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Mathematical modelling of the dynamical system of military population, focusing on the impact of welfare

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Abstract

In this work, we explored the concept of an improved welfare package for military personnel in Nigeria. This proposed package aims to address traditional needs while also meeting modern demands, such as mental health support. We recognise the critical importance of security jobs and the prompt payment of special operation allowances, as these factors provide military personnel with a sense of reassurance and stability. Key elements of the welfare package include upgrading qualified lower ranks to officer status, ensuring timely payment of death benefits to the families of military personnel who have died in active service, procuring standard and adequate weapons, and promoting equal treatment of all military personnel without bias. Poor welfare packages contribute significantly to dissatisfaction among Nigerian military personnel, which can hinder productivity and lead to unethical behaviour. Conversely, enhancing the welfare package can substantially boost productivity and reduce dissatisfaction. We formulated a deterministic military model to assess the impact of improved welfare on the productivity of Nigerian military personnel and to prevent dissatisfaction and misconduct. The model computes the disgruntlement reproduction number, R_m, which is used to demonstrate the local and global stability of both disgruntled and non-disgruntled equilibrium states. To further analyze the model, we conducted a global sensitivity analysis using Latin Hypercube Sampling (LHS) and the Partial Rank Correlation coefficient on R_m and military populations to identify the most sensitive parameters affecting military dynamics. Based on the results of this analysis, we extended the model by introducing a standard improved welfare package to control the most sensitive parameter. Numerical simulations showed that if an improved welfare package is intentionally implemented and maintained, dissatisfaction and misconduct among military personnel will drastically decrease, while productivity will significantly increase.

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1. Introduction

The Nigerian Armed Forces (Nigerian Military) comprise the Army, the Navy, and the Air Force. It is noteworthy to mention that at the point of independence, Nigeria inherited from

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the British colonial administration a military force made up of the Army and the Navy until 1964 when the Air Force joined the Forces [1]. According to Ref. [2], the specific roles of the Nigerian Armed Forces are as follows:

- To defend Nigeria from external aggression;
- To maintain its territorial integrity and secure its borders from violation by land, sea, and air;
- To suppress insurrection and act in aid of civil authorities to restore order when called upon to do so by the President prescribed by an Act of the National Assembly;
- To perform such other functions as may be prescribed by an act of the National Assembly.

As of 2022, the strength of the Nigerian Military is about $223,000$, which is inadequate compared to the numerous security challenges confronting the Nation [3]. In line with the proclamation of the Nigeria Constitution on the roles of the Armed Forces of Nigeria (AFN) [2], it could be adjudged that personnel of the Nigeria Armed Forces are among the most vulnerable sub-population in the nation because of the hazards or dangers associated with the profession and the rapid and dynamic increase in the security challenges in Nigeria as a nation. This centres on the fact that Nigeria's military is involved in counter-terrorism, insurgency, kidnapping, cattle rustling, banditry, farmers and herders clashes, secessionism, and separatism operations across all the regions of the country; these operations above have resulted in the loss of lives, permanent disabilities and other forms of hardship to the operatives of the Nigerian Armed Forces [4]. By extension, these have caused suffering to their families on several occasions when their duly entitlements are not accessible [5]. Therefore, for every insurgency and other security challenge that threatens the peace and security of a nation, there is always a motivating force behind its persistence. On the other hand, for every successful military counterinsurgency or counter-operation, there is always a motivating force propelling the strategies adopted and the goals accomplished. This motivating force in the context of this study is what has been termed the welfare of the Nigerian military personnel. Welfare in this context is the capacity of a group's members to maintain belief in an institution or goal, particularly in the face of opposition or hardship. It is a state of mind that either encourages or impedes action. The greatest combat commanders have always understood that improved welfare reflects their troops' mental, moral, and physical condition. These conditions, in turn, directly relate to the troops' courage, confidence, discipline, not to compromise or be disgruntled, enthusiasm, and willingness to endure the sacrifices and hardships of military duty [6].

The welfare package in this work includes consistency in promotion as at when due, prompt payment of special operations allowances, upgrading of qualified other ranks to officer cadre, a special reward system for the extraordinary performance of military personnel, prompt payment of death benefits to the families of military personnel that died in active service, effective and efficient medical services, proper and adequate feeding of military personnel at the front-line, procurement of standard and sufficient weapons for the military personnel especially for those on the battlefields, fair rotation or transfer of military personnel without lobbying, equal treatment of military personnel without sentiment among others.

Notably, corruption can impede the welfare of the military personnel and, by extension, their productivity, especially by the higher authorities within the system. For instance, Ref. [10] stated that one reason why the respected Nigerian military has not won the war against bandits/terrorists in Nigeria is the corruption of the endemic species in the highest echelon of the federal ministry of defence and other different segments of the military. Kleptocratic and selfish military Generals have been fingered, indicted, and constitutionally punished for groundbreaking crimes of corruption and crude theft of resources that are meant for the purchase of vital military weapons to defeat terrorists. The said corruption is also responsible for the purchase of substandard weapons by Nigerian military leaders, thereby exposing the military personnel at the battle front-line to the dangers of being consistently overpowered and killed by terrorists. The danger of allowing this corruption in the military is that the institution of the military or the Nigerian Armed Forces will be hindered from actualizing their mandate of protecting the territorial integrity of Nigeria [10].

Also, Ref. [13] mentioned that a group was accusing Nigerian military officers of corruption and further explained that during the 2023 elections, funds made available for the welfare of soldiers were siphoned by the Army headquarters, adding that those monies were shared with their commanders. Although the front-line military personnel on the battlefield might have taken an oath to serve the nation with all their strength and defend her unity, it is, however, obligatory for the government and military leaders to ensure that their safety and welfare are given adequate attention [14], hence, the reason why the impact of welfare is being considered as a motivating factor in the Nigerian Armed Force in this study. The term Welfare Package (WP) is defined as a planned, coordinated arrangement towards improving the social conditions of personnel in the Armed forces, which, in return, makes military personnel deliver their services optimally [6–8]. This is the essence of mathematical modelling of the impact of welfare on Nigerian military personnel. Mathematical modelling has also been defined as a detailed process of representing real-world phenomena in terms of mathematical equations and extracting helpful information for understanding and prediction from them [15].

Several authors have employed mathematical modelling in different subject areas of life, such as behaviour [16], racism [17], xenophobia [18], and smoking [19]. While Oigo & Jacko [16] used mathematical modelling to analyse mathematical anxiety behaviour concerning mathematics performance in Kenya, Kotola & Teklu [17] employed mathematical modelling to analyse racism and corruption as mind infections that affect the public and government sectors. Also, Gweryina *et al.* [18] formulated mathematical modelling to analyse policies essential for the coexistence of foreigners and xenophobes, while Alkhudhari *et al.* [19] used mathematical modelling to demonstrate the dynamics of smoking.

Ref. [4] discussed the importance of enhancing Nigerian Army welfare to improve personnel motivation, and by extension, this is also applicable to the Nigerian military in general. Ref. [6] also worked on the morale of Nigeria's military, which further explained that the welfare of the military has to be improved for effective performance. Ref. [8] also considered the role of employee welfare in improving work productivity in service companies in their work. Wokoma and Obasi [9] also worked on the welfare package in an organization and how it enhances employees' job satisfaction. Moreover, Poi [11] researched on the same subject with a case study on Rivers State, Nigeria. So is Ref. [12] on a case study of an industry. In addition, Ampong [40] researched on the impact of welfare packages on employees and how highly motivated employees contribute value to the organization by effectively accomplishing their goals and objectives. These studies were not done with the concept of mathematical modelling. Hence, to the best of our knowledge, no one has developed and analysed a mathematical model on the impact of welfare on Nigerian military personnel.

Consequently, our proposed model is formulated with the concept of the $SEIR$ model to analyse the dynamics and impact of improved welfare on the Nigerian military population, using a deterministic compartmental model of the form, Potential military Population, $S_{mp}(t)$, Susceptible military population, $T_{mp}(t)$, Disgruntled military population or military population at compromising state, $D_{mp}(t)$ and Productive military population, $P_{mp}(t)$. Therefore, in this proposed model, we aim to assess and explore the dynamics of dissatisfaction among military personnel, particularly concerning improving welfare. Specifically, our objective is to investigate how inadequate welfare packages can negatively impact the productivity of Nigerian military personnel, potentially leading to compromises in their performance. Conversely, we will also examine how enhancements in welfare can boost their productivity.

The structure of this study is organized as follows: In Section 2, we will describe and formulate a compartmental mathematical model that examines the impact of welfare on Nigerian military personnel. Section 3 focuses on model analysis, which includes assessing the model's invariant region, ensuring the positivity of the system solutions, identifying equilibrium points, computing the Disgruntled Military Reproduction Number, R_m, and conducting stability analysis. In Section 4, we present the global sensitivity of the basic model. Section 5 includes numerical simulations and a discussion of the results. Finally, Section 6 provides conclusions and recommendations.

2. Model description and formulation

In this section, we formulate a deterministic military model that explores the influence of welfare on military personnel. Given the large population of military members and the discontent expressed by some, we have opted for a deterministic modelling approach. The following assumptions have been considered:

- All recruitment into the military population must be between the ages 18 to 30 for both commissioned and non-commissioned officers [21, 22].
- Recruitment into the military population is of choice (voluntarily) and not compulsory [21, 22].
- Natural death rate and military work-related death rate are the only ways military personnel exit the military population.

Therefore, the total human population, $N(t)$ at any time t, is divided into sub-populations: Potential military Population, $S_{mp}(t)$, Susceptible military population liable to productivity or disgruntlement, $T_{mp}(t)$, Disgruntled military population or military population at compromising state, $D_{mp}(t)$ and Productive military population, $P_{mp}(t)$. The total human population at any given time, t is given by

$$N(t) = S_{mp} + T_{mp} + D_{mp} + P_{mp}.$$

The potential military population is generated via recruitment at the rate Λ. It decreases when they are recruited into the total military population at the rate γ, and by the natural death rate μ, which is assumed for all the sub-populations. The Susceptible military population T_{mp} is generated by recruiting the potential military population at the rate γ. This military population decreases due to the proportion σ of the military population that progresses to the productive military population at the rate ω, the military work-related death rate for military personnel δ_1 and the natural death rate μ. The disgruntled military population D_{mp} is formed by the proportion $(1-\sigma)$ of the total military population that compromises at the rate β. It decreases by the progress rate due to welfare (improved welfare) to the productive military population at the rate τ, military work-related death rate δ_2, and also by the natural death rate μ. Meanwhile, the productive military population P_{mp} is generated via proportion (σ) of the Susceptible military population (T_{mp}) due to welfare, ω, and the improved welfare of the Disgruntled military population (D_{mp}) at the rate τ. This population, P_{mp}, decreases due to the military work-related death rate δ_3 and by the natural death rate μ.

Based on the simplified military population diagram in Figure 1, we formulated the following system of nonlinear differential equations for the military population. Below is the model flow diagram. The mathematical equations obtained from the diagram above are as follows:

$$\begin{aligned}
\frac{dS_{mp}}{dt} &= \Lambda - (\gamma+\mu)S_{mp},\\
\frac{dT_{mp}}{dt} &= \gamma S_{mp} - (1-\sigma)\beta T_{mp}D_{mp} - \sigma\omega T_{mp} - (\mu+\delta_1)T_{mp},\\
\frac{dD_{mp}}{dt} &= (1-\sigma)\beta T_{mp}D_{mp} - \tau D_{mp} - (\mu+\delta_2)D_{mp},\\
\frac{dP_{mp}}{dt} &= \sigma\omega T_{mp} + \tau D_{mp} - (\mu+\delta_3)P_{mp},
\end{aligned} \tag{1}$$

with non-negative initial conditions, $S_{mp}(0), T_{mp}(0)$, $D_{mp}(0)$, $P_{mp}(0)$. The model parameters are assumed to be non-negative except for human recruitment, which is strictly positive.

Table 1: Description and parameter values of the model.

Parameter	Description	Unit	Value	Source
Λ	Recruitment rate	$\frac{\text{Human population}}{\text{year}}$	600	Assumed
β	Influence rate for military population at comprising state	$year^{-1}$	0.001	Assumed
γ	Progress rate from potential military population to susceptible military population	$year^{-1}$	0.728	[23]
$\sigma \in (0,1)$	Proportion of military population that progresses from susceptible military population to productive military population	Nil	0.345	Assumed
ω	Progress rate due welfare from susceptible military population to productive military population	$year^{-1}$	0.3	Assumed
τ	Progress rate due welfare from disgruntled military population to productive military population	$year^{-1}$	0.25	Assumed
μ	Natural death rate for all humans	$year^{-1}$	0.0179	[24]
δ_1	Military work-related death rate for military personnel in T_{mp}	$year^{-1}$	0.06278	Assumed
δ_2	Military work-related death rate for military personnel in D_{mp}	$year^{-1}$	0.05621	Assumed
δ_3	Military work-related death rate for military personnel in P_{mp}	$year^{-1}$	0.06278	Assumed

Table 2: State variables of the model.

Parameter	Interpretation	Unit
$S_{mp}(t)$	Potential military population at time, t	Human population
$T_{mp}(t)$	Susceptible military population liable to productivity or disgruntlement at time, t	Military Human population
$D_{mp}(t)$	Disgruntled military population or military population at compromising state at time, t	Military Human population
$P_{mp}(t)$	Productive military population at time, t	Military Human population

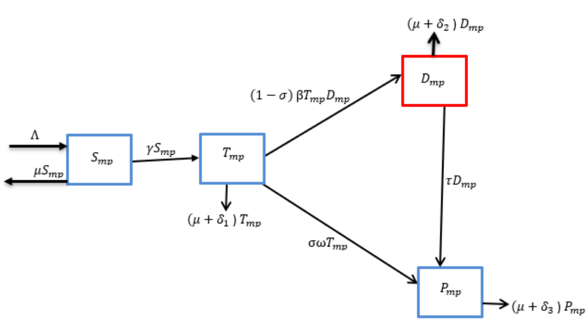


Figure 1: Flow diagram of the basic military model.

3. Mathematical analysis of the basic Nigerian military model

3.1. Invariant region

We show that the model system (equation (1)) has a region where the model solutions are uniformly bounded as used by Madubueze *et al.* [20], Odeh *et al.* [38], Tijani *et al.* [39] and Nelson *et al.* [44].

Theorem 3.1. All feasible solutions of the model are uniformly bounded in a proper subset $\Omega = D_H$, where $D_H = \left\{(S_{mp}, T_{mp}, D_{mp}, P_{mp}) \in \Re^4_+ : N(t) \le \frac{\Lambda}{\mu}\right\}$, is a subset for the population.

Proof 3.1. As adopted by [25], the total human population, $N(t)$ is given by $N = S_{mp} + T_{mp} + D_{mp} + P_{mp}$ with initial conditions $N(0) = N_0$ such that $\frac{dN}{dt} = \Lambda - \mu N - \delta_1 T_{mp} - \delta_2 D_{mp} - \delta_3 P_{mp}$ from (1). In the absence of work-related death rate, that is, $\delta_1 = \delta_2 = \delta_3 = 0$, we have $\frac{dN}{dt} \le \Lambda - \mu N$, which, by the method of integrating factor with the initial condition, $N(0) = N_0$ gives:

$$N(t) \le \frac{\Lambda}{\mu} + \left(N_0 - \frac{\Lambda}{\mu}\right) e^{-\mu t}. \tag{2}$$

As $t \to \infty$ in equation (2), we have $N(t) \le \frac{\Lambda}{\mu}$. This means that the feasible solutions of the model for the human military population are in the region, D_H.This completes the proof.

Theorem 1 shows that the model of system (1) is well posed mathematically. Therefore, it is sufficient to study the dynamics of the model (equation) (1) in the region $\Omega = D_H$.

3.2. Positivity of the system solutions

We show that all the state variables in the military system of equations (1) are non-negative at all time, $t > 0$, indicating that the model is significant epidemiologically and mathematically. Hence, we state and prove the following theorem.

Theorem 3.2. Let $\Omega = \left\{S_{mp}, T_{mp}, D_{mp}, P_{mp} \in \Re^4_+\right\}$, be solution set such that $S_{mp}(0) = S_{mp0}, T_{mp}(0) = T_{mp0}, D_{mp}(0) = D_{mp0}$ and $P_{mp} = P_{mp0}$ are positive, then the elements of the solution set Ω are all positive for $t \ge 0$.

Proof 3.2. Using the approach in Ref. [35], we have from the first equation of the model system (1) that

$$\frac{dS_{mp}}{dt} = \Lambda - (\gamma + \mu)S_{mp} \ge -(\gamma + \mu)S_{mp}. \tag{3}$$

Integrating (3) with initial conditions $S_{mp}(0) = S_{mp0}$ yields $S_{mp} \ge S_{mp0}e^{-(\gamma+\mu)t} \ge 0$.

In the same way, the rest of the equations of the model system (1) with their initial conditions give:
$T_{mp} \ge T_{mp0}e^{-(\sigma\omega+\mu+\delta_1)t} \ge 0$, $D_{mp} \ge D_{mp0}e^{-(\mu+\delta_2)t} \ge 0$, $P_{mp} \ge P_{mp0}e^{-(\mu+\delta_3)t} \ge 0$.

Therefore, the solution set $\left\{S_{mp}(t), T_{mp}(t), D_{mp}(t), P_{mp}(t)\right\}$, of the system of equations (1) is positive for all $t \ge 0$ since their exponential functions as well as their initial conditions are positive. This result guarantees the system's solutions' positivity and the solutions' global existence.

3.3. Disgruntled military generation number, R_m and equilibrium states

We first derived the gisgruntled-free equilibrium to compute the Disgruntled military reproduction number, R_m.

3.3.1. Disgruntled-free equilibrium (DFE)

The disgruntled-free equilibrium state is established when there is no disgruntled or compromised military population in the military (*i.e.* $D_{mp} = 0$). This occurs when the military population is pleased and contented with their welfare packages. That is, the disgruntled state variable is zero.

Hence, solving the military system (1) at the equilibrium state simultaneously when $D_{mp} = 0$ gives DFE,

$$E_0 = \left[S^0_{mp}, T^0_{mp}, D^0_{mp}, P^0_{mp}\right] = \left[\frac{\Lambda}{k_1}, \frac{\Lambda\gamma}{k_1k_2}, 0, \frac{\sigma\omega\Lambda\gamma}{k_1k_2k_4}\right],$$

where

$$k_1 = (\gamma + \mu),\ k_2 = (\sigma\omega + \mu + \delta_1),\ a = (1 - \sigma), \tag{4}$$

$$k_3 = (\tau + \mu + \delta_2),\ k_4 = (\mu + \delta_3). \tag{5}$$

3.3.2. Disgruntled military reproduction number, R_m

For the computation R_m, we have from the third equation of the model system (1) that $D_{mp} < 0$, if

$$\frac{dD_{mp}}{dt} = a\beta T_{mp}D_{mp} - \tau D_{mp} - (\mu + \delta_2)D_{mp} < 0.$$

For D_{mp} to decrease, it means that, $a\beta T_{mp} - k_3 < 0$, giving $\frac{a\beta T_{mp}}{k_3} < 1$. At DFE, $T_{mp} = \frac{\Lambda\gamma}{k_1k_2}$, so that $\frac{a\beta\Lambda\gamma}{k_1k_2k_3} < 1$.

So, this implies that $\frac{a\beta\Lambda\gamma}{k_1k_2k_3}$, is the threshold quantity that determines the poor status of welfare in the military population which we represent as disgruntled military reproduction number, R_m,

$$R_m = \frac{a\beta\Lambda\gamma}{k_1k_2k_3}. \tag{6}$$

The disgruntled military reproduction number, R_m, is a threshold quantity that predicts the extent of the disgruntled or compromised personnel in the military.

3.4. Disgruntled-present equilibrium (DPE), E^*

The disgruntled-present equilibrium state (DPE) occurs when there is a presence of the disgruntled military population in the military population (*i.e.* $D_{mp} \neq 0$). To obtain the DPE, we solve the model system (1) at equilibrium state simultaneously, that is $\frac{dS_{mp}}{dt} = 0$, $\frac{dT_{mp}}{dt} = 0$, $\frac{dD_{mp}}{dt} = 0$, $\frac{dP_{mp}}{dt} = 0$ and get

$$\begin{aligned} E^* &= \left[S^*_{mp}, T^*_{mp}, D^*_{mp}, P^*_{mp}\right] \\ &= \left[\frac{\Lambda}{k_1}, \frac{k_3}{a\beta}, \frac{\Lambda a\beta\gamma - k_1k_2k_3}{a\beta k_1k_3}, \frac{\Lambda a\beta\gamma\tau + k_1k_3^2\sigma\omega - k_1k_2k_3\tau}{a\beta k_1k_3k_4}\right]. \end{aligned}$$

With $R_m = \frac{a\beta\Lambda\gamma}{k_1k_2k_3}$, the disgruntled-present equilibrium will be represented in terms of R_m as

$$S^*_{mp} = \frac{\Lambda}{k_1},\ T^*_{mp} = \frac{\Lambda\gamma}{R_mk_1k_2},\ D^*_{mp} = \frac{(R_m - 1)\Lambda\gamma}{R_m\,k_1k_3}, \tag{7}$$

$$P^*_{mp} = \frac{\Lambda\gamma\left(k_3\sigma\omega + k_2\tau(R_m - 1)\right)}{R_mk_1k_2k_3k_4}. \tag{8}$$

This implies that for the disgruntled-present equilibrium to be positive, R_m must be greater than one, meaning there will be persistence of the disgruntled military individuals among the military personnel.

3.5. Stabilities of the equilibrium states of the model system

3.5.1. Local stability of the disgruntled-free equilibrium State of the model

We established the local stability of the disgruntled-free equilibrium state, E_0 of the model equation (1) in terms of R_m, by computing the linearized Jacobian matrix $J(E_0)$, that is evaluated at DFE.

Theorem 3.3. The disgruntled-free equilibrium, E_0, of the model equation (1) is locally asymptotically stable if $R_m < 1$; otherwise, it is unstable.

Proof 3.3. The local stability of the DFE, E_0 is proven by showing that the Jacobian matrix $J(E_0)$, of the model equation (1) has negative eigenvalues as used by [26][35]. Jacobian Matrix of the model system (1) at DFE, E_0 is given as:

$$J(E_0) = \begin{pmatrix} -k_1 & 0 & 0 & 0 \\ \gamma & -k_2 & a\beta T^0_{mp} & 0 \\ 0 & 0 & a\beta T^0_{mp} - k_3 & 0 \\ 0 & \sigma\omega & \tau & -k_4 \end{pmatrix}. \tag{9}$$

The eigenvalues of $J(E_0)$ are $-k_1, -k_2, -k_4$ and $a\beta T^0_{mp} - k_3$. So, $J(E_0)$ has all negative eigenvalues

$$a\beta T^0_{mp} - k_3 = k_3(\frac{a\beta T^0_{mp}}{k_3} - 1) = k_3(R_m - 1) < 0,$$

that is if $R_m < 1$. Hence, the DFE, E_0 is locally asymptotically stable if when $R_m < 1$, otherwise it is unstable whenever $R_m > 1$. The implication is that the act of disgruntlement or compromise in the military population can be reduced to the barest minimum, provided that $R_m < 1$, which happens when their welfare is improved upon.

3.5.2. Local asymptotic stability of disgruntled-present equilibrium state of the model

We adopt the concept of Ref. [28] to determine the local stability of the disgruntled present equilibrium state using the linearization method as in the case of the disgruntled-free equilibrium state.

Theorem 3.4. If $R_m > 1$, the disgruntled-present equilibrium E^* is locally asymptotically stable; otherwise, it is unstable.

Proof 3.4. Based on the system (1), we have Jacobian Matrix at disgruntled-present equilibrium state E^*, $J(E^*)$ given as:

$$J(E^*) = \begin{pmatrix} -k_1 & 0 & 0 & 0 \\ \gamma & \frac{-\Lambda a\beta\gamma - k_1k_2k_3}{k_1k_3} - k_2 & -k_3 & 0 \\ 0 & \frac{\Lambda a\beta\gamma - k_1k_2k_3}{k_1k_3} & 0 & 0 \\ 0 & \sigma\omega & \tau & -k_4 \end{pmatrix}.$$

Solving the eigenvalues of the Jacobian matrix, $J(E^*)$, gives the solutions of the characteristics equations in terms of R_m in (10) as

$$\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D = 0, \tag{10}$$

where

$$\begin{aligned} A &= k_3\Big(R_m + \frac{1}{k_2}(k_1 + k_4)\Big),\\ B &= k_2k_3(R_m - 1) + k_1(k_2R_m + k_4),\\ C &= k_1k_2k_3(R_m - 1) + k_2k_3k_4(R_m - 1) + k_1k_2k_4R_m,\\ D &= k_1k_2k_3k_4(R_m - 1). \end{aligned} \tag{11}$$

According to Ref. [29], the characteristic equation (10) has all negative real part solutions if the coefficients of the characteristic equation are all positive. So from the characteristic Equation (10), we have $A, B, C, D > 0$ if $R_m > 1$. Therefore, the Jacobian Matrix, $J(E^*)$ has negative real eigenvalues if $R_m > 1$. Thus, the disgruntled-present equilibrium, E^*, is locally asymptotically stable if $R_m > 1$. This completes the proof.

3.6. Local bifurcation analysis

In this section, we study bifurcation analysis. Bifurcation phenomena happen when a small change in the system's parameter values causes topological changes in the system's behaviour. Small changes in the parameter values could cause the exchange of stability of the equilibrium points of dynamical systems. There are different types of bifurcation considered in the dynamical system, among them are saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation, Hopf bifurcation. In this work, we investigate the following bifurcation of the model (equation) (1): forward bifurcation, Backward bifurcation and Hopf bifurcation.

3.6.1. Hopf bifurcation

Using Theorems 2 and 3 (coefficient criterion for four-dimensional Hopf bifurcation) of Asada and Yoshida (2003) [36] for determining the occurrence of Hopf bifurcation, the model system (1) does not exhibit a hopf bifurcation at endemic equilibrium, E^* when $R_m > 1$ as it fails to satisfy the Hopf bifurcation condition that

$$ABC - A^2D - C^2 = 0,$$

since

$$\begin{aligned} ABC - A^2D - C^2 = \frac{k_3}{k_2}(R_m + k_1 + k_4)\Big[\Big(&k_2^2k_3^2(R_m - 1)^2(k_1 + k_4)\\ &+ k_1k_2^2k_3R_m(R_m - 1)(2k_4 + k_1)\\ &+ k_1k_2k_3k_4(R_m - 1)(k_4 + k_1)\\ &+ k_1^2k_2k_4R_m(k_2 + k_4R_m)\Big)\\ &- k_1k_3^2k_4(R_m - 1)(k_1 + k_4 + R_mk_2)\Big]\\ &- k_3k_2^2(k_1 + k_4)(R_m - 1)[k_3(k_1 + k_4) + k_1k_4(R_m - 1)]\\ &+ k_1^2k_2^2k_4^2R_m^2 \neq 0, \end{aligned}$$

at the bifurcation parameter $\beta^{**} = \frac{k1+k_2+k_3+k_4}{(1-\sigma)(T^*_{mp} - D^*_{mp})}$, where A, B, C, and D are defined in (11).

The implication is that neither equilibrium oscillates, meaning there are no periodic rises or falls of the equilibrium states.

3.6.2. Backward and forward bifurcation

Using the concept of bifurcation analysis as adopted from [26], we have at $R_m = 1$ that $\beta^* = \frac{k_3}{(1-\sigma)T^0_{mp}}$, which is chosen as bifurcation parameter with k_3 already defined in equation (5).

At $R_m = 1$, we have from the Jacobian matrix (9) that three negative eigenvalues exist and a simple zero eigenvalue. Therefore, we compute the Jacobian matrix's left and right eigenvalues at $R_m = 1$.

For left eigenvalues, $\mathbf{w} = (w_1, w_2, w_3, w_4)$, we multiply the $J(E_0)$ with w and equate to zero, to get:

$$\mathbf{w} = \left[0, \frac{-(1-\sigma)\beta T^0_{mp} w_3}{k_2}, w_3, \frac{\sigma\omega w_2 + \sigma w_3}{k_4}\right], \text{ where } w_3 > 0.$$

Computing the right eigenvalues $\mathbf{v} = (v_1, v_2, v_3, v_4)$, we transpose the Jacobian matrix, $J(E_0)$ (in equation (9)) and multiply with $\mathbf{v}$ and equate to zero. This yields: $\mathbf{v} = (0, 0, v_3, 0)$.

Next, we compute the bifurcation coefficients, a and b by finding the non-zero second partial derivatives of the model system (1) at DFE using the third equation $f_3 = (1-\sigma)\beta T_{mp} D_{mp} - k_3 D_{mp}$ since $v_1 = v_2 = v_4 = 0$ and $v_3 \neq 0$.

Denoting $x_1 = S_{mp}$, $x_2 = T_{mp}$, $x_3 = D_{mp}$, $x_4 = P_{mp}$, we have:

$$\frac{\partial^2 f_3(E_0)}{\partial x_2 \partial x_3} = (1-\sigma)\beta, \quad \frac{\partial^2 f_3(E_0)}{\partial x_3 \partial \beta} = (1-\sigma)\beta T^0_{mp}.$$

The bifurcation coefficient, a, for a non-zero second partial derivative is given by:

$$a = v_3\left[w_2 w_3 \frac{\partial^2 f_3(E_0)}{\partial x_2 \partial x_3}\right].$$

Upon substitution, we have:

$$a = -\frac{v_3 w_3 (1-\sigma)^2 \beta^2 T^0_{mp}}{k_2} < 0.$$

For the coefficient b, we have:

$$b = v_3\left[w_3 \frac{\partial^2 f_3(E_0)}{\partial x_3 \partial \beta}\right],$$

which substituting yields:

$$b = v_3 w_3 (1-\sigma) T^0_{mp} > 0.$$

Since $\mathbf{v}.\mathbf{w} = 1$, it means that $v_3 w_3 = 1$. Therefore, we have a forward bifurcation at $R_m = 1$ since $a < 0$ and $b > 0$ and no backward forward exists. This indicates that, regardless of how many military personnel are initially dissatisfied, their discontent can be eliminated if the value of $R_m < 1$. However, if $R_m > 1$, the dissatisfaction will become widespread among military personnel, leading to minimal productivity. It is essential

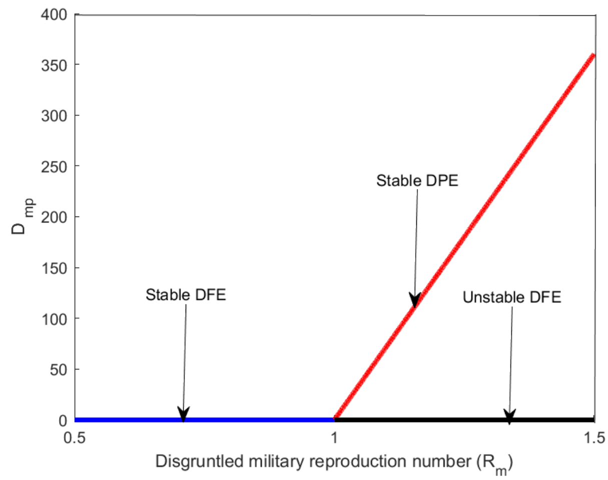


Figure 2: Forward bifurcation plot for the disgruntled military population at $R_m = 1$. The parameter values are from Table 1.

to implement measures such as improved welfare packages to prevent this situation.

The forward bifurcation is illustrated in Figure 2. With the forward bifurcation of the model at $R_m = 1$,, the equilibrium states of the model system (1) can be globally asymptotically stable. We, therefore, establish the global stability of the equilibrium states.

3.7. Global stability of the equilibrium states

In this subsection, the global stability of the equilibrium states is presented.

3.7.1. Global stability of the disgruntled-free equilibrium state of the model

We employ the Comparison Theorem by [27] to the model system (1) to prove the global stability of DFE, E_0 by stating the following theorem.

Theorem 3.5. Provided that $R_m < 1$ and $\delta_1 = 0$, $\delta_2 = 0$, $\delta_3 = 0$, the disgruntled-free equilibrium state,

$$E_0 = \left[\frac{\Lambda}{k_1}, \frac{\Lambda\gamma}{k_1 k_2}, 0, \frac{\sigma\omega\Lambda\gamma}{k_1 k_2 k_4}\right],$$

of the model system (1) is globally asymptotically stable, otherwise it is unstable when the Disgruntled military reproduction number, $R_m > 1$ with k_1, k_2, k_3 defined in Equation (5).

Proof 3.5. From the first condition DFE, E_0, we have the non-disgruntled compartments given by

$$\left.\begin{aligned} \frac{dS_{mp}}{dt} &= \Lambda - k_1 S_{mp}, \\ \frac{dT_{mp}}{dt} &= \gamma S_{mp} - k_2 T_{mp}, \\ \frac{dP_{mp}}{dt} &= \sigma\omega T_{mp} - k_4 P_{mp}, \end{aligned}\right\} \tag{12}$$

with $[S_{mp}, T_{mp}, P_{mp}]$ as the contented and satisfied state compartments, and the disgruntled compartment D_{mp} is zero. Thus, the Jacobian matrix of system (equation) (12) is given as

$$J_u = \begin{pmatrix} -k_1 & 0 & 0 \\ \gamma & -k_2 & 0 \\ 0 & \sigma\omega & -k_4 \end{pmatrix}.$$

The J_u matrix has negative eigenvalues, $-k_1, -k_2$, and $-k_4$; this shows that the non-disgruntled system (equation) (12) is globally asymptotically stable at the DFE, E_0.

For the second condition of the Comparison Theorem, $G(x,y) = Ay - \hat{G}(x,y)$. Applying the Comparison theorem to the Disgruntled compartment of the model, we have $A = a\beta T^0_{mp} - k_3$, $\hat{G}(x,y) = a\beta(T^0_{mp} - T_{mp})D_{mp}$, so that $\hat{G}(x,y) \geq 0$ if $T^0_{mp} = \frac{\Lambda\gamma}{k_1 k_2} > T_{mp}$. With $\hat{G}(x,y) \geq 0$, it implies that $G(x,y) \leq A\,y$. Here, $A = a\beta T^0_{mp} - k_3 = k_3(R_m - 1)$ is an M-matrix (with negative eigenvalue) if $R_m < 1$. This means that the extended model satisfies the two conditions of the Comparison theorem. Hence, DFE, E_0 is globally asymptotically stable if $R_m < 1$.

The implication of the theorem is that irrespective of how many military personnel are initially compromised or disgruntled, provided that the disgruntled military reproduction number, R_m, is less than unity, the number of disgruntled military population will be drastically reduced in the military population and thereby increase the number of the productive military population.

3.7.2. Global stability of the disgruntled-present equilibrium (DPE) of the model

We state and prove the global stability of DPE using the Volterra-Lyapunov approach.

Theorem 3.6. The disgruntled-present equilibrium, E^*, of the model of the system of equations (1) is globally asymptotically stable if $R_m > 1$ in Φ.

Proof 3.6. The global stability of the disgruntled present equilibrium is proved by constructing the Volterra-Lyapunov function given as:

$$\begin{aligned} V(t) = &\left(S_{mp} - S^*_{mp} - S^*_{mp} \ln \frac{S_{mp}}{S^*_{mp}}\right) \\ &+ \left(T_{mp} - T^*_{mp} - T^*_{mp} \ln \frac{T_{mp}}{T^*_{mp}}\right) \\ &+ \left(D_{mp} - D^*_{mp} - D^*_{mp} \ln \frac{D_{mp}}{D^*_{mp}}\right). \end{aligned}$$

Taking the derivatives of $V(t)$ along the solutions of the model equations (1) gives:

$$\begin{aligned} \frac{dV}{dt} = &\left(1 - \frac{S^*_{mp}}{S_{mp}}\right)\frac{dS_{mp}}{dt} + \left(1 - \frac{T^*_{mp}}{T_{mp}}\right)\frac{dT_{mp}}{dt} \\ &+ \left(1 - \frac{D^*_{mp}}{D_{mp}}\right)\frac{dD_{mp}}{dt} \end{aligned}$$

$$= \left(1 - \frac{S^*_{mp}}{S_{mp}}\right)\left(\Lambda - (\mu+\gamma)S_{mp}\right) + \left(1 - \frac{T^*_{mp}}{T_{mp}}\right)\left(\gamma S_{mp} - a\beta T_{mp}D_{mp} - k_2 T_{mp}\right) + \left(1 - \frac{D^*_{mp}}{D_{mp}}\right)\left(a\beta T_{mp}D_{mp} - k_3 D_{mp}\right). \quad (13)$$

Substituting at disgruntled-present equilibrium state, DPE,

$\Lambda = (\mu+\gamma)S^*_{mp}$, $k_2T^*_{mp} = \gamma S^*_{mp} - a\beta T^*_{mp}D^*_{mp}$, $k_3D^*_{mp} = a\beta T^*_{mp}D^*_{mp}$,

into equation (13) and simplifying yields:

$$\frac{dV}{dt} = \mu S^*_{mp}\left(2 - \frac{S^*_{mp}}{S_{mp}} - \frac{S_{mp}}{S^*_{mp}}\right) + \gamma S^*_{mp}\left(3 - \frac{S^*_{mp}}{S_{mp}} - \frac{T_{mp}}{T^*_{mp}} - \frac{S_{mp}}{S^*_{mp}}\frac{T^*_{mp}}{T_{mp}}\right). \quad (14)$$

By Arithmetic-Geometric theorem on Equation (14), we have

$$2 \le \frac{S^*_{mp}}{S_{mp}} + \frac{S_{mp}}{S^*_{mp}}, \quad 3 \le \frac{S^*_{mp}}{S_{mp}} + \frac{T_{mp}}{T^*_{mp}} + \frac{S_{mp}}{S^*_{mp}}\frac{T^*_{mp}}{T_{mp}},$$

which implies that $\frac{dV}{dt} \le 0$. Hence, the disgruntled-present equilibrium, DPE, is globally asymptotically stable for $R_m > 1$ since $\frac{dV}{dt} \le 0$.

The global stability of the DFE and DPE implies that with different initial conditions, the model will always converge to the disgruntled-free and disgruntled-present equilibrium states when $R_m < 1$ and $R_m > 1$, respectively.

We further examine the most sensitive parameters of the model (equation) (1) using the approach of LHS to compute the PRCC of the parameters.

4. Global sensitivity analysis

Global sensitivity analysis (GSA) is examined in this subsection to determine the most sensitive parameters on R_m, T_{mp}, D_{mp} and P_{mp} as multiple points entry. It determines the behaviour and degree of each parameter of the compartments. In this work, Latin Hypercube Sampling (LHS) sampling-based method with Partial Rank Correlation Coefficient (PRCC) is used to analyse GSA by generating 2000 samples from a uniform distribution of each parameter range [30–34]. The parameter values used for the sensitivity analysis are presented in Table 1. The graphical presentations of PRCCs for the parameters on R_m, the state variables, S_{mp}, T_{mp}, D_{mp} and P_{mp} and the outcome of 2000 sample of parameter sets are shown in Figures 3, 4, 5,6 and 7.

Figure 3 shows that the military influence rate, (β), the progress rate from disgruntled military population to productive military population, τ, and the progress rate due to welfare from susceptible military population to productive military population, ω have significant impacts on the disgruntled military reproduction number R_m. While, in Figures 5, 6 and 7, the military influence rate, (β), and the progress rate due to welfare from disgruntled military population to productive military

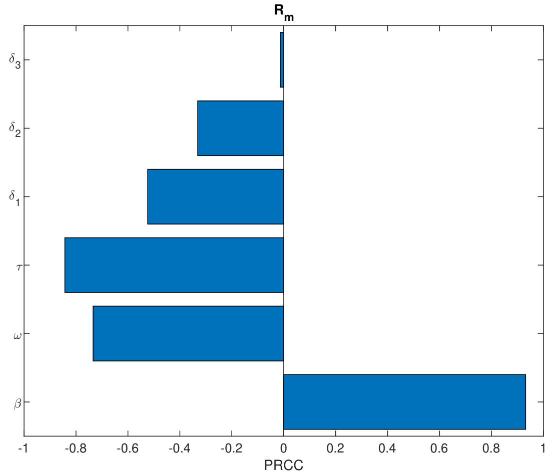


Figure 3: Partial rank correlation coefficient (PRCCs) for important parameters of R_m.

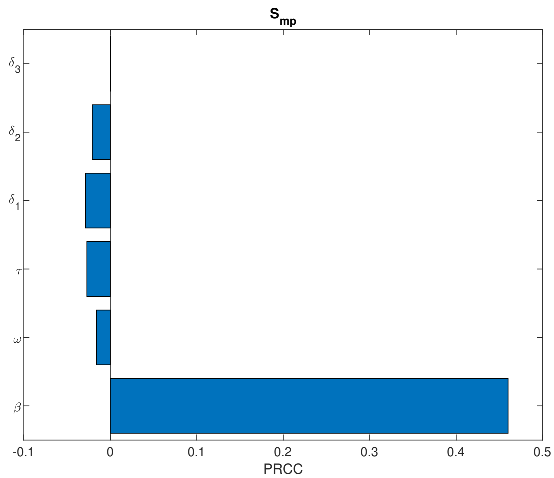


Figure 4: Partial rank correlation coefficient(PRCCs) for important parameters of S_{mp}.

population, τ, have significant effects among the military personnel while other parameters are insignificant based on their monotonicity, PRCCs< |0.5|.

The signs (+ and -) of PRCC indicate the definite qualitative relationship between the parameters and output of the disgruntled generation number, R_m and the state variables, T_{mp}, D_{mp} and P_{mp}. The parameter with positive PRCC implies that increasing it increases the value of R_m, T_{mp}, D_{mp} and P_{mp}. At the same time, the parameters with PRCC negative values indicate that their output decreases whenever their values increase. Figures 8 and 9 show the impact of τ and β on disgruntled and productive compartments, respectively. In Figure 8, whenever β is increased, it increases the disgruntled military population, while the disgruntled military population decreases as τ increases. Also, in Figure 9, the productive military population increases as τ and β increase.

For the population, S_{mp} in Figure 4, there are no significant

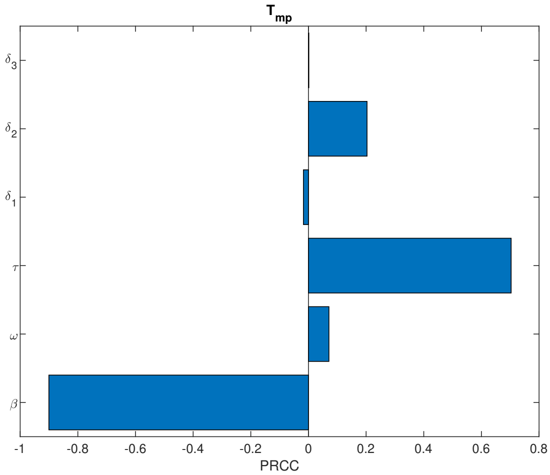


Figure 5: Partial rank correlation coefficient (PRCCs) for T_{mp}.

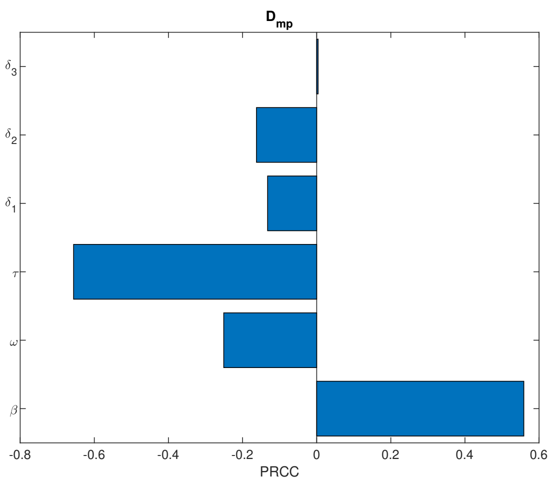


Figure 6: Partial rank correlation coefficient (PRCCs) for D_{mp}.

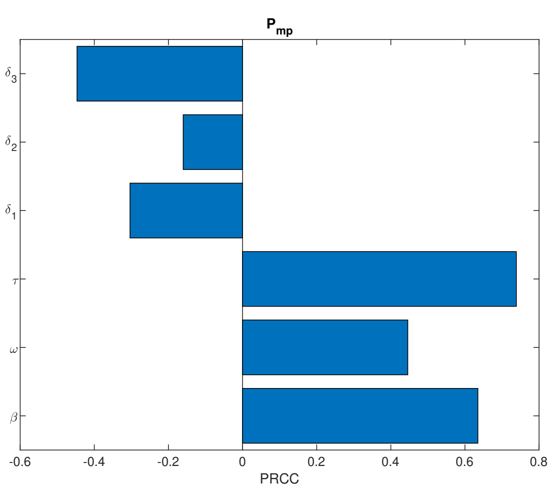


Figure 7: Partial rank correlation coefficient (PRCCs) for P_{mp}.

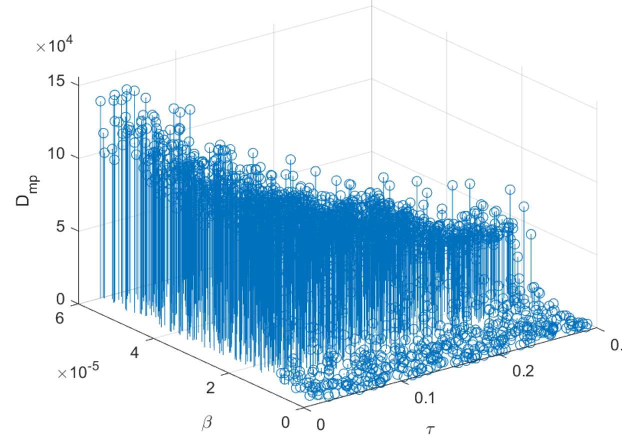


Figure 8: Frequency plot for τ and β on D_{mp}.

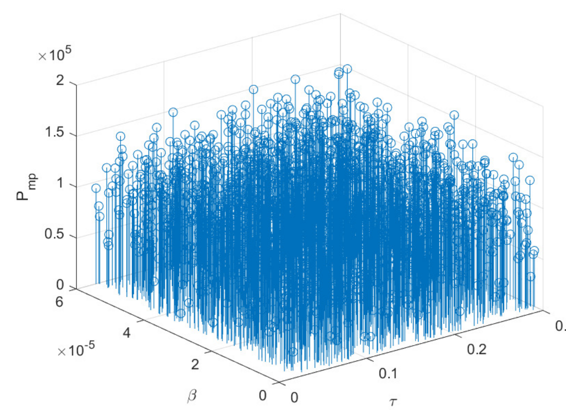


Figure 9: Frequency plot for τ and β on P_{mp}.

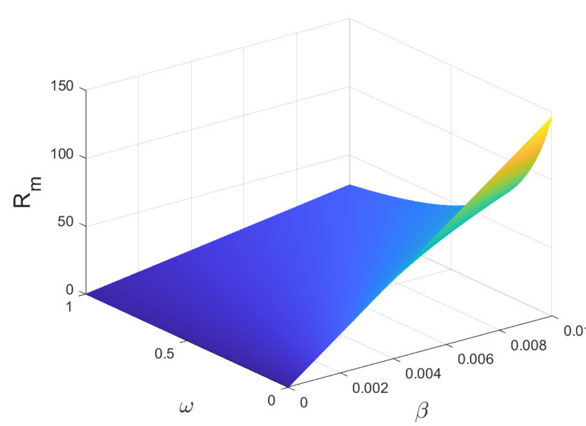


Figure 10: 3D plot for β and ω on R_m.

parameters affecting it as the PRCCs< |0.5|.

Figure 10 shows the effect of β and ω on R_m, as β increases, R_m increases, this means that this lead to increase in the

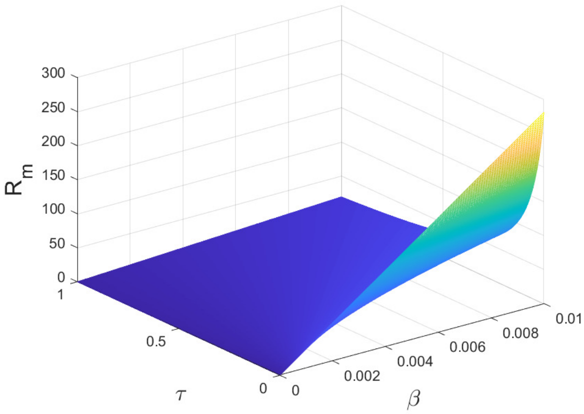


Figure 11: 3D plot for β and τ on R_m.

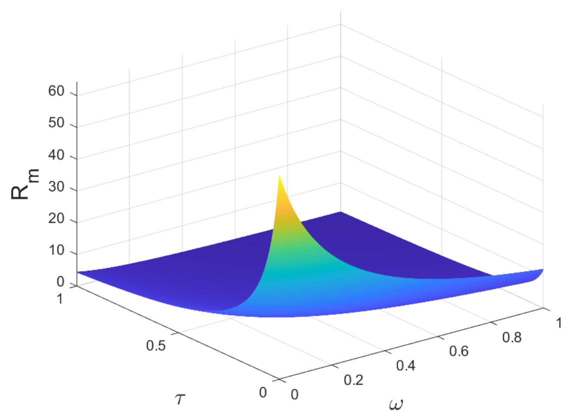


Figure 12: 3D plot for ω and τ on R_m.

level of disgruntlement among the military personnel whereas increase in ω implies decrease in R_m by extension, the level of disgruntlement decreases. Also, Figure 11 shows that as β increases, R_m increases, which means the disgruntlement increases, while as τ increases, R_m decreases, which implies that disgruntlement among the military personnel decreases. In addition, Figure 12 depicts that as both ω (welfare) and τ (welfare) increase, R_m decreases, and this leads to the reduction of disgruntlement among the military personnel; otherwise, it will increase the level of disgruntlement among the military personnel.

5. Numerical simulations with discussion

The numerical simulations of the systems of equations (1) are carried out. The simulations are further carried out on the two compartments D_{mp} and P_{mp} of the basic model (equation) (1) to examine the dynamics of disgruntlement on each compartment before and after the application of control, that is, when the parameter τ in model (equation) (1) is replaced with $\tau(1 + u)$ and without $\tau(1 + u)$. This is to compare, investigate the impact, the efficiency, and effectiveness of the control $(1+u)$ and without control $(1 + u)$ on the system (1). The fourth-order Runge-Kutta method is used for the simulations. The parameter values used for the simulations are presented in Table 2, while the initial conditions are as follows: $S_{mp}(0) = 233,856$, $T_{mp}(0) = 21,338$, $D_{mp}(0) = 1$, $P_{mp}(0) = 342$.

Using the global sensitivity analysis results, the effects of the most significant parameter on the populations of D_{mp} and P_{mp} are implemented to show their importance on the dynamics of military welfare.

The solutions to the dynamic military population model are illustrated in Figure 13, which includes Figures 13a, 13b, 13c, and 13d. Figure 13 displays the recruitment trends within the military population, as well as the effects of disgruntlement and welfare on this group. It shows that the potential military population, denoted as S_{mp}, eligible to join the Armed Forces, continues to decline as individuals transition into the susceptible military population, T_{mp}. This trend stabilizes at a minimum after approximately eight years.

Figure 13a illustrates the recruitment of potential military personnel into the military population, showing a gradual decline that approaches a minimum by year ten. In contrast, Figure 13b depicts the dynamics of the total military population, highlighting shifts toward either dissatisfaction or increased productivity due to improved welfare. Notably, there is a significant drop to a minimal level around year two.

Figure 13c portrays the trends of disgruntled military personnel over ten years. At first, the number of disgruntled personnel is relatively low. However, after approximately one and a half years, the disgruntled population increases, peaking at around 130,000 before declining to about 25,000 by the tenth year. This gradual decline is likely linked to improved welfare measures. Finally, Figure 13d illustrates the dynamics of the productive military population, which starts to increase around the same time, reaching a peak of 115,000 by year ten, also attributed to improved welfare.

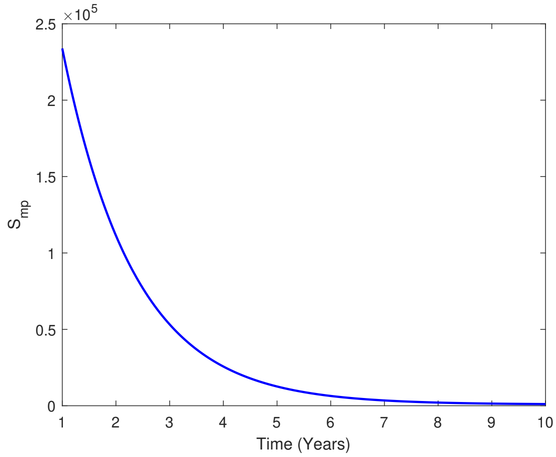
5.1. *The effect of varying the parameter values on the military welfare dynamics*

Figures 14 and 15 shows the effect of varying the parameters β, ω, τ on the disgruntled military population, D_{mp} and productive military population, P_{mp}, respectively.

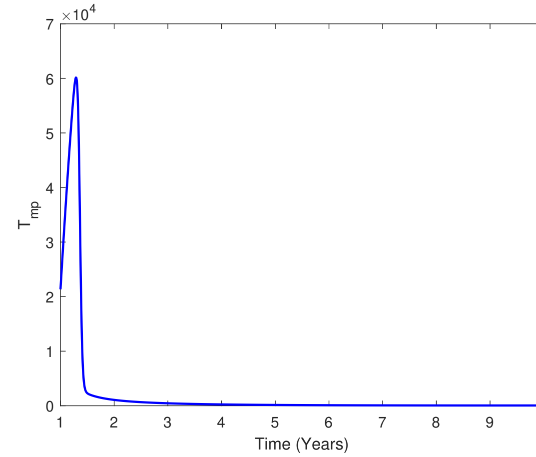
Figure 14 comprises three sub-figures: Figure 14a, Figure 14b, and Figure 14c. These figures illustrate the varying effects of the influence rate β, the welfare improvement ω, and the progress rate due to welfare τ on the disgruntled military population, D_{mp}, as depicted in Figure 14a, Figure 14b, and Figure 14c, respectively.

Figure 14a demonstrates that the influence rate β has a minimal significant impact on the disgruntled military population D_{mp}. Although there is a slight effect of β on this group around one and a half years in, it peaks at approximately 130,000 by year three before gradually declining to about 25,000 by year ten. This trend indicates that disgruntled military personnel are mainly influenced early on.

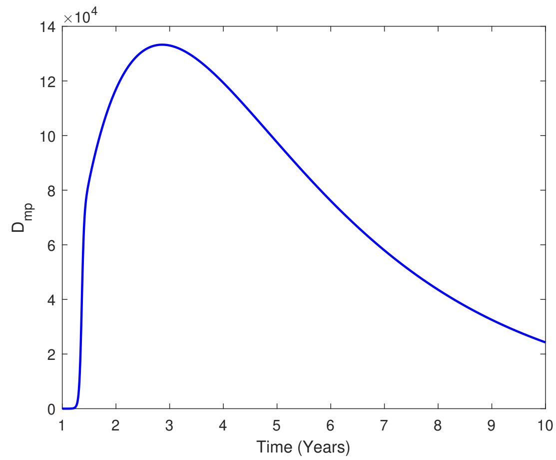
Figure 14b shows the varying effect of ω on the disgruntled military population. It similarly impacts the population, although the improved welfare represented by ω does not directly



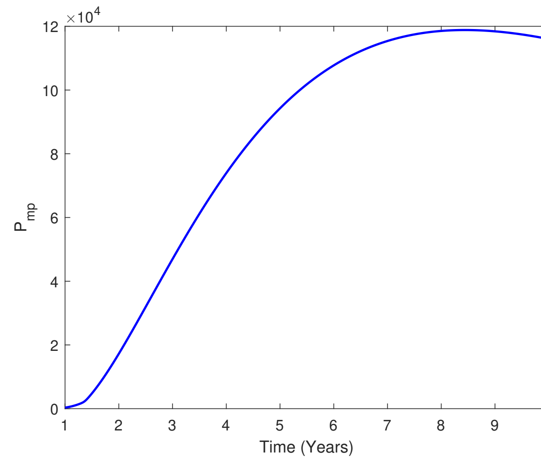
(a) Plot for the S_{mp} population of the model (equation) (1).



(b) Plot for the T_{mp} population of the model (equation) (1).



(c) Plot for the D_{mp} population of the model (equation) (1).



(d) Plot for the P_{mp} population of the model (equation) (1).

Figure 13: Simulations showing solutions of the model (equation) (1).

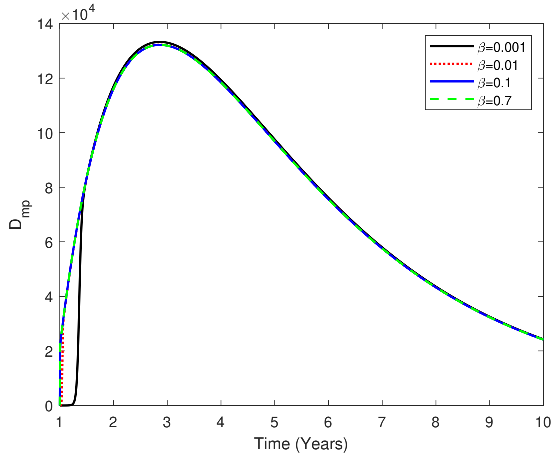
benefit the servicemen. Figure 14c reveals the significant role of the progress rate due to welfare τ in reducing the number of disgruntled military personnel. As τ increases, the population of disgruntled military personnel decreases, approaching zero around year ten when $\tau = 0.7$. This suggests that to eliminate discontented military personnel, it is essential to introduce enhanced welfare programs. This finding validate the conclusions of previous studies [4, 6–9, 11, 40–43] regarding the impact of improved welfare.

Figure 15 illustrates the varying effects of β, ω, and τ on the productive military population P_{mp}. Figure 15a shows that regardless of the influence rate β, the productive military population P_{mp} continues to increase as military personnel receive improved welfare. This indicates that they cannot be easily influenced to become disgruntled. Similarly, Figure 15b demonstrates that as the progress rate of welfare ω increases, moving individuals from the susceptible military population to the productive military population, P_{mp} also rises, irrespective of the specific value of ω. Lastly, Figure 15c illustrates the significant impact of τ on reducing the number of disgruntled military personnel while simultaneously increasing the productive military population. Therefore, as τ increases, the productive military population also grows. This suggests that in order to eliminate disgruntlement among military personnel and enhance overall productivity, it is essential to strengthen and sustain improved welfare initiatives. These results are consistent with the findings of Refs. [4, 6–9, 11, 40, 42] regarding the effects of improved welfare.

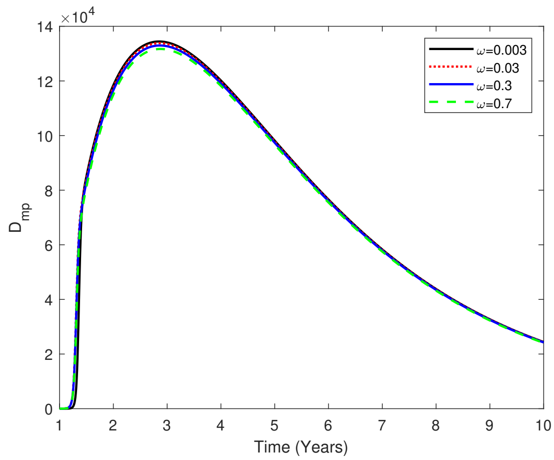
5.2. The impact of implementing control measure on the military welfare dynamics

The basic model for the military population in system (equation) (1) is extended by introducing constant control called the standard improved welfare package control, u, based on the result of the basic model simulations, and τ been the most significant parameter on the disgruntled military generation number R_m and by extension, on each of the sub-military populations, T_{mp}, D_{mp}, P_{mp}. This is done by replacing the parameter τ in model (equation) (1) with $\tau(1 + u)$. The effect of the control is displayed in Figure 16.

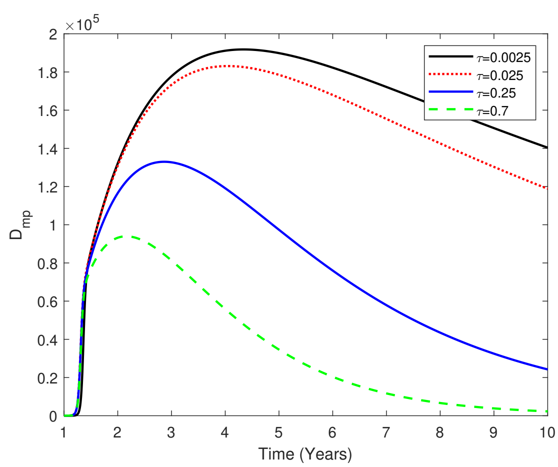
Figure 16 illustrates the dynamics of the disgruntled military population. The Figure 16a depicts the impact of control-



(a) Plot for the effect of β on D_{mp} subpopulation.

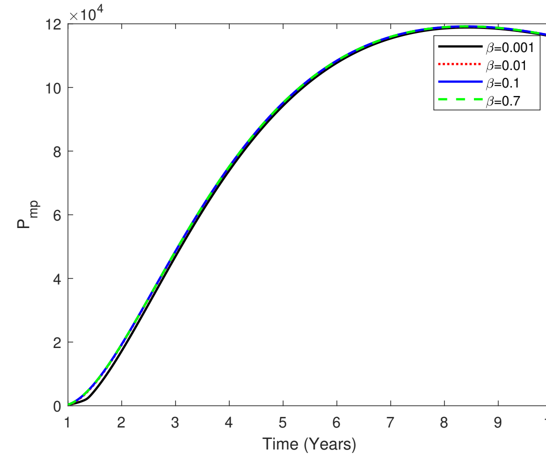


(b) Plot for the effect of ω on D_{mp} subpopulation.

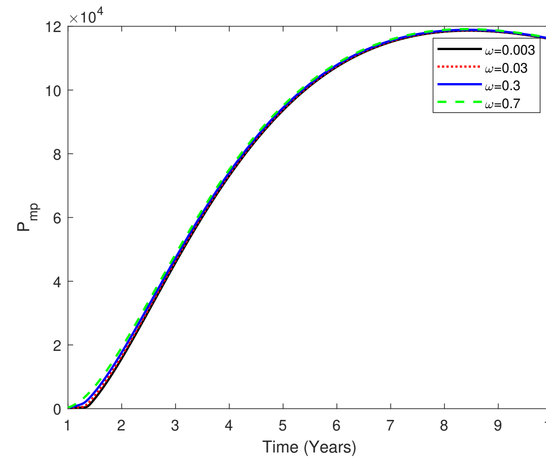


(c) Plot for the effect of τ on D_{mp} subpopulation.

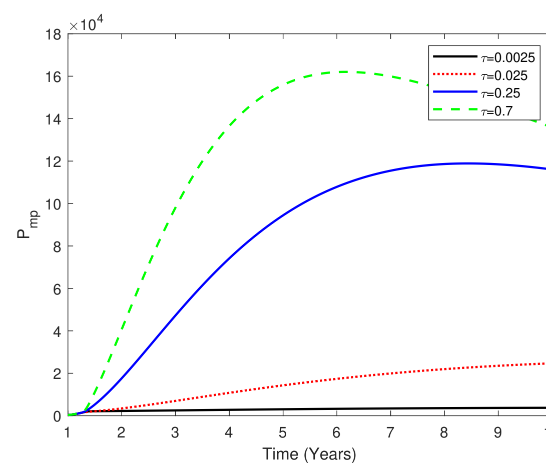
Figure 14: Simulations showing the effects of varying the parameters β, ω, τ on the dynamics of disgruntled military population D_{mp} subpopulation.



(a) Plot for the effect of β on P_{mp} subpopulation.



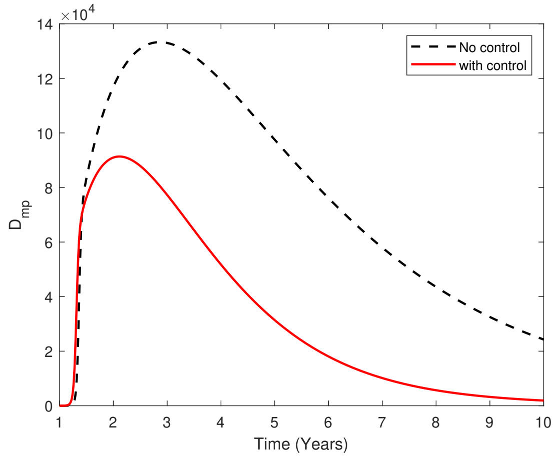
(b) Plot for the effect of ω on P_{mp} subpopulation.



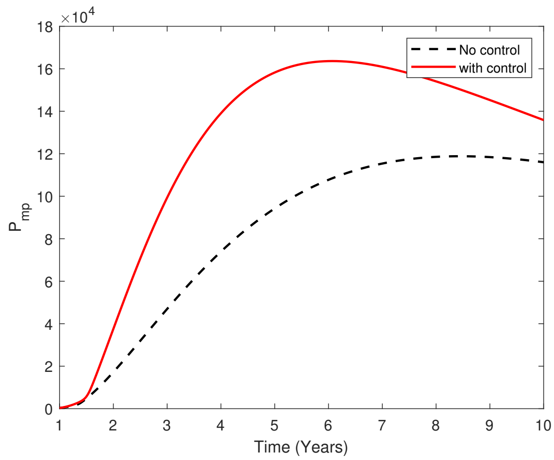
(c) Plot for the effect of τ on P_{mp} subpopulation.

Figure 15: Simulations showing the effects of varying the parameters β, ω, τ on the dynamics of productive military population P_{mp} subpopulation.

ling measures on the disgruntled military population, denoted as D_{mp}. With the implementation of the standard improved welfare package, referred to as u, the disgruntled military population D_{mp} begins to increase around one year and two months, peaking at approximately 8×10^4 at around two years and two months. After this peak, it gradually decreases to nearly zero



(a) Plot for the impact of control on D_{mp} subpopulation.



(b) Plot for the impact of control on P_{mp} subpopulation.

Figure 16: Simulations showing the impact of control on the disgruntled military compartment,D_{mp} and productive military compartment, P_{mp} dynamics.

over ten years. In contrast, without the control measures, D_{mp} also starts to rise at about one year and two months, reaching a peak of approximately 14×10^4. This figure then declines gradually, bottoming out at about 2.5×10^4 around year ten.

The Figure 16b illustrates the effects of the improved welfare package control u on the productive military population, P_{mp}. The implementation of this package increases the productive military population to a maximum level of around 16×10^4 at approximately year five. At the same time, it drops to a minimum of about 1.3×10^4 at around year three.

These findings underscore the importance of the standard improved welfare package in alleviating discontent among military personnel and its potential to enhance productivity within the military ranks. This aligns with the findings of [4, 6–8, 11, 12, 40, 43] regarding the importance of welfare.

6. Conclusion

In this paper, we investigated the impact of welfare on the Nigerian military population and the influence of disgruntlement among military personnel by developing a deterministic mathematical model. We derived the disgruntled military reproduction number, denoted as R_m. Additionally, we analyzed the model's disgruntled-free equilibrium (DFE) and disgruntled present equilibrium (DPE). We conducted a local stability analysis of the model, implementing three bifurcation analyses: forward bifurcation, backward bifurcation, and Hopf bifurcation. We also established the global stability of the system and performed a global sensitivity analysis. The global sensitivity analysis revealed that the influence rate (β), the progress rate from the disgruntled military population to a productive military population (τ), and the progress rate due to welfare that transitions members of the susceptible military population from potential disgruntlement to productivity (ω) have significant impacts on the disgruntled military reproduction number R_m and the level of disgruntlement among the military population.

Subsequently, these findings were used to formulate a control model that incorporates a constant control parameter, referred to as the standard improved welfare package control (u), applied to the most significant parameter, τ, of the basic model. We conducted numerical simulations to examine the impact of these control measures. The results indicate that:

- to minimize the level of discontent and compromise among military operatives, it is essential to implement an improved welfare package.
- to enhance productivity among military personnel, a sustained welfare package must be maintained. This aligns with previous findings.

To enhance the productivity of military personnel and prevent dissatisfaction and compromises among the military operatives, standard improved welfare packages must be established within military formations by the appropriate authorities. This principle applies to all armed forces as well. The positive impact of improved welfare on worker productivity aligns with the statement made by the current President of Nigeria: "A happy worker is a productive worker, and society depends on the productivity of happy workers" [37]. This study acknowledges certain limitations, such as the need to address factors like absence without leave (AWOL), voluntary and compulsory retirement, government instability, and external crises as means of exiting the military population. Future research will explore these factors.

Data availability

All the Data (parameter values) used in this work have been cited appropriately in Table 1.

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