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Numerical solution of fractional advection-diffusion equation with generalized Caputo derivative using shifted ultraspherical collocation method

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Abstract

In this article, we develop an approximate technique for solving a fractional advection-diffusion equation with a generalized Caputo derivative via a finite difference scheme. This technique employed the properties of shifted ultraspherical polynomials, which combine the finite difference scheme for the temporal discretization derivative and express the approximate solution in terms of shifted ultraspherical polynomials. The convergence and the stability of the proposed technique are proved. Numerical examples are considered, and the obtained results are compared with the analytical results and those obtained in the literature to establish the accuracy of the technique. The proposed techniques, based on the generalized Caputo derivative and the properties of shifted ultraspherical polynomials, produce robust results and encompass those obtained using several other families of orthogonal polynomials as demonstrated in the results of the examples considered.

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1. Introduction

Fractional Calculus (FC), which involves the study of derivatives and integrals with arbitrary orders, has gained significant attention in recent years for its ability in handling a wide range of real-world problems in fields such as fluid mechanics, physics, biology, and engineering [1]. Research has demonstrated that Fractional Derivatives (FD) can offer a more precise representation of various physical systems. The applications of fractional calculus span many scientific disciplines, as evidenced by the work of numerous researchers [2–6]. FC has contributed significantly in the study of Fractional Differential Equation (FDE) due to the widespread application of FDE across different fields such as COVID-19 modeling, where fractional-order Partial Differential Equations (PDE) have outperformed traditional models in predicting cases and describing the dynamics of the virus [7–9], modeling turbulent flow [10], groundwater contaminant transport [11], to mention but few. Another form of FDE is Space Fractional Diffusion Equations (SFDE), which derived by replacing the second-order spatial derivative in the classical diffusion equation with an inverse Riesz potential of order $\gamma > 0$ [9].

An extension of SFDE concept is the space-time fractional diffusion equation, several numerical techniques have been developed to solve this problem including shifted Gegenbaeur-

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finite difference technique [12], Chebyshev-finite difference technique [13–15], Compact difference schemes [16, 17]. A more robust form of FDE is fractional advection-diffusion equation (FADE) and recently attracted considerable attention with different approaches developed to solve FADE [18–22]. However, the FADE has not yet been explored in the context of the generalized Caputo derivative via finite difference scheme (FDS), despite numerous studies using other FD to solve it. Therefore, the primary aim of this study is to investigate the space fractional advection-diffusion equation (SFADE) using Caputo-Katugampola (also known as generalized Caputo) derivative of the form:

$$\begin{cases} \frac{\partial \mathscr{U}(\mathtt{x},\mathtt{t})}{\partial \mathtt{t}} + \Upsilon {}^{\mathsf{C}}\mathscr{D}_{0^+}^{\gamma-1,\delta}\mathscr{U}(\mathtt{x},\mathtt{t}) = \lambda {}^{\mathsf{C}}\mathscr{D}_{0^+}^{\gamma,\delta}\mathscr{U}(\mathtt{x},\mathtt{t}) + \chi(\mathtt{x},\mathtt{t}), \\ \mathscr{U}(\mathtt{x},\mathbb{0}) = \psi(\mathtt{x}), \quad 0 < \mathtt{x} < \mathcal{L}, \\ \mathscr{U}(0,\mathtt{t}) = \phi_0(\mathtt{t}), \mathscr{U}(\mathcal{L},\mathtt{t}) = \phi_1(\mathtt{t}), \quad 0 < \mathtt{t} \le \mathcal{T}, \end{cases} \tag{1}$$

where ${}^{\mathsf{C}}\mathscr{D}_{0^+}^{\gamma-1,\delta}\mathscr{U}(\mathtt{x},\mathtt{t})$ and ${}^{\mathsf{C}}\mathscr{D}_{0^+}^{\gamma,\delta}\mathscr{U}(\mathtt{x},\mathtt{t})$ are generalized Caputo fractional derivatives operators, Υ and $\lambda > 0$ are the coefficients of advection and diffusion, respectively, $\delta > 0, 1 < \gamma < 2$.. $\chi(\mathtt{x},\mathtt{t})$ is source term function and $\psi(\mathtt{x}), \phi_0(\mathtt{t})$ and $\phi_1(\mathtt{t})$ are given functions.

The structure of the paper is as follows: Section 2 covers the properties of the generalized fractional derivative and the Shifted Ultraspherical Polynomials (SUP) used throughout the study. Section 3 outlines the algorithm of the proposed scheme. In Subsection 3.1, we analyze the convergence and the stability of the proposed technique. Section 4 presents numerical experiments to demonstrate the effectiveness and accuracy of the method. Finally, Section 5 present the conclusions of the findings.

2. Preliminaries

This section present relevant definitions and mathematical preliminaries related to the FD and the properties of the SUP that are crucial for the discussion in this paper.

2.1. Description of GCFD

Suppose $\gamma \in \mathbb{R}$ and $\mathscr{U} \in C^1([a,b])$, then the GCFD is defined as follows [23]:

$$ {}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\mathscr{U}(\mathtt{x}) = \frac{\delta^{\gamma-i+1}}{\Gamma(i-\gamma)} \int_a^{\mathtt{x}} \frac{y^{(\delta-1)(1-i)}}{(\mathtt{x}^\delta - y^\delta)^{\gamma-m+1}} \mathscr{U}^{(i)}(y)dy, \tag{2}$$

and

$$ {}^{\mathsf{C}}\mathscr{D}_{b^-}^{\gamma,\delta}\mathscr{U}(\mathtt{x}) = \frac{(-1)^i\delta^{\gamma-i+1}}{\Gamma(i-\gamma)} \int_{\mathtt{x}}^{b} \frac{y^{(\delta-1)(1-i)}}{(\mathtt{x}^\delta - y^\delta)^{\gamma-m+1}} \mathscr{U}^{(i)}(y)dy, \tag{3}$$

where $\delta > 0$ and $i-1 < \gamma < i, \ \lfloor\gamma\rfloor + 1$. Note, if $\gamma \to 0^+$ or $\gamma \to 1$, then the GCFD becomes Caputo-Hadamard fractional derivatives or Caputo-fractional derivative, respectively.

The linearity property of GCFD is given by:

$$\mathscr{D}_{a^+}^{\gamma,\delta}(\sigma\mathscr{U}(\mathtt{x}) + \varsigma\mathscr{V}(\mathtt{x})) = \sigma\mathscr{D}_{a^+}^{\gamma,\delta}\mathscr{U}(\mathtt{x}) + \varsigma\mathscr{D}_{a^+}^{\gamma,\delta}\mathscr{V}(\mathtt{x}), \tag{4}$$

where, σ and ς are constants.

The following property of GCFD was proved in Ref. [24] for $\gamma \in (0,1)$ and $\delta > 0$:

$$ {}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\mathtt{t}^j = \begin{cases} 0, \quad j < \lceil\gamma\rceil, \ \left(\gamma - \frac{j}{\delta}\right) \in \mathbb{N} \\ \dfrac{j\delta^{\gamma-1}\Gamma\left(\frac{j}{\delta}\right)}{\Gamma\left(\frac{j}{\delta}+1-\gamma\right)}\mathtt{t}^{j-\gamma\delta}, \quad j \ge \lceil\gamma\rceil, \ \left(\gamma - \frac{j}{\delta}\right) \notin \mathbb{N}. \end{cases} \tag{5}$$

Adopting the approach in Ref. [24], we obtain the following result for $\gamma \in (1,2)$, and $\delta > 0$:

$$ {}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\mathtt{t}^j = \begin{cases} 0, \quad j < \lceil\gamma\rceil, \ \left(\gamma - \frac{j}{\delta}\right) \in \mathbb{N} \\ \dfrac{j(j-1)\delta^{\gamma-2}\Gamma\left(\frac{j}{\delta}-1\right)}{\Gamma\left(\frac{j}{\delta}+1-\gamma\right)}\mathtt{t}^{j-\gamma\delta}, \quad j \ge \lceil\gamma\rceil, \ \left(\gamma - \frac{j}{\delta}\right) \notin \mathbb{N}. \end{cases} \tag{6}$$

2.2. Ultraspherical polynomials (UP) $\mathbb{C}_i^{(\alpha)}(\mathtt{x})$

In this study, Ultraspherical polynomials $\{\mathbb{C}_i^{(\alpha)}(\mathtt{x}), \ i = 0,1,\dots\}$ is employed because it generalize other types of orthogonal polynomials. Specifically, when $\alpha = \frac{1}{2}$, the ultraspherical polynomials simplify to the Legendre polynomials, and when $\alpha = 1$, it reduce to the second-kind Chebyshev polynomials.

UP $\left(\mathbb{C}_i^{(\alpha)}(\mathtt{x})\right)$ is an orthogonal polynomials of degree i in $\mathtt{x} \in [-1, \ 1]$ with respect to weight function $\omega(\mathtt{x}) = (1-\mathtt{x}^2)^{(\alpha-\frac{1}{2})}$ is defined as follows [25]:

$$\mathbb{C}_i^{(\alpha)}(\mathtt{x}) = \sum_{j=0}^{i} \frac{(-1)^j\Gamma(2\alpha+2i-j)\Gamma(\alpha+\frac{1}{2})}{(i-j)!\Gamma(j+1)\Gamma(i-j+\alpha+\frac{1}{2})\Gamma(2\alpha)}\mathtt{x}^{i-j}. \tag{7}$$

The recurrence form is given by:

$$\mathbb{C}_i^{(\alpha)}(\mathtt{x}) = \frac{1}{i}\left[2(i+\alpha-1)\mathtt{x}\mathbb{C}_{i-1}^{(\alpha)}(\mathtt{x}) - (i+2\alpha-2)\mathbb{C}_{i-2}^{(\alpha)}(\mathtt{x})\right], \quad i \ge 2, \tag{8}$$

where $\mathbb{C}_0^{(\alpha)}(\mathtt{x}) = 1, \ \mathbb{C}_1^{(\alpha)}(\mathtt{x}) = 2\alpha\mathtt{x}$.

The corresponding shifted form is defined as:

$$\mathfrak{C}_i^{(\alpha)}(\mathtt{x}) = \frac{1}{i}\Big[2(i+\alpha-1)\left(\frac{2\mathtt{x}-(a+b)}{b-a}\right)\mathfrak{C}_{i-1}^{(\alpha)}(\mathtt{x}) - (i+2\alpha-2)\mathfrak{C}_{i-2}^{(\alpha)}(\mathtt{x})\Big], \ i \ge 2, \mathtt{x} \in [a,b], \tag{9}$$

where $\mathfrak{C}_0^{(\alpha)}(\mathtt{x}) = 1, \ \mathfrak{C}_1^{(\alpha)}(\mathtt{x}) = 2\alpha\left(\frac{2\mathtt{x}-(a+b)}{b-a}\right)$. The explicit analytic form for the SUP $\mathfrak{C}_i^{(\alpha)}(\mathtt{x})$ of degree i defined in the interval $[0, \ 1]$ is given as:

$$\mathfrak{C}_i^{(\alpha)}(\mathtt{x}) = \sum_{j=0}^{i} \frac{(-1)^j\Gamma(2\alpha+2i-j)\Gamma(\alpha+\frac{1}{2})}{(i-j)!\Gamma(j+1)\Gamma(i-j+\alpha+\frac{1}{2})\Gamma(2\alpha)}\mathtt{x}^{i-j}. \tag{10}$$

The orthogonality condition corresponding to the interval $\mathtt{x} \in [a, b]$ is defined as follows:

$$\langle C_i^{(\alpha)}(\mathtt{x}),\ C_j^{(\alpha)}(\mathtt{x})\rangle = \begin{cases} \int_{-1}^{1}\left(1-\mathtt{x}^2\right)^{(\alpha-\frac{1}{2})}\mathbb{C}_i^{(\alpha)}(\mathtt{x})\mathbb{C}_j^{(\alpha)}(\mathtt{x})d\mathtt{x} = \begin{cases} 0, \text{ for } i \neq j \\ \frac{\pi 2^{1-2\alpha}\Gamma(i+2\alpha)}{i![\Gamma(\alpha)]^2(i+\alpha)}, \text{ for } i = j \end{cases} \\ \int_{0}^{1}\left(\mathtt{x}-\mathtt{x}^2\right)^{(\alpha-\frac{1}{2})}\mathfrak{C}_i^{(\alpha)}(\mathtt{x})\mathfrak{C}_j^{(\alpha)}(\mathtt{x})d\mathtt{x} = \begin{cases} 0, \text{ for } i \neq j \\ \frac{\pi 2^{1-4\alpha}\Gamma(i+2\alpha)}{i![\Gamma(\alpha)]^2(i+\alpha)}, \text{ for } i = j \end{cases} \\ \int_{0}^{2}\left(2\mathtt{x}-\mathtt{x}^2\right)^{(\alpha-\frac{1}{2})}\mathscr{C}_i^{(\alpha)}(\mathtt{x})\mathscr{C}_j^{(\alpha)}(\mathtt{x})d\mathtt{x} = \begin{cases} 0, & \text{for } i \neq j \\ \frac{\pi 2^{1-2\alpha}\Gamma(i+2\alpha)}{i![\Gamma(\alpha)]^2(i+\alpha)}, & \text{for } i = j. \end{cases} \end{cases} \tag{11}$$

Let $\mathscr{U}(\mathtt{x})$ be a square integrable function in the interval $[a,\ b]$, then

$$\mathscr{U}(\mathtt{x}) = \sum_{i=0}^{\infty}\beta_i C_i^{(\alpha)}(\mathtt{x}) \tag{12}$$

where the coefficients $\beta_n,\ n = 0,\ 1, \ldots, \mathcal{N}$ is defined as:

$$\beta_i = \begin{cases} \dfrac{i!\,[\Gamma(\alpha)]^2\,(i+\alpha)}{\pi 2^{1-2\alpha}\Gamma(i+2\alpha)}\displaystyle\int_{-1}^{1}\left(1-\mathtt{x}^2\right)^{\alpha-\frac{1}{2}}\mathscr{U}(\mathtt{x})\mathbb{C}_i^{(\alpha)}(\mathtt{x})\,d\mathtt{x}, & \mathtt{x} \in [-1, 1], \\ \dfrac{i!\,[\Gamma(\alpha)]^2\,(i+\alpha)}{\pi 2^{1-4\alpha}\Gamma(i+2\alpha)}\displaystyle\int_{0}^{1}\left(\mathtt{x}-\mathtt{x}^2\right)^{\alpha-\frac{1}{2}}\mathscr{U}(\mathtt{x})\mathfrak{C}_i^{(\alpha)}(\mathtt{x})\,d\mathtt{x}, & \mathtt{x} \in [0, 1], \\ \dfrac{i!\,[\Gamma(\alpha)]^2\,(i+\alpha)}{\pi 2^{1-2\alpha}\Gamma(i+2\alpha)}\displaystyle\int_{0}^{2}\left(2\mathtt{x}-\mathtt{x}^2\right)^{\alpha-\frac{1}{2}}\mathscr{U}(\mathtt{x})\mathscr{C}_i^{(\alpha)}(\mathtt{x})\,d\mathtt{x}, & \mathtt{x} \in [0, 2]. \end{cases} \tag{13}$$

depending on the interval of Eq. (1).
Only the first $(\mathcal{N}+1)-$terms of shifted ultraspherical polynomials are needed in the approximation. Therefore, Eq. (12) becomes

$$\mathscr{U}(\mathtt{x}) \approx \mathcal{U}_{\mathcal{N}}(\mathtt{x}) = \sum_{n=0}^{\mathcal{N}}\beta_n C_n^{(\alpha)}(\mathtt{x}). \tag{14}$$

Theorem 2.1. Let $\mathcal{U}_{\mathcal{N}}(\mathtt{x})$ be an approximate solution of Eq. (1), then the GCFD of $\mathcal{U}_{\mathcal{N}}(\mathtt{x})$ is expressed as:

$${}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\left(\mathcal{U}_{\mathcal{N}}(\mathtt{x})\right) = \sum_{n=\lceil\gamma\rceil}^{\mathcal{N}}\sum_{k=0}^{n-\lceil\gamma\rceil}\beta_n\mathscr{H}_{n,k}\mathtt{x}^{n-k-\gamma\delta}, \quad 0 < \gamma < 1, \tag{15}$$

and

$${}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\left(\mathcal{U}_{\mathcal{N}}(\mathtt{x})\right) = \sum_{n=\lceil\gamma\rceil}^{\mathcal{N}}\sum_{k=0}^{n-\lceil\gamma\rceil}\beta_n\mathcal{H}_{n,k}\mathtt{x}^{n-k-\gamma\delta}, \quad 1 < \gamma < 2 \tag{16}$$

where

$$\mathscr{H}_{n,k} = \frac{(-1)^k\Gamma(2\alpha+2n-k)\Gamma(\alpha+\frac{1}{2})\delta^{\gamma-1}\Gamma\left(\frac{n-k}{\delta}\right)}{(n-k-1)!k!\Gamma(n-k+\alpha+\frac{1}{2})\Gamma(2\alpha)\Gamma\left(\frac{n-k}{\delta}+1-\gamma\right)}, \tag{17}$$

and

$$\mathcal{H}_{n,k} = \frac{(-1)^k\Gamma(2\alpha+2n-k)\Gamma(\alpha+\frac{1}{2})\delta^{\gamma-2}\Gamma\left(\frac{n-k}{\delta}-1\right)}{(n-k-1)!k!\Gamma(n-k+\alpha+\frac{1}{2})\Gamma(2\alpha)\Gamma\left(\frac{n-k}{\delta}+1-\gamma\right)}, \tag{18}$$

Proof: Using Caputo-Katugampola linearity property in Eq. (5) and Eq. (14), we obtain:

$${}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\mathcal{U}_{\mathcal{N}}(\mathtt{x}) = \sum_{n=0}^{\mathcal{N}}\beta_n{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\left(\mathfrak{C}_n^{(\alpha)}(\mathtt{x})\right). \tag{19}$$

Employing the linearity property of the GCFD, we obtain:

$${}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\left(\mathfrak{C}_n^{(\alpha)}(\mathtt{x})\right) = 0, \quad n = 0,\ 1, \ldots, \lceil\gamma\rceil-1,\ \gamma > 0 \tag{20}$$

$${}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\left(\mathfrak{C}_n^{(\alpha)}(\mathtt{x})\right) = \sum_{k=0}^{n}\frac{(-1)^k\Gamma(2\alpha+2n-k)\Gamma(\alpha+\frac{1}{2})}{(n-k)!k!\Gamma(n-k+\alpha+\frac{1}{2})\Gamma(2\alpha)} \times {}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\mathtt{x}^{n-k},\ n \geq \lceil\gamma\rceil. \tag{21}$$

Using Eq. (5), Eq. (21) becomes:

$${}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\left(\mathfrak{C}_n^{(\alpha)}(\mathtt{x})\right) = \sum_{k=0}^{n-\lceil\gamma\rceil}\frac{(-1)^k\Gamma(2\alpha+2n-k)\Gamma(\alpha+\frac{1}{2})\delta^{\gamma-1}\Gamma\left(\frac{n-k}{\delta}\right)}{(n-k-1)!k!\Gamma(n-k+\alpha+\frac{1}{2})\Gamma(2\alpha)\Gamma\left(\frac{n-k}{\delta}+1-\gamma\right)}\mathtt{x}^{n-k-\gamma\delta}. \tag{22}$$

Substituting Eq. (22) in Eq. (19) we obtain:

$${}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\left(\mathcal{U}_{\mathcal{N}}(\mathtt{x})\right) = \sum_{n=\lceil\gamma\rceil}^{\mathcal{N}}\sum_{k=0}^{n-\lceil\gamma\rceil}\beta_n\frac{(-1)^k\Gamma(2\alpha+2n-k)\Gamma(\alpha+\frac{1}{2})\delta^{\gamma-1}\Gamma\left(\frac{n-k}{\delta}\right)}{(n-k-1)!k!\Gamma(n-k+\alpha+\frac{1}{2})\Gamma(2\alpha)\Gamma\left(\frac{n-k}{\delta}+1-\gamma\right)}\mathtt{x}^{n-k-\gamma\delta}, \tag{23}$$

therefore, Eq. (23) can be expressed as:

$${}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\left(\mathcal{U}_{\mathcal{N}}(\mathtt{x})\right) = \sum_{n=\lceil\gamma\rceil}^{\mathcal{N}}\sum_{k=0}^{n-\lceil\gamma\rceil}\beta_n\mathscr{H}_{n,k}\mathtt{x}^{n-k-\gamma\delta}, \tag{24}$$

hence the proof. Using the same approach outlined earlier and Eq. (6), we can verify equation (18).

It can be verified that when $\delta = 1$, the results from Ref. [12] are obtained.

3. Description of the numerical algorithm

In this section, we employ the combination of FDS and shifted ultraspherical collocation method to find the numerical solution of Eq. (1), by first obtaining the time discrete scheme depending on FDS. To achieve this, we define the mesh points as $\mathtt{t}_m = m\delta\mathtt{t}, m = 0, 1, \ldots, \mathcal{M}$ with $\delta\mathtt{t} = \frac{\mathcal{T}}{\mathcal{M}}$, and we use collocation points $\mathfrak{C}_{\mathcal{N}-\lceil\gamma\rceil+1}^{(\alpha)}(\mathtt{x}_i), i = 0, 1, \ldots, (\mathcal{N}-\lceil\gamma\rceil+1)$.

Let

$$\mathcal{U}_{\mathcal{N}}(\mathtt{x},\mathtt{t}) = \sum_{n=0}^{\mathcal{N}}\beta_n(\mathtt{t})\mathfrak{C}_n^{(\alpha)}(\mathtt{x}), \tag{25}$$

be the approximate solution of Eq. (1), therefore Eq. (1) becomes:

$$\frac{\partial\mathcal{U}_{\mathcal{N}}(\mathtt{x},\mathtt{t})}{\partial\mathtt{t}} + \Upsilon^{\mathsf{C}}\mathscr{D}_{0^+}^{\gamma-1,\delta}\mathcal{U}_{\mathcal{N}}(\mathtt{x},\mathtt{t}) = \lambda^{\mathsf{C}}\mathscr{D}_{0^+}^{\gamma,\delta}\mathcal{U}_{\mathcal{N}}(\mathtt{x},\mathtt{t}) + \chi(\mathtt{x},\mathtt{t}), \tag{26}$$

rewriting Eq. (26), using FDS, we have:

$$\mathcal{U}_{\mathcal{N}}^{m} + \delta\mathtt{t}\Upsilon^{\mathsf{C}}\mathscr{D}_{0^+}^{\gamma-1,\delta}\mathcal{U}_{\mathcal{N}}^{m}(\mathtt{x},\mathtt{t}) - \delta\mathtt{t}\lambda^{\mathsf{C}}\mathscr{D}_{0^+}^{\gamma,\delta}\mathcal{U}_{\mathcal{N}}^{m}(\mathtt{x},\mathtt{t}) = \mathcal{U}_{\mathcal{N}}^{m-1} + \delta\mathtt{t}\chi(\mathtt{x},\mathtt{t}), \tag{27}$$

substituting Eqs. (24) and Eqs. (25) in Eq. (27), we obtain:

$$\begin{aligned}&\sum_{n=0}^{\mathcal{N}}\beta_n^m\mathfrak{C}_n^{(\alpha)}(\mathtt{x})+\delta\mathtt{t}\Upsilon\sum_{n=\lceil\gamma-1\rceil}^{\mathcal{N}}\sum_{k=0}^{n-\lceil\gamma-1\rceil}\beta_n(t)\mathscr{H}_{n,k}^{(\alpha,\gamma-1)}\mathtt{x}^{n-k-(\gamma-1)\delta}\\&-\delta\mathtt{t}\lambda\sum_{n=\lceil\gamma\rceil}^{\mathcal{N}}\sum_{k=0}^{n-\lceil\gamma\rceil}\beta_n\mathscr{H}_{n,k}^{(\alpha,\gamma)}\mathtt{x}^{n-k-\gamma\delta}\\&=\sum_{n=0}^{\mathcal{N}}\beta_n^{m-1}\mathfrak{C}_n^{(\alpha)}(\mathtt{x})+\delta\mathtt{t}\chi(\mathtt{x},\mathtt{t})\end{aligned}\tag{28}$$

Collocate Eq. (28) at $\mathtt{x}=\mathtt{x}_i, i=0,1,\ldots,\mathcal{N}-\lceil\gamma\rceil+1$, and simplify we obtain:

$$\begin{aligned}&\beta_0^m+\beta_1^m\left(\mathfrak{C}_1^{(\alpha)}(\mathtt{x}_i)+\mathtt{x}_i^{1-(\gamma-1)\delta}\delta\mathtt{t}\Upsilon\mathscr{H}_{1,0}^{(\alpha,\gamma-1)}\right)\\&+\beta_2^m\Big(\mathfrak{C}_2^{(\alpha)}(\mathtt{x}_i)-\mathtt{x}_i^{2-\gamma\delta}\delta\mathtt{t}\lambda\mathscr{H}_{2,0}^{(\alpha,\gamma)}\\&+\delta\mathtt{t}\Upsilon\left(\mathtt{x}_i^{2-(\gamma-1)\delta}\mathscr{H}_{2,0}^{(\alpha,\gamma-1)}+\mathtt{x}_i^{1-(\gamma-1)\delta}\mathscr{H}_{2,1}^{(\alpha,\gamma-1)}\right)\Big)\\&+\beta_3^m\Big(\mathfrak{C}_3^{(\alpha)}(\mathtt{x}_i)-\delta\mathtt{t}\lambda\left(\mathtt{x}_i^{3-\gamma\delta}\mathscr{H}_{3,0}^{(\alpha,\gamma)}+\mathtt{x}_i^{2-\gamma\delta}\mathscr{H}_{3,1}^{(\alpha,\gamma)}\right)\\&+\delta\mathtt{t}\Upsilon\Big(\mathtt{x}_i^{3-(\gamma-1)\delta}\mathscr{H}_{3,0}^{(\alpha,\gamma-1)}+\mathtt{x}_i^{2-(\gamma-1)\delta}\mathscr{H}_{3,1}^{(\alpha,\gamma-1)}\\&+\mathtt{x}_i^{1-(\gamma-1)\delta}\mathscr{H}_{3,2}^{(\alpha,\gamma-1)}\Big)\Big)+\cdots\\&=\beta_0^{m-1}+\beta_1^{m-1}\mathfrak{C}_1^{(\alpha)}(\mathtt{x}_i)+\beta_2^{m-1}\mathfrak{C}_2^{(\alpha)}(\mathtt{x}_i)\\&+\beta_3^{m-1}\mathfrak{C}_3^{(\alpha)}(\mathtt{x}_i)+\cdots+\delta\mathtt{t}\chi_i^m.\end{aligned}\tag{29}$$

In addition, by substituting the prescribed boundary conditions of Eq. (1) into Eq. (25), together with the evaluation of the SUP terms ($\mathbb{C}_i^{(\alpha)}(\mathtt{x}=0)$ and $\mathbb{C}_i^{(\alpha)}(\mathtt{x}=\mathcal{L})$), we obtain

$$\begin{aligned}\mathcal{U}_{\mathcal{N}}(0,\mathtt{t})&=\sum_{n=0}^{\mathcal{N}}\frac{(-1)^n\Gamma(n+2\alpha)}{n!\Gamma(2\alpha)}\beta_n(\mathtt{t})=\phi_0(\mathtt{t}),\\\mathcal{U}_{\mathcal{N}}(\mathcal{L},\mathtt{t})&=\sum_{n=0}^{\mathcal{N}}\frac{\Gamma(n+2\alpha)}{n!\Gamma(2\alpha)}\beta_n(\mathtt{t})=\phi_1(\mathtt{t}).\end{aligned}\tag{30}$$

This results to $\mathcal{N}+1$ linear algebraic equations, which are solved to determine the unknowns coefficients $\beta_n^m, n=1,2,\ldots,\mathcal{N}$. Note, the initial values of $\beta_n^0, n=1,2,\ldots,\mathcal{N}$ are obtained from Eq. (13).

3.1. Convergence and stability analysis

To examine the stability and convergence of the proposed technique, let Θ be an open and bounded domain in $\mathbb{R}^2$ and $L_2^{\gamma,\delta}(\Theta)$ be a Hilbert space with the inner product

$$\langle\mathscr{F}(\mathtt{x}),\mathscr{G}(\mathtt{x})\rangle=\int_\Theta\mathscr{F}(\mathtt{x})\mathscr{G}(\mathtt{x})d\mathtt{x},\tag{31}$$

Euclidean norm $\|\mathscr{F}(\mathtt{x})\|=\langle\mathscr{F}(\mathtt{x}),\mathscr{F}(\mathtt{x})\rangle^{\frac{1}{2}}$ and Sobolev space as

$$H^{\gamma,\delta}(\Theta)=\left\{\mathscr{F}\in L_2^{\gamma,\delta}(\Theta),\mathscr{D}^{\gamma,\delta}\in L_2^{\gamma,\delta}(\Theta)\right\}.\tag{32}$$

In the following, we present key lemmas that are crucial for analyzing the stability and convergence of the proposed scheme.

Lemma 3.1. For any $\mathscr{F},\mathscr{G}\in H^{\frac{\gamma}{2},\delta}(\Theta)$, then

$$\begin{aligned}&\langle{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\mathscr{F},\mathscr{G}\rangle=\langle{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\frac{\gamma}{2},\delta}\mathscr{F},\ {}^{\mathsf{C}}\mathscr{D}_{b^-}^{\frac{\gamma}{2},\delta}\mathscr{G}\rangle\\&\langle{}^{\mathsf{C}}\mathscr{D}_{b^-}^{\gamma,\delta}\mathscr{F},\ \mathscr{G}\rangle=\langle{}^{\mathsf{C}}\mathscr{D}_{b^-}^{\frac{\gamma}{2},\delta}\mathscr{F},\ {}^{\mathsf{C}}\mathscr{D}_{a^+}^{\frac{\gamma}{2},\delta}\mathscr{G}\rangle,\quad\text{for }1<\gamma<2\\&\langle{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\mathscr{F},{}^{\mathsf{C}}\mathscr{D}_{b^-}^{\gamma,\delta}\mathscr{F}\rangle=\cos(\gamma\pi)\|{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\mathscr{F}\|^2\\&=\cos(\gamma\pi)\|{}^{\mathsf{C}}\mathscr{D}_{b^-}^{\gamma,\delta}\mathscr{F}\|^2\end{aligned}$$

Proof: The proof follows a similar approach to the one presented in Ref. [26].

Lemma 3.2. Given the functions $\mathscr{F}(\mathtt{x})$ and ${}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\in H^{\gamma,\delta}(\Theta)$, there exist a sufficiently small $\delta\mathtt{t}$ such that:

$$\|\mathscr{F}(\mathtt{x})\|\le\|\mathscr{F}(\mathtt{x})+\delta\mathtt{t}\Upsilon{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma-1,\delta}\mathscr{F}(\mathtt{x})-\delta\mathtt{t}\lambda{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\mathscr{F}(\mathtt{x})\|\tag{33}$$

Proof: Using lemma 3.1, we obtain:

$$\begin{aligned}&\|\mathscr{F}(\mathtt{x})+\delta\mathtt{t}\Upsilon{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma-1,\delta}\mathscr{F}(\mathtt{x})-\delta\mathtt{t}\lambda{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\mathscr{F}(\mathtt{x})\|^2\\&=\langle\mathscr{F}(\mathtt{x})+\delta\mathtt{t}\Upsilon{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma-1,\delta}\mathscr{F}(\mathtt{x})-\delta\mathtt{t}\lambda{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\mathscr{F}(\mathtt{x}),\mathscr{F}(\mathtt{x})\\&+\delta\mathtt{t}\Upsilon{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma-1,\delta}\mathscr{F}(\mathtt{x})-\delta\mathtt{t}\lambda{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\mathscr{F}(\mathtt{x})\rangle\\&=\|\mathscr{F}(\mathtt{x})\|^2+2\delta\mathtt{t}\Upsilon\langle{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\frac{\gamma-1}{2},\delta}\mathscr{F}(\mathtt{x}),{}^{\mathsf{C}}\mathscr{D}_{b^-}^{\frac{\gamma-1}{2},\delta}\mathscr{F}(\mathtt{x})\rangle\\&-2\delta\mathtt{t}\lambda\langle\mathscr{D}_{a^+}^{\frac{\gamma}{2},\delta}\mathscr{F}(\mathtt{x}),\mathsf{C}\mathscr{D}_{b^-}^{\frac{\gamma}{2},\delta}\mathscr{F}(\mathtt{x})\rangle\\&\quad+\delta\mathtt{t}^2\Big(\Upsilon^2\|{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\frac{\gamma-1}{2},\delta}\mathscr{F}(\mathtt{x})\|^2+\lambda^2\|{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\frac{\gamma}{2},\delta}\mathscr{F}(\mathtt{x})\|^2\\&-\Upsilon\lambda\langle{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma-1,\delta}\mathscr{F}(\mathtt{x}),{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\mathscr{F}(\mathtt{x})\rangle\Big)\\&=\|\mathscr{F}(\mathtt{x})\|^2+2\delta\mathtt{t}\Upsilon\cos\left(\frac{\gamma-1}{2}\pi\right)\|{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\frac{\gamma-1}{2},\delta}\mathscr{F}(\mathtt{x})\|^2\\&-2\delta\mathtt{t}\lambda\cos\left(\frac{\gamma}{2}\pi\right)\|{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\frac{\gamma}{2},\delta}\mathscr{F}(\mathtt{x})\|^2\\&\quad+\delta\mathtt{t}^2\Big(\Upsilon^2\|{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\frac{\gamma-1}{2},\delta}\mathscr{F}(\mathtt{x})\|^2+\lambda^2\|{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\frac{\gamma}{2},\delta}\mathscr{F}(\mathtt{x})\|^2\\&-2\Upsilon\lambda\langle\mathscr{D}_{a^+}^{\gamma-1,\delta}\mathscr{F}(\mathtt{x}),{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\mathscr{F}(\mathtt{x})\rangle\Big).\end{aligned}\tag{34}$$

It is obvious that $\cos\left(\frac{\gamma-1}{2}\pi\right)>0$ and $\cos\left(\frac{\gamma}{2}\pi\right)<0$ for $1<\gamma<2$, and for sufficiently small $\delta\mathtt{t}$, we have $\delta\mathtt{t}^2\to 0$ therefore Eq. (34) becomes:

$$\begin{aligned}&\|\mathscr{F}(\mathtt{x})+\delta\mathtt{t}\Upsilon{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma-1,\delta}\mathscr{F}(\mathtt{x})-\delta\mathtt{t}\lambda{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\mathscr{F}(\mathtt{x})\|^2\\&=\|\mathscr{F}(\mathtt{x})\|^2+2\delta\mathtt{t}\Upsilon\cos\left(\frac{\gamma-1}{2}\pi\right)\|{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\frac{\gamma-1}{2},\delta}\mathscr{F}(\mathtt{x})\|^2\\&-2\delta\mathtt{t}\lambda\cos\left(\frac{\gamma}{2}\pi\right)\|{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\frac{\gamma}{2},\delta}\mathscr{F}(\mathtt{x})\|^2\ge 0,\end{aligned}$$

therefore,

$$\|\mathscr{F}(\mathtt{x})\|\le\|\mathscr{F}(\mathtt{x})+\delta\mathtt{t}\Upsilon{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma-1,\delta}\mathscr{F}(\mathtt{x})-\delta\mathtt{t}\lambda{}^{\mathsf{C}}\mathscr{D}_{a^+}^{\gamma,\delta}\mathscr{F}(\mathtt{x})\|,$$

hence the proof. For comparison see Ref. [12].

Lemma 3.3. Let $\mathcal{U}^\ell\in H^{\gamma,\delta}(\Theta), \ell=1,2,\ldots,\mathscr{M}$ and $\mathcal{U}^0$ be the solution of Eq. (27) and be it's corresponding initial condition respectively, then:

$$\|\mathcal{U}^\ell\|\le\|\mathcal{U}^{\ell-1}\|+\max_{0\le n\le N}\delta\mathtt{t}\|\chi_n^\ell\|,\tag{35}$$

where $\mathcal{U}^\ell=\mathcal{U}(\mathtt{x},\mathtt{t}_\ell)$.

Table 1: $\xi_{\mathscr{A}}$ for Example 4.1 at $\gamma = 1.8$ and $\delta = 1$ [12].

x	$\alpha = 0.5$	$\alpha = 1$ [13]	$\alpha = 1.5$
0.1	5.33×10^{-6}	5.46×10^{-6}	4.70×10^{-6}
0.2	8.06×10^{-6}	8.51×10^{-6}	16.66×10^{-6}
0.3	8.72×10^{-6}	9.60×10^{-6}	6.50×10^{-6}
0.4	7.84×10^{-6}	9.18×10^{-6}	4.87×10^{-6}
0.5	5.96×10^{-6}	7.69×10^{-6}	2.42×10^{-6}
0.6	3.59×10^{-6}	5.60×10^{-6}	2.35×10^{-6}
0.7	1.29×10^{-6}	3.33×10^{-6}	2.44×10^{-6}
0.8	4.32×10^{-7}	1.34×10^{-6}	2.56×10^{-6}
0.9	1.04×10^{-6}	8.39×10^{-8}	2.96×10^{-6}

Proof: Employing mathematical induction on ℓ.

Put $m \equiv \ell = 1$ in Eq. (27), we have:

$$\mathcal{U}^1 + \delta\mathtt{t}\Upsilon{}^{C}\mathscr{D}_{a^+}^{\gamma-1,\delta}\mathcal{U}^1(\mathtt{x},\mathtt{t}) - \delta\mathtt{t}\lambda{}^{C}\mathscr{D}_{a^+}^{\gamma,\delta}\mathcal{U}^1(\mathtt{x},\mathtt{t}) = \mathcal{U}^0 + \delta\mathtt{t}\chi^1, \tag{36}$$

multiply Eq. (36) by $\mathcal{U}^1$ and integrate the resulting equation on Θ, we obtain

$$\begin{aligned}&\| \mathcal{U}^1 \|^2 + \delta\mathtt{t}\Upsilon\langle{}^{C}\mathscr{D}_{a^+}^{\gamma-1,\delta}\mathcal{U}^1,\ \mathcal{U}^1\rangle - \delta\mathtt{t}\lambda\langle{}^{C}\mathscr{D}_{a^+}^{\gamma,\delta}\mathcal{U}^1,\ \mathcal{U}^1\rangle\\ &= \langle\mathcal{U}^0,\mathcal{U}^1\rangle + \delta\mathtt{t}\langle\chi^1,\mathcal{U}^1\rangle,\end{aligned} \tag{37}$$

Using Lemma 3.1, we have $\langle{}^{C}\mathscr{D}_{a^+}^{\gamma,\delta}\mathcal{U}^1,\ \mathcal{U}^1\rangle < 0$ and $\langle{}^{C}\mathscr{D}_{a^+}^{\gamma-1,\delta}\mathcal{U}^1,\ \mathcal{U}^1\rangle > 0$.

$$\begin{aligned}\| \mathcal{U}^1 \|^2 \leq \| \mathcal{U}^1 \|^2 &+ \delta\mathtt{t}\Upsilon\langle{}^{C}\mathscr{D}_{a^+}^{\gamma-1,\delta}\mathcal{U}^1,\ \mathcal{U}^1\rangle\\ &- \delta\mathtt{t}\lambda\langle{}^{C}\mathscr{D}_{a^+}^{\gamma,\delta}\mathcal{U}^1,\ \mathcal{U}^1\rangle.\end{aligned} \tag{38}$$

From Eqs. (37) and (38), we obtain

$$\| \mathcal{U}^1 \| \leq \| \mathcal{U}^0 \| + \max_{0\leq n\leq N} \delta\mathtt{t} \| \chi_n^1 \| .$$

Assume Eq. (35) is true for all values of $\rho = 1, 2, \ldots, \mathcal{M} - 1$, we have

$$\| \mathcal{U}^\rho \| \leq \| \mathcal{U}^{\rho-1} \| + \max_{0\leq n\leq N} \delta\mathtt{t} \| \chi_n^\rho \| . \tag{39}$$

Again, multiply Eq. (27) by $\mathcal{U}^\ell$ and integrate the resulting equation on Θ, we get

$$\begin{aligned}&\| \mathcal{U}^\ell \|^2 + \delta\mathtt{t}\Upsilon\langle{}^{C}\mathscr{D}_{a^+}^{\gamma-1,\delta}\mathcal{U}^\ell,\ \mathcal{U}^\ell\rangle - \delta\mathtt{t}\lambda\langle{}^{C}\mathscr{D}_{a^+}^{\gamma,\delta}\mathcal{U}^\ell,\ \mathcal{U}^\ell\rangle\\ &= \langle\mathcal{U}^{\ell-1},\mathcal{U}^\ell\rangle + \delta\mathtt{t}\langle\chi^\ell,\mathcal{U}^\ell\rangle.\end{aligned} \tag{40}$$

Applying similar procedure, described in Eq. (37), we have

$$\| \mathcal{U}^\ell \| \leq \| \mathcal{U}^{\ell-1} \| + \max_{0\leq n\leq N} \delta\mathtt{t} \| \chi_n^\ell \| . \tag{41}$$

This produces the desired outcome of the proof.

Theorem 3.1. The numerical scheme presented in Section 3, specifically in Eq. (27), is unconditionally stable for $\gamma > 0$.

Table 2: $\xi_{\mathscr{A}}$ for Example 4.1 at $\gamma = 1.5, \alpha = 0.5, \delta\mathtt{t} = \frac{1}{100}, \mathcal{N} = 3$.

x	$\delta = 0.5$	$\delta = 1$
0.1	5.90×10^{-7}	3.90×10^{-7}
0.2	7.37×10^{-6}	4.92×10^{-6}
0.3	2.07×10^{-5}	1.38×10^{-5}
0.4	3.63×10^{-5}	2.42×10^{-5}
0.5	5.10×10^{-5}	3.40×10^{-5}
0.6	6.16×10^{-5}	4.11×10^{-5}
0.7	6.49×10^{-5}	4.33×10^{-5}
0.8	5.79×10^{-5}	3.86×10^{-5}
0.9	3.73×10^{-5}	2.49×10^{-5}

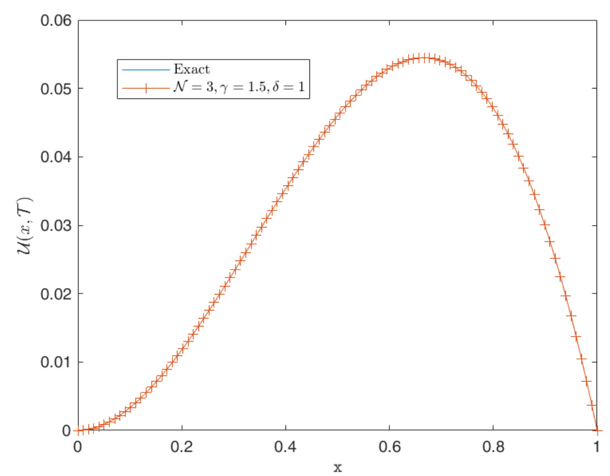

Figure 1: Exact solution and its corresponding $\mathcal{U}_{\mathcal{N}}(\mathtt{x}, \mathcal{T} = 1), \mathcal{N} = 3$.

Proof: Suppose $\mathcal{U}_n^\ell = \mathcal{U}_{\mathcal{N}}^\ell(\mathtt{x}_n, \mathtt{t}_\ell), \ell = 1, 2, \ldots, \mathscr{M}$ represents the ultraspherical approximate solution to Eq. (27) with the boundary condition $\mathcal{U}_n^0 = \mathcal{U}_{\mathcal{N}}^0(\mathtt{x}_n, \mathtt{t}_0)$, then $\xi^\ell = \mathscr{U}(\mathtt{x}_n, \mathtt{t}_\ell) - \mathcal{U}_n^\ell$ satisfies

$$\xi^\ell + \delta\mathtt{t}\Upsilon{}^{C}\mathscr{D}_{a^+}^{\gamma-1,\delta}\xi^\ell - \delta\mathtt{t}\lambda{}^{C}\mathscr{D}_{a^+}^{\gamma,\delta}\xi^\ell = \xi^{\ell-1}.$$

Based on Lemma 3.3, we have:

$$\| \xi^\ell \| \leq \| \xi^{\ell-1} \| .$$

This completes the proof of the scheme's unconditional stability.

4. Numerical experiments

This section presents two numerical examples to validate and demonstrate the effectiveness of the method discussed in Section 3 for solving SFADE. To assess the accuracy of the proposed method, the absolute error, $\xi_{\mathscr{A}}$, is computed, where

$$\xi_{\mathscr{A}} = \max_{0\leq \mathtt{i}\leq 100} |\mathscr{U}(\mathtt{x_i}, \mathcal{T}) - \mathcal{U}_{\mathcal{N}}(\mathtt{x_i}, \mathcal{T})| . \tag{42}$$

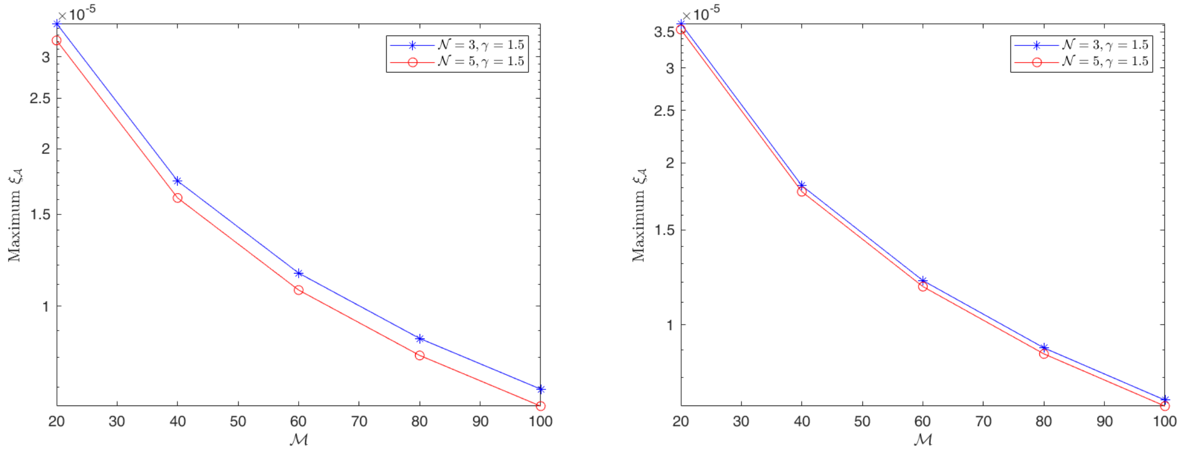

Figure 2: $\xi_{\mathscr{A}}$ for Example 4.1 at $\gamma = 1.5, \alpha = 0.5$ (left-side), $\gamma = 1.5, \alpha = 1$ (right-side).

Example 4.1

Consider the SFADE below, along with its initial and boundary conditions:

$$\begin{cases} \frac{\partial \mathscr{U}(\mathtt{x},\mathtt{t})}{\partial \mathtt{t}} + \Upsilon {}^{C}\mathscr{D}_{0^+}^{\gamma-1,\delta}\mathscr{U}(\mathtt{x},\mathtt{t}) = \lambda {}^{C}\mathscr{D}_{0^+}^{\gamma,\delta}\mathscr{U}(\mathtt{x},\mathtt{t}) + \chi(\mathtt{x},\mathtt{t}), \\ \mathscr{U}(\mathtt{x},\mathbb{0}) = \mathtt{x}^2(1-\mathtt{x}), \quad 0<\mathtt{x}<1, \\ \mathscr{U}(0,\mathtt{t}) = \mathscr{U}(1,\mathtt{t}) = 0, \quad 0<\mathtt{t}\leq \mathcal{T}. \end{cases}$$

The exact solution is $\mathscr{U}(\mathtt{x},\mathtt{t}) = (\mathtt{x}^2 - \mathtt{x}^3)\exp(-\mathtt{t})$ and the functions $\chi(\mathtt{x},\mathtt{t})$ and λ are given as:

$$\chi(\mathtt{x},\mathtt{t}) = (\mathtt{x}^3 - \mathtt{x}^2)\exp(-\mathtt{t}) - \lambda\exp(-\mathtt{t})\Big[\frac{2\delta^{\gamma-2}\Gamma\left(\frac{2}{\delta}-1\right)}{\Gamma\left(\frac{2}{\delta}-\gamma+1\right)}\mathtt{x}^{2-\gamma\delta} - \frac{6\delta^{\gamma-2}\Gamma\left(\frac{3}{\delta}-1\right)}{\Gamma\left(\frac{3}{\delta}-\gamma+1\right)}\mathtt{x}^{3-\gamma\delta}\Big] + \Upsilon\exp(-\mathtt{t})\left[\frac{2\delta^{\gamma-2}\Gamma\left(\frac{2}{\delta}\right)}{\Gamma\left(\frac{2}{\delta}-\gamma+2\right)}\mathtt{x}^{2-(\gamma-1)\delta} - \frac{3\delta^{\gamma-2}\Gamma\left(\frac{3}{\delta}\right)}{\Gamma\left(\frac{3}{\delta}-\gamma+2\right)}\mathtt{x}^{3-(\gamma-1)\delta}\right],$$
$$\lambda(\mathtt{x}) = \Gamma(1.2)\mathtt{x}^{\gamma}.$$

Table 1 presents the values of $\xi_{\mathscr{A}}$ obtained for $\gamma = 1.8$ and $\delta = 1$, which correspond to the results reported in Ref. [12] and column 3 correspond to the results obtained in Ref. [13], by putting $\Upsilon = 0$, while Table 2 presents the values of $\xi_{\mathscr{A}}$ obtained for various values of δ, which show the robustness of the technique. Figure 1 shows the relationship between the exact solution $\mathscr{U}(\mathtt{x},\mathcal{T})$ and its corresponding approximate $\mathcal{U}_{\mathcal{N}}(\mathtt{x},\mathcal{T})$ at $\mathcal{N} = 3$ while Figure 2 shows the comparison of $\xi_{\mathscr{A}}$ at various values of $\mathcal{N}$.

Table 3: $\xi_{\mathscr{A}}$ for Example 4.1 at $\gamma = 1.8, \alpha = 0.5, \mathcal{N} = 3$

x	$\alpha = 0.5$	$\alpha = 1$ [13]
0.1	3.35×10^{-7}	1.51×10^{-7}
0.2	1.02×10^{-6}	1.10×10^{-6}
0.3	1.90×10^{-6}	2.53×10^{-6}
0.4	2.81×10^{-6}	4.14×10^{-6}
0.5	3.59×10^{-6}	5.61×10^{-6}
0.6	4.09×10^{-6}	6.63×10^{-6}
0.7	4.13×10^{-6}	8.38×10^{-6}
0.8	3.58×10^{-7}	6.08×10^{-6}
0.9	2.25×10^{-6}	3.89×10^{-8}

Table 4: $\xi_{\mathscr{A}}$ for Example 4.2 at $\gamma = 1.5, \alpha = 1, \mathcal{N} = 3$

x	$\delta = 0.5$	$\delta = 1$
0.1	7.14×10^{-7}	9.72×10^{-7}
0.2	1.12×10^{-6}	1.52×10^{-6}
0.3	1.27×10^{-6}	1.72×10^{-6}
0.4	1.22×10^{-6}	1.65×10^{-6}
0.5	1.03×10^{-6}	1.40×10^{-6}
0.6	7.61×10^{-7}	1.04×10^{-6}
0.7	4.66×10^{-7}	6.34×10^{-7}
0.8	2.03×10^{-7}	2.76×10^{-7}
0.9	2.80×10^{-8}	3.80×10^{-8}

Example 4.2

Consider the SFADE below, along with its initial and boundary conditions:

$$\begin{cases} \frac{\partial \mathscr{U}(\mathtt{x},\mathtt{t})}{\partial \mathtt{t}} + \Upsilon {}^{C}\mathscr{D}_{0^+}^{\gamma-1,\delta}\mathscr{U}(\mathtt{x},\mathtt{t}) = \lambda {}^{C}\mathscr{D}_{0^+}^{\gamma,\delta}\mathscr{U}(\mathtt{x},\mathtt{t}) + \chi(\mathtt{x},\mathtt{t}), \\ \mathscr{U}(\mathtt{x},\mathbb{0}) = \mathtt{x}^3, \quad 0<\mathtt{x}<1, \\ \mathscr{U}(0,\mathtt{t}) = 0, \quad \mathscr{U}(1,\mathtt{t}) = \exp(-\mathtt{t}), \quad 0<\mathtt{t}\leq \mathcal{T}. \end{cases}$$

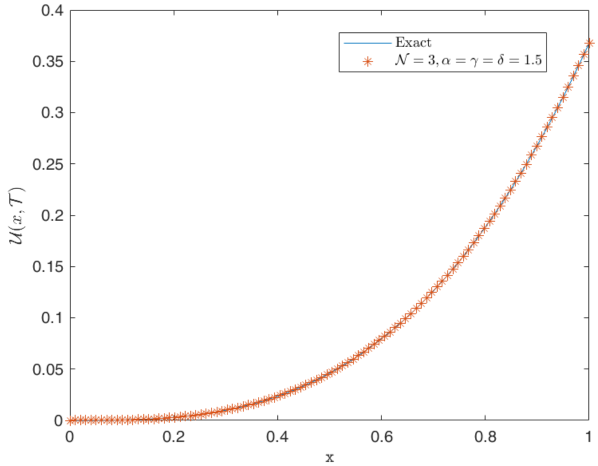

Figure 3: Exact solution and its corresponding $\mathcal{U}_{\mathcal{N}}(\mathtt{x},\mathcal{T}=1), \mathcal{N}=3$ for Example 4.2.

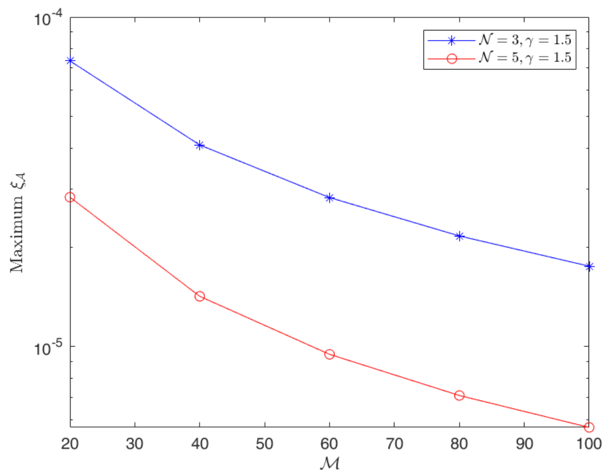

Figure 4: Maximum $\xi_{\mathscr{A}}$ for Example 4.2 at $\gamma = 1.5, \alpha = 1.5$.

The exact solution is $\mathscr{U}(\mathtt{x},\mathtt{t}) = \mathtt{x}^3 \exp(-\mathtt{t})$ and the functions $\chi(\mathtt{x},\mathtt{t})$ and $\lambda(\mathtt{x})$ are given as:

$$\chi(\mathtt{x},\mathtt{t}) = 3\Upsilon \exp(-\mathtt{t}) \frac{\delta^{\gamma-2}\Gamma\left(\frac{3}{\delta}\right)}{\Gamma\left(\frac{3}{\delta}-\gamma+2\right)} \mathtt{x}^{3-(\gamma-1)\delta}$$
$$- 6\lambda \exp(-\mathtt{t}) \frac{\delta^{\gamma-2}\Gamma\left(\frac{3}{\delta}-1\right)}{\Gamma\left(\frac{3}{\delta}-\gamma+1\right)} \mathtt{x}^{3-\gamma\delta} - \mathtt{x}^3 \exp(-\mathtt{t}),$$
$$\lambda(\mathtt{x}) = \Gamma\left(\frac{2.2}{6}\right) \mathtt{x}^{1+\gamma}.$$

Table 3 displays the values of $\xi_{\mathscr{A}}$ for $\gamma = 1.5$ and $\delta = 1$, which align with the results reported in Ref. [12] when $\Upsilon = 0$. Table 4 shows the values of $\xi_{\mathscr{A}}$ for different values of δ. Figure 3 illustrates the relationship between the exact solution $\mathscr{U}(\mathtt{x},\mathcal{T})$ and its corresponding approximate solution $\mathcal{U}_{\mathcal{N}}(\mathtt{x},\mathcal{T})$ at $\mathcal{N} = 3$, while Figure 4 compares the values of $\xi_{\mathscr{A}}$ at various values of $\mathcal{N}$.

5. Conclusion

This study presents an efficient numerical scheme for the solution of a class of SFADE using GCFD. The scheme combines FDS with shifted ultraspherical polynomials. We have shown that the approach is both unconditionally stable and convergent. We validate the effectiveness of the scheme by performing several numerical tests. The results are computed for different values of δ, γ, and α and compared with the exact solution $\mathscr{U}(\mathtt{x},\mathcal{T})$ as well as the results from Refs. [12, 13] for $\delta = 1$, and $\mathcal{T} = 1$. The numerical results demonstrate that our scheme is robust and achieves high accuracy as $\mathcal{N}$ and $\mathcal{M}$ increases.

Data availability

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

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