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Computational optimization of auctioneer revenue in modified discrete Dutch auctions with cara risk preferences

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Abstract

This research presents a computational optimization framework designed to maximize auctioneer revenue in modified discrete Dutch auctions by explicitly incorporating bidders' risk preferences—modeled independently of wealth through the Constant Absolute Risk Aversion (CARA) utility function—thus enabling the analysis of risk-averse, risk-neutral, and risk-loving behaviors within the auction context. The study models bidders with three distinct risk profiles–risk-loving, risk-neutral, and risk-averse–employing nonlinear programming techniques to optimize expected revenues for the discrete bid levels. Discrete optimization methods are applied to analyze the impact of varying risk preferences, revealing that auctioneer revenue grows nonlinearly with bidder participation. For risk-neutral bidders ($\alpha \to 0$), revenue increases sharply from $\mathscr{R}^* = 0.3849$ for $n = 2$ to $\mathscr{R}^* = 0.8179$ for $n = 20$ (a 112.5% increase), but the rate of growth declines significantly beyond $n = 30$, with revenue plateauing near $\mathscr{R}^* = 0.9454$ for $n = 100$ (a mere 9.5% increase from $n = 30$ to $n = 100$). Similar patterns hold for risk-averse ($\alpha > 0$) and risk-loving ($\alpha < 0$) bidders, though the magnitudes differ. Moreover, risk-loving bidders (for $\alpha = -0.5$) yield $\mathscr{R}^* = 1.2122$ for $n = 100$, a 28% higher revenue than risk-neutral case with $\mathscr{R}^* = 0.9454$ and a 61% higher than risk-averse case (for $\alpha = 0.5$) with $\mathscr{R}^* = 0.7519$. This nonlinearity suggests diminishing marginal returns to additional bidders, a critical insight for auction design. The findings suggest that for larger bidder groups, fewer bid levels are sufficient for revenue maximization, with risk-averse behavior decreasing expected returns and risk-loving behavior amplifying them. This computational approach highlights the critical role of risk preferences in auction design, offering a robust mathematical model that can be adapted for broader applications in algorithmic auction mechanisms.

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1. Introduction

Dutch auctions (DA), also known as descending-price auctions, are a unique auction format where the auctioneer starts with a high asking price and progressively lowers it until a bidder accepts the current price. This type of auction is particularly effective for selling perishable goods and time-sensitive items due to its rapid transaction process. In most of the auction models, it is assumed that bidders are risk-neutral, meaning they do not consider risk when making decisions. However, in reality, bidders exhibit varying degrees of risk preferences, which can

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significantly influence their bidding behavior and the auction outcomes.

This research investigates how to optimize the auctioneer's revenue in a modified discrete Dutch auction (DDA) - a format where prices descend in fixed discrete steps rather than continuously. The modification considered here incorporates bidder risk preferences through the Constant Absolute Risk Aversion (CARA) utility function and it is defined as $U(x) = \frac{1-e^{-\alpha x}}{\alpha}$, where α represents the risk parameter. This function is widely used in economic modeling because it implies that a bidder's attitude toward risk remains constant regardless of their wealth level. It enables consistent modeling of risk-averse, risk-neutral, and risk-loving behaviors in a mathematically tractable way. Three cases are explored in this study: risk-loving bidders ($\alpha < 0$), risk-neutral bidders ($\alpha \to 0$), and risk-averse bidders ($\alpha > 0$). Risk-loving bidders prefer uncertainty and are more likely to bid early at higher prices; risk-neutral bidders evaluate outcomes solely based on expected value without regard to uncertainty; risk-averse bidders prefer certainty and tend to delay bidding, waiting for lower prices to avoid potential losses. This study establishes a computational framework for optimizing Dutch auction design by explicitly modeling bidder risk profiles, enabling auctioneers to strategically select bid levels and pricing structures to maximize revenue outcomes. The derived analytical insights reveal how optimal auction parameters vary parametrized by risk attitude, bidder count, and bid-level granularity, providing actionable guidance for tailoring auction formats to specific bidder populations while balancing revenue potential against implementation complexity.

This study advances auction theory by extending the discrete Dutch auction (DDA) model through the integration of bidders' diverse risk preferences, modeled using the CARA utility function. This study develops a computational optimization framework based on nonlinear programming, which systematically determines the set of discrete bid levels that maximize the auctioneer's expected revenue under varying bidder risk preferences independent of their wealth, modeled using the CARA utility function. Implemented through parametric simulations in R, the model quantifies how revenue outcomes shift with changes in number of bid levels, bidder count and risk profiles—risk-averse, risk-neutral, and risk-loving. This enables auctioneers to anticipate revenue performance under different market conditions and tailor bid level granularity accordingly, thus offering a practical decision-support tool for auction design. This research fills a critical gap in the literature by incorporating CARA risk preferences into the DDA model, providing a mathematically rigorous framework that reflects the complexities of real-world bidding dynamics.

2. Literature review

DAs have been the subject of extensive study in auction theory and continue to be a prominent area of research in various fields [1, 2]. Under certain assumptions, Vickrey [3] established the revenue equivalence between DA and first-price sealed-bid auctions. Over time, numerous studies have challenged the concept of revenue equivalence, demonstrating that it holds only

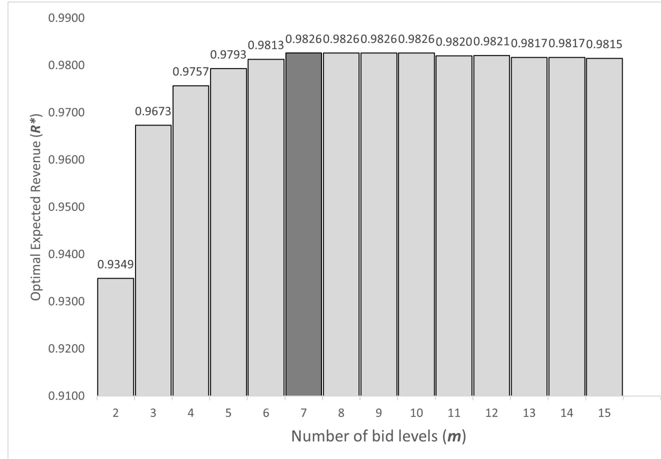


Figure 1: The maximum expected revenue of the auctioneer vs. number of bid levels with $n = 80$ and $\alpha \to 0$.

under certain assumptions and does not apply universally [4–7]. Subsequent research has examined various aspects of DAs, including optimal reserve prices [8], asymmetric bidders [9], and multi-unit auctions [10]. While traditional DA involve continuous price decrements, DDA use discrete bid levels to simplify implementation. The distinction between DA and DDA lies in the granularity of price decrements, with DDA offering more structured bidding increments, which can influence bidder strategies and auction outcomes.

Li and Kuo [11] were among the first to explore revenue maximization in DDAs, formulating the problem as a nonlinear program (NLP) and deriving closed-form expressions for optimal bid levels under the assumption of uniformly distributed valuations. Their results demonstrate that expected revenue increases with the number of bid levels. Building on this, Li and Kuo [12] extended the DDA model by considering the number of bidders as a Poisson random variable, yielding similar outcomes. Li *et al.* [13] further expanded this work by incorporating time constraints and specifying salvage values, revealing that DDA models generate higher average revenues per unit of time for the auctioneer compared to models without time considerations. These studies underscore the critical role of bid levels and time constraints in optimizing auction outcomes.

Most existing models of DAs assume risk-neutral bidders [6, 12–16]. However, bidders often exhibit risk aversion or risk-seeking behavior, significantly impacting auction outcomes. Cox *et al.* [17] were among the early works to incorporate risk aversion in auction models, demonstrating its effects on bidding strategies and revenues. Hu *et al.* [18] explored the influence of risk aversion on equilibrium strategies in continuous DAs, and Makui *et al.* [19] conducted an experimental study on auction behavior with risk preferences. Recently, Shamim and Ali [20] explored bidders' emotional attachment in DDA by incorporating a lognormal valuation distribution and optimizing the formulated nonlinear programming model.

The CARA utility function is widely used in the literature to model risk preferences [21, 22] and has also been employed by some researchers to study bidders' risk preferences in auc-

Table 1: Auctioneer's maximum expected revenue for risk-neutral bidders (i.e. $\alpha \to 0$) for $\bar{v} = 1$, and $m \in \{2, 3, \ldots, 15\}$.

n	$\mathscr{R}^*_{m=2}$	$\mathscr{R}^*_{m=3}$	$\mathscr{R}^*_{m=4}$	$\mathscr{R}^*_{m=5}$	$\mathscr{R}^*_{m=6}$	$\mathscr{R}^*_{m=7}$	$\mathscr{R}^*_{m=8}$	$\mathscr{R}^*_{m=9}$	$\mathscr{R}^*_{m=10}$	$\mathscr{R}^*_{m=11}$	$\mathscr{R}^*_{m=12}$	$\mathscr{R}^*_{m=13}$	$\mathscr{R}^*_{m=14}$	$\mathscr{R}^*_{m=15}$
2	0.3849	0.4908	0.5394	0.5671	0.5850	0.5975	0.6067	0.6137	0.6193	0.6238	0.6275	0.6307	0.6334	**0.6357**
5	0.5824	0.6935	0.7377	0.7611	0.7754	0.7851	0.7920	0.7972	0.8013	0.8045	0.8072	0.8094	0.8113	**0.8129**
10	0.7153	0.8110	0.8449	0.8618	0.8718	0.8784	0.8830	0.8865	0.8891	0.8912	0.8929	0.8943	0.8955	**0.8965**
15	0.7793	0.8619	0.8892	0.9024	0.9101	0.9150	0.9185	0.9211	0.9230	0.9246	0.9258	0.9268	0.9277	**0.9285**
20	0.8179	0.8906	0.9136	0.9244	0.9306	0.9346	0.9374	0.9395	0.9410	0.9422	0.9432	0.9441	0.9447	**0.9452**
25	0.8440	0.9092	0.9291	0.9383	0.9435	0.9469	0.9492	0.9509	0.9522	0.9532	0.9540	0.9542	0.9543	**0.9547**
30	0.8631	0.9222	0.9398	0.9478	0.9523	0.9552	0.9572	0.9587	0.9598	**0.9607**	0.9607	0.9607	0.9607	0.9607
40	0.8891	0.9394	0.9536	0.9601	0.9637	0.9659	0.9675	0.9686	0.9692	0.9688	**0.9695**	0.9688	0.9688	0.9687
60	0.9185	0.9577	0.9682	0.9728	0.9753	0.9769	0.9771	**0.9780**	0.9780	0.9776	0.9779	0.9780	0.9772	0.9770
80	0.9349	0.9673	0.9757	0.9793	0.9813	**0.9826**	0.9826	0.9826	0.9826	0.9820	0.9821	0.9817	0.9817	0.9815
100	0.9454	0.9733	0.9803	0.9833	**0.9850**	0.9850	0.9850	0.9850	0.9850	0.9850	0.9850	0.9850	0.9850	0.9850

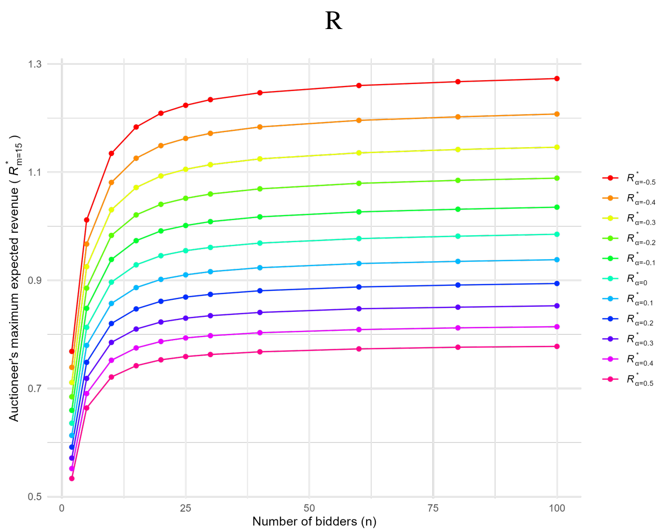


Figure 2: The maximum expected revenue of the auctioneer ($\mathscr{R}^*_{m=15}$) vs. number of bidders (n) where $m = 15$ and $\alpha \in \{-0.5, 0.4, \ldots, 0.5\}$.

tion theory [9, 23–25]. With CARA utility, a bidder's degree of risk aversion is captured by a single parameter α. Positive α indicates risk aversion, negative α indicates risk-seeking behavior and $\alpha \to 0$ reduces to the risk-neutral case. While CARA utility has been applied to analyze various auction formats [9, 23], its implications for DDAs have not been thoroughly explored. Other applications of CARA utility extend beyond auctions, being utilized in financial risk management and insurance modeling, where understanding risk preferences is crucial for decision-making [26, 27].

Bidders' risk preferences can significantly affect auction outcomes, particularly in DAs. Risk-loving bidders may bid earlier at higher prices, potentially increasing auctioneer revenue, while risk-averse bidders may wait for lower prices, risking the item being sold to another bidder. Understanding these dynamics is crucial for optimizing auction design and maximizing revenue. Studies have shown that incorporating risk preferences into auction models can lead to different equilibrium strategies and revenue outcomes, emphasizing the need for auctioneers to consider these factors in auction design [8].

Li and Kuo [11] and Li and Kuo [12] claimed to address risk aversion in their studies. However, in their subsequent work [13], they presented a similar model for risk-neutral bidders, introducing additional parameters such as salvage value and focusing on maximizing revenue per unit of time instead of total revenue. They suggested that future research could explore risk preferences. Despite these claims, none of the studies—Li and Kuo [11], Li and Kuo [12], and Li *et al.* [13]—included terms explicitly representing bidders' risk preferences, nor did they explore the effects of increasing risk-aversion or risk-loving behaviors. Therefore, a more comprehensive model is developed here that accurately captures these preferences. The model in this study introduces bidders whose risk preferences are modeled using the CARA utility function.

Most traditional models of DDAs have assumed risk-neutral bidders, focusing primarily on the effects of bid level granularity, reserve prices, and bidder asymmetries on auction outcomes. Seminal works by Li and Kuo [11–13] formulated revenue-maximizing DDA frameworks under the assumption of uniformly distributed, risk-neutral bidder valuations, finding that expected revenue generally increases with the number of bid levels and bidders. However, this risk-neutral assumption does not capture the diversity of real-world bidding behavior, where participants often exhibit varying degrees of risk aversion or risk-seeking tendencies. Early research that incorporated risk preferences into auction models—such as the use of the CARA utility function—demonstrated that risk attitudes can significantly alter equilibrium strategies and auction revenues, yet these studies were largely limited to continuous auction formats or did not explicitly address DDAs. The CARA utility function, defined as $U(x) = \frac{1-e^{-\alpha x}}{\alpha}$, is widely used in auction and economic theory to capture risk attitudes independent of wealth, with positive α indicating risk aversion and negative α indicating risk-seeking behavior. Despite its theoretical appeal, the implications of CARA-based risk preferences for DDA outcomes have remained underexplored. Recent literature suggests that incorporating such preferences can lead to substantial differences in predicted revenues and optimal auction design, highlighting the necessity of moving beyond risk-neutral models to better reflect the complexities of bidder behavior. By addressing this gap, the present study extends the DDA literature through the explicit integration of CARA-modeled risk preferences, offering a more realistic and

Table 2: Auctioneer's maximum expected revenue for risk-averse bidders (i.e. $\alpha \in \{0.1, 0.2, \ldots, 0.5\}$) for $\overline{v} = 1$, and $m \in \{2, 3, \ldots, 15\}$.

n	$\mathscr{R}^*_{m=2}$	$\mathscr{R}^*_{m=3}$	$\mathscr{R}^*_{m=4}$	$\mathscr{R}^*_{m=5}$	$\mathscr{R}^*_{m=6}$	$\mathscr{R}^*_{m=7}$	$\mathscr{R}^*_{m=8}$	$\mathscr{R}^*_{m=9}$	$\mathscr{R}^*_{m=10}$	$\mathscr{R}^*_{m=11}$	$\mathscr{R}^*_{m=12}$	$\mathscr{R}^*_{m=13}$	$\mathscr{R}^*_{m=14}$	$\mathscr{R}^*_{m=15}$
2	0.3741	0.4755	0.5218	0.5482	0.5651	0.5770	0.5857	0.5923	0.5976	0.6019	0.6054	0.6084	0.6109	**0.6131**
5	0.5625	0.6677	0.7094	0.7313	0.7447	0.7537	0.7602	0.7651	0.7688	0.7719	0.7743	0.7764	0.7781	**0.7796**
10	0.6879	0.7777	0.8094	0.8250	0.8343	0.8404	0.8446	0.8478	0.8503	0.8522	0.8538	0.8551	0.8562	**0.8571**
15	0.7478	0.8249	0.8503	0.8625	0.8695	0.8741	0.8773	0.8797	0.8815	0.8829	0.8840	0.8850	0.8858	**0.8864**
20	0.7838	0.8515	0.8727	0.8827	0.8884	0.8921	0.8946	0.8965	0.8979	0.8991	0.9000	0.9007	0.9014	**0.9016**
25	0.8081	0.8686	0.8869	0.8954	0.9002	0.9033	0.9054	0.9070	0.9082	0.9091	0.9098	**0.9105**	0.9103	0.9100
30	0.8257	0.8806	0.8967	0.9041	0.9083	0.9109	0.9128	0.9141	0.9151	0.9159	**0.9160**	0.9160	0.9160	0.9159
40	0.8498	0.8963	0.9094	0.9153	0.9186	0.9207	0.9221	0.9231	0.9235	0.9237	**0.9239**	0.9233	0.9233	0.9232
60	0.8769	0.9131	0.9227	0.9269	0.9292	0.9307	0.9317	0.9317	0.9313	0.9311	**0.9317**	0.9316	0.9309	0.9309
80	0.8920	0.9219	0.9296	0.9329	0.9347	**0.9358**	0.9358	0.9358	0.9358	0.9355	0.9352	0.9351	0.9349	0.9349
100	0.9017	0.9274	0.9338	0.9365	**0.9380**	0.9380	0.9380	0.9380	0.9380	0.9380	0.9380	0.9380	0.9380	0.9380
(a) For	$\alpha = 0.1$													
2	0.3741	0.4755	0.5218	0.5482	0.5651	0.5770	0.5857	0.5923	0.5976	0.6019	0.6054	0.6084	0.6109	**0.6131**
5	0.5625	0.6677	0.7094	0.7313	0.7447	0.7537	0.7602	0.7651	0.7688	0.7719	0.7743	0.7764	0.7781	**0.7796**
10	0.6879	0.7777	0.8094	0.8250	0.8343	0.8404	0.8446	0.8478	0.8503	0.8522	0.8538	0.8551	0.8562	**0.8571**
15	0.7478	0.8249	0.8503	0.8625	0.8695	0.8741	0.8773	0.8797	0.8815	0.8829	0.8840	0.8850	0.8858	**0.8864**
20	0.7838	0.8515	0.8727	0.8827	0.8884	0.8921	0.8946	0.8965	0.8979	0.8991	0.9000	0.9007	0.9014	**0.9016**
25	0.8081	0.8686	0.8869	0.8954	0.9002	0.9033	0.9054	0.9070	0.9082	0.9091	0.9098	**0.9105**	0.9103	0.9100
30	0.8257	0.8806	0.8967	0.9041	0.9083	0.9109	0.9128	0.9141	0.9151	0.9159	**0.9160**	0.9160	0.9160	0.9159
40	0.8498	0.8963	0.9094	0.9153	0.9186	0.9207	0.9221	0.9231	0.9235	0.9237	**0.9239**	0.9233	0.9233	0.9232
60	0.8769	0.9131	0.9227	0.9269	0.9292	0.9307	0.9317	0.9317	0.9313	0.9311	**0.9317**	0.9316	0.9309	0.9309
80	0.8920	0.9219	0.9296	0.9329	0.9347	**0.9358**	0.9358	0.9358	0.9358	0.9355	0.9352	0.9351	0.9349	0.9349
100	0.9017	0.9274	0.9338	0.9365	**0.9380**	0.9380	0.9380	0.9380	0.9380	0.9380	0.9380	0.9380	0.9380	0.9380
(b) For	$\alpha = 0.2$													
2	0.3637	0.4609	0.5051	0.5301	0.5463	0.5575	0.5657	0.5720	0.5770	0.5811	0.5844	0.5872	0.5896	**0.5916**
5	0.5437	0.6434	0.6825	0.7031	0.7156	0.7241	0.7301	0.7346	0.7382	0.7410	0.7433	0.7452	0.7468	**0.7482**
10	0.6620	0.7463	0.7758	0.7903	0.7989	0.8045	0.8085	0.8114	0.8137	0.8155	0.8169	0.8181	0.8191	**0.8200**
15	0.7181	0.7901	0.8137	0.8249	0.8314	0.8356	0.8386	0.8407	0.8424	0.8437	0.8447	0.8456	0.8463	**0.8470**
20	0.7516	0.8147	0.8343	0.8435	0.8488	0.8521	0.8545	0.8562	0.8575	0.8585	0.8594	0.8600	0.8606	**0.8611**
25	0.7742	0.8304	0.8473	0.8551	0.8596	0.8624	0.8643	0.8657	0.8668	0.8677	**0.8684**	0.8689	0.8689	0.8689
30	0.7906	0.8415	0.8563	0.8631	0.8669	0.8693	0.8710	0.8722	0.8732	0.8739	**0.8740**	0.8740	0.8740	0.8739
40	0.8129	0.8559	0.8680	0.8734	0.8764	0.8782	0.8795	0.8805	**0.8812**	0.8812	0.8810	0.8808	0.8810	0.8806
60	0.8379	0.8712	0.8801	0.8839	0.8860	0.8874	**0.8883**	0.8883	0.8883	0.8880	0.8883	0.8878	0.8876	0.8876
80	0.8518	0.8793	0.8863	0.8893	0.8910	**0.8920**	0.8920	0.8920	0.8920	0.8920	0.8920	0.8916	0.8920	0.8912
100	0.8607	0.8843	0.8901	0.8926	**0.8940**	0.8940	0.8940	0.8940	0.8940	0.8940	0.8940	0.8940	0.8940	0.8940
(c) For	$\alpha = 0.3$													
2	0.3444	0.4338	0.4740	0.4967	0.5112	0.5213	0.5287	0.5344	0.5388	0.5425	0.5454	0.5479	0.5501	**0.5519**
5	0.5088	0.5983	0.6330	0.6511	0.6621	0.6695	0.6747	0.6787	0.6818	0.6842	0.6862	0.6879	0.6893	**0.6905**
10	0.6143	0.6886	0.7142	0.7268	0.7341	0.7390	0.7423	0.7448	0.7468	0.7483	0.7495	0.7506	0.7514	**0.7521**
15	0.6635	0.7264	0.7466	0.7562	0.7618	0.7653	0.7678	0.7697	0.7711	0.7722	0.7730	0.7738	0.7744	**0.7749**
20	0.6927	0.7474	0.7641	0.7719	0.7764	0.7792	0.7812	0.7826	0.7837	0.7846	0.7853	0.7859	0.7864	**0.7868**
25	0.7122	0.7607	0.7751	0.7817	0.7854	0.7878	0.7894	0.7906	0.7915	0.7922	0.7928	**0.7933**	0.7933	0.7933
30	0.7263	0.7701	0.7827	0.7884	0.7916	0.7936	0.7950	0.7960	0.7968	0.7974	**0.7978**	0.7976	0.7976	0.7975
40	0.7454	0.7822	0.7924	0.7969	0.7994	0.8010	0.8021	0.8029	**0.8035**	0.8035	0.8032	0.8031	0.8034	0.8030
60	0.7666	0.7951	0.8025	0.8057	0.8075	0.8086	0.8093	**0.8094**	0.8093	0.8093	0.8093	0.8093	0.8093	0.8088
80	0.7784	0.8018	0.8077	0.8102	0.8116	**0.8124**	0.8124	0.8124	0.8124	0.8124	0.8122	0.8124	0.8124	0.8120
100	0.7860	0.8060	0.8108	0.8129	0.8140	**0.8141**	0.8141	0.8141	0.8141	0.8141	0.8141	0.8141	0.8141	0.8141
(d) For	$\alpha = 0.4$													
2	0.3354	0.4212	0.4595	0.4811	0.4949	0.5045	0.5115	0.5169	0.5211	0.5246	0.5274	0.5298	0.5318	**0.5335**
5	0.4927	0.5775	0.6102	0.6271	0.6374	0.6443	0.6493	0.6529	0.6558	0.6581	0.6600	0.6615	0.6628	**0.6639**
10	0.5923	0.6621	0.6860	0.6976	0.7045	0.7089	0.7121	0.7144	0.7161	0.7176	0.7187	0.7196	0.7204	**0.7211**
15	0.6384	0.6972	0.7160	0.7248	0.7299	0.7332	0.7355	0.7372	0.7385	0.7395	0.7403	0.7410	0.7416	**0.7420**
20	0.6656	0.7166	0.7321	0.7393	0.7434	0.7460	0.7478	0.7491	0.7501	0.7509	0.7515	0.7521	0.7525	**0.7529**
25	0.6838	0.7289	0.7422	0.7482	0.7516	0.7538	0.7553	0.7564	0.7572	0.7579	0.7584	0.7588	**0.7589**	0.7589
30	0.6968	0.7375	0.7491	0.7543	0.7573	0.7591	0.7604	0.7613	0.7620	0.7626	0.7627	**0.7631**	0.7627	0.7627
40	0.7145	0.7486	0.7580	0.7621	0.7644	0.7659	0.7669	0.7676	0.7681	**0.7682**	0.7680	0.7681	0.7679	0.7677
60	0.7342	0.7604	0.7672	0.7701	0.7718	0.7728	0.7734	**0.7735**	0.7735	0.7735	0.7735	0.7735	0.7735	0.7731
80	0.7450	0.7666	0.7719	0.7742	0.7755	0.7762	**0.7763**	0.7763	0.7763	0.7763	0.7763	0.7763	0.7762	0.7762
100	0.7519	0.7704	0.7748	0.7767	0.7777	0.7778	0.7778	**0.7780**	0.7778	0.7778	0.7778	0.7778	0.7778	0.7777
(e) For	$\alpha = 0.5$													

nuanced understanding of how risk attitudes shape auction outcomes and providing new insights for auction design and revenue optimization.

While our model focuses on maximizing revenue by optimizing bid levels based on risk preferences, it also inherently considers the risk of overpayment. Risk-averse bidders, for instance, may wait for lower prices, reducing the likelihood of overpayment. Conversely, risk-loving bidders may bid earlier at higher prices, potentially leading to overpayment. Our framework allows auctioneers to balance these risks by adjusting bid levels according to the prevailing risk attitudes among bidders, thereby mitigating the risk of overpayment while maximizing

Table 3: Auctioneer's maximum expected revenue for risk-loving bidders (i.e. $\alpha \in \{-0.1, -0.2, \ldots, -0.5\}$) for $\overline{v} = 1$, and $m \in \{2, 3, \ldots, 15\}$.

n	$\mathscr{R}^*_{m=2}$	$\mathscr{R}^*_{m=3}$	$\mathscr{R}^*_{m=4}$	$\mathscr{R}^*_{m=5}$	$\mathscr{R}^*_{m=6}$	$\mathscr{R}^*_{m=7}$	$\mathscr{R}^*_{m=8}$	$\mathscr{R}^*_{m=9}$	$\mathscr{R}^*_{m=10}$	$\mathscr{R}^*_{m=11}$	$\mathscr{R}^*_{m=12}$	$\mathscr{R}^*_{m=13}$	$\mathscr{R}^*_{m=14}$	$\mathscr{R}^*_{m=15}$
2	0.3963	0.5068	0.5579	0.5871	0.6059	0.6191	0.6288	0.6362	0.6421	0.6469	0.6509	0.6542	0.6570	**0.6595**
5	0.6032	0.7206	0.7677	0.7926	0.8079	0.8183	0.8257	0.8313	0.8356	0.8391	0.8420	0.8444	0.8464	**0.8481**
10	0.7442	0.8463	0.8827	0.9009	0.9117	0.9188	0.9238	0.9275	0.9304	0.9326	0.9345	0.9360	0.9373	**0.9384**
15	0.8126	0.9011	0.9306	0.9449	0.9532	0.9585	0.9623	0.9651	0.9672	0.9689	0.9703	0.9714	0.9724	**0.9732**
20	0.8541	0.9322	0.9571	0.9688	0.9756	0.9799	0.9830	0.9852	0.9869	0.9882	0.9893	0.9902	0.9910	**0.9912**
25	0.8822	0.9524	0.9739	0.9839	0.9896	0.9933	0.9958	0.9977	0.9991	1.0002	1.0011	1.0014	**1.0019**	1.0012
30	0.9028	0.9665	0.9856	0.9944	0.9993	1.0025	1.0046	1.0062	1.0075	1.0084	**1.0085**	1.0085	1.0084	1.0084
40	0.9309	0.9852	1.0008	1.0078	1.0117	1.0142	1.0159	1.0172	**1.0175**	1.0174	1.0173	1.0172	1.0172	1.0172
60	0.9627	1.0052	1.0167	1.0217	1.0245	1.0263	**1.0275**	1.0270	1.0270	1.0268	1.0275	1.0265	1.0265	1.0264
80	0.9806	1.0158	1.0250	1.0289	1.0311	**1.0325**	1.0325	1.0325	1.0325	1.0319	1.0315	1.0317	1.0314	1.0313
100	0.9921	1.0224	1.0301	1.0333	**1.0352**	1.0351	1.0352	1.0351	1.0352	1.0352	1.0351	1.0352	1.0351	1.0351
(a) For	$\alpha = -0.1$													
2	0.4082	0.5237	0.5773	0.6080	0.6279	0.6418	0.6521	0.6600	0.6662	0.6713	0.6755	0.6790	0.6820	**0.6846**
5	0.6253	0.7493	0.7994	0.8260	0.8424	0.8534	0.8614	0.8674	0.8720	0.8758	0.8788	0.8814	0.8836	**0.8854**
10	0.7748	0.8836	0.9227	0.9423	0.9540	0.9617	0.9671	0.9711	0.9742	0.9767	0.9787	0.9804	0.9818	**0.9829**
15	0.8480	0.9427	0.9746	0.9900	0.9990	1.0049	1.0090	1.0120	1.0143	1.0162	1.0176	1.0189	1.0199	**1.0208**
20	0.8925	0.9764	1.0033	1.0161	1.0235	1.0282	1.0315	1.0339	1.0358	1.0372	1.0384	1.0394	1.0402	**1.0404**
25	0.9228	0.9983	1.0217	1.0326	1.0388	1.0428	1.0456	1.0476	1.0492	1.0504	1.0514	**1.0516**	1.0516	1.0515
30	0.9450	1.0137	1.0344	1.0440	1.0494	1.0528	1.0552	1.0570	1.0583	1.0593	**1.0594**	1.0594	1.0594	1.0594
40	0.9754	1.0341	1.0510	1.0587	1.0630	1.0657	1.0676	1.0689	**1.0692**	1.0691	1.0691	1.0690	1.0691	1.0690
60	1.0099	1.0560	1.0685	1.0740	1.0771	1.0790	1.0799	1.0801	**1.0802**	1.0794	1.0796	1.0792	1.0793	1.0792
80	1.0293	1.0676	1.0776	1.0819	1.0843	**1.0858**	1.0858	1.0858	1.0852	1.0858	1.0849	1.0848	1.0847	1.0847
100	1.0418	1.0748	1.0832	1.0868	**1.0888**	1.0887	1.0887	1.0888	1.0887	1.0887	1.0888	1.0887	1.0888	1.0888
(b) For	$\alpha = -0.2$													
2	0.4208	0.5415	0.5978	0.6302	0.6511	0.6658	0.6767	0.6850	0.6916	0.6970	0.7014	0.7051	0.7083	**0.7110**
5	0.6485	0.7796	0.8329	0.8613	0.8788	0.8907	0.8992	0.9056	0.9106	0.9146	0.9179	0.9207	0.9230	**0.9250**
10	0.8073	0.9233	0.9653	0.9864	0.9990	1.0073	1.0131	1.0175	1.0209	1.0236	1.0257	1.0275	1.0290	**1.0303**
15	0.8855	0.9870	1.0214	1.0381	1.0478	1.0542	1.0587	1.0620	1.0645	1.0665	1.0681	1.0694	1.0706	**1.0715**
20	0.9333	1.0235	1.0526	1.0665	1.0745	1.0796	1.0832	1.0859	1.0879	1.0895	1.0908	1.0919	**1.0928**	1.0928
25	0.9660	1.0473	1.0726	1.0845	1.0913	1.0956	1.0987	1.1009	1.1026	1.1039	1.1050	1.1052	1.1052	1.1051
30	0.9899	1.0640	1.0865	1.0969	1.1028	1.1066	1.1092	1.1111	1.1126	1.1137	1.1137	**1.1138**	1.1138	1.1137
40	1.0228	1.0863	1.1047	1.1130	1.1177	1.1207	1.1228	1.1243	**1.1245**	1.1245	1.1244	1.1244	1.1243	1.1244
60	1.0602	1.1101	1.1238	1.1298	1.1332	1.1353	1.1356	**1.1364**	1.1361	1.1362	1.1357	1.1356	1.1356	1.1355
80	1.0812	1.1228	1.1338	1.1385	1.1412	**1.1428**	1.1428	1.1428	1.1423	1.1428	1.1418	1.1416	1.1415	1.1417
100	1.0949	1.1308	1.1399	1.1439	1.1460	**1.1461**	1.1460	1.1461	1.1461	1.1461	1.1460	1.1460	1.1461	1.1460
(c) For	$\alpha = -0.3$													
2	0.4340	0.5602	0.6193	0.6535	0.6756	0.6911	0.7026	0.7114	0.7184	0.7241	0.7288	0.7327	0.7361	**0.7390**
5	0.6731	0.8117	0.8684	0.8987	0.9174	0.9301	0.9393	0.9462	0.9515	0.9558	0.9594	0.9623	0.9648	**0.9669**
10	0.8416	0.9654	1.0105	1.0332	1.0468	1.0558	1.0621	1.0669	1.0705	1.0734	1.0758	1.0777	1.0794	**1.0808**
15	0.9253	1.0341	1.0712	1.0893	1.0999	1.1068	1.1116	1.1152	1.1180	1.1201	1.1219	1.1234	1.1246	**1.1256**
20	0.9767	1.0736	1.1051	1.1202	1.1289	1.1345	1.1384	1.1413	1.1435	1.1453	1.1467	1.1479	1.1488	**1.1489**
25	1.0118	1.0994	1.1269	1.1398	1.1472	1.1520	1.1553	1.1577	1.1596	1.1610	1.1622	**1.1625**	1.1623	1.1622
30	1.0376	1.1177	1.1421	1.1534	1.1599	1.1640	1.1669	1.1689	1.1705	**1.1718**	1.1718	1.1718	1.1713	1.1716
40	1.0733	1.1419	1.1620	1.1711	1.1762	1.1795	1.1817	1.1834	**1.1836**	1.1836	1.1836	1.1835	1.1834	1.1834
60	1.1138	1.1680	1.1829	1.1895	1.1932	1.1955	**1.1971**	1.1971	1.1967	1.1971	1.1963	1.1960	1.1957	1.1956
80	1.1367	1.1819	1.1939	1.1991	1.2020	**1.2038**	1.2038	1.2038	1.2038	1.2030	1.2028	1.2026	1.2022	1.2021
100	1.1516	1.1906	1.2006	1.2049	1.2073	1.2073	**1.2074**	1.2074	1.2074	1.2074	1.2073	1.2074	1.2073	1.2073
(d) For	$\alpha = -0.4$													
2	0.4479	0.5799	0.6420	0.6780	0.7014	0.7178	0.7299	0.7393	0.7467	0.7527	0.7576	0.7618	0.7654	**0.7685**
5	0.6990	0.8456	0.9059	0.9383	0.9583	0.9719	0.9818	0.9892	0.9949	0.9996	1.0034	1.0065	1.0092	**1.0115**
10	0.8780	1.0101	1.0585	1.0831	1.0977	1.1074	1.1143	1.1194	1.1234	1.1265	1.1291	1.1312	1.1330	**1.1345**
15	0.9676	1.0842	1.1242	1.1438	1.1553	1.1628	1.1681	1.1720	1.1750	1.1773	1.1793	1.1809	1.1822	**1.1833**
20	1.0228	1.1270	1.1611	1.1775	1.1869	1.1931	1.1973	1.2005	1.2029	1.2048	1.2064	1.2076	1.2087	**1.2088**
25	1.0607	1.1550	1.1848	1.1989	1.2070	1.2122	1.2158	1.2184	1.2204	1.2220	1.2233	**1.2234**	1.2234	1.2234
30	1.0885	1.1749	1.2014	1.2138	1.2208	1.2253	1.2284	1.2307	1.2325	1.2338	**1.2339**	1.2339	1.2335	1.2339
40	1.1271	1.2013	1.2231	1.2331	1.2387	1.2423	1.2448	1.2466	**1.2467**	1.2466	1.2467	1.2466	1.2466	1.2466
60	1.1711	1.2298	1.2461	1.2533	1.2574	1.2599	1.2609	**1.2617**	1.2617	1.2603	1.2602	1.2601	1.2600	1.2600
80	1.1960	1.2451	1.2581	1.2639	1.2670	1.2673	1.2673	**1.2690**	1.2690	1.2679	1.2675	1.2674	1.2674	1.2671
100	1.2122	1.2546	1.2655	1.2703	1.2729	1.2729	1.2729	**1.2730**	1.2730	1.2729	1.2729	1.2729	1.2729	1.2729
(e) For	$\alpha = -0.5$													

revenue.

This research addresses the gap in the literature by incorporating CARA risk preferences into a DDA model. By extending the nonlinear programming approach of Li and Kuo [11] and Li *et al.* [13] to account for heterogeneous bidder risk attitudes, the study analyzes how different distributions of risk preferences impact the optimal auction design and expected revenues. It provides a more realistic representation of DAs and yields insights into revenue maximization strategies when faced with risk-averse or risk-seeking bidders.

In summary, this paper contributes to both auction theory and computational mathematics by:

1. Extending the DDA model through the integration of diverse bidder risk preferences, utilizing the CARA utility function.
2. Applying nonlinear programming techniques to analyze the impact of these risk preferences on optimal bid levels and expected revenues, demonstrating the role of mathematical optimization in understanding auction dynamics.
3. Offering a computational framework that provides theo-

Table 4: Risk-neutral (i.e. $\alpha \to 0$) optimal bid levels for $m = 15$ and $\overline{v} = 1$.

n	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
2	0	0.1351	0.2341	0.318	0.393	0.462	0.5265	0.5874	0.6455	0.7012	0.7548	0.8067	0.857	0.9059	0.9535
5	0	0.3108	0.4448	0.5345	0.6037	0.6611	0.7104	0.7541	0.7934	0.8293	0.8625	0.8934	0.9223	0.9496	0.9754
10	0	0.4884	0.6208	0.6959	0.7487	0.7898	0.8235	0.8523	0.8775	0.8999	0.9201	0.9386	0.9556	0.9714	0.9862
15	0	0.5911	0.7111	0.7735	0.8155	0.8472	0.8728	0.8942	0.9127	0.929	0.9436	0.9568	0.9689	0.98	0.9903
20	0	0.5853	0.7394	0.8044	0.8443	0.8729	0.8953	0.9136	0.9292	0.9427	0.9547	0.9654	0.9752	0.9841	0.9923
25	0	0.0008	0.043	0.7163	0.8141	0.8606	0.8905	0.9125	0.9298	0.9442	0.9564	0.9671	0.9766	0.9851	0.9929
30	0	0.0013	0.0015	0.0017	0.128	0.7638	0.8514	0.8923	0.9185	0.9375	0.9525	0.9649	0.9754	0.9846	0.9927
40	0	0.0044	0.0048	0.0054	0.0086	0.0168	0.2649	0.8235	0.8951	0.9279	0.9485	0.9635	0.9752	0.9848	0.9929
60	0	0.0008	0.0013	0.0019	0.0041	0.007	0.0074	0.0093	0.2946	0.8795	0.9356	0.9597	0.9745	0.9851	0.9933
80	0	0.0015	0.008	0.0102	0.0113	0.0116	0.0139	0.0177	0.0193	0.4267	0.9113	0.9556	0.9743	0.9856	0.9937
100	0	0.0015	0.0019	0.0024	0.0028	0.0053	0.0086	0.0087	0.0106	0.0582	0.9191	0.9619	0.9783	0.988	0.9948

retical guidance for auctioneers in setting bid levels to maximize revenue under various bidder risk profiles.

By addressing these aspects, this research enhances the application of mathematical computing to auction design and optimization, offering valuable insights into designing more efficient auction mechanisms. This contribution employs nonlinear programming and discrete optimization to solve a real-world auction problem, thereby advancing both auction theory and computational optimization.

Further, the paper is structured as follows. Section 3 covers the development of the DDA model, considering bidders with CARA risk preferences. Section 4 presents the results and discussion, where the non-linear programming model from Section 3 is optimized using R software. Finally, Section 5 provides the conclusion.

3. Model development

This study examines the effects of the bidders' risk preferences on the revenue of a DA characterized by discrete bidding increments within an independent private value (IPV) framework with symmetric information. In this context, each participant is aware of their own valuation of the item up for auction which is taken from the uniform distribution, and this valuation is not influenced by or known to other bidders [13, 28, 29]. This study examines scenarios where bidders demonstrate risk-averse, risk-neutral, or risk-loving behavior. In each case, a bidder is expected to place a bid when the asking price first drops to or below their valuation.

The discrete bid levels taken in this setting are $b_1 < b_2 < \cdots < b_m$, where $m \geq 1$. Initially, the auctioneer opens the bidding process at a very high bid level b_{m+1} where nobody is willing to bid, and then the price decreases to $b_m, b_{m-1}, \cdots, b_2, b_1$ after each preset interval of time until a bidder bids to buy the item at bid level b_k for any $k \in \{1, 2, \cdots, m\}$. In the DA setting, the item is sold at a price b_i if and only if there exist q number of bidders having their valuations in the interval $[b_i, b_{i+1})$ and nobody is willing to buy it for the price higher than b_{i+1}. Also, the remaining $n-q$ bidders' valuations lie below b_i, $i = 1, 2, \cdots, m$. If only one bidder has the valuation in the interval $[b_i, b_{i+1})$, then the object is sold to him/her and if there are two or more such bidders, the one who stops the clock first or calls out 'mine' first will get the item.

If $n \geq 2$ participants are participating in the auction then the probability of the item to be sold at the price level b_i, $i = 1, 2, \cdots, m$ is $P(b_i)$ and is given by Li *et al.* [11, 13];

$$\begin{aligned} P(b_i) &= \sum_{q=1}^{n} \binom{n}{q} F(b_i)^{n-q} \left[F(b_{i+1}) - F(b_i)\right]^q \\ &= F(b_{i+1})^n - F(b_i)^n, \end{aligned} \tag{1}$$

where the valuations of bidders are drawn from a distribution whose cumulative distribution function (c.d.f.) is $F(.)$ and probability distribution function (p.d.f.) is $f(.)$.

To account for the risk preferences of the bidders, whether they are risk-loving, risk-neutral, or risk-averse, their utility of accepting a bid at the price level b_i is represented using the CARA utility function $U(b_i) = \frac{1-e^{-\alpha b_i}}{\alpha}$, where α is the constant of absolute risk aversion [19, 24, 29–31]. Therefore, the revenue expected by the auctioneer in a DDA considering the risk preferences is given by;

$$\mathscr{R} = \sum_{i=1}^{m} U(b_i) P(b_i). \tag{2}$$

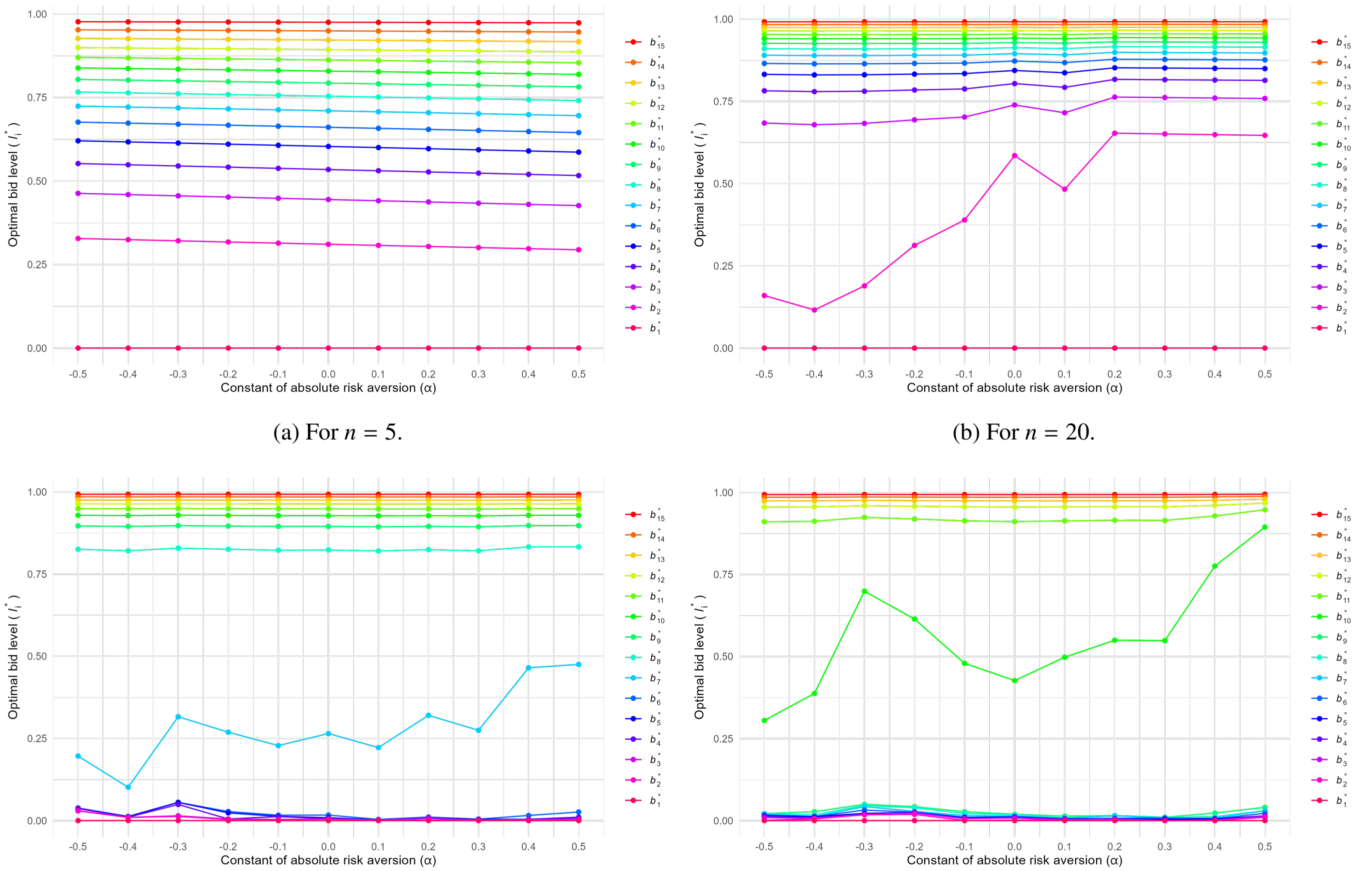
In light of equation (1), the equation (2) becomes:

$$\mathscr{R} = \sum_{i=1}^{m} \frac{1 - e^{-\alpha b_i}}{\alpha} (F(b_{i+1})^n - F(b_i)^n), \tag{3}$$

where α is the coefficient of constant absolute risk aversion which determines the level of risk and v is the bidder's valuation.

Here, it is assumed that the valuation of each bidder j is v_j, $j = 1, 2, \cdots, n$, which is drawn from a uniform distribution defined on $[0, \overline{v}]$ with c.d.f. $F(.)$ and p.d.f $f(.)$. Also, the study defines $b_1 = 0$, $b_{m+1} = \overline{v}$, $F(b_1) = 0$ and $F(b_{m+1}) = \overline{v}$ without any loss of generality [12, 32, 33]. It means that the highest asking price is $\overline{v}$ and the least asking price is 0 indicating that the item is ultimately given away for free and $F(b_i) = \frac{b_i}{\overline{v}}$, $i = 1, 2, \cdots, m$. Hence, the seller's expected revenue $\mathscr{R}$ can be expressed as follows;

$$\mathscr{R} = \sum_{i=1}^{m} \frac{1 - e^{-\alpha b_i}}{\alpha} \Big[(\frac{b_{i+1}}{\overline{v}})^n - (\frac{b_i}{\overline{v}})^n \Big]. \tag{4}$$



(a) For $n = 5$.

(b) For $n = 20$.

(c) For $n = 40$.

(d) For $n = 80$.

Figure 3: Constant of absolute risk aversion (α) vs. Optimal bid levels (b_i^*) when $m = 15$.

Therefore, the formulated model as an NLP in decision variables $b_1, b_2, \cdots, b_m$ and the parameters α, m and n is given as;

Maximize

$$\mathscr{R} = \sum_{i=1}^{m} \frac{1 - e^{-\alpha b_i}}{\alpha}\Big[(\frac{b_{i+1}}{\overline{v}})^n - (\frac{b_i}{\overline{v}})^n\Big]. \tag{5}$$

subject to:

$$b_{i+1} \geq b_i, \quad i = 1, 2, \cdots, m, \\ b_1 \geq 0, \\ b_{m+1} = \overline{v}. \tag{6}$$

In the problem (equation (5)), it is crucial to recognize that as α approaches 0, it signifies the risk-neutral case. This is due to the fact that $\lim_{\alpha \to 0} \frac{1-e^{-\alpha b_i}}{\alpha} = b_i$, which leads to the reduction of the NLP (equation (6)) to the model described by Li and Kuo [11], which does not account for the risk preferences of the bidders, as that model lacks any parameters to define risk behaviors. Moreover, positive α indicates risk-averse bidders, and negative α indicates risk-seeking behavior of the bidders [24, 29, 30].

This paper focuses on maximizing the NLP (equation (5)) while adhering to the specified constraints, and R software is utilized to accomplish this.

4. Results and discussion

In this section, a set of problem instances is addressed to explore the behavior of the developed model as a function of certain parameters. The number of bid levels is denoted as $m \in \{2, 3, \ldots, 15\}$, the number of bidders as $n \in \{2, 5, 10, 15, 20, 25, 30, 40, 60, 80, 100\}$, $\overline{v} = 1$ (as in Ref. [13]) and the risk parameter as $\alpha \in \{-0.5, -0.4, \ldots, 0.5\}$. According to equation (5), the NLPs are set up and solved for the various combinations of m, n, and α by running a program on the R software. In the subsequent discussion, $\mathscr{R}^*_{m=\gamma}$ denotes the auctioneer's maximum expected revenue with γ bid levels.

To initiate further discussions, Table 1 summarizes the auctioneer's expected revenues for all values of m mentioned, under the assumption of risk-neutral bidders. When the bidders are risk-neutral ($\alpha \to 0$), the model (equation (6)) simplifies to the revenue model described in the studies of Li and Kuo [11], and the results presented in Table 1 align with those reported by Li and Kuo [11]. However, it is important to highlight that their analysis was conducted with $\overline{v} = 10$ and limited to $n = 20$, whereas this study adopts $\overline{v} = 1$ and explores a wider range of n values, extending up to 100. By setting $\overline{v} = 10$, all the results of Li and Kuo [11] can be verified.

It can be seen in Table 1 that with the increase in the num-

Table 5: Risk-averse (i.e. $\alpha \in \{0.1,\ 0.2, \ldots,\ 0.5\}$) optimal bid levels for $m = 15$ and $\overline{v} = 1$.

n	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
2	0	0.1327	0.2304	0.3135	0.3881	0.4569	0.5213	0.5824	0.6408	0.6969	0.751	0.8035	0.8544	0.9041	0.9526
5	0	0.3075	0.4411	0.5309	0.6003	0.6579	0.7075	0.7515	0.7911	0.8274	0.8609	0.8921	0.9213	0.949	0.9751
10	0	0.4854	0.6182	0.6937	0.7469	0.7882	0.8222	0.8512	0.8765	0.8991	0.9195	0.9381	0.9553	0.9712	0.986
15	0	0.5885	0.7093	0.7721	0.8143	0.8462	0.872	0.8935	0.9121	0.9285	0.9432	0.9565	0.9687	0.9799	0.9903
20	0	0.4831	0.7156	0.7929	0.8372	0.8681	0.8917	0.911	0.9272	0.9412	0.9535	0.9646	0.9746	0.9838	0.9922
25	0	0.007	0.0071	0.4533	0.7556	0.8325	0.8735	0.9011	0.9218	0.9384	0.9523	0.9642	0.9747	0.984	0.9924
30	0	0.0008	0.0009	0.0062	0.184	0.7656	0.8518	0.8924	0.9184	0.9375	0.9525	0.9649	0.9754	0.9846	0.9927
40	0	0.0017	0.0017	0.0018	0.0024	0.0035	0.2222	0.8204	0.894	0.9272	0.9481	0.9632	0.975	0.9847	0.9929
60	0	0.0021	0.0028	0.0046	0.0047	0.0048	0.0054	0.007	0.4422	0.8847	0.937	0.9603	0.9749	0.9852	0.9934
80	0	0.0017	0.0021	0.004	0.0041	0.0062	0.0075	0.0085	0.0136	0.4979	0.9136	0.9562	0.9745	0.9858	0.9938
100	0	0.0001	0.0003	0.0004	0.001	0.0019	0.0058	0.0069	0.0076	0.0556	0.9185	0.9617	0.9783	0.988	0.9948
(a) For	$\alpha = 0.1$														
2	0	0.1303	0.2267	0.3091	0.3832	0.4518	0.5162	0.5774	0.6361	0.6925	0.7472	0.8002	0.8519	0.9023	0.9516
5	0	0.3042	0.4375	0.5273	0.5969	0.6547	0.7047	0.7489	0.7889	0.8254	0.8592	0.8908	0.9204	0.9483	0.9748
10	0	0.4823	0.6157	0.6916	0.745	0.7866	0.8208	0.85	0.8755	0.8983	0.9188	0.9376	0.9549	0.9709	0.9859
15	0	0.5859	0.7075	0.7707	0.8132	0.8453	0.8711	0.8928	0.9116	0.9281	0.9429	0.9562	0.9685	0.9797	0.9902
20	0	0.6535	0.7633	0.8172	0.8524	0.8785	0.8993	0.9166	0.9314	0.9443	0.9559	0.9663	0.9757	0.9844	0.9925
25	0	0.0024	0.0028	0.6747	0.7974	0.851	0.8841	0.9079	0.9265	0.9417	0.9546	0.9658	0.9757	0.9846	0.9926
30	0	0.0001	0.0012	0.0044	0.199	0.7652	0.8513	0.8921	0.9182	0.9373	0.9523	0.9648	0.9753	0.9845	0.9927
40	0	0.0011	0.0065	0.007	0.0073	0.0108	0.3202	0.8245	0.8952	0.9278	0.9484	0.9634	0.9751	0.9847	0.9929
60	0	0	0.0054	0.0062	0.0083	0.0134	0.0173	0.0184	0.5638	0.8905	0.9388	0.9611	0.9752	0.9855	0.9935
80	0	0.0005	0.0046	0.0046	0.0051	0.0071	0.0146	0.0153	0.0154	0.5498	0.9153	0.9566	0.9747	0.9858	0.9938
100	0	0.0029	0.0093	0.0144	0.027	0.0274	0.0277	0.0292	0.0305	0.1508	0.9194	0.9619	0.9783	0.988	0.9948
(b) For	$\alpha = 0.2$														
2	0	0.128	0.2231	0.3047	0.3784	0.4467	0.5111	0.5724	0.6313	0.6881	0.7432	0.7969	0.8492	0.9005	0.9507
5	0	0.3009	0.4338	0.5237	0.5935	0.6515	0.7017	0.7463	0.7866	0.8234	0.8576	0.8894	0.9193	0.9476	0.9744
10	0	0.4792	0.6131	0.6894	0.7431	0.785	0.8194	0.8488	0.8745	0.8974	0.9181	0.9371	0.9545	0.9707	0.9858
15	0	0.5833	0.7056	0.7692	0.812	0.8443	0.8703	0.8922	0.911	0.9276	0.9425	0.9559	0.9683	0.9796	0.9902
20	0	0.6512	0.7619	0.8161	0.8516	0.8778	0.8987	0.9161	0.931	0.944	0.9556	0.9661	0.9756	0.9844	0.9925
25	0	0.0007	0.1156	0.7156	0.8127	0.8593	0.8894	0.9116	0.9291	0.9436	0.956	0.9668	0.9764	0.985	0.9928
30	0	0.0003	0.0006	0.0006	0.1746	0.7622	0.85	0.8913	0.9176	0.9369	0.9521	0.9646	0.9752	0.9844	0.9926
40	0	0.0015	0.0017	0.0018	0.0042	0.0043	0.2747	0.821	0.8939	0.9271	0.9479	0.9631	0.9749	0.9846	0.9929
60	0	0.0014	0.007	0.0081	0.021	0.0284	0.0284	0.0307	0.7932	0.912	0.9466	0.9648	0.9773	0.9864	0.9938
80	0	0	0.0001	0.0006	0.004	0.008	0.0082	0.01	0.01	0.5486	0.9149	0.9565	0.9746	0.9858	0.9938
100	0	0.0001	0.0034	0.0036	0.0036	0.0046	0.0053	0.0088	0.0202	0.1532	0.919	0.9618	0.9783	0.988	0.9948
(c) For	$\alpha = 0.3$														
2	0	0.1257	0.2196	0.3004	0.3735	0.4416	0.5059	0.5674	0.6265	0.6837	0.7393	0.7935	0.8466	0.8986	0.9497
5	0	0.2976	0.4302	0.5201	0.5901	0.6483	0.6988	0.7436	0.7842	0.8214	0.8559	0.8881	0.9183	0.9469	0.9741
10	0	0.4762	0.6106	0.6872	0.7413	0.7833	0.818	0.8476	0.8735	0.8966	0.9175	0.9365	0.9541	0.9704	0.9857
15	0	0.5807	0.7038	0.7678	0.8108	0.8433	0.8695	0.8915	0.9104	0.9272	0.9421	0.9557	0.968	0.9795	0.9901
20	0	0.649	0.7605	0.8151	0.8507	0.8772	0.8982	0.9157	0.9306	0.9437	0.9554	0.9659	0.9755	0.9843	0.9924
25	0	0.0105	0.1086	0.7134	0.8115	0.8585	0.8888	0.9111	0.9287	0.9433	0.9558	0.9666	0.9762	0.9849	0.9928
30	0	0.0006	0.0022	0.0036	0.2471	0.7658	0.851	0.8917	0.9178	0.937	0.9521	0.9646	0.9752	0.9844	0.9926
40	0	0.0009	0.003	0.0034	0.004	0.0156	0.4648	0.8327	0.8976	0.9289	0.949	0.9637	0.9753	0.9848	0.993
60	0	0.0019	0.0098	0.0109	0.0112	0.0121	0.0122	0.0123	0.6338	0.8946	0.9399	0.9616	0.9755	0.9856	0.9935
80	0	0.0001	0.0015	0.0018	0.0051	0.0054	0.0085	0.0118	0.023	0.7758	0.9286	0.9608	0.9765	0.9866	0.9941
100	0	0.0015	0.0047	0.0053	0.0059	0.0062	0.0208	0.0227	0.0293	0.1855	0.9191	0.9618	0.9782	0.988	0.9948
(d) For	$\alpha = 0.4$														
2	0	0.1235	0.2161	0.2961	0.3688	0.4366	0.5008	0.5623	0.6216	0.6792	0.7353	0.7901	0.8438	0.8967	0.9487
5	0	0.2944	0.4266	0.5165	0.5866	0.6451	0.6958	0.741	0.7819	0.8194	0.8542	0.8867	0.9173	0.9462	0.9737
10	0	0.4731	0.608	0.685	0.7394	0.7817	0.8166	0.8464	0.8725	0.8957	0.9168	0.936	0.9537	0.9702	0.9855
15	0	0.5781	0.7019	0.7663	0.8096	0.8423	0.8687	0.8908	0.9099	0.9267	0.9417	0.9554	0.9678	0.9793	0.99
20	0	0.6468	0.7591	0.8141	0.8499	0.8765	0.8976	0.9152	0.9303	0.9434	0.9552	0.9657	0.9753	0.9842	0.9924
25	0	0.002	0.0814	0.7096	0.8098	0.8574	0.888	0.9105	0.9283	0.943	0.9555	0.9664	0.9761	0.9848	0.9927
30	0	0.0065	0.0095	0.0113	0.3012	0.7688	0.8518	0.892	0.918	0.937	0.9521	0.9646	0.9752	0.9844	0.9926
40	0	0.0037	0.0037	0.0085	0.0101	0.026	0.4752	0.8329	0.8976	0.9288	0.9489	0.9636	0.9752	0.9848	0.9929
60	0	0.0003	0.0057	0.0097	0.0114	0.0114	0.0156	0.0225	0.68	0.8982	0.9411	0.9622	0.9758	0.9857	0.9936
80	0	0.01	0.0127	0.0142	0.0143	0.0222	0.0298	0.0298	0.0403	0.8948	0.9474	0.9678	0.9798	0.9883	0.9948
100	0	0.0025	0.0056	0.0064	0.0114	0.0122	0.0122	0.0147	0.0153	0.1442	0.9178	0.9614	0.9781	0.9879	0.9948
(e) For	$\alpha = 0.5$														

ber of bidders n, the expected revenue consistently increases for each value of the number of bid levels m from 2 to 15. Also, the expected revenue increases with the increase in the value of m corresponding to each value of n. This observation is con-

Table 6: Risk-loving (i.e. $\alpha \in \{-0.1, -0.2, \ldots, -0.5\}$) optimal bid levels for $m = 15$ and $\overline{v} = 1$.

n	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
2	0	0.1376	0.2379	0.3225	0.398	0.4671	0.5316	0.5924	0.6502	0.7055	0.7586	0.8099	0.8595	0.9076	0.9544
5	0	0.3142	0.4484	0.5381	0.6071	0.6642	0.7133	0.7566	0.7956	0.8313	0.8641	0.8947	0.9233	0.9502	0.9758
10	0	0.4915	0.6233	0.698	0.7505	0.7913	0.8249	0.8535	0.8785	0.9007	0.9208	0.9391	0.956	0.9717	0.9863
15	0	0.5937	0.7129	0.7749	0.8166	0.8482	0.8736	0.8949	0.9133	0.9295	0.9439	0.9571	0.9691	0.9801	0.9904
20	0	0.3895	0.7026	0.7881	0.8348	0.8667	0.8909	0.9104	0.9268	0.941	0.9534	0.9645	0.9746	0.9837	0.9922
25	0	0.0035	0.0101	0.2978	0.7411	0.8275	0.871	0.8996	0.9209	0.9379	0.9519	0.964	0.9745	0.9839	0.9923
30	0	0.0045	0.0049	0.0071	0.1264	0.7651	0.8521	0.8928	0.9188	0.9378	0.9528	0.9651	0.9755	0.9847	0.9928
40	0	0.0026	0.0033	0.0123	0.0129	0.0164	0.2282	0.8227	0.8951	0.9279	0.9485	0.9635	0.9752	0.9848	0.9929
60	0	0.0047	0.0056	0.0092	0.0096	0.0107	0.0145	0.0191	0.3813	0.8832	0.9367	0.9603	0.9748	0.9853	0.9934
80	0	0.0004	0.006	0.0062	0.0088	0.01	0.0152	0.0214	0.0269	0.4794	0.9137	0.9563	0.9746	0.9858	0.9938
100	0	0.0041	0.0067	0.009	0.0094	0.0113	0.0114	0.0131	0.0132	0.0629	0.9197	0.9621	0.9784	0.9881	0.9948
(a) For	$\alpha = -0.1$														
2	0	0.1402	0.2417	0.3271	0.403	0.4723	0.5367	0.5974	0.6548	0.7097	0.7623	0.813	0.8619	0.9093	0.9553
5	0	0.3176	0.4521	0.5417	0.6105	0.6673	0.7161	0.7591	0.7978	0.8331	0.8657	0.8959	0.9242	0.9509	0.9761
10	0	0.4946	0.6259	0.7001	0.7524	0.7929	0.8262	0.8546	0.8794	0.9015	0.9215	0.9396	0.9564	0.9719	0.9864
15	0	0.5963	0.7147	0.7763	0.8178	0.8491	0.8744	0.8955	0.9138	0.9299	0.9443	0.9573	0.9693	0.9803	0.9905
20	0	0.3123	0.6941	0.7849	0.8332	0.8657	0.8903	0.9101	0.9266	0.9408	0.9533	0.9645	0.9745	0.9837	0.9922
25	0	0.0168	0.0169	0.3169	0.7441	0.8289	0.8719	0.9003	0.9214	0.9382	0.9522	0.9642	0.9747	0.984	0.9924
30	0	0.0021	0.0022	0.004	0.0815	0.7644	0.8522	0.893	0.919	0.9379	0.9529	0.9651	0.9756	0.9847	0.9928
40	0	0.0033	0.0047	0.0049	0.0236	0.0275	0.2689	0.8256	0.8962	0.9285	0.9489	0.9637	0.9754	0.9849	0.993
60	0	0.0023	0.0083	0.0096	0.0117	0.0239	0.0243	0.0461	0.4688	0.8873	0.938	0.9609	0.9751	0.9854	0.9934
80	0	0.0189	0.0228	0.0245	0.025	0.0258	0.0283	0.0392	0.0424	0.6144	0.9193	0.9579	0.9753	0.9861	0.9939
100	0	0.0058	0.0065	0.0132	0.0266	0.0282	0.0475	0.0493	0.0561	0.1643	0.9216	0.9626	0.9786	0.9882	0.9949
(b) For	$\alpha = -0.2$														
2	0	0.1428	0.2456	0.3317	0.408	0.4775	0.5419	0.6023	0.6595	0.7139	0.766	0.816	0.8643	0.9109	0.9561
5	0	0.3211	0.4558	0.5452	0.6139	0.6704	0.7189	0.7616	0.8	0.835	0.8672	0.8972	0.9252	0.9515	0.9764
10	0	0.4977	0.6284	0.7022	0.7542	0.7945	0.8276	0.8558	0.8804	0.9023	0.9221	0.9401	0.9567	0.9721	0.9865
15	0	0.5989	0.7165	0.7777	0.8189	0.8501	0.8751	0.8962	0.9144	0.9304	0.9447	0.9576	0.9695	0.9804	0.9905
20	0	0.1892	0.6833	0.7811	0.8312	0.8646	0.8896	0.9096	0.9263	0.9406	0.9531	0.9643	0.9745	0.9837	0.9922
25	0	0.0005	0.0088	0.228	0.7383	0.8271	0.871	0.8998	0.9211	0.9381	0.9521	0.9641	0.9746	0.984	0.9924
30	0	0.0015	0.0084	0.0085	0.0292	0.7636	0.8522	0.8931	0.9191	0.938	0.9529	0.9652	0.9757	0.9847	0.9928
40	0	0.0127	0.0145	0.049	0.0554	0.0554	0.316	0.8289	0.8974	0.9291	0.9493	0.964	0.9755	0.985	0.993
60	0	0.0019	0.0077	0.008	0.0158	0.0159	0.0189	0.0418	0.4108	0.8854	0.9375	0.9607	0.975	0.9854	0.9934
80	0	0.0178	0.0186	0.0186	0.0219	0.032	0.0427	0.0461	0.0496	0.6991	0.9243	0.9597	0.9762	0.9865	0.9941
100	0	0.0096	0.01	0.0271	0.0272	0.0363	0.0386	0.0386	0.0446	0.0752	0.9209	0.9624	0.9786	0.9882	0.9949
(c) For	$\alpha = -0.3$														
2	0	0.1455	0.2496	0.3364	0.413	0.4826	0.547	0.6072	0.664	0.718	0.7696	0.819	0.8666	0.9125	0.9569
5	0	0.3246	0.4595	0.5488	0.6172	0.6734	0.7216	0.7641	0.8022	0.8369	0.8688	0.8984	0.9261	0.9521	0.9767
10	0	0.5008	0.6309	0.7043	0.7559	0.796	0.8289	0.8569	0.8813	0.9031	0.9227	0.9406	0.9571	0.9724	0.9866
15	0	0.6015	0.7183	0.7791	0.82	0.851	0.8759	0.8968	0.9149	0.9308	0.945	0.9579	0.9697	0.9805	0.9906
20	0	0.1159	0.6792	0.78	0.8308	0.8644	0.8896	0.9096	0.9263	0.9407	0.9532	0.9644	0.9745	0.9837	0.9922
25	0	0.0006	0.0007	0.0686	0.7299	0.8244	0.8697	0.8991	0.9207	0.9378	0.9519	0.964	0.9746	0.9839	0.9923
30	0	0.0002	0.0007	0.0036	0.0046	0.7136	0.8361	0.8849	0.9142	0.9349	0.9508	0.9638	0.9747	0.9842	0.9925
40	0	0.0096	0.0098	0.0103	0.0119	0.0127	0.1014	0.8211	0.895	0.9281	0.9487	0.9637	0.9753	0.9849	0.993
60	0	0.0002	0.0003	0.0009	0.0121	0.0151	0.0156	0.0167	0.2354	0.8805	0.9362	0.9601	0.9748	0.9852	0.9934
80	0	0.0051	0.0068	0.0081	0.0122	0.0123	0.0125	0.0176	0.027	0.3874	0.9122	0.956	0.9745	0.9858	0.9938
100	0	0.0002	0.0002	0.0095	0.0105	0.0234	0.0247	0.0248	0.0248	0.0888	0.9216	0.9626	0.9787	0.9882	0.9949
(d) For	$\alpha = -0.4$														
2	0	0.1482	0.2536	0.3411	0.4181	0.4878	0.5521	0.6121	0.6685	0.7221	0.7731	0.822	0.8689	0.9141	0.9578
5	0	0.3281	0.4632	0.5524	0.6205	0.6765	0.7244	0.7665	0.8043	0.8387	0.8703	0.8996	0.927	0.9527	0.977
10	0	0.5039	0.6334	0.7064	0.7577	0.7975	0.8302	0.858	0.8823	0.9039	0.9233	0.9411	0.9575	0.9726	0.9867
15	0	0.604	0.72	0.7805	0.8212	0.8519	0.8767	0.8975	0.9154	0.9312	0.9454	0.9582	0.9699	0.9806	0.9907
20	0	0.1598	0.6845	0.7825	0.8325	0.8657	0.8905	0.9103	0.9269	0.9411	0.9536	0.9647	0.9747	0.9838	0.9922
25	0	0.0013	0.0015	0.11	0.7339	0.8262	0.8708	0.8999	0.9213	0.9382	0.9523	0.9643	0.9747	0.984	0.9924
30	0	0.0091	0.0168	0.017	0.1048	0.77	0.8549	0.8947	0.9202	0.9388	0.9535	0.9656	0.9759	0.9849	0.9929
40	0	0.0295	0.0301	0.0369	0.0373	0.0383	0.1964	0.8256	0.8966	0.9289	0.9492	0.964	0.9755	0.985	0.993
60	0	0.002	0.0065	0.007	0.0084	0.0099	0.01	0.01	0.1893	0.88	0.9361	0.9601	0.9747	0.9852	0.9934
80	0	0.0008	0.009	0.0134	0.015	0.0173	0.0187	0.0214	0.0214	0.3052	0.9108	0.9556	0.9743	0.9857	0.9938
100	0	0.0002	0.003	0.0079	0.0097	0.0174	0.0292	0.0336	0.0378	0.0519	0.9217	0.9627	0.9787	0.9882	0.9949
(e) For	$\alpha = -0.5$														

sistent with the existing results in the literature [11, 13, 34]. However, for $n \geq 30$, the expected revenue initially rises as m increases, up to a certain value of m. Beyond this point, the revenue either begins to decline or ceases to grow further. The highest optimum values for expected revenue corresponding to each value of n are boldfaced in Table 1. For example, in the table, the highest revenues for $n = 30, 40, 60, 80, 100$ are observed at $m = 11, 12, 9, 7, 6$ respectively. The scenario with $n = 80$ is illustrated in Figure 1, showing that the optimum revenue $\mathscr{R}^*_{m=2}$ increases with m, peaking at 0.9826 when the least value of m is 7. This suggests that the auctioneer can maximize expected revenue with only 12 or fewer bid levels when n is greater than or equal to 30, and for $n \geq 100$, only 6 bid levels are sufficient. A similar trend is observed in the study by Li *et al.* [13], where it was shown that only three or fewer bid levels are needed to maximize the auctioneer's expected revenue per unit of time. It is important to note that Li *et al.* [13] focused on maximizing revenue per unit of time, whereas maximizing total revenue requires comparatively more bid levels.

Table 2 presents the auctioneer's maximum expected revenue for various bid levels m and numbers of bidders n under conditions of risk-averse bidders, specifically for $\alpha \in \{0.1, 0.2, \ldots, 0.5\}$, where a higher value of α signifies greater risk aversion [35, 36]. The table shows that as α increases, reflecting increased risk-aversion, the auctioneer's expected revenue decreases corresponding to each value of m and n. The higher the value of α, the greater the risk-aversion, leading bidders to bid less aggressively and wait for the price to drop due to their increased tendency to avoid potential losses. This behavior results in a decrease in the auctioneer's maximum expected revenue as α increases. Moreover, comparison between the results of Table 1 for risk-neutral bidders and the results of Table 2 for risk-averse bidders, shows that the revenue for risk-neutral bidders is greater than the revenue for risk-averse bidders. As risk-averse bidders tend to bid less aggressively to avoid potential losses, resulting in a decrease in the auctioneer's maximum expected revenue compared to risk-neutral bidders [35, 37]. In summary, risk-averse bidders are inclined to adopt strategies that prioritize minimizing potential losses over maximizing expected gains. Consequently, this behavior of bidders results in lower bids, directly impacting and reducing the auctioneer's expected revenue [38, 39]. Furthermore, in Table 2a-2e the highest expected revenue corresponding to each value of n is boldfaced and it shows that for a higher number of bidders, a lesser number of bid levels are needed to maximize the auctioneer's expected revenue. The maximum number of bid levels to maximize the auctioneer's expected revenue also increases with the increase in the risk-aversion coefficient α. For instance, in the case of $n = 100$ for $\alpha = 0.1$ only 6 bid levels are sufficient to maximize the expected revenue but for $\alpha = 0.5$, 9 bid levels are needed.

Table 3 presents the auctioneer's maximum expected revenue for various bid levels m and numbers of bidders n under conditions of risk-loving/risk-seeking bidders, specifically for $\alpha \in \{-0.1, -0.2, \ldots, -0.5\}$, where a more negative value of α signifies greater risk-seeking behavior [35, 36]. The table demonstrates that as α becomes more negative, reflecting increased risk-loving behavior, the auctioneer's expected revenue rises for each value of m and n. This increased risk-loving behavior leads bidders to bid more aggressively, avoiding delays for price drops due to their higher tendency to seek risks. Consequently, the auctioneer's maximum expected revenue increases as α decreases. Moreover, when comparing the results of Table 1 for risk-neutral bidders with those of Table 3 for risk-loving bidders, it is evident that the revenue for risk-neutral bidders is lower than that for risk-loving bidders. Risk-loving bidders, driven by their propensity to take risks, tend to bid more aggressively, resulting in higher maximum expected revenue for the auctioneer compared to risk-neutral bidders [35, 37]. In summary, risk-loving bidders prioritize strategies that maximize expected gains over minimizing potential losses. This behavior leads to higher bids, which directly increase the auctioneer's expected revenue [38, 39]. Furthermore, in Tables 3a-3e, the highest expected revenue for each value of n is boldfaced, indicating that a higher number of bidders requires fewer bid levels to maximize the auctioneer's expected revenue. Additionally, the maximum number of bid levels needed to maximize revenue increases as the risk parameter α becomes more negative. For instance, in the case of $n = 100$, only 6 bid levels are sufficient to maximize expected revenue for $\alpha = -0.1$, whereas 9 bid levels are required for $\alpha = -0.5$.

From Table 1 to Table 3, it is evident that the inequality $\mathscr{R}_{rl} > \mathscr{R}_{rn} > \mathscr{R}_{ra}$ holds consistently for each value of m and n, where $\mathscr{R}_{rl}$, $\mathscr{R}_{rn}$, and $\mathscr{R}_{ra}$ denote the auctioneer's expected revenue for risk-loving, risk-neutral, and risk-averse bidders, respectively. These results for $m = 15$ are also illustrated in Figure 2, where the expected revenue $\mathscr{R}^*_{m=15}$ increases consistently as the value of α decreases, confirming the aforementioned inequality. Furthermore, Figure 2 demonstrates that as the number of bidders increases, the revenue initially grows rapidly but the rate of increase slows down after that. Almost similar results can be observed for other values of m.

Tables 4, 5, and 6 present the optimal bid levels for $m = 15$ with $\overline{v} = 1$ for risk-neutral ($\alpha \to 0$), risk-averse ($\alpha > 0$), and risk-loving ($\alpha < 0$) bidders, respectively. In each table, $b_1 = 0$ indicates that the lowest bid level is zero, meaning the item will be given away for free if not sold by that point [11]. This is an assumption in the developed model. Additionally, b_{m+1}, the highest asking price, is set to 1, with all other bid levels determined by optimizing the NLP (equations (5) and (6)). Figure 3 illustrates the relationship between the constant of absolute risk aversion α and the optimal bid levels l_i from Table 4-6. Specifically, Figure 3a and 3b represent the case for $n = 5$ and $n = 20$ respectively, where it is evident that for a small number of bidders, the auctioneer must set each bid level distinctly to maximize expected revenue for each value of α. However, as the number of bidders increases, the lines representing b_i^* become closer, as depicted in 3c and 3d, indicating that the auctioneer can skip several bid levels, as represented by nearly coincident lines, and still maximize revenue for each value of α. These graphs also demonstrate that fewer bid levels are sufficient to maximize the auctioneer's expected revenue as the number of bidders significantly increases.

Although this study has not presented the optimal solution of the NLP (equation (6)) for the parameters chosen in Li and Kuo [11], by selecting the same parameter values and setting $\alpha \to 0$, those results can easily be verified . This demonstrates the superiority of the model developed in this study, as it not only validates existing findings in the literature but also addresses the impact of bidders' risk preferences on the auctioneer's expected revenue in DDAs, a topic that has not been explored before.

5. Conclusion

This study presents a novel approach to modeling the DDA through a nonlinear program that maximizes the auctioneer's expected revenue while accounting for bidders' risk preferences. The developed model extends previous research by incorporating the CARA utility function to represent bidders' risk attitudes, with α as the risk parameter. The results of the extensive numerical experiments yield several significant insights:

Findings of the study indicate that as the number of bidders increases, the auctioneer's expected revenue in DDA also rises. Initially, when the number of bidders increases, the expected revenue experiences rapid growth; however, as the number of bidders becomes larger, the rate of revenue growth slows. For smaller values of n, the auctioneer must set each bid level distinctly from the others to maximize revenue. In contrast, for higher values of n, some bid levels can be omitted without a decrease in optimum revenue. Additionally, with the increase in the number of bid levels m, the maximum expected revenue of the auctioneer increases up to some point and for a substantial number of bidders ($n \geq 30$), the study determined that 12 or fewer bid levels are sufficient to maximize expected revenue, with only 6 bid levels needed when $n \geq 100$. These insights are valuable for auctioneers in estimating potential revenues based on the number of participants. Also, these results have practical implications for auction design, suggesting that auctioneers can simplify their processes without compromising revenue.

As the risk aversion coefficient α increases, the auctioneer's expected revenue decreases. This trend is attributed to risk-averse bidders' tendency to bid less aggressively, prioritizing loss avoidance over potential gains. Conversely, as α becomes more negative (indicating increased risk-loving behavior), the auctioneer's expected revenue rises. This is due to risk-loving bidders' propensity to bid more aggressively, prioritizing potential gains over loss avoidance. It is = consistently observed that $\mathscr{R}_{rl} > \mathscr{R}_{rn} > \mathscr{R}_{ra}$, where $\mathscr{R}_{rl}$, $\mathscr{R}_{rn}$, and $\mathscr{R}_{ra}$ represent the auctioneer's expected revenue for risk-loving, risk-neutral, and risk-averse bidders, respectively. This finding underscores the substantial influence of bidders' risk attitudes on auction outcomes.

While the developed model extends beyond previous research by incorporating risk preferences, it successfully reproduces results from earlier studies when $\alpha \to 0$, confirming its validity and broader applicability.

These findings significantly advance the understanding of DDA and provide practical insights for auction design. However, this study has certain limitations. The absence of real-world data for validation, the assumption of a zero minimum selling price, and the exclusive use of the CARA utility function alongside uniform bidder valuations underscore areas for future exploration. Subsequent research could address these gaps by examining the effects of setting a non-zero minimum price, optimizing revenue per unit of time, exploring alternative probability distributions for bidder valuations, and considering other risk utility functions. Furthermore, validating the model with empirical data, if available, would enhance its practical applicability. While our model assumes that risk aversion is independent of wealth, which simplifies the analysis, we recognize that this assumption may not hold universally. In reality, wealth can influence risk attitudes, with wealthier individuals potentially being more risk-tolerant. Future research could explore how incorporating wealth-dependent risk aversion affects auction outcomes, potentially leading to more nuanced models of bidder behavior.

In conclusion, this research enhances the understanding of DDA by incorporating bidders' risk preferences using a computational optimization framework, offering a more nuanced and realistic model for auction outcomes. By applying nonlinear programming to analyze the impact of risk preferences, this study not only contributes to auction theory but also advances the application of mathematical computing in auction design. The insights gained from this work provide a foundation for developing more efficient and effective auction mechanisms, with implications for various economic and computational contexts.

Data availability

We do not have any research data outside this manuscript.

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