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Relativistic correction on bottomia within the gaussian basis function method

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Abstract

In this theoretical research to describe the latest experimental results from the Large Hadron Collider, Belle II, and heavy-ion collisions obtained in high energy hadronic physics, we include relativistic corrections to improve predictions of the mass spectra of Bottomia resonance states using the Gaussian basis function within the tanh-shaped hyperbolic plus a linear confinement potential in the framework of nonrelativistic and relativistic quantum mechanics under the Schrödinger equation. The relativistic effects of bound states in high energy physics must be described within the framework of approximations or perturbation methods and specific relativistic equations, but in this paper, we provided a mathematically relativistic correction on the bound states mass spectrum; and considering the applied relativistic corrections to the Schrödinger equation using quantum mechanics and quantum field theory principles and high energy approximation. Also, we have helped predict and solve one of the most critical issues of resonance states in particle physics. Current theoretical work is focused on studying this problem for the mass spectra of Bottomia within the refined hybrid potential and based on the modified Schrödinger equation. The mass spectra results agree closely with the experimental and other theoretical data.

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1. Introduction

Resonance states of Bottomia ($b\tilde{b} : (nS)$, $n = 1, 2, 3, \cdots$) are strong bound states of a bottom quark and its antiquark ($\Upsilon(1S)$ was first observed in 1977 at Fermilab) in the higher excited states. Bottomia represents an important aspect in hadronic physics research and fills an important gap by describing the quark model in hadronic physics proposed by Murray Gell-Mann and George Zweig in 1964 to classify the structure of hadrons [1, 2]. $\Upsilon(1S)$ and other $b\tilde{b} : (nS)$, open a new window to explore the dynamics of the strong interaction in both nonperturbative (exact, strongly coupled, non-linear behavior) and perturbative (approximate, analytical, series expansion-based) regimes. Hadronic physics and the standard model were evaluated under the $b\tilde{b} : (nS)$ resonance states' behaviors and effects on decay and interaction of other particles.

This bound state becomes ideal for modeling different potential forms and nonrelativistic quantum chromodynamics (QCD) in effective field theories. $b\tilde{b} : (nS)$ has a variety of radial and orbital excitation states that can describe how quarks interact via the exchange of gluons, and confinement operates at high energy physics. Also, the importance of research on

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$b\tilde{b}$: (nS) mass spectrum includes behaviors of QCD, relativistic corrections, and quark confinement frameworks [3, 4]. As we know, the heavy mass of the bottom quark-antiquark in $b\tilde{b}$: (nS) states are less affected by relativistic corrections compared to lighter mesons, but in the resonance states the relativistic behavior and spins interaction are as a relativistic main subject and have very significant effects on the bound states in theoretical and experimental research. So, in the study of $\Upsilon(nS)$ bound states within the theoretical calculation and modeling, the relativistic corrections can't be neglected. Hence, in recent studies, Bottomia resonance states remain a focal point in hadronic investigations. These results are fundamental to a better understanding QCD and strong interaction properties and characteristics. It is widely recognized as an important probe for the physics of hybrids and exotic hadronic bound states. As we know, the importance of these investigations in future years includes: Probing quark-gluon plasma:$b\tilde{b}$: (nS) are utilized as probes to investigate the properties of quark-gluon plasma, where $b\tilde{b}$: (nS) in-medium masses can exist at finite temperatures in the excited states $b\tilde{b}$: (nS).

This approach connects theoretical potentials directly to experimental observables, enhancing our understanding of $M_{b\tilde{b}}$ of bound states behaviors in ultra-hot environments; Theoretical modeling of potential interactions: the studies in theoretical models, including relativistic potential, tanh-shaped hyperbolic potential, and relativistic screened potential approaches, will be employed to predict $b\tilde{b}$: (nS) properties [5]. This theoretical research on Bottomia resonance states allows us to understand the $b\tilde{b}$: (nS) mass spectra and properties within various conditions; Heavy-ion collisions: recent research on $M_{b\tilde{b}}$ describes the quantum processes that reform from unbound quark pairs within the quark-gluon plasma; the size of $b\tilde{b}$: (nS) in xenon-xenon and lead-lead collisions: at the Large Hadron Collider and Belle II, the scale and size of Bottomia resonance states is important in experimental research due to predict behaviors of other hadrons and new aspects of QCD theory. Lead nuclei are larger than xenon nuclei, but xenon nuclei are heavy enough to provide valuable results from inside the ultra-hot medium and present us with their forms and evolution. Hence, this type of investigation on nuclear systems in high energy collision determines how Bottomia resonance states react on the ultra-hot medium and at extreme conditions, therefore we can investigate and improve prediction on properties, dynamics, standard model's new perspectives, new phenomena in QCD, and completely improve theoretical knowledge of quark-gluon plasm and the core of a star based on the behavior of $M_{b\tilde{b}}$ states.

For this reason, some of the great significance of $M_{b\tilde{b}}$ mass spectra included in the determination of parameters in the Cabibbo-Kobayashi-Maskawa matrix, which is a fundamental concept in the standard model, the violation of the combined symmetry of charge conjugation and parity, refinement of heavy quark effective theories, B-meson physics, relativistic corrections, and perturbative behavior. It is necessary to point out, that spin interactions, including spin-spin, spin-orbit, and tensor components, are important for evaluating the mass spectra of hadronic bound states. In this study, we focus on relativistic effects and the relativistic correction of mass, as well as the optimization of Schrödinger equation into its relativistic form. Spin effects have been neglected to simplify calculations, as their contribution is estimated to be less than $\sim (2-3)\%$. However, the presented framework can be extended to include spin interaction effects by incorporating the Breit-Fermi interaction terms in the modified Schrödinger equation. This would allow for a computational analysis of spin contributions within the mass spectrum and relativistic mass corrections using the same methodology outlined in this paper. In future work, we will include this calculation in full.

Therefore, $b\tilde{b}$: (nS) resonance states will be the main bound state for testing the principles of particle physics and the relativistic effects of the strong interaction. The relativistic modified Schrödinger equations were chosen to define the relativistic corrections of mass in the $b\tilde{b}$: (nS) bound states. The start point is included in the asymptotic behavior of n-point Green's function, and we obtain the relativistic correction using the Feynman functional path integral in the external vector field at $x \to \infty$ within the exponentiated model [6, 7]. n-point Green's function in the path integral form helps us to explain bound states properties using QM, QCD, and quantum field theory (QFT). We start our analytical techniques from the vacuum closed loop function based on the physical behaviors of a system through the Feynman functional path integral method in the applied gauge field [5, 6] using the QFT perspective and framework [8]. This study is organized in the following manner: in Sect. 2-4, the important behavior of tanh-shaped hyperbolic potential, and the potential form of the bound state are explained and describe the creation of $b\tilde{b}$: (nS) resonance states, then the modified radial Schrödinger equation (MRSE) and an analytical approach for obtaining the relativistic corrections are described.

The canonical variables at a higher order by the method of conversion of symplectic space and description of the oscillator representation method are applied to obtain $b\tilde{b}$: (nS) mass spectrum and relativistic correction of mass and energy eigenvalue within modified tanh-shaped hyperbolic, and the article is concluded. In Sect. 5-6, mass spectra of $M_{b\tilde{b}}$ based on the relativistic Schrödinger equation under the Bernoulli modification, tanh-shaped hyperbolic potential and linear confinement term are solved and calculated. The computational program was performed in the open-source Octave 7.1 software, which is licensed under the GNU General Public License.

2. Materials and methods

2.1. Aims and methodology

This theoretical study aims to improve predictions of Bottomia resonance mass spectra by integrating relativistic corrections into QFT and QM models. It introduces a refined tanh-shaped hyperbolic plus linear confinement potential within a modified Schrödinger framework, enabling explanation of heavy quark interactions in nonperturbative QCD regimes. The methodology begins with Green's functions to model two particles propagation and bound states, transitions to Feynman path integrals for computing gauge field interactions and polarization tensors, and then using a harmonic oscillator representation to solve the modified Schrödinger equation under Gaussian basis functions. In this formalism Green's Functions used to define the propagator for two charged scalar particles in an external gauge field within interaction potential, explaining the response to a delta function that is fundamental for field equations; Feynman Path Integrals represent the propagator as a sum over all possible paths using functional integral techniques under proper-Time formalism; proper-time allows reparameterization of propagators within a Schwinger proper-time variable which makes nonperturbative techniques more efficiently then worldline methods used to define the path integral over trajectories based on QM principles. To calculation mass spectrum and relativistic behaviors on mass of highly resonance states, harmonic oscillator representation and Gaussian function formalism used to enable simplifications when evaluating Gaussian path integrals. This multi-formalism approach allows for analytical and numerical determination of mass spectra aligned with experimental results from LHC and Belle II, and offers predictions for higher radial excitation in future collider experiments.

2.2. *Tanh-shaped hyperbolic potential*

In the last study of hadronic bound states, the general tanh-shaped hyperbolic potentials became a good candidate to analytically and effectively determine the mass spectra and energy eigenvalues of $M_{b\tilde{b}}$ system:

$$V(r) = -\frac{V_0}{\cosh^2(\alpha r)},$$

where V_0 is a potential strength parameter (controls the depth of the potential), and α is a parameter that controls the range of the potential (related to the size of the interaction region) [9]. During 2023-2024, using the hyperbolic potential model and the Dirac equation, the mass spectra of $b\tilde{b} : (nS)$ are defined. The results showed us that the hyperbolic potentials provide good agreement with experimental data, which achieves in predicting radial and orbital excited states of hadrons and $M_{b\tilde{b}}$.

This ability allows us to describe our method in this research using the complex relativistic interactions characteristic of $b\tilde{b} : (nS)$ based on the modified Schrödinger equation that has the relativistic kinetic and hybrid potential and confinement parts correction. Our presented method on the relativistic correction to the mass of $b\tilde{b} : (nS)$, allows us to predict future theoretical and experimental research of quarks, gluons, and hybrids bound states in extreme conditions, such as the quark-gluon plasma, and core of stars where we have to explain $b\tilde{b} : (nS)$ mass spectra using QCD and QFT. As we know, in strong interaction theory based on QCD principles, the potential of interaction has two terms: hard (Coulomb-like forces) and soft (confinement effects) terms that refer to two separate behaviors of hadronic interactions. These terms describe how particles and filed interact at high energy conditions or in different energy scales. Hard and soft concepts in QCD are essential for studying the behavior of $b\tilde{b} : (nS)$ or other quarkonium states. As we know hard concepts related to high energy physics, where the interaction between quarks and gluons (particles and fields) is dominated by local interactions and high momentum exchange effects (short-distance effects on the order of $10^{-18}m$ or smaller), in such a situation (which is the typical phase in high energy particle collisions like those at the Large Hadron Collider), due to asymptotic freedom, the strong coupling constant α_s becomes small. These characteristics of hard QCD indicate that the interaction between particles and fields is weaker as the interacting objects converge at ultra-relativistic energies (small b-$\tilde{b}$ separation). This type of interaction, based on relativistic corrections on the mass of bound states in the Schrödinger equation, is the Coulomb-like force, and it can be determined using perturbation theory in QCD.

Contrary to this phase, the soft concept (confinement effects) as the low-energy phase is also described under the perturbation theory, where the interaction between quarks and gluons (particles and fields) includes long-distance effects [10]. Hence, strong coupling constantα_s becomes large contrary to the hard phase and the momentum of particles becomes small. Therefore, we have to use nonperturbative methods and techniques to describe the soft interactions. Now, considering the research topic, which includes $b\tilde{b} : (nS)$ bound states, so we will examine the interaction behavior in both above-described phases. In this research we consider $b\tilde{b} : (nS)$ is composed of a bottom quark and its antiquark with equal mass $m_b = m_{\tilde{b}} \sim 4.823\, GeV$, this system plays a main role in discovering QCD in perturbative and nonperturbative regimes, i.e., in hard and soft QCD [11]. Hence, in this research, we choose a tanh-shaped hyperbolic with linear confinement potential potentials to describe the mass spectra of $\Upsilon(nS)$ resonance states based on relativistic corrections on mass. The presented potential is the modified tanh-shaped hyperbolic potential $V(r) = -V_0 \tanh(\alpha r) + \sigma r$, where V_0 and α - constant in the hard phase of interaction, σ- constant governs the confinement effect at the soft phase. The tanh-shaped hyperbolic potential analytically allows us to use the modified Schrödinger equations, which leads to predicting the mass spectra and other properties of $M_{b\tilde{b}}$ hadronic bound state [9].

2.3. *Bound states in quantum field theory*

Now, we have to obtain the relativistic mass of $M_{b\tilde{b}}$ formed by b, $\tilde{b}$ quarks that are provided by a high energy interacting potential in a gauge field. We are interested in the creation mechanism of bound states, to form stable $M_{b\tilde{b}}$ resonance states. A gauge boson interacts with virtual particle-antiparticle pairs in QFT, and a polarization tensor arises in a vacuum state [12]. The polarization loop operator (the polarization tensor or the self-energy) is an operator that describes quark-antiquark interaction. This operator for a quark with the proper mass m, the momentum k without interactions and in the free state reads

$$\Pi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \Pi(p),$$

where

$$\Pi(p) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\varepsilon}.$$

Hence, the polarization tensor (polarization loop operator) presents our framework based on QFT and QM. The polarization tensor describes the modification of the field propagator, due to the quantum correction that is extracted from the loop integral in Feynman diagrams and involves an integral over the internal momentum of the propagating b, $\tilde{b}$ quarks. The self-interactions of gluons (the gauge bosons) in QCD and photons (the gauge bosons) in quantum electrodynamics (QED) are described by the polarization tensor [13, 14]. It helps us to determine and understand the effects of color confinement and asymptotic behavior (asymptotic freedom) within the interactions between b, $\tilde{b}$ quarks and gluons in QFT allow us to calculate physical observables like mass spectrum, energy eigenvalue, scattering amplitudes, cross-sections, and decay rates. Therefore, the representation of the polarization loop operator in the form of a Feynman functional integral is introduced, and we investigated its asymptotic behavior for the soft QCD limit [15]. We start the definition with the Klein-Gordon equation for a scalar particle with the proper mass m and under the coupling constant g in an applied gauge field A that is localized at y in the natural unit form ($\hbar = c = 1$):

$$\left[\left(i\partial_\mu + gA(x)\right)(i\partial^\mu + gA(x)) + m^2 + V(x)\right]\phi(x) = \delta(x-y). \quad (1)$$

Equation (1) describes how the scalar field $\phi(x)$ reacts to a disturbance at y. But we need equation that describes how disturbances propagate in an external gauge filed within interaction potential V(x). In this case, the propagator $G(x, y \mid A, V)$ (Green's function) is presented by equation (1) [8, 12] then the Green's function for a scalar particle reads:

$$G(x, y \mid A, V) = \left\langle 0\middle|T\left[\phi(x)\phi^*(y)\right]\middle|0\right\rangle_{A,V}.$$

This function in any gauge field background helps us compute solutions for the scalar field $\phi(x)$, which is a nonphysical field, but we can use it to describe how propagation in an applied gauge field creates, thus equation (1) reads

$$\begin{aligned}&\left[(i\partial_\mu + gA_\mu(x))(i\partial^\mu + gA^\mu(x)) + m^2 + V(x)\right]\\ &\times \quad G(x, y \mid A, V)\\ &= \delta(x-y).\end{aligned} \quad (2)$$

Now, we are dealing with a particle in an applied gauge field in QFT, and to have an effective interaction for creating the bound states over all possible configurations of the gauge field, we need to average over the external field, i.e., we integrate over the field:

$$\phi(x)_A = \int \mathfrak{D}AP[A]\,\phi(x, A). \quad (3)$$

The notation $\langle\cdot\rangle_A$ indicates an average (expectation value) over the field. Thus, to obtain the scalar field $\phi(x)$, we apply equation (2) and integrate function $G(x, y)$ using equation (3) [12]:

$$e^{i\int dx \mathcal{J}_\mu(x)A(x)_A} = e^{-\frac{1}{2}\int\int dxdy\, \mathcal{J}_\mu(x)\mathfrak{D}_{\mu\nu}(x-y)\mathcal{J}_\nu(y)},$$

where $\mathcal{J}_\mu(x)$, $\mathcal{J}_\nu(y)$ are the real currents. $\mathfrak{D}_{\mu\nu}(x-y)$ is the propagator of the applied filed A_μ. So, function $\mathfrak{D}_{\mu\nu}(x-y)$ reads:

$$\mathfrak{D}_{\mu\nu}(x-y) = A_\mu(x)A_\nu(x) = \partial_\mu\partial_\nu D(x-y) + \mathfrak{D}_1(x-y),$$

and then:

$$\mathfrak{D}_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\widehat{\mathfrak{D}}(k^2)e^{ik(x-y)},$$

where function $D(x-y)$ describes the properties of the interaction and is represented by a field A_μ. Based on the nonAbelian gauge theories, the field A_μ within noncommutative symmetry groups for QCD is described. We can consider functions $D(x-y)$ and $\mathfrak{D}_1(x-y)$ are the main functions that define the gauge of $A_\mu(x)$. As we know, the physical properties of the field and medium are independent of the function $\mathfrak{D}_1(x-y)$ [12].

Thus, in hadronic physics, if the local gauge transformation $\Lambda(x)$ acts on $\phi(x)$ or $A_\mu(x)$, it is typically written in the form $A_\mu \to A_\mu + g\partial_\mu\Lambda + \Lambda A_\mu\Lambda^{-1} - ig\Lambda\partial_\mu\Lambda\Lambda^{-1}$ and the propagator $G(x, y \mid A, V)$ transforms in a specific context that reads:

$$G\left(x, y \mid A_\mu(x) \to A_\mu(x) + g\partial_\mu\Lambda(x)\right) = e^{ig\Lambda(x)}G(x, y \mid A)e^{-ig\Lambda(x)}.$$

The $G(x, y \mid A, V)$ with the polarization tensor is used to describe the full propagator of a particle in the external gauge field based on the Dyson equation, and these two functions are connected by the Dyson equation as follows:

$$G(x, y \mid A, V) = \frac{1}{G(p \mid A, V) - \Pi(p \mid A, V)},$$

where $G(p \mid A, V)$ is the Green's function of motion in the free field, and the effects of interactions (self-interactions, interactions with other particles, etc.) are included in the function $G_{full}(p \mid A, V)$. Now, the two-particle bound state creation mechanism is our main goal in this research, and so the polarization tensor of two scalar particles with proper masses m_1, m_2 reads:

$$\Pi(x-y) = \langle G_1(x, y \mid A, V)\bullet G_2(x, y \mid A, V)\rangle_A. \quad (4)$$

In the context of quantum chromodynamics, the soft phase involves long-range forces $(x - y)^2 \to \infty$, such as confinement and mass generation processes, i.e., $E_{bin} = M - (m_1+m_2)$ creating a bound state system with the mass M, with the non-perturbative interactions, thus the behavior of the polarization tensor is particularly in the exponential form [12]:

$$\Pi(x-y) = Ce^{-M\sqrt{(x-y)^2}}. \quad (5)$$

Equation (5) can be realized as follows:

(a) $M < \infty$ and $M = m_1 + m_2$, a coupled mass formation M can be created.

(b) $M = m_1 + m_2$, the strength of the interaction is very weak, so a bound state can not be created, and particles are two independent objects with no binding interaction.

Thus, we must solve to determine the coupled mass spectrum. The mathematical solution of equation (1) is presented as a functional integral. In theoretical physics, QM and QFT, the Green's function method is a powerful mathematical approach used to solve this type of differential equation, hence, the propagator of two particles in the applied gauge field A_μ and interaction $V(x)$ reads:

$$G(x,y|A,V) = \int_0^\infty d\tau e^{-\tau m^2} e^{\tau\left(\partial_\mu - igA_\mu\ (x)\right)^2} e^{-\tau V(x)} \delta(x-y)$$
$$= \int_0^\infty d\tau \int de^{-\tau m^2} e^{\tau\left(\partial_\mu - igA_\mu\ (x)\right)^2} e^{-\tau V(x)}$$
$$\times \Xi(\xi) \int_0^\infty \frac{d\tau}{(4\tau\pi^2)^2} e^{-\tau m^2 - \frac{(x-y)^2}{4\tau}}. \quad (6)$$

Equation (6) is similar to the Schwinger representation of propagators, and the propagator in an applied field A_μ is written as an integral over proper time τ. $\Xi(\xi)$ in equation (6) is defined as follows:

$$\Xi(\xi) = \int d\sigma_\nu e^{ig\int_0^1 d\xi \frac{\partial Z_\mu(\xi)}{\partial \xi} A_\mu\ (\xi)},$$

where

$$Z_{\mu(\xi)} = (x-y)_\mu \xi + y_\mu - 2\sqrt{\tau} B_\mu(\xi),$$

$$d\sigma_\nu = C\delta B_\mu(\xi)\, e^{-0.5\int_0^1 d\xi B_\mu^2(\xi)},$$

and $B_\mu(\xi)$ is some auxiliary field, C is the normalization constant, and we choose a specific normalization condition $B_\mu(0) = B_\mu(1) = 0$, $\int d\sigma_\nu = 1$. These conditions certify that a probability distribution, wavefunction, Green's function, or functional integral is consistent with physical laws in hadronic and particle physics, i.e., the correlation function (Green's function) is renormalized at a specific reference point within the functional integral method. Parameter τ acts as a proper time parameter in the path-integral formulation of QFT and parameterizes the evolution of the propagator. The term $e^{-\tau m^2}$ is the higher-energy contributions and reflects mass dependence in the correlation function, and $\delta(x-y)$ function confirms localization at the given points. ξ is a worldline parameter that, in functional integral formalism, runs from 0 to 1, parametrizing the path of the particles within the interaction in the external gauge field A_μ. We have to integrate over the worldline parameter for all possible trajectories of moving particles in equation (6). Function $\frac{\partial Z_\mu(\xi)}{\partial \xi}$ is the velocity along ξ and determines how the particle interacts with A_μ [12]. Thus, due to define the simplest form of interaction between x, y points in spacetime coordinate under the Gaussian correlator, we have to average over A_μ using the 2-point correlator:

$$G\left(x,y \mid A_\mu\right) = \phi(x,A_\mu)\ \ \phi(y,A_\mu)\ .$$

The Gaussian correlator depends only on the difference between two points x and y [12]. Then the polarization tensor of two interacting particles in an applied field A_μ is defined by averaging over all possible configurations of A_μ in the form of a path integral as follows:

$$\Pi(x-y \mid A,V) = \int_0^\infty \int_0^\infty \frac{d\mu_1\, d\mu_2}{\left(8\pi^2(x-y)\right)^2} J(\mu_1,\mu_2)$$
$$\times \exp\left[-\frac{|x-y|}{2}\left(\frac{m_1^2}{\mu_1} + \mu_1 + \frac{m_2^2}{\mu_2} + \mu_2\right)\right]. \quad (7)$$

From equation (7), the functional integral $J(\mu_1,\mu_2)$ in non-relativistic QM and QFT, it appears similar to the Feynman path integral formalism under the Minkowski spacetime coordinate for the motion of two particles or fields with proper masses m_1, m_2 [12]. The interaction term of these particles includes the functional integral:

$$J(\mu_1,\mu_2) = C_1 C_2 \int\int dxdy e^{-V(x,y)} \times$$
$$e^{-0.5\int_0^x d\tau\left[\mu_1 \hat{x}^2(\tau) + \mu_2 \hat{y}^2(\tau)\right]}, \quad (8)$$

where

$$e^{-V(x,y)} = \exp\left[\frac{g^2}{2}(-1)^{i+j}\int_0^\infty\int_0^\infty d\tau_1\, d\tau_2 \times\right.$$
$$\left.\hat{Z}_\mu^i(\tau_1)\, D^{\mu\nu}\left(Z^i(\tau_1) - Z^j(\tau_2)\right)\hat{Z}_\nu^j(\tau_2)\right]. \quad (9)$$

Hence, $V(x,y) = V_{11} + V_{22} - 2V_{12}$ presents the local and nonlocal interactions. The interaction of the constituent particles with the gauge field is the function V_{ij} within the Feynman diagram formalism. The interaction of particles with each other is V_{ii}. Then, by a saddle point of the integral in the functional form of equation (6), in the context of the soft phase (long-range forces: $(x-y)^2 \to \infty$), as described above, the polarization tensor of two interacting particles in an applied field A_μ is determined. $J(\mu_1,\mu_2)$ the functional integral interaction reads[12, 16]:

$$J(\mu_1,\mu_2) = Ce^{-(x-y)\ E(\mu_1,\mu_2)}. \quad (10)$$

Function $E(\mu_1,\mu_2)$ is being independent of m_1, m_2 and dependent on μ_1, μ_2 and g. The parameters μ_1, μ_2, have the same dimensional units as mass $m_1\ m_2$. Hence, $b\tilde{b} : (nS)$ bound states mass M is defined as:

$$M = \lim_{(x-y)^2\to\infty} -\frac{ln\Pi(x-y)}{|x-y|}\ . \quad (11)$$

After some mathematical adjustments, it becomes:

$$M = 0.5 \min_{\mu_1,\mu_2}\left(\frac{\mu_2 m_1^2 + \mu_1 m_2^2}{\mu_1\mu_2} + \mu_1 + \mu_2 + 2E(\mu_1,\mu_2)\right). \quad (12)$$

In the bound state, the constituent masses of particles with the rest masses m_1, m_2 are μ_1, μ_2. All interaction potential functions, such as nonperturbative behavior, local, and nonlocal interactions that are included in the hadronic bound states described by the function $V(x)$, and relativistic behavior defined by integrating over the time component of the function $J(\mu_1,\mu_2)$ in the Minkowski spacetime coordinate. Hence, based on being independent of the fourth component of two particles in the Minkowski spacetime coordinate, the functional integral $J(\mu_1,\mu_2)$ is determined by the Feynman path integral method and describes the motion of two particles with masses μ_1, μ_2 in the nonrelativistic formalism of interaction. Hamiltonian of this type of interaction reads:

$$H = -\frac{\hat{p}_1^2}{2\mu_1} - \frac{\hat{p}_2^2}{2\mu_2} + V(x, y). \tag{13}$$

Using equation (10) and equation (13) in the context of Feynman path integral in QFT, we can see that in equation (10) function $E(\mu_1, \mu_2)$ is the energy eigenvalue of the Schrödinger equation $H\Psi = E(\mu_1, \mu_2)\Psi$. Minimizing equation (12) for μ_1, μ_2 we can define equations [12]:

$$2\frac{dE(\mu_1, \mu_2)}{d\mu_1} = \frac{m_1^2}{\mu_1} - \mu_1, \tag{14}$$

$$2\frac{dE(\mu_1, \mu_2)}{d\mu_2} = \frac{m_2^2}{\mu_2} - \mu_2. \tag{15}$$

These relations present the constituent mass of particles in the bound states.

3. Modified radial Schrödinger equation

$b\tilde{b} : (nS)$ bound states mass spectra under modified radial Schrödinger equation (MRSE) under the Feynman path integral method with the constituent mass μ_1, μ_2 becomes:

$$H\Psi(r) = E(\mu_1, \mu_2)\Psi(r). \tag{16}$$

The kinetic energy and potential energy of $M_{b\tilde{b}}$ hadronic system in high energy physics is on the left side of equation (16). This equation will present the relativistic behavior [17]. We know that the total relativistic energy of a system with the rest mass m_0 and radial momentum p_r in special relativity of a free system is $E = \sqrt{m_0^2 + p_r^2}$, therefore equation (7) up to equation (11) can be adjusted within the path of integral formalism and "minimizing technique". This technique in theoretical physics is a useful method that allows one to rewrite the square root in a simpler form using an auxiliary parameter. An approximation (effective mass approximation) in the form of

$$\sqrt{m^2 + \hat{p}_r^2} \approx \min_{\mu} \frac{1}{2}\left(\mu + \frac{m^2 + \hat{p}_r^2}{\mu}\right),$$

gives the optimal choice of this parameter μ. This expression represents an approximation for the relativistic bound state's energy in the context of high energy physics [12, 17]. We need to use this approximation to represent the relativistic kinetic energy in functional integral forms based on the path integrals and QFT principles, and the minimization technique in the variational methods helps us approximate hadronic bound state problems. In summary, this issue is useful for describing a system's heavy quark-bound states, mass spectra, and energy eigenvalues. Now, in the Schrödinger equation (16), the classical kinetic energy is replaced by the relativistic energy, and within the "minimizing technique", MRSE or the relativistic form of the radial Schrödinger equation is defined and reads [17]:

$$\sqrt{m^2 + \hat{p}_r^2} \approx \min_{\mu} \frac{1}{2}\left(\mu + \frac{m^2 + \hat{p}_r^2}{\mu}\right).$$

The relativistic kinetic energies of the constituent b, $\tilde{b}$ particles in $M_{b\tilde{b}}$ with using the "minimization technique" reads:

$$\sqrt{m_b^2 + \hat{p}_r^2} \approx \min_{\mu_b} \frac{1}{2}\left(\mu_b + \frac{m_b^2 + \hat{p}_r^2}{\mu_b}\right), \tag{17}$$

$$\sqrt{m_{\tilde{b}}^2 + \hat{p}_r^2} \approx \min_{\mu_{\tilde{b}}} \frac{1}{2}\left(\mu_{\tilde{b}} + \frac{m_{\tilde{b}}^2 + \hat{p}_r^2}{\mu_{\tilde{b}}}\right), \tag{18}$$

Then, substituting equation (17) and (18) in the MRSE (16) and defining:

$$\left(\sqrt{m_b^2 + \hat{p}_r^2} + \sqrt{m_{\tilde{b}}^2 + \hat{p}_r^2} + V(r)\right)\Psi(r) = \left[\min_{\mu_b}\left(\frac{1}{2}\left(\mu_b + \frac{(m_b^2 + \hat{p}_r^2)}{\mu_b}\right)\right) + \min_{\mu_{\tilde{b}}}\left(\frac{1}{2}\left(\mu_{\tilde{b}} + \frac{(m_{\tilde{b}}^2 + \hat{p}_r^2)}{\mu_{\tilde{b}}}\right)\right)\right]\Psi(r) + V(r)\Psi(r) = E(\mu_b, \mu_{\tilde{b}})\Psi(r), \tag{19}$$

where

$$E(\mu_b, \mu_{\tilde{b}}) = E(\mu), |\hat{p}_{rb}^2| = |\hat{p}_{r\tilde{b}}^2| = |\hat{p}_r^2|,$$

μ_b and $\mu_{\tilde{b}}$ are the relativistic mass correction of $b, \tilde{b}$ particles in $M_{b\tilde{b}}$ bound states (constituent mass of particles) based on quantum field theory and Feynman path integral formalism. Equation (19), after some mathematical modifications, provides an important equation that is a good mathematical way to define relativistic characteristics of hadronic bound states at high energy physics of ($b, \tilde{b}$ quarks move with relativistic velocity v) as follows:

$$\left[\frac{\hat{p}_r^2}{2\mu} + V(r) - E_\ell(\mu)\right]\Psi(r) = 0, \tag{20}$$

where $\ell$is the orbital quantum number, μ is the reduced mass of $b\tilde{b} : (nS)$ bound state $\frac{1}{\mu} = \frac{1}{\mu_b} + \frac{1}{\mu_{\tilde{b}}}$. The constituent mass of$b\tilde{b}$ quarks in the $M_{b\tilde{b}}$ bound state using equation (14) and equation (15) are determined as[12]:

$$\mu_b = \sqrt{m_b^2 - 2\mu^2\frac{dE_\ell(\mu)}{d\mu}}, \tag{21}$$

$$\mu_{\tilde{b}} = \sqrt{m_{\tilde{b}}^2 - 2\mu^2\frac{dE_\ell(\mu)}{d\mu}}. \tag{22}$$

4. Solving the modified Schrödinger equation

In this section, to solve the modified radial Schrödinger equation, the "Gaussian basis function" method is used. We determined the mass spectra energy eigenvalues and wave functions of the hadronic bound state in the relativistic formalism of QM and QFT [12, 18]. We are involved with complicated differential equations to determine systems' mass and energy spectra in theoretical physics and relativistic QFT. However, the Gaussian basis function method provides an easier way to solve these complicated differential equations in the quantum harmonic oscillator form and approximate the properties of these states. This method is useful for finding relativistic properties. $M_{b\tilde{b}}$ bound state problems in QCD, meson spectroscopy, and high energy physics because of these contexts: $M_{b\tilde{b}}$ bound states are a quantum harmonic oscillator system; it involves the creation and annihilation operators; directly presents the quantization of wavefunctions in an auxiliary space; and is extended to n-body QFT formalism for hadronic physics [19].

The Gaussian basis function method presents properties of wave functions or fields in the context of bound states in theoretical physics, described by complex variables. Mathematically, the Fock space does not require complex variables. At the same time, it can easily be simplified under complex variables to simplify physical problems in QM and QFT, for example, in gauge theory, conformal field theory, Grassmannian and supersymmetry, path integral formulations, complex scalar fields, and variational methods. Therefore, using the Gaussian basis function method to describe $M_{b\tilde{b}}$ bound states are a powerful quantum mechanics and quantum field theory method to simplify physical problems in high energy hadronic physics. Now we present the Gaussian basis function method based on the quantum harmonic oscillator representation of $b\tilde{b} : (nS)$ where the dynamics of $b\tilde{b} : (nS)$ the system is described in terms of a quantum harmonic oscillator. The behavior of $b\tilde{b} : (nS)$ can be approximated or modeled by harmonic oscillators using this representation, which deals with hadronic systems such as quarks and gluons in a specific potential. The method is established from the quantum harmonic oscillator representation, and it is associated with the second quantization method or quantum field quantization, where canonical operators $(\hat{x}, \hat{p})$ and fields are quantized by creation $(\hat{a}^+)$ and annihilation $(\hat{a})$ operators, and satisfy the commutation relation $[\hat{a}, \hat{a}^+] = 1$ and $[\hat{x}, \hat{p}] = i\hbar$ [20]. Thus, canonical operators can be presented in the natural unit form $(\hbar = c = 1)$ [16, 17] by:

$$\hat{x} = \sqrt{\frac{1}{2m\omega}}\left(\hat{a}^+ + \hat{a}\right), \hat{p} = i\sqrt{\frac{m\omega}{2}}\left(\hat{a}^+ - \hat{a}\right),$$

where m is the proper mass of the particle, and ω is the angular frequency of the quantum harmonic oscillator. Angular frequency is an important parameter to determine the dynamics of the quantum harmonic system. By converting symplectic space under the quantum harmonic oscillator form in the Gaussian basis for the radial part of the wavefunction in a spherical symmetry coordinate system, we obtain canonical variables at a higher order [20]. To use this method, we follow the transformation of the radial variable r from the Cartesian coordinate R^n to the auxiliary space R^d (d is dimension):

$$r = q^{2\rho}, \; p_r \to p_q,$$

$$\Psi(r) \to \Psi\left(q^2\right),$$

$$\Psi\left(q^2\right) = q^{2\rho\ell}\Phi\left(q^2\right),$$

for determining $M_{b\tilde{b}}$, the properties of the bound state are presented as a hadronic quantum oscillator in the context of QFTs. This transformation $r = q^{2\rho}$ is mapping quantum numbers onto harmonic oscillator-like quantum levels and defining the wave function with the simpler component $\Psi\left(q^2\right)$ and $q^{2\rho\ell}$. The auxiliary d space introduced in this method as a symplectic space can be interpreted. Under transformation, variables provide a canonical way to connect the auxiliary d with the main physical space and make calculations adjustable.

In the radial coordinate transformation formalism ρ is a scaling parameter. It controls the distribution of the coordinate grid over the spatial domain. It also adjusts the density of grid points in the specific regions of radius r. This can be important for the behavior of the wave function and the potential structure. The parameter ρ is a variational parameter, and using ρ we achieve the physical features of the asymptotic behavior of the bound state. It is defined by the condition $\frac{dE_\ell(\mu)}{d\rho} = 0$, which will be explained in the next paragraph. In summary, this method helps us to understand $b\tilde{b} : (nS)$ bound state properties and define the exact mass spectrum and energy eigenvalue solutions. The transformation $r = q^{2\rho}$ simplifies the Hamiltonian in the process of solving $M_{b\tilde{b}}$ bound states mass spectra. We have to represent the radial Laplacian operator in the auxiliary d space, to solve MRSE equation (16), defining the quadratic operators $\hat{p}_q^2$ (momentum), $\hat{q}^2$(position) of the $b\tilde{b} : (nS)$ system. The radial Laplacian operator in the auxiliary space, under the transformation $r = q^{2\rho}$ [12] reads:

$$\Delta_r \to \Delta_q,$$

$$\frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr} \to \frac{d^2}{dq^2} + \frac{d-1}{q}\frac{d}{dq},$$

ℓ -the angular momentum quantum number, and ρ - a variational parameter that is used to determine the behavior of the wave function, and then we define the quadratic operators $\hat{p}_q^2$, $\hat{q}^{2\tau}$ in the normal ordering method for the growing potential form [20, 21]:

$$p_q^{2\tau} = \omega_\ell^\tau \frac{\Gamma(\frac{d}{2}+\tau)}{\Gamma(\frac{d}{2})} + : p_q^{2\tau} :,$$

$$q^{2\tau} = \frac{1}{\omega_\ell^\tau}\frac{\Gamma(\frac{d}{2}+\tau)}{\Gamma(\frac{d}{2})} + : q^{2\tau} :,$$

where $\tau = 1, 2, 3, 4, \ldots$, the notation ": ■ :" is the normal ordering, which reorders operators in a specific format: the creation operators are placed to the left of the annihilation operators, and to confirm the correct vacuum expectation values, it is used in QFT. Now using the minimizing technique for the $b\tilde{b} : (nS)$ bound state with constituent quarks b, $\tilde{b}$. Let us define equation (19) with $|\hat{p}_{rb}^2| = |\hat{p}_{r\tilde{b}}^2| = |\hat{p}_r^2|$ in the form of:

$$\left(\min_{\mu_b}\frac{1}{2}\left(\mu_b + \frac{(m_b^2 + \hat{p}_r)^2}{\mu_b}\right) + \min_{\mu_{\tilde{b}}}\frac{1}{2}\left(\mu_{\tilde{b}} + \frac{(m_{\tilde{b}}^2 + \hat{p}_r)^2}{\mu_{\tilde{b}}}\right)\right)\Psi(r) + V(r)\Psi(r) = E_\ell(\mu)\Psi(r), \tag{23}$$

and then inserting formulas of the quadratic operators p_q^2, q^2 in the equation (19) with $|p_{qb}^2| = |p_{q\tilde{b}}^2| = \left|p_q^2\right|$, and we determine

Hamiltonin of the $b\tilde{b}:(nS)$ bound states in the form of normal ordering operators reads as follows[12]:

$$\left(\frac{p_q^2}{2}+V\left(q^{2\rho}\right)-E_\ell(\mu)\right)\Phi\left(q^2\right)=0 \tag{24}$$

The operator momentum p_q^2 in the auxiliary d space, acts on the ground state wavefunction $\Phi\left(q^2\right)$ and depends only on the $q^{\alpha\rho\ell}$. The energy eigenvalue refers to the set of possible energies $\varepsilon_\ell\left(E_\ell(\mu)\right)$ in the form of:

$$\varepsilon_\ell\left(E_\ell(\mu)\right)\Phi\left(q^2\right)= \left(\frac{p_q^2}{2}+V\left(q^{2\rho}\right)-E(\mu)\right)\Phi\left(q^2\right)=0c, \tag{25}$$

can be described. This means that $\varepsilon_\ell\left(E_\ell(\mu)\right)$ of the modified Schrödinger equation based on equation (25) in some region in the auxiliary d space satisfy the energy spectrum equal to $\varepsilon_\ell\left(E_\ell(\mu)\right)=0$ [12]. This relation might indicate a ground state energy solution in the specific situation in the auxiliary $d-$ dimension space, that we consider this situation as a $M_{b\tilde{b}}$ bound state of $b,\tilde{b}$-quarks. On the other words, $\varepsilon_\ell\left(E_\ell(\mu)\right)=0$, the minimum energy level of the bound state$n=0,1,2,\ldots$, is formed in the auxiliary space under the second quantization method. Under the second quantization method, we define the radial ($n_r=\frac{n-\ell}{2}$) and orbital (ℓ) excited states, the minimum energy level of $n=0$, identify as the energy $\varepsilon_0\left(E_0(\mu)\right)=\varepsilon_0$ of the ground state in the zeroth approximation. Hence, the Hamiltonian interaction in the total form using the normal ordering method and the quadratic operators $\hat{p}_q^{2n}$, $\hat{q}^{2n}$, in the modified Schrödinger equation reads:

$$H=H_0+H_q+\varepsilon_0\left(E_0(\mu)\right), \tag{26}$$

where H_0 is the Hamiltonian of uncoupled harmonic and independent oscillators, i.e., refers to a free oscillation $H_0\left(\hat{a}^+\hat{a}\right)$ of a system, H_q - Hamiltonian of interactions has nonquadratic canonical operators, and $\varepsilon_0\left(E_0(\mu)\right)$ is the ground state energy. In this method, all quadratic form of the normal ordering of canonical operators includes the principal operator (Hamiltonian) for the unbound oscillator, and this condition lets us consider the specific condition $\frac{d\varepsilon_0(E_0(\mu))}{d\omega}=0$, to determine the oscillator frequency $\omega_\ell=\omega_0$ which describes the dynamics of the quantum harmonic $b\tilde{b}:(nS)$ bound state. Using equation equation (24) up to equation equation (26), we can define the ground energy equation by:

$$\varepsilon_\ell\left(E_\ell(\mu)\right)\,\Phi(q^2)= \left(\frac{d\,\omega_\ell}{4}+4\mu\rho^2q^{4\rho-2}V\left(q^{2\rho}\right)-E_\ell(\mu)+2n_r\,\omega_\ell\right)\Phi(q^2)=0. \tag{27}$$

Then from conditions $\varepsilon_0\left(E_0(\mu)\right)=0$ and $\frac{d\varepsilon_0(E_0(\mu))}{d\omega}=0$, we can define the energy eigenvalue $E_0(\mu)$ and the oscillator frequency ω_0 for the ground state. Thus, by determining $\varepsilon_\ell\left(E_\ell(\mu)\right)$ and ω_ℓ for $n=0,\ \ell=0,\ n_r=0$, and inserting in equation (12) and using equation (21) and equation (22), we determine within the specific potential form, the mass spectrum of the $b\tilde{b}:(nS)$ hadronic bound state in the high energy and relativistic conditions.

5. $b\tilde{b}:(nS)$ within the refined tanh-shaped hyperbolic potential

In this section, based on the phenomenological model and due to creation $M_{b\tilde{b}}$ hadronic bound state, we choose the effective force that represents the net interaction between the $b-\tilde{b}$ in this way, an attractive term at short distances is described by a hyperbolic function that is strongly peaked at $r\to 0$ and falls off as $r\to\infty$. These limitations help us to bind $b-\tilde{b}$ together with a strong short-range attractive force. The function σr with the constant repulsiveσ is a confining component at long distances. It describes the long-range confining behavior and keeps $b-\tilde{b}$ together as a bound state. Hence, in this research, the effective interaction potential is presented within the tanh-shaped hyperbolic [9] with a linear confinement term in the form of:

$$V(r)=-V_0\tanh(\alpha r)\,+\sigma r. \tag{28}$$

$V_0\succ 0$ - an attractive potential constant, σ - the confinement constant, α- control the range of tanh-shaped hyperbolic potential. The value of $V_0,\alpha,\ \sigma$ are adjusted to match the experimental data of $b\tilde{b}:(nS)$ energy levels. We use the Bernoulli series [22, 23]:

$$\tanh(\alpha r)\;=\int_{i=1}^{\infty}\frac{2^{2i}(2^{2i}-1)(B_{2i}(\alpha r))^{2i-1}}{(2i)!} \tag{29}$$

where $B_{2i}:\ B_0=1, B_1=-\frac{1}{2}, B_2=1/6,\cdots$ are the Bernoulli numbers, and refine the tanh-shaped hyperbolic potential using third-order terms from (28) and (29):

$$V(r)=(\sigma-\alpha V_0)\,r+\frac{\alpha^3V_0}{3}r^3. \tag{30}$$

To solve MRSE using the second quantization method in the symplectic space, using the results in Section 5 and expanding under q^{2n} and simplifying equation (30), we define the tanh-shaped hyperbolic potential $V(q)$ as follows:

$$V(q)=(\sigma-\alpha V_0)\,q^{2\rho}+\frac{\alpha^3V_0}{3}q^{6\rho}. \tag{31}$$

Then the equation (25) under equation (27) and equation (31) inauxiliary d space defines the energy $\varepsilon_\ell\left(E_\ell(\mu)\right)$ of the bound state. Energy eigenvalue for the ground state $n=0$ and excited states n_r in the zeroth approximation i.e., reads:

$$\varepsilon_n\left(E_0(\mu)\right)=\frac{p_q^2}{2}+4\mu\rho^2q^{4\rho-2}\left(aq^{2\rho}+bq^{6\rho}\right) -4\mu\rho^2q^{4\rho-2}E_n(\mu)+2n_r\omega_n=0, \tag{32}$$

where $a=\sigma-\alpha V_0,\ \ b=\frac{\alpha^3V_0}{3}$. Simplifying, one can define the energy eigenvalue equation:

$$E_n(\mu)=\frac{c'}{\mu\rho^2}\frac{\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d}{2}+2\rho+1)}\omega_n^{2\rho} +a\frac{\Gamma(\frac{d}{2}+3\rho+1)}{\Gamma(\frac{d}{2}+2\rho+1)}\omega_n^{-\rho}+b\frac{\Gamma(\frac{d}{2}+5\rho+1)}{\Gamma(\frac{d}{2}+2\rho+1)}\omega_n^{-3\rho}, \tag{33}$$

Table 1: Nonrelativistic mass spectra, reduced mass, and oscillator frequency of resonance states of Bottomia under parameters $V_0 = 1.1,\ \alpha = 0.1,\ \sigma = 0.25$, all parameters and results are in units of GeV.

State	ρ	$M_{b\bar{b}}$	μ	ω_n	$M_{b\bar{b}}^{\text{Theor. } i}$	$M_{b\bar{b}}^{\text{Theor. } ii}$	$M_{b\bar{b}}^{\text{Theor. } iii}$	$M_{b\bar{b}}^{\text{Exp. } iv}$
1s	0.5266	9.2692	2.4115	0.4682	9.46030	9.4600	9.4603	9.46030
2s	0.4935	10.2354	2.4115	0.1704	10.02326	10.0230	10.0232	10.0323
3s	0.4806	10.3616	2.4115	0.1082	10.35520	10.3550	10.3552	10.3522
4s	0.4727	10.4581	2.4115	0.0804	10.40540	10.5800	10.5794	10.5794

i Ref. [25], ii Ref. [26], iii Ref. [27], iv Ref. [11]

Table 2: Relativistic mass spectra, reduced mass, constituent mass, and oscillator frequency of resonance states of Bottomia under parameters $V_0 = 1.1,\ \alpha = 0.1,\ \sigma = 0.25$, all parameters and results are in units of GeV.

State	ρ	$M_{b\bar{b}}$	μ	μ_b	ω_n	$M_{b\bar{b}}^{\text{Theor.Rel.}}$	$M_{b\bar{b}}^{\text{Theor.Rel. } i}$	$M_{b\bar{b}}^{\text{Theor.Rel. } ii}$	$M_{b\bar{b}}^{\text{Exp. } iii}$
1s	0.4895	9.4582	2.4285	4.8390	0.3911	9.42539	9.4603	9.46030	9.460
2s	0.4616	10.3068	2.4342	4.8899	0.1832	9.91895	9.9519	10.0232	10.074
3s	0.4538	10.4569	2.4433	4.9292	0.0859	10.2464	10.2792	10.3552	10.359
4s	0.4497	10.5741	2.4474	4.9501	0.0649	10.4452	10.4452	10.5794	10.683

i Ref. [23], ii Ref. [28], iii Ref. [11]

Table 3: Mass spectra of higher resonance states of Bottomia under parameters $V_0 = 1.1,\ \alpha = 0.1,\ \sigma = 0.25$, all parameters and results are in units of GeV.

State	$M_{b\bar{b}}^{\text{Theor.Nonrel.}}$	$M_{b\bar{b}}^{\text{Theor.Rel.}}$	$M_{b\bar{b}}^{\text{Theor.}}$ Ref. [23]	$M_{b\bar{b}}^{\text{Exp.}}$ Ref. [24]
1s	0.4895	9.4582	2.4285	0.3911
2s	0.4616	10.3068	2.4342	0.1832
3s	0.4538	10.4569	2.4433	0.0859
4s	0.4497	10.5741	2.4474	0.0649

and the oscillator frequency ω_n equation:

$$\omega_n{}^{5\rho} - \frac{a\mu\rho^2}{2c'} \frac{\Gamma\left(\frac{d}{2}+3\rho+1\right)}{\Gamma\left(\frac{d}{2}+1\right)} \omega_n{}^{2\rho} - \frac{3b\mu\rho^2}{2c'} \frac{\Gamma\left(\frac{d}{2}+5\rho+1\right)}{\Gamma\left(\frac{d}{2}+1\right)} = 0, \tag{34}$$

where $c' = \frac{1}{8} + \frac{n}{d},\ n_r = \frac{n-\ell}{2} = 0, 1, 2, \cdots$. Now, to continue our calculations, we set the frequency to the minimum and define:

$$\omega_{nmin} = \left(\frac{a\mu\rho^2}{5c'} \frac{\Gamma\left(\frac{d}{2}+3\rho+1\right)}{\Gamma\left(\frac{d}{2}+1\right)} \right)^{\frac{1}{3\rho}}, \tag{35}$$

for the ground state $n = 0,\ \ell = 0,\ n_r = 0$ under the Gaussian basis method, which describes acting on the ground state (lowest energy state) by creation and annihilation operators represented by canonical variables, corresponding to the quantum harmonic oscillator representation.

6. Results

6.1. Determination of Bottomia mass spectra

Throughout this section, we will determine the mass spectrum of $M_{b\tilde{b}}$ in the ground state within the specific parameters of refined tanh-shaped hyperbolic potential $b - \tilde{b}$ bound states, and consider the ground state. Taking into account the Gaussian basis method and the pure interaction at minimum oscillation level $n = 0,\ \ell = 0,\ n_r = 0$, the ground and $n,\ \ell,\ n_r$ excited state energies are determined (neglected spin-spin, spin-orbit, and tensor effects and interactions). The results of numerical calculations are presented in Table 1. Using specific parameters for the bottom quark [11] and tanh-shaped hyperbolic potential interaction. The Gaussian wavefunction at the origin of the coordinate is considered, and we determined the mass spectra of ground and resonance states$b\tilde{b}$: (nS) of $b - \tilde{b}$ quarks with nonrelativistic and relativistic corrections to the mass are calculated. We calculate the mass spectra of $b\tilde{b}$: (nS) mesons consisting of radial excitation $n_r = 0, 1, 2, \cdots$ and with the orbital quantum value $\ell = 0$. From equation (33), when the spin interactions are denied, only the orbital quantum number will affect the mass spectra among the quantum space parameters and change the properties of ground and excited bound states. Using mathematical definitions and calculations, the equation for determining the oscillator frequencies ω_n and mass spectra of $b\tilde{b}$: (nS) resonance states from equation (34) and equation (35) are defined and calculated. The results of numerical calculations for $m_b = 4.823(GeV)$, are introduced in Table 1 and Table 2.

6.2. Predicates of Bottomia mass spectra of higher radially excited states

Future discoveries and research on the $b\tilde{b}$: (nS) meson resonance states behavior, properties, and decay characteristics have been conducted at several high energy collider facilities and hadronic laboratories at the Large Hadron Collider through experiments CMS, LHCb, and HLCs. In recent years the relativistic models of the $b\tilde{b}$: (nS) mesons have been analyzed and refined mass spectra. Additionally, for detecting new $b\tilde{b}$: (nS) resonance states researchers try to predict and refine their experimental techniques and find the best ways for observing higher resonance states such as $7s$, $8s$, $9s$, $10s$, due to understanding the formation and interactions of heavier $b\tilde{b}$: (nS) under quark-antiquark approaches and within ultra hot medium i [29, 30]. Hence, the importance and application of mass spectra and mesic characteristics of $b\tilde{b}$: (nS) resonance states at high resonances in future experiments that will be launched around 2027 into the 2030s are used in the following cases:

(a) Study $b\tilde{b}$: (nS) resonance decay processes and search for new physics beyond the standard model.

(b) allowing for a detailed potential discovery model of higher states.

(c) Study experimental data of proton-antiproton collisions.

(d) Study phenomena with high luminosity of $b\tilde{b}$: (nS).

(e) Study the structure of matter and stars based on QCD.

Therefore, due to the great importance of hadronic physics for understanding the structure and dynamics of quarks and the strong interaction in QCD, we defined and calculated the masses of $b\tilde{b}$: (nS) mesons in higher excited states such as $7S$, $8S$, $9S$, and $10S$. Hence, we theoretically predicted $b\tilde{b}$: (nS) mass spectra under the relativistic behavior of mass with the radial excitation $n_r = 7, 8, 9, 10$ and orbital quantum value $\ell = 0$. The results of numerical calculations of highly resonance states of $b\tilde{b}$: (nS) are introduced in Table 3.The variational parameter ρ is usually determined from the equations derived by minimizing the energy function $\frac{dE(\mu)}{d\rho} = 0$, and to calculate the numerical value of the variation, the equation involving the digamma function $\Psi(x) = \frac{d}{dx}(\ln\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}]$ was solved using MATLAB's built-in functions.

7. Conclusion

The polarization loop function for b, $\tilde{b}$ quarks in a quantum gauge interaction in this study are defined and concentrated on describing the asymptotic behavior of the wavefunction and MRSE under nonrelativistic and relativistic correction properties. The interaction Hamiltonian of Bottomia resonance bound states within the tanh-shaped hyperbolic with a linear confinement term, is formulated $(V(r) = -V_0 \tanh(\alpha r) + \sigma r)$. The classical kinetic energies of proper masses $m_b = m_{\tilde{b}}$ using the constituent mass of b, $\tilde{b}$ quarks are defined. Then, the exchange of quantum gauge interaction through Feynman diagrams is characterized. The results of the mass spectra show the relation of constituent mass $\mu_b$of b, $\tilde{b}$ quarks arise in bound states under the relation $M_{b\tilde{b}} = 2\mu_b + \mu_b E'(\mu_b) + E(\mu_b)$. Numerical results are compared with experimental and theoretical resonances obtained from other models and methods for relativistic and nonrelativistic limits. Due to future research at high energy collider facilities and hadronic laboratories at the Large Hadron Collider, we predicted higher resonances $7s$, $8s$, $9s$, $10s$ mass spectra using the tanh-shaped hyperbolic with a linear confinement term potential model $V(r) = -V_0 \tanh(\alpha r) + \sigma r$. Those that have not yet been confirmed are defined and presented. The Theoretical defined result of resonance states of $b\tilde{b}$: (nS) demonstrated good agreement with theoretical and experimental data presented in this research. The theoretical method presented in this study describing relativistic effect on mass and modified Schrödinger equation with other interaction potentials form under the Gaussian basis and harmonic oscillator framework is applicable for hadronic bound states, from light mesons to heavy quarkonia, glueballs, pomerons, charmonium hybrids, gluonic excitations and exotic atoms such as π-meson or μ-meson bound states. Its flexibility to use both perturbative and nonperturbative QCD regimes using variational techniques, path-integral formalism, and oscillator quantization, which are valuable for bound states with strong coupling and nonlocal interactions. This approach also offers a unified theoretical tool for modeling bound state phenomena in short-range and long-range quark-gluon dynamics in hadronic bound states.

Data availability

All data presented in this article were obtained from the analytical equations discussed herein and are available upon request by contacting the corresponding authors.

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References

[1] J. Yoh, "The discovery of the b quark at Fermilab in 1977: The experiment coordinator's story", in *Proceedings of AIP*, AIP, New York, United States of America, 1998, pp. 29–42. https://doi.org/10.1063/1.55114.

[2] D. M. Asner, "Belle II executive summary", arXiv 2203.10203v2 [hep-ex] (2022). https://doi.org/10.48550/arXiv.2203.10203.

[3] N. Brambilla, S. Eidelman, C. Hanhart & A. Vairo, "The XYZ states: experimental and theoretical status and perspectives", Physics Reports **873** (2020) 154. https://doi.org/10.1016/j.physrep.2020.05.001.

[4] C. Bokade & A. Bhaghyesh, "Predictions for bottomonia from a relativistic screened potential model", arXiv 2501.03147v1 [hep-ph] (2025). https://arxiv.org/pdf/2501.03147v1.

[5] Z. Tang, B. Wu, A. Hanlon & A. Rothkopf, "Quarkonium spectroscopy in the quark-gluon plasma", arXiv 2502.09044v1 [nucl-th] (2025). https://doi.org/10.48550/arXiv.2502.09044.

[6] D. Fujiwara, *Rigorous time slicing approach to Feynman path integrals*, Springer International Publishing, Berlin, Germany, 2017, pp. 39–164. https://doi.org/10.1007/978-4-431-56553-6.

[7] J. Struckmeier, "Relativistic generalization of Feynman's path integral on the basis of extended Lagrangians", arXiv 2406.06530v1 [quant-ph] (2024). https://doi.org/10.48550/arXiv.2406.06530.

[8] G. Semenoff, *Quantum field theory*, Springer International Publishing, Berlin, Germany, 2023, pp. 307–355. https://doi.org/10.1007/978-981-99-5410-0.

[9] V. Badalov & S. Badalov, "Generalised tanh-shaped hyperbolic potential: Klein–Gordon equation's bound state solution", Communications in Theoretical Physics **75** (2023) 075003. https://doi.org/10.1088/1572-9494/acd441.

[10] V. Kher, R. Chaturvedi, N. Devlani & A. K. Rai, "Bottomonia spectroscopy using Coulomb plus linear (Cornell) potential", European Physical Journal Plus **137** (2022) 357. https://doi.org/10.1140/epjp/s13360-022-02538-5.

[11] R. L. Workman, "Particle Data Group", Progress of Theoretical and Experimental Physics **2022** (2022) 083C01. https://doi.org/10.1093/ptep/ptac097.

[12] M. Dienykhan, G. Efimov, G. Ganbolds & A. N. Sissakian, *Oscillator representation in quantum physics*, Springer International Publishing, Berlin, Germany, 1995, pp. 196–225. https://doi.org/10.1007/978-3-540-49186-6.

[13] W. Greiner, S. Schramm & W. Stein, *Quantum chromodynamics*, Springer International Publishing, Berlin, Germany, 2007, pp. 230–441. https://doi.org/10.1007/978-3-540-48535-3.

[14] R. Gould, *Quantum electrodynamics: Electromagnetic processes*, Springer International Publishing, Berlin, Germany, 2020, pp. 75–130. https://doi.org/10.1515/9780691215846.

[15] S. Albeverio, R. Høegh-Krohn & S. Mazzucchi, *Mathematical theory of Feynman path integrals*, Springer International Publishing, Berlin, Germany, 2008, pp. 19–51. https://doi.org/10.1007/978-3-540-76956-9.

[16] E. Copson, *Asymptotic expansions: The saddle-point method*, Cambridge University Press, Cambridge, United Kingdom, 2009, pp. 63–99. https://doi.org/10.1017/CBO9780511526121.

[17] K. Barley, J. Vega-Guzmán, A. Ruffing & A. D. Alhaidari, "Discovery of the relativistic Schrödinger equation", Journal Uspekhi Fizicheskikh Nauk Russian Academy of Sciences **65** (2022) 90. https://doi.org/10.3367/ufne.2021.06.039000.

[18] W. Kutzelnigg, "Theory of the expansion of wave functions in a Gaussian basis", Quantum Chemistry Journal **51** (1994) 447. https://doi.org/10.1002/qua.560510612.

[19] M. Rushka & J. Freericks, "A completely algebraic solution of the simple harmonic oscillator", American Journal of Physics **88** (2019) 976. https://doi.org/10.1119/10.0001702.

[20] J. Kelley & J. Leventhal, "Ladder operators for the harmonic oscillator", in *Problems in classical and quantum mechanics*, Springer International Publishing, Berlin, Germany, 2017, pp. 149–188. https://doi.org/10.1007/978-3-319-46664-4_7.

[21] D. H. Lehmer, "On the maxima and minima of Bernoulli polynomials", American Mathematical Monthly **47** (1940) 533. https://doi.org/10.1080/00029890.1940.11991015.

[22] Z. Sun & H. Pan, "Identities concerning Bernoulli and Euler polynomials", arXiv:math/0409035 [math.NT] (2004). https://doi.org/10.48550/arXiv.math/0409035.

[23] A. Ahmadov, K. Abasova & M. Orucova, "Bound state solution of Schrödinger equation for extended Cornell", Advances in High Energy Physics **2021** (2021) 1861946. https://doi.org/10.1155/2021/1861946.

[24] A. Zyla, "Particle Data Group", Progress of Theoretical and Experimental Physics **2020** (2020) 083C01. https://doi.org/10.1093/ptep/ptaa104.

[25] V. Badalov, A. I. Ahmadov, E. A. Dadashov & A. N. Gadzhiev, "Dirac equation solution with generalized tanh-shaped hyperbolic potential: Application to charmonium and bottomonia mass spectra", arXiv 2409.15538v3 [hep-ph] (2025). https://doi.org/10.48550/arXiv.2409.15538.

[26] M. Abu-Shady, "Heavy quarkonia and bc* mesons in the Cornell potential with harmonic oscillator potential in the N-dimensional", International Journal of Applied Mathematics and Theoretical Physics **2** (2016) 16. https://doi.org/10.11648/j.ijamtp.20160202.11.

[27] B. Bukor & J. Tekel, "On quarkonium masses in 3D noncommutative space", European Physical Journal Plus **138** (2023) 499. https://doi.org/10.1140/epjp/s13360-023-04049-3.

[28] K. Purohit & A. Rai, "Quarkonium spectroscopy of the linear plus modified Yukawa potential", Physica Scripta **97** (2022) 044002. https://doi.org/10.1088/1402-4896/ac5bc2.

[29] Q. Li, M. Allosh, G. S. Hassan & A. Upadhyay, "Canonical interpretation of $Y(10750)$ and $Y(10860)$ in the Y family", European Physical Journal C **80** (2020) 59. https://doi.org/10.1140/epjc/s10052-020-7626-2.

[30] J. Wang & X. Liu, "Identifying a characterized energy level structure of higher charmonium well matched to the peak structures in $e^+e^- \to \pi + D0D^*$", Physics Letters B **849** (2024) 138456. https://doi.org/10.48550/arXiv.2306.14695.