, \beta : \zeta) = \zeta^{n+2} - \left(1 + \alpha + 2\alpha^2 + \frac{4}{3}\alpha^3\right)\zeta^n$$

$$-\left(2\beta + 2\beta^2 + \frac{4}{3}\beta^3 + 4\alpha\beta + 4\alpha^2\beta + 4\alpha\beta^2\right)$$

When $p = 4$, the stability polynomial for EPMM (2, 4) is given as

$$S(\alpha, \beta : \zeta) = \zeta^{n+2} - \left(1 + \alpha + 2\alpha^2 + \frac{4}{3}\alpha^3 + \frac{2}{3}\alpha^4\right)\zeta^n$$

$$-\left(2\beta + 2\beta^2 + \frac{4}{3}\beta^3 + \frac{2}{3}\beta^4 + 2\alpha\beta + 4\alpha^2\beta + 4\alpha\beta^2 + \frac{8}{3}\alpha\beta^3 + \frac{8}{3}\alpha^3\beta + 4\alpha^2\beta^2\right)$$

The stability regions of EPMM (2, 2), EPMM (2, 3) and EPMM (2, 4) are given in Figures 1 -3.
In a similar manner, we can obtain the stability polynomials and their corresponding regions of EPMM with r-step and of any order p.

4. Numerical Examples

Example 1: (Stiff linear system with multiple delays)

$$y_1'(t) = -\frac{1}{2}y_1(t) - \frac{1}{2}y_2(t-1) + f_1(t),$$

$$y_2'(t) = -y_2(t) - \frac{1}{2}y_1\left(t - \frac{1}{2}\right) + f_2(t), 0 \le t \le 1$$

with initial conditions

$$y_1(t) = e^{-t/2}, \frac{-1}{2} \le t \le 0,$$

$$y_2(t) = e^{-t}, -1 \le t \le 0$$

Table 1. Comparison of Absolute Error Results in EPMM and RMM2 for Example 1

Time (t)	Y	EPMM (2, 2)	RMM2 (2, 2)	EPMM (2, 3)	RMM2 (2,3)	EPMM (2, 4)	RMM2(2, 4)
0.2	y_1	1.52e-06	7.54e-07	3.80e-09	1.26e-09	7.60e-12	9.06e-07
	y_2	3.26e-04	3.31e-04	3.26e-04	3.26e-04	3.15e-04	3.32e-04
0.4	y_1	2.75e-06	1.36e-06	6.88e-09	2.28e-09	1.38e-11	1.64e-06
	y_2	5.62e-04	5.71e-04	5.62e-04	5.62e-04	5.43e-04	5.72e-04
0.6	y_1	3.73e-06	1.85e-06	9.33e-09	3.10e-09	1.87e-11	2.22e-06
	y_2	7.27e-04	7.38e-04	7.27e-04	7.27e-04	7.04e-04	7.40e-04
0.8	y_1	4.50e-06	2.23e-06	1.13e-08	3.73e-09	2.25e-11	2.68e-06
	y_2	8.36e-04	8.48e-04	8.36e-04	8.36e-04	8.12e-04	8.51e-04
1.0	y_1	5.09e-06	2.53e-06	1.27e-08	4.22e-09	2.55e-11	3.04e-06
	y_2	9.03e-04	9.16e-04	9.03e-04	9.03e-04	8.78e-04	9.18e-04

Table 2. Comparison of Absolute Error Results in EPMM and RMM2 for Example 2

Time (t)	EPMM (2, 2)	RMM2 (2, 2)	EPMM (2, 3)	RMM2 (2, 3)	EPMM (2, 4)	RMM2 (2, 4)
1.1	1.82e-06	4.04e-06	1.31e-06	1.46e-06	1.35e-06	9.58e-06
1.2	9.26e-07	2.51e-06	1.25e-06	1.20e-06	2.75e-06	8.03e-06
1.3	1.50e-06	3.23e-08	2.22e-06	4.13e-06	2.26e-07	8.41e-06
1.4	3.50e-06	5.92e-06	2.70e-06	3.29e-06	2.99e-06	1.19e-05
1.5	4.95e-05	1.96e-06	3.13e-06	6.16e-06	4.39e-06	6.83e-06

Table 3. Comparison of Absolute Error Results in EPMM and RMM2 for Example 3

Time(t)	EPMM (2, 2)	RMM2 (2, 2)	EPMM (2, 3)	RMM2 (2, 3)	EPMM (2, 4)	RMM2 (2, 4)
0.2	1.33e-05	1.35e-05	1.97e-09	2.36e-07	4.92e-09	4.87e-07
0.4	2.60e-05	2.80e-05	2.46e-08	5.62e-07	1.76e-09	5.46e-06
0.6	3.78e-05	4.48e-05	5.67e-08	7.45e-07	3.03e-09	3.58e-05
0.8	4.79e-05	6.56e-05	9.93e-08	8.72e-07	2.45e-09	2.32e-04
1.0	5.63e-05	9.31e-05	2.85e-06	2.85e-06	3.24e-09	3.24e-04

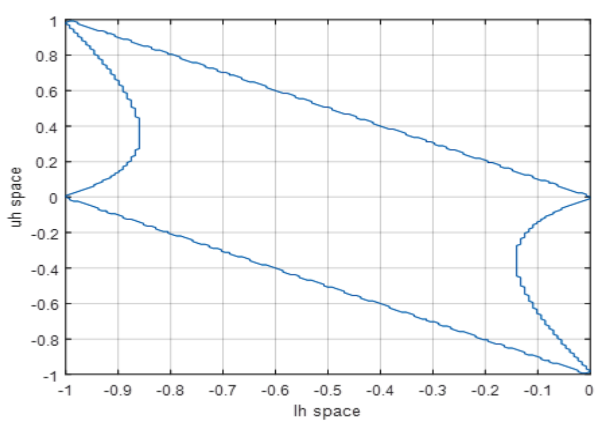

Figure 1. Stability Region of 2-step Second Order EPMM

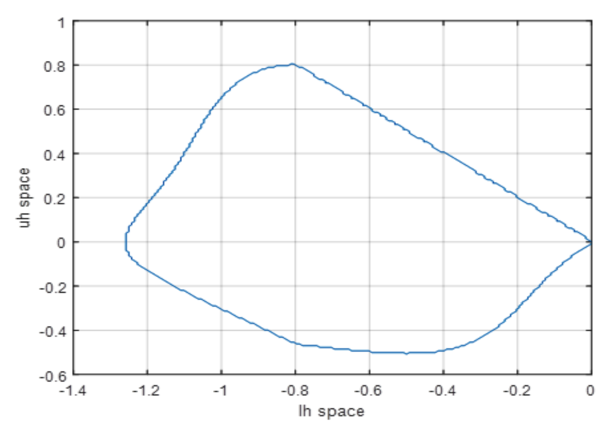

Figure 2. Stability Region of 2-step Third Order EPMM

where

$$f_1(t) = \frac{1}{2}e^{-(t-1)}$$

and

$$f_2(t) = \frac{1}{2}e^{-(t-1/2)/2}$$

The exact solution is given by

$$y_1(t) = e^{-t/2},\ y_2(t) = e^{-t}$$

Example 2: (Time-dependent delay)

$$y'(t) = \frac{t-1}{t}y(\ln(t) - 1)y(t),\ 1 \leq t \leq \frac{3}{2}$$

With initial condition

$$y(t) = 1,\ 0 \leq t \leq 1$$

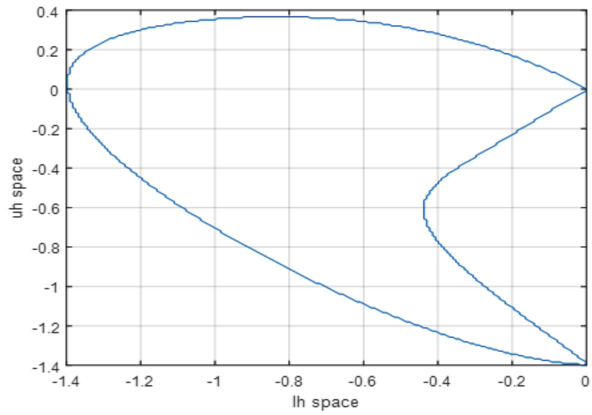

Figure 3. Stability Region of 2-step Fourth Order EPMM

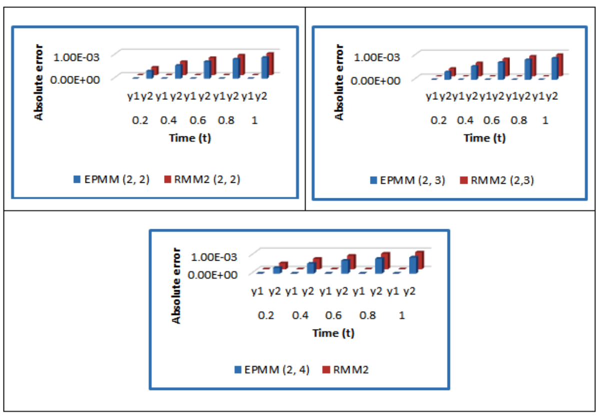

Figure 4. Comparison of Error Graph of y_1 and y_2 in Example 1

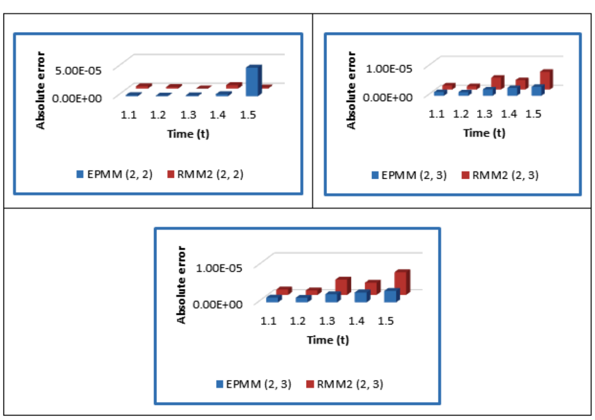

Figure 5. Comparison of Absolute Error Graph of y in Example 2

and the exact solution is given by

$$y(t) = \exp(t - \ln(t) - 1),\ 1 \le t \le \frac{3}{2}$$

Example 3: (State-dependent delay)

$$y'(t) = \cos(t) y(y(t) - 2),\ t \ge 0$$

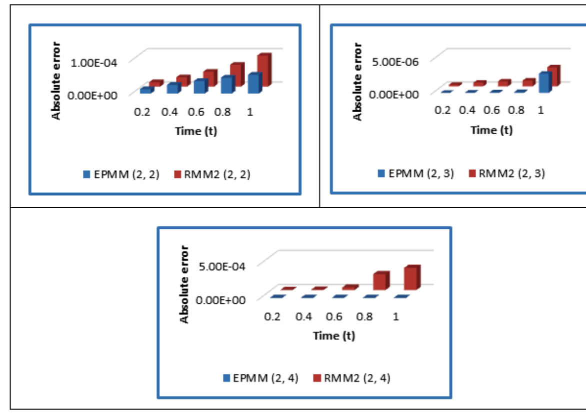

Figure 6. Comparison of Absolute Error Graph of y in Example 3

With initial condition,

$$y(t) = 1,\ t \le 0$$

and the exact solution is given by

$$y(t) = \sin(t) + 1,\ 0 \le t \le 1$$

By taking the step-size h = 0.01in the above examples, the absolute errors by using EPMM and RMM2 are given in Tables 1 – 3 and their corresponding error graphs are shown in Figures 4 – 6.

5. Conclusion

In this paper, the new multi-step method of r-step and p-th order that are based on interpolating functions which consists of both polynomial and exponential function is presented for solving DDEs. The local truncation errors have been determined. The stability polynomials of EPMM (2, p) where $p = 2, 3, 4$ are derived and their corresponding stability regions are obtained and shown in Figures 1–3. The delay argument is approximated using Lagrange interpolation. Numerical examples of DDEs with constant delay, time dependent delay and state dependent delays have been considered to demonstrate the efficiency of the proposed method. The comparative absolute error analyses of EPMM (2,p) in the context of RMM2 (2,p) for Examples 1, 2 and 3 were shown in Tables 1, 2 and 3, respectively. From the Figures 4 – 6, it is evident that the newly proposed method gives results with good accuracy than the existing RMM2. Hence, it is concluded that the proposed EPMM (2, p) is suitable for solving DDEs.

Acknowledgments

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