Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 7 (2025) 2569

Journal of the Nigerian Society of Physical Sciences

A Dai-Kou-type method with image de-blurring application

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Abstract

This paper proposes a three-term Dai-Kou-type conjugate gradient method for solving constrained nonlinear monotone equations. This is achieved by exploiting nice attributes of three-term conjugate gradient (TTCG) methods, which includes satisfying the vital condition for global convergence, easy implementation as well as the efficiency of the classical Dai-Kou scheme. The proposed method combines a modified Dai-Kou search direction with the projection strategy, where a hyperplane, which separates the current iterate from the required solution point is constructed. The projection strategy ensures global convergence of the algorithm by projecting the current point onto the hyperplane. The derivative-free structure of the method makes it ideal for solving large-scale and nonsmooth problems. The method also converge globally under mild assumptions. An important contribution of the scheme is its application in image recovery problems, where experiments with some standard images show that it de-blurs noisy images better than some methods in the literature. Furthermore, test results of some numerical experiments suggests that the proposed approach outperforms three recent schemes for convex constrained nonlinear monotone equations

DOI:10.46481/jnsps.2025.2569

Keywords: Nonlinear monotone systems, Spectral parameter, Image de-blurring, Convex constraint

Article History :
Received: 13 December 2025
Received in revised form: 15 March 2025
Accepted for publication: 17 April 2025
Published: 21 May 2025

Communicated by: Pankaj thakur

1. Introduction

In recent decades, practical applications in areas such as science, engineering, industry etc. have been modelled as systems of nonlinear equations. Some Instances of these include the Chandrasekhar integral equation [1], which is vital in radiactive transfer and transport theory [2], the economic equilibrium problems studied in Ref. [3, 4] as well as signal and image processing [5, 6] problems in compressed sensing. For more practical applications of the concept, the reader may refer to [7, 8].

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Typically, nonlinear system of equations is formulated as

$$F(x) = 0, \quad x \in \mathbb{R}^n, \tag{1}$$

where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a nonlinear mapping, which is continuous, and in some cases monotone, namely it satisfies the inequality

$$(F(x) - F(y))^T (x - y) \geq 0, \quad \forall x, y \in \mathbb{R}^n. \tag{2}$$

In this work, we consider the constrained version of Eq. (1) in which the solution exists in a nonempty, closed convex set $C \subset \mathbb{R}^n$, and is formulated as

$$F(x) = 0, \quad x \in C \subset \mathbb{R}^n. \tag{3}$$

Of all the iterative methods for solving Eq. (1) and its constrained version, the Newton's and quasi-Newton's methods [9–11] are the most popular, due to their rapid local convergence properties [12, 13]. These schemes, however, require huge matrix storage to implement at each iteration making them unsuitable for large-scale problems.

By virtue of its less memory requirement, derivative-free structure, and strong global convergence properties [14], the conjugate gradient (CG) method is the most ideal iterative scheme for solving Eq. (1) and Eq. (3) with large dimensions. The methods is primarily developed for solving the minimization problem

$$\min_{x \in \mathbb{R}^n} f(x), \tag{4}$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ represents a nonsmooth nonlinear function with gradient given by $g(x_k) = \nabla f(x) = g_k$. From an initial starting point $x_0 \in \mathbb{R}^n$, iterates of the CG method is generated via the recursive formula

$$x_{k+1} = x_k + \alpha_k d_k, \quad k = 0, 1, ..., \tag{5}$$

where x_k is the k^{th} iterate, $\alpha_k > 0$ is a step-size determined by a line search procedure, and d_k is the scheme's search direction defined by

$$d_{k+1} = -g_{k+1} + \beta_k d_k, \quad d_0 = -g_0, \quad k = 0, 1, ..., \tag{6}$$

in which $g_{k+1} = g(x_{k+1})$ and β_k is the CG update parameter, which defines the type of CG method and is crucial in the scheme's performance. The classical β_k parameters are defined as follows:

$$\beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2} [15], \quad \beta_k^{CD} = \frac{\|g_{k+1}\|^2}{-d_k^T g_k} [16], \quad \beta_k^{DY} = \frac{\|g_{k+1}\|^2}{d_k^T (g_{k+1} - g_k)} [17]. \tag{7}$$

$$\beta_k^{HS} = \frac{g_{k+1}^T (g_{k+1} - g_k)}{d_k^T (g_{k+1} - g_k)} [18], \quad \beta_k^{PRP} = \frac{g_{k+1}^T (g_{k+1} - g_k)}{\|g_k\|^2} [19, 20], \quad \beta_k^{LS} = \frac{g_{k+1}^T (g_{k+1} - g_k)}{-d_k^T g_k} [21]. \tag{8}$$

where $\|.\|$ denotes $\ell_2 - norm$.

A CG scheme implemented using Eq. (4) and Eq. (5) is said to satisfy the descent condition if

$$d_k^T g_k < 0, \quad k \geq 0. \tag{9}$$

In most cases, it suffices to possess the descent condition Eq. (9), however, it is crucial, especially in analyzing global convergence, for CG schemes to satisfy the following sufficient descent condition:

$$g_{k+1}^T d_{k+1} \leq -\vartheta \|g_{k+1}\|^2, \quad \forall\ k \geq 0, \quad \vartheta > 0. \tag{10}$$

Now, even though the CG methods for solving Eq. (4) exhibit some nice attributes, non of the update parameters satisfy Eq. (9) and Eq. (10). To address this shortcoming, their modifications have been proposed in the past decades, see Refs. ([14, 22–28]) and the references therein.

As stated earlier, due to the appealing properties exhibited by CG methods, their modifications for solving Eq. (1) and Eq. (3) have been proposed in recent decades. This includes the work of Cheng in Ref. [29], where the author developed a PRP-type method for solving Eq. (1) by combining the unmodified PRP scheme [19, 20] with the projection method [9]. Under simple assumptions, the author proved global convergence of the scheme.

By applying Powell's strategy on the unmodified Liu-Storey method [21], Liu *et al.* [30] proposed an LS-type scheme for solving Eq. (3). The scheme satisfies the vital descent condition and converges globally under appropriate conditions.

Inspired by the scheme in Ref. [31], Wang *et al.* [23] proposed a three-term CG method for solving Eq. (3), where the method's update parameter is a modified variant of the HS scheme [18]. In Ref. [32], Liu and Wang extended the FR-type scheme developed in [25], to propose its modification for solving symmetric nonlinear equations. Global convergence of the method was shown by employing appropriate conditions.

Liu and Li [33] also combined the DY scheme [34] with the multivariate spectral gradient method developed in Ref. [35] to propose a spectral DY-type method for solving Eq. (3). The authors proved global convergence of the method by applying some mild conditions.

Ahmed *et al.* [36] proposed a modified variant of the HS method by conducting a singular value study of the search direction of a modified scheme for constrained monotone nonlinear system equations with applications. Ahmed *et al.* [37] also proposed an extension of the method in [36] by developing its two-parameter variant for solving Eq. (3) with applications to signal and image deblurring. Only recently, Ahmed *et al.* [38] proposed two RMIL-type CG methods for solving Eq. (3) with compressed sensing applications.

For other methods and recent studies, see the works in Refs. [10, 38–52].

By exploiting the self-scaling memoryless BFGS method by Perry [53] and Shanno [54], an essential CG method that has not gained much attention from researchers was proposed by Dai and Kou [55], with the following formula for β_k:

$$\beta_k^{DK} = \frac{g_{k+1}^T y_k}{d_k^T y_k} - \left(\tau_k + \frac{\|y_k\|^2}{s_k^T y_k} - \frac{s_k^T y_k}{\|s_k\|^2}\right) \frac{g_{k+1}^T s_k}{d_k^T y_k}, \tag{11}$$

where τ_k represents a scaling parameter similar to that of the scaled memoryless BFGS method. The authors in Ref. [55] also presented the truncated version of Eq. (11) as

$$\beta_k^+(\tau_k) = \max\left\{\beta_k^{DK}, \eta \frac{g_{k+1}^T d_k}{\|d_k\|^2}\right\}, \quad \eta \in [0, 1]. \tag{12}$$

An important attribute of the Dai-Kou search direction with the updates Eq. (11) and Eq. (12) is that it satisfies the sufficient descent condition Eq. (10) for all $k \geq 1$. For different choices of

τ_k that have been designed and implemented with the truncated version Eq. (12), see Refs. [56–58] and the references therein.

In [55], the authors employed the parameter defined by Oren and Spedicato in Refs. [57, 59], namely

$$\tau_k = \frac{y_k^T s_k}{\|s_k\|^2}, \tag{13}$$

as their choice of τ_k. However, as an open problem, the authors further stated that other more effective approximations for τ_k are possible using different approaches and techniques. In line with this, Huang and Liu [22] presented a Dai-Kou-type method for solving the optimality condition $\nabla f(x) = 0$, where the scheme utilizes only gradient information. The authors suggested the following choice for τ_k:

$$\tau_k = \zeta\tau_k^{(A)} + (1-\zeta)\tau_k^{(B)}, \quad \zeta \in [0,1],$$

where

$$\tau_k^{(A)} = \frac{\|y_k\|^2}{s_k^T y_k}, \quad \tau_k^{(B)} = \frac{y_k^T s_k}{\|s_k\|^2}.$$

Other choices of the parameter τ_k in Eq. (11) have been proposed over the years, including those in Refs. [60–66]. Though these choices have been employed to implement the schemes, they have some limitations. For example, the choice of τ_k derived in [65] may not always be positive, which can affect the numerical performance of the scheme. Similarly, the two choices for τ_k in [66] require frequent restart procedures when they fail to meet the defined conditions.

Here, by employing a different approach from those attempted in Refs. [22, 65, 66], we develop a three-term structure for the Dai-Kou method in Eq. (11) with some modifications to the choice of τ_k in Eq. (13), and present an effective Dai-Kou-type method for solving Eq. (3).

The main contribution of this paper is to present a more effective Dai-Kou-type method that avoids some of the problems of the methods in Refs. [65, 66] for solving the constrained problem Eq. (3), with application in image deblurring problems. We are motivated by the work of Narushima *et al.* [67] and the unmodified Dai-Kou method [55] for unconstrained optimization.

The rest of the paper is outlined as follows: the motivation, derivations, and algorithm of the scheme are presented in Section 2. Global convergence of the proposed algorithm is established in Section 3. Numerical results and discussions are presented in Section 4. Application of the proposed method in image deblurring is discussed in Section 5. Conclusions are given in Section 6.

2. A three-term modified Dai-Kou method and its Algorithm

In this section, we present some three-term methods proposed for unconstrained optimization and use the idea of their structure to formulate our method for solving Eq. (3). We begin with some notations that will be useful in this section of the work. We consider each vector $v \in \mathbb{R}^n$ to be a column vector with its norm denoted as $\|v\| = \sqrt{v^T v}$. Now, directions of three-term CG methods for solving Eq. (1), Eq. (3) and Eq. (4) are generally formulated as

$$d_{k+1} = -g_{k+1} + \beta_k d_k + \theta_k y_k, \quad \forall\, k \geq 0, \tag{14}$$

where β_k is any of the classical update parameters defined in Eq. (7), Eq. (8) or their modified versions, while θ_k is a parameter defined to ensure the sufficient descent condition Eq. (10) holds. It is known that the classical PRP method [19, 20] does not satisfy Eq. (10). As a result, by employing the idea in Eq. (14), its modified versions with that attribute have been proposed. One of this methods is the three-term scheme proposed by Zhang *et al.* [26] with search direction defined by

$$d_{k+1} = -g_{k+1} + \frac{g_{k+1}^T y_k}{g_k^T g_k} d_k - \frac{g_{k+1}^T d_k}{g_k^T g_k} y_k, \quad \forall\, k \geq 0. \tag{15}$$

It can clearly be observed that Eq. (15) satisfies the inequality Eq. (10) independent of the line search procedure employed. By employing the idea in Eq. (14) Zhang *et al.* [26] proposed a three-term HS scheme with search direction defined as follows:

$$g_{k+1} = -g_{k+1} + \frac{g_{k+1}^T y_k}{s_k^T y_k} s_k - \frac{g_{k+1}^T s_k}{s_k^T y_k} y_k, \quad \forall\, k \geq 0. \tag{16}$$

Here also, the method satisfies Eq. (10) irrespective of the line search technique used. The two schemes were also shown to converge under mild assumptions.

In another development, based on the descent property exhibited in Refs. [26] and [27], Zhang *et al.* [68] proposed a descent three-term extension of the Dai-Liao method [69], with the search direction defined as

$$d_{k+1} = -g_{k+1} + \frac{g_{k+1}^T (y_k - t s_k)}{s_k^T y_k} s_k - \frac{g_{k+1}^T s_k}{s_k^T y_k}(y_k - t s_k), \quad \forall k \geq 0,\ t \geq 0. \tag{17}$$

This method also satisfies the inequality Eq. (10).

Additionally, a three-term conjugate gradient (CG) method was presented by Narushima *et al.* [67], where the scheme's search direction is defined by

$$d_{k+1} = \begin{cases} -g_0, & \text{if } k = 0 \text{ or } g_{k+1}^T \varphi_k = 0; \\ -g_{k+1} + \beta_k d_k - \beta_k \frac{g_{k+1}^T d_k}{g_{k+1}^T \varphi_k} \varphi_k, & \text{otherwise,} \end{cases} \tag{18}$$

in which φ_k is a vector in $\mathbb{R}^n$ and β_k is an update parameter of the scheme. A careful inspection reveals that the scheme also satisfies Eq. (10) irrespective of the line search strategy employed and the choice of the vector $\varphi_k \in \mathbb{R}^n$. Other three-term CG schemes can be found in Refs. [70–72] and the references therein.

Considering that the efficiency and numerical performance of the Dai-Kou method depends heavily on the choice of the parameter τ_k, and given that the few existing adaptations of the method for solving Eq. (3) suffer from certain shortcomings attributed to their choices of τ_k, we present a three-term Dai-Kou method that does not require the explicit computation of the Dai-Kou parameter.

Table 1. Test results of the four methods for problems 1-2.

NP	VAR	IG	TTDK				SCGP				HTTCGP				ACGD			
			NIT	FE	PT	Norm	ITN	FE	PT	Norm	ITN	FE	PT	Norm	ITN	FE	PT	Norm
1	1000	a_1	8	10	0.0520	1.32E-08	14	31	0.0334	8.76E-08	15	25	0.0297	1.65E-08	15	33	0.0273	3.08E-08
	1000	a_2	8	10	0.0159	1.32E-08	14	31	0.0143	8.75E-08	15	25	0.0215	1.65E-08	15	33	0.0206	3.08E-08
	1000	a_3	7	10	0.0160	7.30E-08	21	66	0.0256	9.76E-08	15	24	0.0214	5.68E-08	14	31	0.0203	6.53E-08
	1000	a_4	7	10	0.0087	6.17E-08	14	29	0.0124	8.24E-08	15	24	0.0193	4.79E-08	14	31	0.0143	5.51E-08
	1000	a_5	8	10	0.0093	1.14E-08	18	58	0.0267	1.68E-08	15	24	0.0129	8.92E-08	15	33	0.0143	2.66E-08
	1000	a_6	7	10	0.0122	9.06E-08	14	31	0.0135	5.99E-08	15	24	0.0123	7.04E-08	14	31	0.0141	8.10E-08
	50000	a_1	8	10	0.2036	9.38E-08	16	33	0.4553	7.38E-08	17	27	0.4235	5.02E-08	16	35	0.4646	5.67E-08
	50000	a_2	8	10	0.1970	9.38E-08	16	33	0.4262	7.37E-08	17	27	0.4129	5.02E-08	16	35	0.4531	5.67E-08
	50000	a_3	8	10	0.1923	5.16E-08	15	33	0.4509	8.28E-08	15	25	0.4021	6.43E-08	16	35	0.4614	3.12E-08
	50000	a_4	8	10	0.1987	4.36E-08	18	69	0.6253	7.71E-08	15	25	0.3996	5.42E-08	16	35	0.4851	2.63E-08
	50000	a_5	8	10	0.1962	8.10E-08	16	33	0.4354	5.75E-08	17	27	0.4163	4.34E-08	16	35	0.4502	4.90E-08
	50000	a_6	8	10	0.2090	6.40E-08	16	33	0.4318	5.11E-08	15	25	0.3911	7.97E-08	16	35	0.4469	3.87E-08
	100000	a_1	8	11	0.4568	6.63E-08	16	35	0.9663	5.16E-08	17	27	0.9148	7.10E-08	16	35	1.0059	8.02E-08
	100000	a_2	8	11	0.4239	6.63E-08	16	35	0.9468	5.16E-08	17	27	0.8789	7.10E-08	16	35	0.9684	8.02E-08
	100000	a_3	8	10	0.3974	7.30E-08	33	140	2.6886	8.75E-08	15	25	0.8469	9.09E-08	16	35	0.9958	4.41E-08
	100000	a_4	8	10	0.4207	6.16E-08	27	104	2.0297	7.08E-08	15	25	0.8224	7.67E-08	16	35	0.9692	3.73E-08
	100000	a_5	8	11	0.4552	5.73E-08	25	74	1.6842	7.66E-08	17	27	0.9052	6.14E-08	16	35	0.9745	6.93E-08
	100000	a_6	8	10	0.4225	9.05E-08	24	72	1.6736	5.00E-08	17	27	0.9403	4.85E-08	16	35	0.9955	5.47E-08
2	1000	a_1	7	10	0.0065	6.48E-08	13	29	0.0094	6.82E-08	10	16	0.0072	5.19E-08	14	31	0.0104	2.78E-08
	1000	a_2	7	10	0.0070	6.49E-08	13	29	0.0096	6.83E-08	10	16	0.0070	5.30E-08	14	31	0.0149	2.78E-08
	1000	a_3	1	3	0.0041	0	2	5	0.0041	0	2	5	0.0040	0	14	33	0.0123	3.76E-08
	1000	a_4	9	11	0.0073	3.21E-08	14	32	0.0098	6.6E-08	3	6	0.0046	0	14	34	0.0156	3.00E-08
	1000	a_5	1	3	0.0034	0	1	3	0.0031	0	1	3	0.0036	0	14	33	0.0119	7.32E-08
	1000	a_6	9	11	0.0083	4.11E-08	14	32	0.0101	6.14E-08	2	5	0.0058	0	15	37	0.0115	7.36E-08
	50000	a_1	8	10	0.1329	4.58E-08	15	31	0.2541	5.73E-08	10	17	0.1812	5.24E-08	15	33	0.2846	5.15E-08
	50000	a_2	8	10	0.1353	4.58E-08	15	31	0.2434	5.73E-08	10	17	0.1686	5.25E-08	15	33	0.2875	5.15E-08
	50000	a_3	1	3	0.0378	0	2	5	0.0633	0	2	5	0.0486	0	15	35	0.2877	6.93E-08
	50000	a_4	10	12	0.1628	1.13E-08	16	34	0.2625	5.54E-08	3	6	0.0665	0	15	36	0.2733	4.45E-08
	50000	a_5	1	3	0.0266	0	1	3	0.0262	0	1	3	0.0270	0	16	37	0.2990	3.50E-08
	50000	a_6	10	12	0.1637	1.45E-08	15	33	0.2604	4.12E-08	2	5	0.0482	0	17	41	0.3059	3.52E-08
	100000	a_1	8	10	0.2404	6.48E-08	15	31	0.4705	8.11E-08	10	17	0.3173	7.42E-08	15	33	0.5055	7.29E-08
	100000	a_2	8	10	0.2334	6.48E-08	15	31	0.4822	8.11E-08	10	17	0.3465	7.42E-08	15	33	0.5315	7.29E-08
	100000	a_3	1	3	0.0566	0	2	5	0.0835	0	2	5	0.0999	0	15	35	0.5346	9.80E-08
	100000	a_4	10	12	0.2879	1.60E-08	16	34	0.5489	7.84E-08	3	6	0.1191	0	15	36	0.5563	6.29E-08
	100000	a_5	1	3	0.0500	0	1	3	0.0487	0	1	3	0.0493	0	16	37	0.5456	4.95E-08
	100000	a_6	10	12	0.2851	2.05E-08	16	34	0.5271	7.29E-08	2	5	0.0874	0	17	41	0.6133	4.98E-08

By substituting Eq. (13) into Eq. (11), the revised form of Eq. (11) is obtained as

$$\beta_k^{DK} = \frac{g_{k+1}^T y_k}{d_k^T y_k} - \frac{\|y_k\|^2}{s_k^T y_k} \cdot \frac{g_{k+1}^T s_k}{d_k^T y_k}. \tag{19}$$

Motivated by the descent property exhibited by the three-term method in Eq. (18) and the efficiency of Eq. (19), we propose the following search directions:

$$d_{k+1} = \begin{cases} -F_0, & \text{if } k = 0, \\ -\gamma_k F_{k+1} + \beta_k^{DK1} d_k - \beta_k^{DK1} \frac{F_{k+1}^T d_k}{F_{k+1}^T w_k} w_k, & \text{if } F_{k+1}^T w_k \geq \xi \|s_k\| \|F_{k+1}\| > 0,\ \xi > 0, \\ -\gamma_k F_{k+1} + \beta_k^{DK+} d_k, & \text{otherwise}, \end{cases} \tag{20}$$

where $F_{k+1} = F(x_{k+1})$, $F_k = F(x_k)$, and

$$\beta_k^{DK1} := \max\left\{\beta_k^{DK2},\ \mu \frac{F_{k+1}^T d_k}{\|d_k\|^2}\right\}, \quad \mu > 0, \tag{21}$$

with

$$\beta_k^{DK2} := \frac{F_{k+1}^T}{d_k^T w_k}\left(w_k - \frac{\|w_k\|^2}{d_k^T w_k} d_k\right), \quad w_k := y_k + r s_k, \quad \gamma_k := \frac{s_k^T s_k}{s_k^T w_k}, \quad r > 0, \tag{22}$$

$$y_k := F(z_k) - F(x_k), \quad z_k = x_k + \alpha_k d_k, \quad s_k = z_k - x_k,$$

and γ_k is a spectral parameter, which can be considered a modification of the one proposed by Barzilai and Borwein [73]. The term β_k^{DK+} is a modified form of Eq. (19), defined as

$$\beta_k^{DK+} := \gamma_k \frac{F_{k+1}^T w_k}{d_k^T w_k} - \gamma_k \frac{\|w_k\|^2}{s_k^T w_k} \cdot \frac{F_{k+1}^T s_k}{d_k^T w_k}. \tag{23}$$

We now state the following assumptions required for the next proposition and to establish the global convergence of the proposed method.

Assumption 1. The solution set of F, which is defined as $\bar{C}$, is nonempty.

Assumption 2. The function F in Eq. (1) is Lipschitz continuous; i.e., there exists $L > 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in \mathbb{R}^n. \tag{24}$$

Proposition 1. Suppose that F is monotone. Then the search direction defined by Eq. (20), Eq. (21), Eq. (22), and Eq. (23) satisfies the inequality

$$d_{k+1}^T F_{k+1} \leq -\vartheta \|F_{k+1}\|^2, \quad \forall k \geq 0, \quad \vartheta > 0. \tag{25}$$

Proof. From Eq. (20), for $k = 0$, we have $d_0^T F_0 = -\|F_0\|^2$, which implies that Eq. (25) holds with $\vartheta = 1$.

Table 2. Test results of the four methods for problems 3-4.

NP	VAR	IG	TTDK				SCGP				HTTCGP				ACGD			
			NIT	FE	PT	Norm	ITN	FE	PT	Norm	ITN	FE	PT	Norm	ITN	FE	PT	Norm
3	1000	a_1	7	10	0.0087	6.48E-08	13	29	0.0091	6.82E-08	7	11	0.0057	8.03E-08	14	31	0.0175	2.78E-08
	1000	a_2	7	10	0.0080	6.49E-08	13	29	0.0131	6.83E-08	7	11	0.0149	8.43E-08	14	31	0.0106	2.78E-08
	1000	a_3	1	3	0.0034	0	2	5	0.0048	0	2	5	0.0041	0	14	33	0.0107	3.76E-08
	1000	a_4	9	11	0.0094	3.21E-08	14	32	0.0101	6.60E-08	3	6	0.0045	0	14	34	0.0105	6.11E-08
	1000	a_5	1	3	0.0033	0	1	3	0.0032	0	1	3	0.0039	0	14	33	0.0148	7.32E-08
	1000	a_6	9	11	0.0066	4.11E-08	14	32	0.0145	6.14E-08	2	5	0.0041	0	15	37	0.0113	7.36E-08
	50000	a_1	8	10	0.1333	4.58E-08	15	31	0.2412	5.73E-08	8	14	0.1474	1.86E-08	15	33	0.2618	5.15E-08
	50000	a_2	8	10	0.1388	4.58E-08	15	31	0.2283	5.73E-08	8	14	0.1648	1.86E-08	15	33	0.2596	5.15E-08
	50000	a_3	1	3	0.0292	0	2	5	0.0450	0	2	5	0.0461	0	15	35	0.2848	6.93E-08
	50000	a_4	10	12	0.1611	1.13E-08	16	34	0.2704	5.54E-08	3	6	0.0647	0	15	36	0.2822	4.65E-08
	50000	a_5	1	3	0.0297	0	1	3	0.0258	0	1	3	0.0263	0	16	37	0.2893	3.50E-08
	50000	a_6	10	12	0.1477	1.45E-08	15	33	0.2526	4.12E-08	2	5	0.0491	0	17	41	0.3250	3.52E-08
	100000	a_1	8	10	0.2258	6.48E-08	15	31	0.5030	8.11E-08	8	14	0.2515	2.63E-08	15	33	0.5008	7.29E-08
	100000	a_2	8	10	0.2378	6.48E-08	15	31	0.4637	8.11E-08	8	14	0.2546	2.63E-08	15	33	0.5088	7.29E-08
	100000	a_3	1	3	0.0498	0	2	5	0.0834	0	2	5	0.0822	0	15	35	0.5501	9.80E-08
	100000	a_4	10	12	0.2901	1.60E-08	16	34	0.5014	7.84E-08	3	6	0.1206	0	15	36	0.5419	6.43E-08
	100000	a_5	1	3	0.0489	0	1	3	0.0428	0	1	3	0.0623	0	16	37	0.5211	4.95E-08
	100000	a_6	10	12	0.2899	2.05E-08	16	34	0.5365	7.29E-08	2	5	0.0925	0	17	41	0.6190	4.98E-08
4	1000	a_1	9	11	0.0116	8.03E-08	***	***	***	***	25	47	0.0216	4.20E-08	79	361	0.0990	6.94E-08
	1000	a_2	9	11	0.0088	3.13E-08	***	***	***	***	13	23	0.0123	2.32E-08	21	78	0.0232	4.63E-08
	1000	a_3	8	11	0.0086	7.08E-08	***	***	***	***	18	32	0.0153	1.19E-08	21	86	0.0255	2.83E-08
	1000	a_4	8	10	0.0081	2.88E-08	***	***	***	***	17	30	0.0147	2.97E-08	15	45	0.0151	3.33E-08
	1000	a_5	9	11	0.0082	3.81E-08	***	***	***	***	18	31	0.0144	7.17E-08	38	196	0.0458	9.15E-08
	1000	a_6	9	11	0.0123	5.38E-08	***	***	***	***	16	30	0.0137	2.33E-08	33	152	0.0540	6.13E-08
	50000	a_1	10	12	0.2331	1.93E-08	***	***	***	***	18	32	0.5149	8.82E-08	19	61	0.5916	3.97E-08
	50000	a_2	10	12	0.2215	5.11E-08	***	***	***	***	18	33	0.4450	2.99E-08	23	82	0.7990	4.04E-08
	50000	a_3	9	11	0.2182	1.50E-08	***	***	***	***	18	34	0.4332	4.78E-08	41	168	1.4467	4.89E-08
	50000	a_4	8	10	0.1914	1.94E-08	***	***	***	***	19	32	0.4751	2.52E-08	17	51	0.5303	2.92E-08
	50000	a_5	9	12	0.2213	4.93E-08	***	***	***	***	23	41	0.5447	5.63E-08	31	136	1.1270	3.62E-08
	50000	a_6	9	12	0.2517	7.46E-08	***	***	***	***	17	31	0.4744	7.18E-08	16	47	0.4855	9.57E-08
	100000	a_1	9	12	0.4528	5.39E-08	***	***	***	***	15	28	0.7908	2.01E-08	26	92	1.8445	8.81E-08
	100000	a_2	10	12	0.4757	4.08E-08	***	***	***	***	19	34	0.9936	9.43E-08	31	144	2.5643	9.44E-08
	100000	a_3	9	11	0.4336	6.04E-08	***	***	***	***	16	30	0.8728	5.17E-08	27	114	2.0955	4.71E-08
	100000	a_4	9	12	0.4525	8.63E-08	***	***	***	***	17	30	0.8704	3.34E-08	46	211	3.6977	9.29E-08
	100000	a_5	9	11	0.4515	3.63E-08	***	***	***	***	15	28	0.7975	7.26E-08	25	89	1.7346	5.82E-08
	100000	a_6	9	12	0.4542	5.07E-08	***	***	***	***	21	37	1.1124	3.71E-08	31	146	2.6017	4.99E-08

Now, for $k \geq 1$ and when $F_{k+1}^T w_k \geq \xi \|s_k\| \|F_{k+1}\|$, we need to ensure that γ_k is well-defined. To do this, it suffices to ensure $s_k^T w_k > 0$. By definition and from Eq. (2), we obtain

$$s_k^T w_k = s_k^T (y_k + r s_k) = s_k^T (F(z_k) - F(x_k)) + r\|s_k\|^2 \geq r\|s_k\|^2 > 0, \tag{26}$$

which ensures that γ_k is well-defined.

Now, applying Assumption 2, the Cauchy–Schwarz inequality, and using Eq. (26), we have:

$$\begin{aligned} w_k^T s_k &= (F(z_k) - F(x_k))^T s_k + r\|s_k\|^2 \leq L\|s_k\|^2 + r\|s_k\|^2 \\ &= (L + r)\|s_k\|^2. \end{aligned} \tag{27}$$

Combining Eq. (26) and Eq. (27), we obtain:

$$r\|s_k\|^2 \leq s_k^T w_k \leq (L + r)\|s_k\|^2. \tag{28}$$

Thus,

$$\frac{1}{L + r} := c \leq \frac{\|s_k\|^2}{s_k^T w_k} \leq \frac{1}{r},$$

which implies

$$c \leq \gamma_k \leq \kappa := \frac{1}{r}.$$

Also, using the fact that $s_k = \alpha_k d_k$, we can similarly deduce:

$$d_k^T w_k = \alpha_k^{-1} s_k^T (F(z_k) - F(x_k) + r s_k) \geq \frac{r}{\alpha_k}\|s_k\|^2, \tag{29}$$

which implies that $d_k^T w_k > 0$ whenever the solution is not attained. Therefore, β_k^{DK1} and β_k^{DK+} are well-defined.

Now, when $F_{k+1}^T w_k \geq \xi \|s_k\| \|F_{k+1}\|$, we substitute into Eq. (20) to get:

$$\begin{aligned} d_{k+1}^T F_{k+1} &= -\gamma_k \|F_{k+1}\|^2 + \beta_k^{DK1} d_k^T F_{k+1} - \beta_k^{DK1} \frac{d_k^T F_{k+1}}{F_{k+1}^T w_k} F_{k+1}^T w_k \\ &= -\gamma_k \|F_{k+1}\|^2. \end{aligned} \tag{30}$$

Since $c \leq \gamma_k \leq \kappa$, it follows that

$$d_{k+1}^T F_{k+1} \leq -c\|F_{k+1}\|^2.$$

Hence, Eq. (25) is satisfied with $\vartheta = c$ in this case.

Next, we consider the case where $F_{k+1}^T w_k < \xi \|s_k\| \|F_{k+1}\|$. From Eq. (20) and using Eq. (23), we obtain

$$\begin{aligned} d_{k+1}^T F_{k+1} &= -\gamma_k \|F_{k+1}\|^2 + \gamma_k \frac{F_{k+1}^T w_k}{s_k^T w_k} F_{k+1}^T s_k - \gamma_k \frac{\|w_k\|^2 (F_{k+1}^T s_k)^2}{(s_k^T w_k)^2} \\ &= \frac{\gamma_k F_{k+1}^T w_k \cdot s_k^T w_k \cdot F_{k+1}^T s_k - \gamma_k \|F_{k+1}\|^2 (s_k^T w_k)^2 - \gamma_k \|w_k\|^2 (F_{k+1}^T s_k)^2}{(s_k^T w_k)^2}. \end{aligned} \tag{31}$$

Table 3. Test results of the four methods for problems 5-6.

NP	VAR	IG	TTDK				SCGP				HTTCGP				ACGD			
			NIT	FE	PT	Norm	ITN	FE	PT	Norm	ITN	FE	PT	Norm	ITN	FE	PT	Norm
5	1000	a_1	32	36	0.0207	8.37E-08	***	***	***	***	21	89	0.0231	6.45E-08	187	1669	0.2657	6.23E-08
	1000	a_2	35	39	0.0281	5.99E-08	***	***	***	***	20	82	0.0265	7.08E-08	40	352	0.0556	2.16E-08
	1000	a_3	26	43	0.0190	9.87E-08	***	***	***	***	22	89	0.0208	8.16E-08	***	***	***	***
	1000	a_4	34	39	0.0244	7.08E-08	***	***	***	***	21	89	0.0210	6.45E-08	***	***	***	***
	1000	a_5	34	52	0.0255	8.90E-08	***	***	***	***	25	106	0.0253	4.77E-08	***	***	***	***
	1000	a_6	35	53	0.0261	7.16E-08	***	***	***	***	25	106	0.0310	9.55E-08	***	***	***	***
	50000	a_1	37	41	0.5826	9.06E-08	***	***	***	***	21	89	0.6070	4.91E-08	***	***	***	***
	50000	a_2	36	41	0.5758	9.13E-08	***	***	***	***	24	102	0.6444	9.77E-08	161	1457	6.5525	8.66E-08
	50000	a_3	30	48	0.5726	9.22E-08	***	***	***	***	27	114	0.7288	6.35E-08	***	***	***	***
	50000	a_4	31	49	0.5583	7.29E-08	***	***	***	***	30	117	0.8255	9.48E-08	***	***	***	***
	50000	a_5	36	56	0.6400	8.98E-08	***	***	***	***	27	114	0.7092	7.08E-08	***	***	***	***
	50000	a_6	38	57	0.6453	7.16E-08	***	***	***	***	27	111	0.7380	9.46E-08	***	***	***	***
	100000	a_1	38	42	1.2951	6.89E-08	***	***	***	***	20	88	1.1858	9.22E-08	116	1060	10.7848	7.73E-08
	100000	a_2	38	42	1.3089	9.15E-08	***	***	***	***	20	84	1.1278	9.45E-08	217	1819	18.5753	9.46E-08
	100000	a_3	30	49	1.1578	7.02E-08	***	***	***	***	27	111	1.5035	5.96E-08	181	1455	14.8590	6.44E-08
	100000	a_4	31	50	1.1869	5.56E-08	***	***	***	***	26	104	1.4388	5.62E-08	64	580	5.8005	3.88E-08
	100000	a_5	38	58	1.3649	6.42E-08	***	***	***	***	29	121	1.6456	7.29E-08	***	***	***	***
	100000	a_6	38	58	1.4028	5.54E-08	***	***	***	***	33	124	1.7751	8.96E-08	***	***	***	***
6	1000	a_1	2	6	0.0042	0	26	192	0.0492	6.27E-08	13	51	0.0129	3.52E-08	11	96	0.0236	0
	1000	a_2	2	6	0.0047	0	20	132	0.0274	4.12E-08	12	47	0.0117	0	16	145	0.0258	0
	1000	a_3	3	10	0.0079	0	3	18	0.0061	0	8	37	0.0104	0	2	15	0.0147	0
	1000	a_4	4	13	0.0066	0	***	***	***	***	20	125	0.0388	0	***	***	***	***
	1000	a_5	1	8	0.0043	0	2	14	0.0052	0	3	17	0.0060	0	1	3	0.0044	0
	1000	a_6	5	15	0.0070	0	1	6	0.0052	0	6	39	0.0149	6.30E-10	***	***	***	***
	50000	a_1	2	6	0.0692	0	13	63	0.4394	1.55E-08	12	47	0.3632	0	15	108	0.6409	2.12E-08
	50000	a_2	2	6	0.0791	0	13	60	0.4417	1.50E-08	12	47	0.3654	0	15	108	0.6488	2.12E-08
	50000	a_3	3	10	0.1284	0	3	18	0.1690	0	8	37	0.3176	0	***	***	***	***
	50000	a_4	4	13	0.1560	0	***	***	***	***	20	125	1.0203	0	***	***	***	***
	50000	a_5	1	8	0.0635	0	2	14	0.1035	0	3	17	0.1389	0	1	3	0.0370	0
	50000	a_6	5	15	0.1751	0	1	6	0.0596	0	6	39	0.2980	4.45E-09	***	***	***	***
	100000	a_1	2	6	0.1335	0	13	62	0.9059	2.16E-08	12	47	0.7396	6.00E-15	15	108	1.3636	3.05E-08
	100000	a_2	2	6	0.1311	0	13	61	0.8918	2.18E-08	12	47	0.7641	5.62E-15	15	108	1.3498	3.05E-08
	100000	a_3	3	10	0.2169	0	3	18	0.2894	0	8	37	0.6147	0	2	15	0.7056	0
	100000	a_4	4	13	0.2920	0	***	***	***	***	20	125	2.0233	0	***	***	***	***
	100000	a_5	1	8	0.1213	0	2	14	0.2017	0	3	17	0.2514	0	1	3	0.0762	0
	100000	a_6	5	15	0.3527	0	1	6	0.1167	0	6	39	0.7012	6.30E-09	***	***	***	***

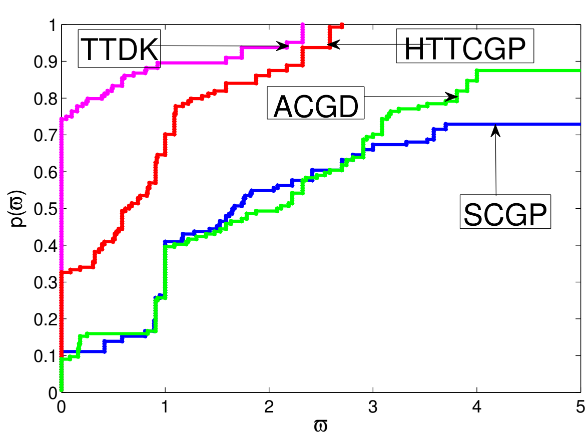

Figure 1. The four method's performance profile for number of iterations.

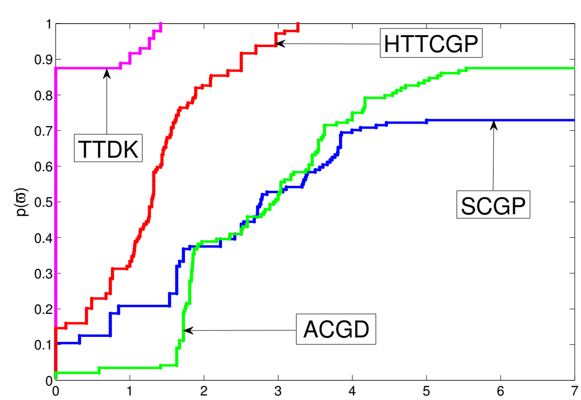

Figure 2. The four method's performance profile for function evaluations.

Now, using the inequality $c_1^T c_2 \le \frac{1}{2}(\|c_1\|^2+\|c_2\|^2)$ on the second equality above, where we set

$$c_1 = \sqrt{\gamma_k}\frac{F_{k+1}(s_k^T w_k)}{\sqrt{2}}, \quad c_2 = \sqrt{2\gamma_k}(F_{k+1}^T s_k) w_k,$$

and obtain

$$\begin{aligned} d_{k+1}^T F_{k+1} &\le \frac{\frac{\gamma_k \|F_{k+1}\|^2 (s_k^T w_k)^2}{4} + \gamma_k \|w_k\|^2 (F_{k+1}^T s_k)^2 - \gamma_k (s_k^T w_k)^2 \|F_{k+1}\|^2 - \gamma_k \|w_k\|^2 (F_{k+1}^T s_k)^2}{(s_k^T w_k)^2} \\ &= \frac{\gamma_k \|F_{k+1}\|^2}{4} - \gamma_k \|F_{k+1}\|^2 \\ &\le -\frac{3c}{4}\|F_{k+1}\|^2. \end{aligned} \tag{32}$$

Table 4. Test results of the four methods for problems 7-8.

NP	VAR	IG	TTDK				SCGP				HTTCGP				ACGD			
			NIT	FE	PT	Norm	ITN	FE	PT	Norm	ITN	FE	PT	Norm	ITN	FE	PT	Norm
7	1000	a_1	13	16	0.0092	8.64E-08	29	138	0.02965	4.69E-08	22	59	0.0196	7.25E-08	13	57	0.0117	4.30E-08
	1000	a_2	13	16	0.0143	8.64E-08	26	110	0.01995	4.23E-08	23	63	0.0147	6.23E-08	13	57	0.0120	4.29E-08
	1000	a_3	15	18	0.0101	3.60E-08	53	287	0.04716	8.06E-08	17	45	0.0123	7.53E-08	17	64	0.0140	5.72E-08
	1000	a_4	15	19	0.0102	5.83E-08	46	215	0.04412	5.41E-08	22	58	0.0143	5.10E-08	17	64	0.0296	5.93E-08
	1000	a_5	15	18	0.0118	8.94E-08	47	251	0.03942	8.95E-08	20	54	0.0323	4.88E-08	17	65	0.0140	6.39E-08
	1000	a_6	16	19	0.0118	3.63E-08	43	192	0.03549	8.99E-08	17	42	0.0112	5.08E-08	16	63	0.0131	5.24E-08
	50000	a_1	15	17	0.2139	9.98E-08	34	142	0.72422	6.68E-08	18	50	0.3386	7.28E-08	14	61	0.3174	7.42E-08
	50000	a_2	15	17	0.2207	9.98E-08	27	112	0.56895	4.39E-08	22	61	0.4175	6.23E-08	14	61	0.3209	7.42E-08
	50000	a_3	16	19	0.2392	6.08E-08	46	224	1.08430	7.41E-08	21	56	0.4254	4.05E-08	18	68	0.3894	9.74E-08
	50000	a_4	16	20	0.2446	9.84E-08	54	272	1.27313	5.16E-08	24	63	0.4208	1.70E-08	18	68	0.4204	7.44E-08
	50000	a_5	17	20	0.2461	3.60E-08	49	265	1.18219	5.00E-08	21	56	0.3895	7.14E-09	19	73	0.4395	2.73E-08
	50000	a_6	17	20	0.2673	6.12E-08	58	280	1.41202	4.94E-08	17	44	0.3181	1.34E-08	17	67	0.3692	7.75E-08
	100000	a_1	15	18	0.4601	4.91E-08	50	234	2.25181	5.83E-08	24	65	0.9052	6.07E-08	15	65	0.7076	2.54E-08
	100000	a_2	15	18	0.4408	4.91E-08	20	63	0.77876	7.79E-08	23	62	0.8818	8.49E-08	15	65	0.7078	2.54E-08
	100000	a_3	16	19	0.4624	8.60E-08	57	328	2.92865	6.44E-08	21	56	0.8188	5.73E-08	19	72	0.8588	3.32E-08
	100000	a_4	17	20	0.4987	9.53E-08	54	289	2.67821	7.11E-08	24	63	0.8943	2.40E-08	19	72	0.8486	2.50E-08
	100000	a_5	17	20	0.5033	5.09E-08	48	243	2.35149	4.85E-08	21	56	0.7630	1.01E-08	19	73	0.8457	3.87E-08
	100000	a_6	17	20	0.4815	8.66E-08	72	400	3.69100	8.53E-08	17	44	0.6333	1.90E-08	18	71	0.8299	2.63E-08
8	1000	a_1	2	3	0.0048	0	23	38	0.01473	5.07E-08	9	15	0.0080	0	12	32	0.0120	9.25E-09
	1000	a_2	2	3	0.0041	0	26	43	0.01834	7.33E-08	9	15	0.0081	0	14	48	0.0151	2.31E-08
	1000	a_3	2	3	0.0039	0	4	5	0.00499	0	3	4	0.0045	0	19	65	0.0189	8.87E-08
	1000	a_4	3	4	0.0047	0	4	5	0.00605	2.88E-13	3	4	0.0150	0	10	15	0.0092	1.73E-10
	1000	a_5	2	3	0.0051	0	2	3	0.00399	5.46E-14	3	4	0.0042	0	6	8	0.0071	1.56E-09
	1000	a_6	3	4	0.0089	0	3	4	0.00459	0	11	17	0.0098	3.49E-08	14	34	0.0214	3.23E-08
	50000	a_1	2	3	0.0446	0	24	43	0.43000	5.10E-08	10	17	0.1933	0	23	94	0.6404	4.96E-08
	50000	a_2	2	3	0.0451	0	24	43	0.43174	5.14E-08	10	17	0.1965	0	26	110	0.7383	2.92E-08
	50000	a_3	2	3	0.0452	0	4	5	0.07553	0	3	4	0.0614	0	15	27	0.3075	5.87E-08
	50000	a_4	3	4	0.0629	0	4	5	0.07986	0	3	4	0.0646	0	15	28	0.3013	8.41E-08
	50000	a_5	2	3	0.0417	0	2	3	0.03941	1.32E-11	3	4	0.0540	0	13	24	0.2405	4.54E-08
	50000	a_6	3	4	0.0635	0	3	4	0.06839	0	11	17	0.2322	5.81E-08	23	64	0.5205	3.60E-08
	100000	a_1	2	3	0.0869	0	24	43	0.88568	6.99E-08	10	17	0.3882	0	17	70	0.9634	0
	100000	a_2	2	3	0.0976	0	24	43	0.84930	7.07E-08	10	17	0.3833	0	18	79	1.0903	0
	100000	a_3	2	3	0.0837	0	4	5	0.15927	0	3	4	0.1172	0	17	32	0.6866	5.45E-08
	100000	a_4	3	4	0.1192	0	4	5	0.14799	4.60E-10	3	4	0.1266	0	13	23	0.4961	7.90E-08
	100000	a_5	2	3	0.0937	0	2	3	0.07667	0	3	4	0.1041	0	13	24	0.4740	6.40E-08
	100000	a_6	3	4	0.1183	0	3	4	0.11290	0	11	17	0.4124	7.87E-08	14	37	0.6510	0

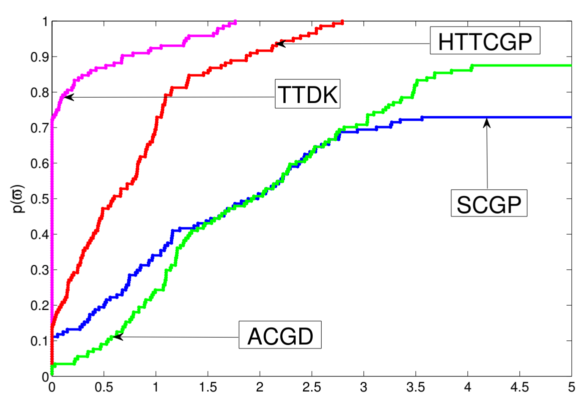

Figure 3. The four method's performance profile for processing time.

By letting $\vartheta := \min\{1, c, \frac{3c}{4}\}$, the proof is established.

Now, we briefly discuss the projection operator. Let $C \subset \mathbb{R}^n$ be as defined in the introduction section. Then, the projection of $x \in \mathbb{R}^n$ onto C is defined as

$$P_C(x) = \arg\min_{y \in C} \|x - y\|,$$

with the properties

$$\|P_C(x) - P_C(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathbb{R}^n,$$

and

$$\|P_C(x) - y\| \leq \|x - y\|, \quad \forall y \in C. \tag{33}$$

Algorithm 1

1. Given $\epsilon > 0$, $x_0 \in C$, $\phi \in (0, 2)$, $\delta \in (0, 1)$, $\zeta > 0$. Set $k = 0$ and $d_0 = -F_0$.
2. Determine $F(x_k)$. If $\|F(x_k)\| \leq \epsilon$, stop; otherwise, go to step 3.
3. Compute $z_k = x_k + \alpha_k d_k$, where $\alpha_k = \delta^{m_k}$ with m_k being the smallest nonnegative integer m for which
$$-F(x_k + \delta^m d_k)^T d_k \geq \zeta \delta^m \|d_k\|^2 \tag{34}$$
is satisfied.
4. If $z_k \in C$ and $\|F(z_k)\| \leq \epsilon$, stop. Otherwise, compute
$$x_{k+1} = P_C\left[x_k - \sigma_k F(z_k)\right],$$
where
$$\sigma_k = \frac{F(z_k)^T (x_k - z_k)}{\|F(z_k)\|^2}. \tag{35}$$

Figure 4. From the left: Original and blurred images (First and second columns), restored images by TTDK, HTTCGP, and MFRM (third, fourth and fifth columns.

5. Obtain direction d_{k+1} using Eq. (20), Eq. (21), and Eq. (22) if $F(x_{k+1})^T w_k \geq \xi \|s_k\| \|F_{k+1}\|$; otherwise, set

   $d_{k+1} = -\gamma F_{k+1} + \beta_k^{DK+} d_k,$

   where γ_k and β_k^{DK+} are defined by Eq. (22) and Eq. (23).
6. Set $k = k + 1$ and return to step 2.

3. Convergence Results

We begin with the following lemma.

Lemma. Let $\{d_{k+1}\}$ be the sequence of search directions obtained by Algorithm 1. Then

$$\vartheta \|F_{k+1}\| \leq \|d_{k+1}\| \leq \psi \|F_{k+1}\|, \quad \forall\, k \geq 0, \tag{36}$$

where $\vartheta := \min\{1, c, \frac{3c}{4}\}$ and $\psi := \max\{\psi_1, \psi_2, \psi_3\}$.

Proof. It is easy from the Cauchy–Schwarz inequality and Eq. (25) to show that the first inequality holds. From Eq. (20), we have $d_0 = -F_0$, which shows $\|d_0\| = \|F_0\|$. Now, we prove that the second inequality in Eq. (36) is satisfied for $k \geq 1$. By the Lipschitz continuity of F, we obtain

$$\|w_k\| = \|y_k + r s_k\| \leq \|y_k\| + r\|s_k\| \leq L\|s_k\| + r\|s_k\| = (L + r)\|s_k\|. \tag{37}$$

Next, we consider three cases for the search directions for k≥1.

Case 1:. From Eq. (20), for $k \geq 1$ and $F_{k+1}^T w_k \geq \xi \|s_k\| \|F_{k+1}\|$. If $\beta_k^{\mathrm{DK2}} > \mu \frac{F_{k+1}^T d_{k-1}}{\|d_{k-1}\|^2}$, then $\beta_k^{\mathrm{DK1}} = \beta_k^{\mathrm{DK2}}$. Using Eq. (20), Eq. (22), Eq. (37), and Cauchy–Schwarz inequality, we get:

$$\begin{aligned}\|d_{k+1}\| &\leq \gamma_k \|F_{k+1}\| + \frac{L+r}{r}\|F_{k+1}\| + \left(\frac{L+r}{r}\right)^2 \|F_{k+1}\| \\ &\quad + \frac{L+r}{r\xi}\|F_{k+1}\| + \frac{(L+r)^3}{\xi r^2}\|F_{k+1}\| \\ &= \psi_1 \|F_{k+1}\|,\end{aligned} \tag{38}$$

where $\psi_1 = \kappa + \frac{L+r}{r} + \left(\frac{L+r}{r}\right)^2 + \frac{L+r}{r\xi} + \frac{(L+r)^3}{\xi r^2}$.

Case 2:. From Eq. (20), for $k \geq 1$ and $F_{k+1}^T w_k \geq \xi \|s_k\| \|F_{k+1}\|$. If $\beta_k^{\mathrm{DK2}} \leq \mu \frac{F_{k+1}^T d_k}{\|d_k\|^2}$, then $\beta_k^{\mathrm{DK1}} = \mu \frac{F_{k+1}^T d_k}{\|d_k\|^2}$. Using similar inequalities, we get:

$$\begin{aligned}\|d_{k+1}\| &\leq \kappa \|F_{k+1}\| + \mu \|F_{k+1}\| + \mu \frac{(L+r)}{\xi}\|F_{k+1}\| \\ &= \psi_2 \|F_{k+1}\|,\end{aligned} \tag{39}$$

where $\psi_2 = \kappa + \mu + \mu\frac{L+r}{\xi}$.

Case 3:. From Eq. (20), for $k \geq 1$ and $F_{k+1}^T w_k < \xi \|s_k\| \|F_{k+1}\|$, then:

$$\begin{aligned}\|d_{k+1}\| &\leq \kappa \|F_{k+1}\| + \frac{\kappa(L+r)}{r}\|F_{k+1}\| + \frac{\kappa(L+r)^2}{r^2}\|F_{k+1}\| \\ &= \psi_3 \|F_{k+1}\|,\end{aligned} \tag{40}$$

where $\psi_3 = \kappa + \frac{\kappa(L+r)}{r} + \frac{\kappa(L+r)^2}{r^2}$.

Thus, by defining $\psi := \max\{\psi_1, \psi_2, \psi_3\}$, we obtain the required result. We use the next Lemma to show that the line-search condition Eq. (34) used in step 2 of Algorithm 1 is well-defined and yields a uniform lower bound of α_k.

Lemma. (1) Let $\{d_k\}$ and $\{x_k\}$ be sequences generated by Algorithm 1. If F is continuous on $\mathbb{R}^n$, for each $k \geq 0$, then a nonnegative integer m_k exists such that Eq. (34) holds.

(2) Let Assumption 2 hold with $\{x_k\}$ and $\{z_k\}$ generated by Algorithm 1. Then, the step-size $\alpha_k > 0$ computed in step 2 of Algorithm 1 satisfies the inequality

$$\alpha_k \geq \alpha := \min\left\{1, \frac{\delta\vartheta}{(L+\zeta)\psi^2}\right\}. \tag{41}$$

Proof. For the first part, suppose $k_0 \geq 0$ exists for which Eq. (34) does not hold in the k_0^{th} iterate for each nonnegative integer m. Then for all $m \geq 0$,

$$-F(x_{k_0} + \delta^m d_{k_0})^T d_{k_0} < \zeta \delta^m \|d_{k_0}\|^2. \tag{42}$$

Since F is continuous on $\mathbb{R}^n$, taking limit in Eq. (42) as $m \to \infty$ yields

$$F(x_{k_0})^T d_{k_0} \leq 0,$$

which contradicts Eq. (25), i.e.,

$$F(x_{k_0})^T d_{k_0} \leq -\vartheta \|F(x_{k_0})\|^2 < 0.$$

Hence, we established the result.

For the proof of (2), assume that Algorithm 1 stops at x_k, then $F(x_k) = 0$ or $F(z_k) = 0$, implying x_k to be a solution. However, if $F(x_k) \neq 0$, by Eq. (25), we have that $d_k \neq 0$. To show that Eq. (34) stops at a finite number of steps, we see from Eq. (34) that if $\alpha_k \neq 1$, then $\bar{\alpha}_k = \delta^{-1}\alpha_k$ does not satisfy Eq. (34), namely,

$$-F(\bar{z}_k)^T d_k < \zeta \bar{\alpha}_k \|d_k\|^2,$$

with $\bar{z}_k = x_k + \bar{\alpha}_k d_k$. From Eq. (24) and Eq. (25), we have

$$\begin{aligned}\vartheta\|F_k\|^2 &\leq -F_k^T d_k \\ &= (F(\bar{z}_k) - F_k)^T d_k - F(\bar{z}_k)^T d_k \\ &\leq L\bar{\alpha}_k \|d_k\|^2 + \zeta\bar{\alpha}_k\|d_k\|^2 \\ &= \delta^{-1}\alpha_k (L+\zeta)\|d_k\|^2.\end{aligned}$$

Table 5. Reported results for image de-blurring.

	TTDK				HTTCGP				MFRM			
Images	PSNR	SNR	SSIM	PT	PSNR	SNR	SSIM	PT	PSNR	SNR	SSIM	PT
Cameraman	27.64	21.82	0.88	31.72	26.96	21.20	0.86	3.09	27.46	21.70	0.87	4.16
Man	30.61	23.00	0.86	582.44	31.17	23.56	0.86	84.44	31.11	23.50	0.86	113.39
Barbara	25.61	19.73	0.80	163.88	24.92	19.34	0.77	16.17	25.32	19.64	0.79	28.25

Hence, we get

$$\begin{aligned}\alpha_k &\geq \frac{\delta\vartheta}{L+\zeta}\cdot\frac{\|F_k\|^2}{\|d_k\|^2}\\ &\geq \frac{\delta\vartheta}{L+\zeta}\cdot\frac{\|F_k\|^2}{\psi^2\|F_k\|^2}\\ &= \frac{\delta}{(L+\zeta)\psi^2},\end{aligned}$$

where the second inequality follows from Eq. (36).

Lemma. Given Assumptions 1 and 2 hold with $\bar{x}$ being an arbitrary solution of Eq. (3) in $\bar{C}$. Then the sequence $\{\|x_k-\bar{x}\|\}$ is convergent and thus the sequence $\{x_k\}$ is bounded. Furthermore, we have that

$$\lim_{k\to\infty}\alpha_k\|d_k\|=0. \tag{43}$$

Proof. First, we show that $\{x_k\}$ and z_k are bounded. From Eq. (34) and the definition of z_k, we get

$$(x_k-z_k)^T F(z_k) \geq \zeta\alpha_k^2\|d_k\|^2. \tag{44}$$

From Eq. (2) and for all $\bar{x}\in\bar{C}$, we have

$$\begin{aligned}(x_k-\bar{x})^T F(z_k) &= (x_k-z_k)^T F(z_k) + (z_k-\bar{x})^T F(z_k)\\ &\geq (x_k-z_k)^T F(z_k) + (z_k-\bar{x})^T F(\bar{x})\\ &= (x_k-z_k)^T F(z_k).\end{aligned} \tag{45}$$

From Eq. (33), Eq. (35), Eq. (44) and Eq. (45), we have

$$\begin{aligned}\|x_{k+1}-\bar{x}\|^2 &= \left\|P_C\left[x_k-\phi\sigma_k F(z_k)\right]-\bar{x}\right\|^2\\ &\leq \|x_k-\phi\sigma_k F(z_k)-\bar{x}\|^2\\ &= \|(x_k-\bar{x})-\phi\sigma_k F(z_k)\|^2\\ &= \|x_k-\bar{x}\|^2-2\phi\sigma_k F(z_k)^T(x_k-\bar{x})+\phi^2\sigma_k^2\|F(z_k)\|^2\\ &\leq \|x_k-\bar{x}\|^2-2\phi\sigma_k F(z_k)^T(x_k-z_k)+\phi^2\sigma_k^2\|F(z_k)\|^2\\ &= \|x_k-\bar{x}\|^2-\phi(2-\phi)\frac{\left(F(z_k)^T(x_k-z_k)\right)^2}{\|F(z_k)\|^2}\\ &\leq \|x_k-\bar{x}\|^2-\phi(2-\phi)\frac{\zeta^2\|x_k-z_k\|^4}{\|F(z_k)\|^2}.\end{aligned} \tag{46}$$

This implies

$$0\leq\|x_{k+1}-\bar{x}\|\leq\|x_k-\bar{x}\|,\quad \forall\, k\geq 0. \tag{47}$$

So, the sequence $\{\|x_k-\bar{x}\|\}$ is decreasing, hence convergent, and $\{x_k\}$ is bounded. Since $\{x_k\}$ is bounded and F is continuous, a constant u_1 exists such that

$$\|x_k\|\leq u_1,\quad \|F(x_k)\|\leq u_1,\quad \forall\, k\geq 0.$$

Using Eq. (2), Cauchy-Schwarz inequality, and Eq. (44), we get

$$u_1\geq\|F_k\|\geq\frac{F_k^T(x_k-z_k)}{\|x_k-z_k\|}\geq\frac{F(z_k)^T(x_k-z_k)}{\|x_k-z_k\|}\geq\zeta\|x_k-z_k\|\geq\zeta\|z_k\|-\zeta u_1,$$

which implies

$$\|z_k\|\leq\frac{u_1+\zeta u_1}{\zeta}.$$

Setting $u_2 := \frac{u_1(1+\zeta)}{\zeta}$, we see that $\{z_k\}$ is bounded. Thus, continuity of F implies existence of constant $\bar{u}$ such that

$$\|F(z_k)\|\leq\bar{u},\quad \forall\, k\geq 0.$$

Using this and Eq. (46), we obtain

$$\zeta^2\|x_k-z_k\|^4\leq\frac{\bar{u}^2}{\phi(2-\phi)}(\|x_k-\bar{x}\|^2-\|x_{k+1}-\bar{x}\|^2). \tag{48}$$

Since $\{\|x_k-\bar{x}\|\}$ converges and $\{F(z_k)\}$ is bounded, taking limit in Eq. (48) as $k\to\infty$ yields

$$\zeta^2\lim_{k\to\infty}\alpha_k^4\|d_k\|^4\leq 0,$$

which implies

$$\lim_{k\to\infty}\alpha_k\|d_k\|=0.$$

Theorem. Let Assumptions 1 and 2 hold with $\{x_k\}$ obtained by Algorithm 1. Then, $\{x_k\}$ converges to a solution of Eq. (3).

Proof. Considering Eq. (41) and Eq. (43), we deduce that $0\leq\alpha\|d_k\|\leq\alpha_k\|d_k\|\to 0$, hence $\lim_{k\to\infty}\|d_k\|=0$. Combining this with Eq. (36), we obtain

$$0\leq\vartheta\|F_k\|\leq\|d_k\|\to 0,$$

implying $\lim_{k\to\infty}\|F_k\|=0$.

Since $\{x_k\}$ is bounded, a cluster point $\tilde{x}\in\bar{C}$ exists. Let $\mathcal{K}\subset\{0,1,2,\ldots\}$ be an index set such that

$$\lim_{k\to\infty,\, k\in\mathcal{K}} x_k=\tilde{x}\in\bar{C}.$$

Then, by continuity of F,

$$0=\lim_{k\to\infty}\|F_k\|=\|F(\tilde{x})\|,$$

which implies $\tilde{x}$ is a solution of Eq. (3). Also, since $\{\|x_k-\bar{x}\|\}$ is convergent by Lemma 3.3, setting $\bar{x}=\tilde{x}$ yields

$$\lim_{k\to\infty}\|x_k-\bar{x}\|=0.$$

Therefore, $\{x_k\}$ converges to $\bar{x}\in\bar{C}$.

Assumption 3. For the solution $\bar{x} \in \bar{C}$, $\tau \in (0,1)$ and $\varsigma > 0$ exist satisfying

$$\tau \text{dist}(x, \bar{C}) \leq \|F(x)\|, \quad \forall x \in \mathcal{M}_{\varsigma}(\bar{x}), \tag{49}$$

with $\mathcal{M}_{\varsigma}(\bar{x})$ being the neighbourhood of $\bar{x}$, which is defined as

$$\mathcal{M}_{\varsigma}(\bar{x}) := \{x \in \mathbb{R}^n : \|x - \bar{x}\| \leq \varsigma\}.$$

Furthermore, $\text{dist}(x, \bar{C})$ stands for the distance from x to $\bar{C}$.

Theorem. Suppose Assumptions 1, 2, and 3 hold and the sequence $\{x_k\}$ is obtained by Algorithm 1, then the sequence $\text{dist}(x, \bar{C})$ converges Q-linearly to 0, implying that $\{x_k\}$ is R-linearly convergent to $\bar{x}$.

Proof. Assuming that $\tilde{x}_k := \arg\min\{\|x_k - \tilde{x}\| : \tilde{x} \in \bar{C}\}$. Then

$$\|x_k - \tilde{x}_k\| = \text{dist}(x_k, \bar{C}). \tag{50}$$

Utilizing $\tilde{x}$ for $\bar{x}$ in Eq. (46), we obtain

$$\|x_{k+1} - \tilde{x}\|^2 \leq \|x_k - \tilde{x}\|^2 - \Lambda \frac{\zeta^2 t_k^4 \|d_k\|^4}{\|F(z_k)\|^2}, \tag{51}$$

where $\Lambda = \phi(2 - \phi)$.

From the definition of $\tilde{x}_k$ and Eq. (24), we have

$$\begin{aligned} \|F(z_k)\| &= \|F(z_k) - F(\tilde{x}_k)\| \leq L\|z_k - \tilde{x}_k\| \\ &\leq L(\|z_k - x_k\| + \|x_k - \tilde{x}_k\|) \\ &= L(t_k\|d_k\| + \|x_k - \tilde{x}_k\|) \\ &\leq L(\|d_k\| + \|x_k - \tilde{x}_k\|) \\ &\leq L(\psi\|F(x_k)\| + \|x_k - \tilde{x}_k\|) \\ &= L(\psi\|F(x_k) - F(\tilde{x}_k)\| + \|x_k - \tilde{x}_k\|) \\ &\leq L(1 + L\psi)\|x_k - \tilde{x}_k\| \\ &= L(1 + L\psi)\,\text{dist}(x_k, \bar{C}). \end{aligned} \tag{52}$$

By the Cauchy-Schwarz inequality and Eq. (25), we obtain

$$\|d_k\| \geq \vartheta\|F_k\|. \tag{53}$$

Thus, since $\tilde{x}_k \in \bar{C}$, from Eq. (49), Eq. (50), Eq. (51), Eq. (52), Eq. (53), and the result in Eq. (41), we obtain

$$\begin{aligned} \text{dist}(x_{k+1}, \bar{C})^2 &= \|x_{k+1} - \tilde{x}_k\|^2 \\ &\leq \text{dist}(x_k, \bar{C})^2 - \Lambda \frac{\zeta^2 t_k^4 \|d_k\|^4}{\|F(z_k)\|^2} \\ &\leq \text{dist}(x_k, \bar{C})^2 - \Lambda \frac{\zeta^2 t^4 \vartheta^4 \|F(x_k)\|^4}{\|F(z_k)\|^2} \\ &\leq \text{dist}(x_k, \bar{C})^2 - \Lambda \frac{\zeta^2 t^4 \vartheta^4 \tau^4 \text{dist}(x_k, \bar{C})^4}{L^2(1 + L\psi)^2 \text{dist}(x_k, \bar{C})^2} \\ &= \left(1 - \Lambda \frac{\zeta^2 t^4 \vartheta^4 \tau^4}{L^2(1 + L\psi)^2}\right) \text{dist}(x_k, \bar{C})^2. \end{aligned}$$

Now, since the positive scalars $\Lambda, \tau, \vartheta, \zeta$ and t are all in $(0,1)$ with $L > 1$, it shows that $\{\text{dist}(x_k, \bar{C})\}$ converges Q-linearly to 0, implying that $\{x_k\}$ is R-linearly convergent to $\bar{x}$.

4. Results of numerical experiments

Some numerical results are reported here by implementing Algorithm 1, labelled simply as TTDK. Three recent effective algorithms in Refs. [74–76], which we label as SCGP, HTTCGP and ACGD, were employed to compare the effectiveness of the proposed scheme. All four algorithms were coded in Matlab $R2015a$ and implemented on a system with the following configuration: ($4GB\ RAM$, $2.30GHZ\ CPU$). The same linesearch described in Eq. (34) was used for each algorithm with parameters for SCGP, HTTCGP, and ACGD set according to each paper. We set parameters for Algorithm 1 as $\zeta = 10^{-2}$, $\delta = 0.4$, $\alpha = 0.1$, $r = 1$, $\xi = 0.26$, and $\phi = 1.9$. Also, the criterion to terminate the program was set as $\|F(x_k)\| \leq 10^{-7}$ or $\|F(z_k)\| \leq 10^{-7}$ or iterations exceeding 1000.

Test problems for Algorithm 1, SCGP, HTTCGP, and ACGD methods, where the mapping F is given as: $F = (f_1(x), f_2(x), ..., f_n(x))^T$ are as follows:

Problem 1. This problem is obtained from Ref. [77] where $C = \mathbb{R}^n_+$ is added to yield.

$$f_1(x) = x_1 - e^{\left(\cos \frac{x_1 + x_2}{n+1}\right)},$$
$$f_i(x) = x_i - e^{\left(\cos \frac{x_{i-1} + x_i + x_{i+1}}{n+1}\right)}, \quad i = 2, 3, \ldots, n-1,$$
$$f_n(x) = x_n - e^{\left(\cos \frac{x_{n-1} + x_n}{n+1}\right)}.$$

Problem 2. This problem is obtained from Ref. [78] where $C = \mathbb{R}^n_+$ is added to yield

$$f_i(x) = 2x_i - \sin x_i, \quad i = 1, 2, \ldots, n.$$

Problem 3. Nonsmooth function obtained from Ref. [78] where $C = \mathbb{R}^n_+$ is added to yield

$f_i(x) = 2x_i - \sin|x_i|, \quad i = 1, 2, \ldots, n.$ This problem is clearly nonsmooth at the point $x = (0, 0, ..., 0)^T$.

Problem 4. Tridiagonal exponential function obtained from Ref. [24].

$$f_1(x) = x_1 - e^{\left(\cos \frac{x_1 + x_2}{2}\right)},$$
$$f_i(x) = x_i - e^{\left(\cos \frac{x_{i-1} + x_i + x_{i+1}}{i+1}\right)}, \quad i = 2, 3, \ldots, n-1,$$
$$f_n(x) = x_n - e^{\left(\cos \frac{x_{n-1} + x_n}{n}\right)}.$$

where $C = \mathbb{R}^n_+$.

Problem 5. This problem is obtained from Ref. [24].

$$f_1(x) = 2x_1 + \sin x_1 - 1,$$
$$f_i(x) = 2x_{i-1} + 2x_i + 2\sin x_i - 1,$$
$$f_n(x) = 2x_n + \sin x_n - 1, \quad i = 2, \ldots, n-1,$$

where $C = \mathbb{R}^n_+$.

Problem 6. This problem is obtained from Ref. [24].

$$f_i(x) = (e^{x_i})^2 + 3\sin(x_i)\cos(x_i) - 1, \quad i = 1, 2, \ldots, n,$$

where $C = \mathbb{R}^n_+$.

Problem 7. This problem is obtained from Ref. [6].

$$f_i(x) = x_i - \sin|x_i - 1|, \quad i = 1, 2, \ldots, n,$$

where $C = \left\{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \leq n, \quad x_i \geq 0, \quad i = 1, 2, \ldots, n\right\}$.

Problem 8. The Logarithmic Function obtained from Ref. [77] where $C = \mathbb{R}^n_+$ is added to yield $f_i(x) = \log(x_i + 1) - \frac{x_i}{n}$, $i = 2, \ldots n$

The above problems were tested with dimensions 1000, 50,000, 100,000, and the following initial guesses: $a_1 = \left(\frac{n-1}{n}, \frac{n-2}{n}, \frac{n-3}{n}, ..., 0\right)^T$, $a_2 = \left(\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, ..., 1\right)^T$, $a_3 = \left(3, 1, 3, ..., -2\frac{[(-1)^n-2]}{2}\right)^T$, $a_4 = \left(4, 2, 4, ..., -2\frac{[(-1)^n-3]}{2}\right)^T$, $a_5 = \left(3, 0, 3, ..., -3\frac{[(-1)^n-1]}{2}\right)^T$, $a_6 = \left(\frac{9}{2}, \frac{3}{2}, \frac{9}{2} ..., -3\frac{[(-1)^n-2]}{2}\right)^T$.

Furthermore, Tables 1 - 4 displays the experiment's results, where the columns marked "NP" and "VAR" stand for problem number and number of variables, "IG" and "ITN" denote initial guess and number of iterations, FE" and "PT" represent function evaluations and processing time obtained. Value of the norm attained at approximate solution and failure to solve a problem are denoted by "*Norm*" and " $***$ ".

Tables 1 - 4 showed that Algorithm 1 performs much better than SCGP, HTTCGP, and ACGD schemes. It can be seen from the Tables that Algorithm 1 solved all the test problems successfully and also yields the best results for less iterations, function evaluations and cpu time. Furthermore, we employ the statistical tool proposed in Ref. [79] to clearly explain performances of each of the four algorithms. Using the idea in Ref. [79], Figures 1 - 3 were drawn for the three aforementioned metrics. The percentage of problems that an algorithm solved with less value of any of the metrics is indicated on the y-axis of each figure; the percentage of successfully solved problems is displayed on the right side of each figure, while the most problems solved in a time that was within a factor ϖ of the best time is given by the algorithm whose curve stays at the top of the ones representing the other algorithms. Figure 1 indicated that Algorithm 1 solved 74% (SCGP-11%, HTTCGP-32%, ACGD-10%) of problems with least iterations. Figure 2 showed that Algorithm 1 recorded 88% (SCGP-10%, HTTCGP-14%, ACGD-2%) with less function evaluations. Also, Figure 3 indicated that Algorithm 1 is the fastest of the algorithms since it solved 72.22% of problems with less processing time compared to (SCGP-11.11%, HTTCGP-13.89%, ACGD-2.78%). It is worth noting that, for the exception Figure 4, the values recorded in Figures 1 and 2 include cases where 2, 3 or all the four algorithms solved the problems with the same value of the metric considered. Also, note that in each of the figures, Algorithm 1 retains the top curve. Based on these analysis, we conclude that Algorithm 1 is effective for solving Eq. (3).

5. Application of algorithm 1

Here, we first describe a concept in compressed sensing problems known as sparse recovery, which involves obtaining spars solutions to ill-conditioned linear systems of equations $\mathcal{A}x = b$, with $\mathcal{A} \in \mathbb{R}^{k\times n} (k \ll n)$ being linear operator, $b \in \mathbb{R}^k$ is an observed value, and $x \in \mathbb{R}^n$ denotes the signal to be reconstructed. The process involves minimizing the $\ell_1 - norm$ regularization problem

$$\min_x \frac{1}{2}\|\mathcal{A}x - b\|_2^2 + \eta\|x\|_1, \tag{54}$$

where $\eta > 0$ is a regularization parameter. A suitable iterative method (see Refs. [80–83] for details) is employed to solve Eq. (54) by reconstructing an original signal, say $\hat{x}$ from a sample of disturbed signals. The scheme proposed in Ref. [83] has been described as the most prominent of the methods. The technique in Ref. [83] involves splitting any vector $x \in \mathbb{R}^n$ in to two parts as follows:

$$x = \upsilon - \nu, \quad \upsilon \geq 0, \quad \nu \geq 0, \tag{55}$$

with $\upsilon \in \mathbb{R}^n$, $\nu \in \mathbb{R}^n$, $\upsilon_i = (x_i)_+$ and $\nu_i = (-x_i)_+$ for all $i = 1, 2, ..., n$, where $(.)_+$ is defined as $(x)_+ = \max\{0, x\}$. Utilizing this, we have $\|x\|_1 = e_n^T\upsilon + e_n^T\nu$, with $e_n = (1, 1, ...,)^T \in \mathbb{R}^n$. So, Eq. (54) can be re-written as

$$\min_{\upsilon,\nu} \frac{1}{2}\|\mathcal{A}(\upsilon - \nu) - b\|_2^2 + \eta e_n^T\upsilon + \eta e_n^T\nu, \quad \upsilon \geq 0, \quad \nu \geq 0. \tag{56}$$

In Ref. [83], it was indicated that Eq. (56) can be further expressed as

$$\min_z \frac{1}{2} z^T D z + \chi^T z, \quad z \geq 0, \tag{57}$$

where

$$z = \begin{pmatrix} \upsilon \\ \nu \end{pmatrix}, \quad \chi = \eta e_{2n} + \begin{pmatrix} -h \\ h \end{pmatrix}, \quad h = \mathcal{A}^T b, \quad D = \begin{pmatrix} \mathcal{A}^T\mathcal{A} & -\mathcal{A}^T\mathcal{A} \\ -\mathcal{A}^T\mathcal{A} & \mathcal{A}^T\mathcal{A} \end{pmatrix}.$$

Being that D represents positive semi-definite matrix, equation Eq. (57) denotes a convex quadratic programming problem [84]. Moreover, by the equivalence of the optimality condition for Eq. (4) and Eq. (1), z in Eq. (57) represents the minimizer of Eq. (57) provided it represents solution of the nonlinear equations

$$F(z) = \min\{z, Dz + \chi\} = 0.$$

In Refs. [84] and [85], the authors proved that F satisfies Eq. (2) and Eq. (24), hence, Eq. (54) can be expressed as Eq. (3), which can be solved using Algorithm 1.

Here, we apply Algorithm 1, HTTCGP [75] and MFRM [86] solvers to de-blur some images that are contaminated by impulse noise. The process is carried out by applying noise suppression strategies, that require minimization of a composite function [87]. To measure restoration quality, we use the signal-to-noise ratio (SNR)

$$SNR = 20 \times \log_{10}\left(\frac{\|\tilde{x}\|}{\|x - \tilde{x}\|}\right),$$

and the peak signal to noise ratio (PSNR)

$$PSNR = 10 \times \log_{10} \frac{V^2}{MSE},$$

with V being the reconstruction's maximum absolute value and MSE denotes mean square error. Furthermore, the processing time (PT) is considered with the structured similarity index (SSIM), which exhibits similarity between actual and recovered images. The SSIM index MATLAB implementation can be assessed at http://www.cns.nyu.edu/lcv/ssim/. The parameters used for Algorithm 1 remain as applied in the first experiment with $r = 1.8$. The HTTCGP and MFRM schemes retain the parameters used in each of the papers. The standard images Cameraman.png (256×256), Man.bmp (512×512), and Barbara.png (512 × 512) were used in the experiment.

Typically, an algorithm having the largest values of SNR, PSNR and SSIM is the most effective. In the experiments, for the exception of the image Man, where HTTCGP recorded much better results, Algorithm 1 has the largest values of the three aforementioned performance metrics (see Table 5). In addition, Table 5 indicated that HTTCGP recorded the least processing time. Also, Figure 4 displayed the original, blurry and recovered images by the three Algorithms. Figure 4 also indicated that all the three Algorithms were able to recover the images almost exactly. Following this analysis, we determine that Algorithm 1 is appropriate for restoring original images from blurry ones considered.

6. Conclusion

In an attempt to add to the few Dai-Kou-type methods for solving constrained system of nonlinear monotone equations, an efficient modified three-term version was presented in this paper. To realize this objective, a version of the classical Dai-Kou method obtained with the most effective choice of the parameter τ_k was used. Furthermore, to improve performance of the new method, a modified version of the popular spectral parameter by Barzilai and Bowein was incorporated in the scheme with the implication that it speeds up convergence by enhancing the distribution of eigenvalues of the method's search direction matrix. Unlike the unmodified Dai-Kou method, our scheme satisfies the condition for global convergence irrespective of the line search procedure employed. Proof of the scheme's global convergence using mild conditions shows that its sequence of iterates converged globally. Also, numerical results of some experiments with eight test problems shows that the scheme outperforms three other methods in the literature. To further highlight the method's effectiveness, it is applied to de-blur some standard images contaminated by impulse noise. As a future research, we intend to develop an improved version of the scheme with its application in signal reconstruction.

Data Availability

No additional data was used beyond those presented in the submitted manuscript.

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