Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 8 (2026) 3105

Journal of the Nigerian Society of Physical Sciences

A hybrid IFR-IDY conjugate gradient algorithm for unconstrained optimization and its application in portfolio selection

Diva Marchandra Mulansari[a], Maulana Malik[a,b,*], Sindy Devila[a,b], Ibrahim Mohammed Sulaiman[c,d], Dian Lestari[a,b], Fevi Novkaniza[a,b], Fida Fathiyah Addini[a,b]

[a]*Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Indonesia, Depok 16424, Indonesia*
[b]*Advanced Risk, Actuarial, and Financial Analitycs Laboratory (ARAFA Lab), Faculty of Mathematics and Natural Sciences, Universitas Indonesia, Depok 16424, Indonesia*
[c]*College of Applied and Health Sciences, A'Sharqiyah University, Ibra 400, Sultanate of Oman*
[d]*Faculty of Education and Arts, Sohar University, Sohar 311, Oman*

Abstract

This study introduces the Improved Fletcher-Reeves (IFR)-Improved Dai-Yuan (IDY) hybrid conjugate gradient method, which combines the strengths of the IFR and IDY parameters through a minimum-operator strategy to enhance robustness in unconstrained optimization. The method is shown to satisfy descent and global convergence properties under the strong Wolfe line search. Numerical experiments on 134 benchmark functions demonstrate that IFR-IDY achieves superior performance, solving 98 problems more than IFR and IDY and exhibiting faster CPU times and fewer iterations in most cases. The method is also used to solve an IDX30 portfolio optimization problem, which results in an optimal allocation with an expected return of 0.00042 and a risk of 0.000050545. These results highlight the efficiency of IFR-IDY and its practical applicability in real-world decision-making.

DOI:10.46481/jnsps.2026.3015

Keywords: Inexact line search, Global convergence, Investment, Numerical optimization, Strong Wolfe

Article History :
Received: 24 August 2025
Received in revised form: 09 December 2025
Accepted for publication: 12 December 2025
Published: 16 February 2026

Communicated by: B. J. Falaye

1. Introduction

Optimization problems are prevalent in everyday life. For example, a driver may seek the fastest route to a destination, a trader may aim to maximize profits, or a manufacturer may strive to design the most efficient production process. Such problems are addressed by identifying the best possible solution to a given task [1]. The optimal solution depends on the objective of the problem, which is mathematically expressed through an objective function. An objective function serves as a quantitative measure—such as time, profit, or other numerical criteria—determined by specific characteristics known as variables [2]. Optimization is achieved by selecting values for these variables such that the objective function attains its optimal value.

*Corresponding author Tel. No: +62-8988072292.
Email address: m.malik@sci.ui.ac.id (Maulana Malik)

The conjugate gradient method is a widely used approach for solving large linear systems and can also be adapted for nonlinear optimization problems [3]. Originally introduced by Hestenes and Stiefel [4] for solving linear systems with positive-definite coefficient matrices, it became known as the linear conjugate gradient method. Later, Fletcher and Reeves [5] extended the method to address large-scale nonlinear optimization problems, leading to the development of the nonlinear conjugate gradient method [2]. Owing to its simple iterative formula and relatively low memory requirements, the conjugate gradient method is regarded as one of the most efficient techniques for unconstrained optimization [3].

To date, numerous variants of the conjugate gradient method have been developed to address optimization problems. According to Andrei [1], the primary differences among these methods lie in two aspects: the strategy used to update the search direction at each iteration and the computational procedure employed to determine the step size along that direction. Nevertheless, all conjugate gradient methods are founded on the same core principle—the search direction is selected to satisfy the descent condition. In general, conjugate gradient methods can be categorized into several classes, including standard conjugate gradient methods, hybrid conjugate gradient methods, spectral conjugate gradient methods, and three-term conjugate gradient methods [1].

Several well-known standard conjugate gradient methods include the Hestenes–Stiefel (HS) [4], Fletcher–Reeves (FR) [5], Polak–Ribiere–Polyak (PRP) [6], Conjugate Descent (CD) of Fletcher [7], Liu–Storey (LS) [8], and Dai–Yuan (DY) [9] methods. Over time, these standard methods have been extensively modified to enhance their convergence properties and numerical performance. Jiang and Jian [10] proposed two such methods, namely the Improved Fletcher–Reeves (IFR) and Improved Dai–Yuan (IDY) methods, as refinements of the standard FR and DY approaches. The main idea behind these improvements was to combine the conjugate parameters of the FR and DY methods with the second inequality condition of the strong Wolfe line search procedure [10]. Both IFR and IDY methods have been proven to satisfy the global convergence and descent conditions under certain assumptions and when using the strong Wolfe line search. However, despite meeting these theoretical properties, both methods exhibit limited efficiency, as some test functions remain unsolved. Further details on modifications of the conjugate gradient methods can be found in [11–17].

Therefore, building upon the above discussion and considering the strengths and limitations of the IFR and IDY conjugate gradient methods, this paper proposes a hybrid IFR–IDY conjugate gradient parameter for solving unconstrained optimization problems. The main contributions of this study are as follows:

- Proposing a novel hybrid IFR–IDY conjugate gradient method that integrates the characteristics of the IFR and IDY methods.
- Demonstrating that the proposed method satisfies the descent property under the strong Wolfe line search procedure.
- Establishing the global convergence of the proposed method under appropriate assumptions.
- Evaluating the efficiency and robustness of the proposed method in comparison with existing conjugate gradient algorithms in terms of computational performance.
- Illustrating the practical applicability of the proposed method through its implementation in stock portfolio selection problems.

The remainder of this paper is organized as follows. Section 2 presents the proposed algorithm incorporating parameter mixing. Section 3 discusses the descent property and global convergence analysis under suitable assumptions. Section 4 reports the results of numerical experiments, while Section 5 demonstrates the application of the proposed method to portfolio selection problems. Finally, Section 6 provides concluding remarks.

2. A hybrid IFR-IDY conjugate gradient algorithm

In this section, we start with the following general minimization problem:

$$\min_{x\in\mathbb{R}^n} f(x) \tag{1}$$

where $f:\mathbb{R}^n\to\mathbb{R}$ is a continously differentiable function with its gradient $\nabla f(x_k)$ by g_k and $\mathbb{R}$ denotes the set of real number. This study focuses on conjugate gradient (CG) methods for solving problem (1). The CG algorithm addresses problem (1) by generating a sequence of iterative points as follows [1]:

$$x_{k+1} = x_k + \alpha_k d_k, \quad k = 1, 2, \ldots, \tag{2}$$

where x_1 is the initial point, x_k is the k-th approximation to the solution, and d_k is the search direction, defined by:

$$d_k = \begin{cases} -g_k, & k = 1, \\ -g_k + \beta_k d_{k-1}, & k > 1, \end{cases} \tag{3}$$

where β_k is the conjugate gradient parameter [2], and $\alpha_k > 0$ is the step length, obtained either by exact or inexact line search. In this paper, we employ an inexact line search, namely the strong Wolfe line search, defined as follows:

$$f(x_k + \alpha_k d_k) \le f(x_k) + \delta\alpha_k g_k^T d_k, \tag{4}$$

$$\left|g(x_k + \alpha_k d_k)^T d_k\right| \le -\sigma g_k^T d_k, \tag{5}$$

where g_k^T is transpose g_k and $0 < \delta < \sigma < 1$ [1].

There are many choices for the parameter β_k in the conjugate gradient method. Among them are β_k^{IFR} and β_k^{IDY}, corresponding to the IFR and IDY conjugate gradient methods, respectively.

The IFR and IDY conjugate gradient methods, introduced by Jiang and Jian [10], are modified versions of the FR and DY conjugate gradient methods, respectively. Al-Baali [18] demonstrated that the FR method satisfies the descent condition and achieves global convergence under the strong Wolfe line search with $0 < \sigma < \frac{1}{2}$.

Table 1: Numerical results of IFR, IDY, and hybrid IFR-IDY conjugate gradient methods.

Fungsi	Dimensi	IFR		IDY		IFR-IDY	
		TCPU	Itr	TCPU	Itr	TCPU	Itr
COSINE	6000	0.4121	134	F	F	0.2947	49
	10000	13.0754	413	F	F	1.4218	36
	80000	F	F	F	F	27.4349	74
DIXMAANA	2000	1.645	117	0.5862	25	0.4593	17
	30000	7.8969	54	4.7393	29	4.4518	22
DIXMAANB	8000	2.4661	54	2.0151	30	1.6669	23
	16000	9.6033	108	2.966	25	5.6459	51
DIXMAANC	900	0.4875	73	0.36	34	0.2261	19
	9000	1.9547	36	1.6753	21	1.4330	15
DIXMAAND	4000	2.3487	94	1.7341	44	1.3340	33
	30000	27.6264	191	F	F	14.8430	75
DIXMAANE	800	4.7696	808	3.3623	463	4.6260	755
	16000	F	F	139.1142	1423	F	F
DIXMAANF	5000	F	F	65.4642	1561	F	F
	20000	F	F	F	F	F	F
DIXMAANG	4000	F	F	46.6299	1794	F	F
	30000	F	F	F	F	F	F
DIXMAANH	2000	F	F	16.36363	1290	F	F
	50000	F	F	F	F	F	F
DIXMAANI	120	F	F	F	F	F	F
	12	0.1601	686	0.1958	423	0.3058	544
DIXMAANJ	1000	F	F	16.3476	1932	F	F
	5000	36.61	972	29.6561	705	64.3916	1344
DIXMAANK	4000	15.8173	630	26.8473	745	37.6879	1032
	40	0.3979	1026	0.633	759	0.3577	963
DIXMAANL	800	2.1393	392	6.0858	903	2.8627	324
	8000	25.026	521	47.585	769	57.6559	815
DIXON3DQ	150	F	F	0.1612	860	F	F
	15	0.0421	466	0.0474	219	0.0834	449
DQDRTIC	9000	0.3728	366	0.5008	210	0.6056	245
	90000	1.7841	280	1.5862	214	2.0224	288
DQRTIC	5000	1.3288	124	1.3373	74	0.5305	30
	150000	43.141	164	94.5834	226	22.6362	100
EDENSCH	7000	3.8466	248	7.4374	423	1.0488	57
	40000	11.3761	148	15.6409	159	4.5358	99
	500000	305.1124	313	1702.5	1619	138.8814	463
EG2	35	F	F	0.1386	346	0.0718	385
	1000	0.0439	430	0.0603	225	0.1094	397
FLETCHCR	1000	0.1148	565	0.0623	165	0.061	168
	50000	4.2824	850	F	F	3.0733	533
	200000	10.7434	579	F	F	4.7304	196
FREUROTH	460	F	F	F	F	F	F
	10	0.1074	1415	0.0848	429	0.1386	867
GENROSE	1000	F	F	F	F	F	F
	100	F	F	0.2445	1409	F	F
HIMMELBG	70000	0.0614	2	0.0878	2	0.089	2
	240000	0.0935	2	0.2289	2	0.2273	2
LIARWHD	15	0.0346	186	0.0245	103	0.0406	164
	1000	0.0329	239	0.0543	192	0.1352	722
PENALTY1	1000	0.6525	232	F	F	F	F

	8000	101.6616	156	25.6781	101	5.9235	34
QUARTC	4000	0.685	95	1.1156	116	0.5065	44
	80000	19.7324	150	51.1821	213	19.7324	80
	500000	470.2391	187	598.165	297	80.4703	120
TRIDIA	300	0.182	1816	0.1117	901	0.3268	1877
	50	0.0574	624	0.0415	343	0.1135	582
WOODS	150000	4.2713	396	6.1109	481	15.9805	1155
	200000	10.8526	760	7.531	487	17.151	1017
BDEXP	5000	0.0296	2	0.0059	2	0.0176	2
	50000	0.0553	2	0.0758	2	0.1398	2
	500000	0.4771	2	0.6053	2	0.9591	2
EXDENSCHNF	90000	1.2392	128	1.918	207	0.3712	24
	280000	2.9712	120	2.2558	78	3.2893	99
	600000	5.8497	119	6.5723	104	1.9089	25
EXDENSCHNB	6000	0.0768	119	0.0269	56	0.0205	16
	24000	0.3082	129	0.1403	77	0.1034	19
	300000	2.6393	129	3.063	124	0.8791	19
GENQUARTIC	300000	0.061	29	0.1087	83	0.0515	15
	9000	1.0657	125	0.8221	92	0.4608	32
	90000	5.6442	142	6.015	122	1.8367	33
BIGGSB1	110	0.1004	1046	0.0541	531	0.2719	1401
	200	F	F	0.1184	1048	F	F
SINE	100000	F	F	F	F	F	F
	50000	F	F	F	F	F	F
FKETCBV2	15	0.0216	97	0.032	75	0.0302	118
	55	F	F	0.0747	542	F	F
NONSCOMP	5000	0.121	240	0.2	191	0.0675	59
	80000	2.1955	309	F	F	F	F
POWER1	150	F	F	0.2939	1820	F	F
	90	F	F	0.2148	1127	F	F
RAYDAN1	500	0.152	659	0.5071	1593	0.134	364
	5000	F	F	F	F	1.4426	1586
RAYDAN2	2000	0.0291	22	0.0547	21	0.0732	31
	20000	0.1856	18	1.0939	51	0.3319	17
	500000	9.3988	45	22.1415	83	9.3988	181
DIAGONAL1	800	F	F	F	F	1.4372	1266
	2000	F	F	F	F	2.8432	1748
DIAGONAL2	100	0.0333	179	0.0608	161	0.0759	193
	1000	0.4238	625	0.3625	274	0.9859	699
DIAGONAL3	500	F	F	F	F	0.5186	888
	2000	F	F	F	F	F	F
BV	2000	4.8518	66	9.7677	119	14.578	174
	20000	1.0148	0	1.2975	0	1.6111	0
IE	500	37.0519	80	15.5645	31	7.7288	14
	1500	790.0997	117	302.2989	65	54.8539	17
SINGX	1000	14.6019	707	14.9872	519	F	F
	2000	200.4983	1449	136.8045	1696	F	F
LIN	100	0.0971	14	0.0697	14	0.0491	14
	500	0.234	13	0.5153	13	0.4844	13
OSB2	100	0.3566	1346	0.3382	765	F	F
PEN1	200	F	F	F	F	0.9877	533
	1000	F	F	F	F	49.2886	1026
PEN2	100	0.2949	869	0.4253	697	0.3162	483
	110	1.1198	1575	F	F	0.4939	245
ROSEX	500	1.8315	269	2.1154	175	7.3636	1104

	1000	10.7184	362	10.42	165	28.6056	794
TRID	500	F	F	F	F	F	F
	50	0.1371	530	0.178	322	0.0928	264
HIMMELH	70000	F	F	11.4835	245	6.7694	112
	240000	16.3869	100	F	F	16.7023	94
BADSCB	2	F	F	F	F	F	F
BD	4	F	F	F	F	0.108	454
BIGGS	6	F	F	0.1056	415	F	F
OSB1	11	F	F	F	F	F	F
WHITE	10	0.1933	636	0.0559	247	0.2346	1061
EXBEALE	5000	6.1376	690	F	F	4.5667	451
	10000	F	F	F	F	F	F
HIMMELBC	500000	8.4679	192	5.2873	117	1.1468	25
	1000000	11.803	147	9.7806	119	8.7409	101
ARWHEAD1	100	F	F	F	F	0.0102	20
	1000	F	F	F	F	F	F
BDQRTIC	10	F	F	F	F	F	F
ENGVAL11	500000	126.3965	1092	F	F	54.1269	856
	1000000	F	F	F	F	144.354	1418
DENSCHNA	500000	179.1522	125	103.6448	60	63.0315	103
	1000000	367.4751	125	358.8993	99	28.6341	29
DENSCHNB	500000	5.3705	144	1.7115	48	1.2008	22
	1000000	6.6871	99	11.3401	141	2.4978	29
DENSCHNC	10	0.0873	237	0.0584	205	0.013	49
	500	0.339	209	F	F	F	F
DENSCHNF	500000	9.5053	178	F	F	7.3281	141
	1000000	F	F	F	F	F	F
ENGVAL8	500000	F	F	F	F	F	F
	1000000	F	F	F	F	226.6371	1126

Table 2. Summary of functions successfully and unsuccessfully solved by the IFR-IDY, IFR, and IDY methods.

Method	Successful	Unsuccessful
IFR-IDY	98	36
IFR	92	42
IDY	93	41

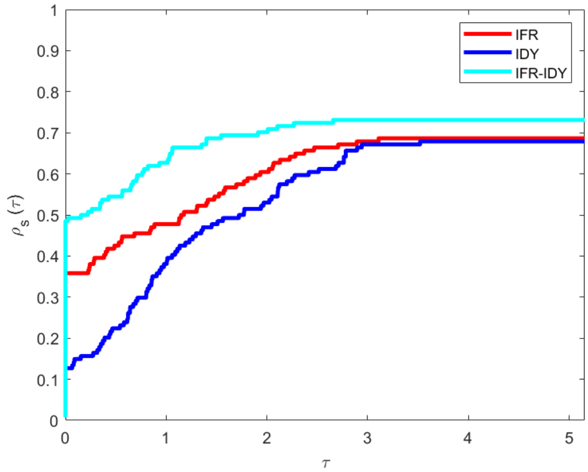

Figure 1. Performance profile curves based on number of iteration.

Similarly, Dai and Yuan [19] showed that the DY method satisfies the descent condition and converges globally under the Wolfe line search. Although both methods possess strong convergence properties, their numerical performance can suffer from stagnation. Motivated by the strong Wolfe line search framework, Jiang and Jian [10] proposed modifications to these methods to enhance numerical performance while preserving their convergence properties. The coefficients β_k for the IFR and IDY methods are defined as follows:

$$\beta_k^{IFR} = \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \cdot \beta_k^{FR} = \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \cdot \frac{||g_k||^2}{||g_{k-1}||^2}. \quad (6)$$

$$\beta_k^{IDY} = \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \cdot \beta_k^{DY} = \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \cdot \frac{||g_k||^2}{d_{k-1}^T (g_k - g_{k-1})}. \quad (7)$$

The limitations of existing conjugate gradient methods continue to drive the development of new modifications aimed at achieving both strong descent and global convergence properties while delivering improved numerical performance. Among the well-known conjugate gradient methods are the LS [9] and CD [5] methods. For general objective functions, the LS method is recognized for its strong convergence properties; however, it is often prone to stagnation in numerical performance. In contrast, the CD method typically exhibits good computational performance but may fail to converge in general cases. To exploit the strengths of both approaches, Yang *et al.* [20] proposed a hybrid LS–CD method in which the β_k coefficients are constructed using maximum and minimum operators. The β_k coefficients for the LS–CD hybrid conjugate gradient method are defined as follows:

$$\beta_k^{LS-CD} = \max\{0, \min\{\beta_k^{LS}, \beta_k^{CD}\}\}, \quad (8)$$

where β_k^{LS} and β_k^{CD} are defined as follows,

$$\beta_k^{LS} = -\frac{g_k^T (g_k - g_{k-1})}{d_{k-1}^T g_{k-1}}, \quad (9)$$

$$\beta_k^{CD} = -\frac{||g_k||^2}{d_{k-1}^T g_{k-1}}. \quad (10)$$

Inspired by the modification process of the LS-CD method, in this study, similar modifications were made to the IFR and IDY gradient conjugate methods. This method is called the IFR-IDY hybrid gradient conjugate method with the parameter β_k given as follows.

$$\beta_k^{IFR-IDY} = \max\{0, \min\{\beta_k^{IFR}, \beta_k^{IDY}\}\}, \quad (11)$$

where β_k^{IFR} and β_k^{IDY} are given by the equations (6) and (7). However, since Jiang & Jian (2018) have shown that the conjugate parameters $\beta_k^{IFR} \geq 0$ and $\beta_k^{IDY} \geq 0$, then the form (11) is equivalent to the following form.

$$\beta_k^{IFR-IDY} = \min\{\beta_k^{IFR}, \beta_k^{IDY}\}. \quad (12)$$

The algorithm of the proposed IFR-IDY hybrid conjugate gradient method is given as follows.

Algorithm 1 The Hybrid IFR-IDY Conjugate Gradient Algorithm

1: Select an initial guess $x_1 \in \mathbb{R}^n$ and choose the Wolfe parameters δ and σ such that $0 < \sigma < \delta < 1$, $d_1 = -g_1$, $\epsilon > 0$.
2: If $||g_k|| \leq \epsilon$, stop; otherwise go to Step 3.
3: Compute the step size α_k using the strong Wolfe line search procedures (4) and (5).
4: Set $x_{k+1} = x_k + \alpha_k d_k$. Compute the gradient $g_{k+1} := g(x_{k+1})$ and the parameter β_{k+1} using (12).
5: Compute the search direction $d_{k+1} = -g_{k+1} + \beta_{k+1} d_k$, set $k := k + 1$, and return to Step 2.

3. Convergence analysis

In this section, we analyze the convergence of the hybrid IFR-IDY conjugate gradient method by showing that this method satisfies the descent condition and global convergence properties. The following assumptions are required in the proof.

Assumption 1. *The level set* $\Lambda = \{x \in \mathbb{R}^n \mid f(x) \leq f(x_1)\}$ *is finite, where* x_1 *is the starting point.*

Assumption 2. *In some neighborhood* U *of* Λ, $g(x)$ *is continuously differentiable and its gradient is Lipschitz continuous; i.e., there exists a constant* $L > 0$ *such that*

$$||g(x) - g(y)|| \leq L||x - y||, \quad \forall x, y \in U.$$

Table 3. List of stocks that have been constituents of the IDX30 Index for the period of February 2022 - January 2024

No.	Code	Stock Name
1	ADRO	Adaro Energy Tbk
2	ANTM	Aneka Tambang Tbk
3	ASII	Astra International Tbk
4	BBCA	PT Bank Central Asia Tbk
5	BBNI	PT Bank Negara Indonesia (Persero) Tbk
6	BBRI	PT Bank Rakyat Indonesia (Persero) Tbk
7	BMRI	PT Bank Mandiri (Persero) Tbk
8	BRPT	Barito Pacific Tbk
9	BUKA	PT Bukalapak.com Tbk
10	CPIN	Charoen Pokphand Indonesia Tbk
11	EMTK	Elang Mahkota Teknologi Tbk
12	INCO	Vale Indonesia Tbk
13	INDF	Indofood Sukses Makmur Tbk
14	KLBF	Kalbe Farma Tbk
15	MDKA	PT Merdeka Copper Gold Tbk.
16	PGAS	PT Perusahaan Gas Negara Tbk.
17	PTBA	Bukit Asam Tbk
18	SMGR	Semen Indonesia (Persero) Tbk
19	TLKM	PT Telkom Indonesia (Persero) Tbk
20	TOWR	Sarana Menara Nusantara Tbk
21	UNTR	United Tractors Tbk
22	UNVR	Unilever Indonesia Tbk

The conjugate gradient method algorithm must guarantee that the search direction or d_k is a descent direction. This is to ensure that the search direction descends towards the optimum point. Based on the form $\beta_k^{IFR-IDY} = \min(\beta_k^{IFR}, \beta_k^{IDY})$, then in proving the descent condition it is divided into two cases, namely $\beta_k^{IFR-IDY} = \beta_k^{IFR}$ and $\beta_k^{IFR-IDY} = \beta_k^{IDY}$.

Therefore, to prove the descent condition of the hybrid IFR-IDY conjugate gradient method, the two theorems below are needed.

The first theorem below establishes the descent condition for the IFR conjugate gradient method. In what follows, we restate the proof in accordance with the original article [10], while providing additional clarification to enhance readability.

Theorem 1. *Suppose Assumptions 1 and 2 are satisfied and $0 < \sigma < \frac{\sqrt{2}}{2}$, and d_k is generated by the IFR conjugate gradient method, then*

$$\frac{g_k^T d_k}{\|g_k\|^2} \leq -1 + \frac{\sigma^2}{1-\sigma^2}, \; \forall k \geq 1. \tag{13}$$

Proof. The proof is done by mathematical induction. Consider again the direction of the IFR gradient conjugate method search on (3). For $k = 1$, by multiplying both sides of (3) by g_1^T the following result is obtained

$$g_1^T d_1 = g_1^T(-g_1) = -\|g_1\|^2 \leq 0.$$

Therefore, for $k = 1$, (13) is satisfied. Furthermore, assuming that (13) is satisfied for $k-1$ ($k > 2$), the following proves that (13) is also satisfied for k.

Next, for $k \geq 2$, by multiplying both sides of (3) by g_k^T and substituting the form (6) we obtain

$$\begin{aligned} g_k^T d_k &= g_k^T(-g_k + \beta_k^{IFR} d_{k-1}) \\ &= -\|g_k\|^2 + g_k^T(\beta_k^{IFR} d_{k-1}) \\ &= -\|g_k\|^2 + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \frac{\|g_k\|^2}{\|g_{k-1}\|^2} g_k^T d_{k-1}. \end{aligned} \tag{14}$$

By dividing both sides of (14) by $\|g_k\|^2$ we get

$$\frac{g_k^T d_k}{\|g_k\|^2} = -1 + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \frac{\|g_k\|^2}{\|g_{k-1}\|^2} \frac{g_k^T d_{k-1}}{\|g_k\|^2}. \tag{15}$$

Then, by using the strong Wolfe search line procedure (5) and (15) we obtain

$$\begin{aligned} \frac{g_k^T d_k}{\|g_k\|^2} &= -1 + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \frac{\|g_k\|^2}{\|g_{k-1}\|^2} \frac{g_k^T d_{k-1}}{\|g_k\|^2} \\ &\leq -1 + \frac{-\sigma g_{k-1}^T d_{k-1}}{-g_{k-1}^T d_{k-1}} \frac{1}{\|g_{k-1}\|^2} g_k^T d_{k-1} \\ &= -1 + \sigma \frac{g_k^T d_{k-1}}{\|g_{k-1}\|^2} \\ &\leq -1 + \sigma \frac{|g_k^T d_{k-1}|}{\|g_{k-1}\|^2}. \end{aligned} \tag{16}$$

Table 4. Selected stocks to form a portfolio optimization model.

No.	Stock Code	Return Expectation
1	ADRO	0.018%
2	ANTM	0.054%
3	ASII	0.029%
4	BRPT	0.029%
5	BUKA	0.192%
6	CPIN	0.082%
7	EMTK	0.354%
8	INCO	0.071%
9	INDF	0.008%
10	KLBF	0.041%
11	MDKA	0.104%
12	PGAS	0.061%
13	PTBA	0.046%
14	SMGR	0.041%
15	TLKM	0.019%
16	TOWR	0.048%
17	UNTR	0.072%
18	UNVR	0.023%

Table 5. Proporsi saham portofolio optimal

No.	Nama Saham	Proporsi	Alokasi Modal
1	ADRO	-2.83%	-Rp28,278,912.01
2	ANTM	3.94%	Rp39,383,536.48
3	ASII	6.87%	Rp68,700,314.02
4	BRPT	0.86%	Rp8,586,129.60
5	BUKA	2.02%	Rp20,182,389.91
6	CPIN	3.96%	Rp39,582,077.47
7	EMTK	1.97%	Rp19,711,542.18
8	INCO	5.73%	Rp57,292,998.83
9	INDF	28.92%	Rp289,191,223.80
10	KLBF	1.45%	Rp14,455,135.91
11	MDKA	-3.17%	-Rp31,745,869.35
12	PGAS	6.29%	Rp62,910,466.58
13	PTBA	4.89%	Rp48,851,283.22
14	SMGR	1.75%	Rp17,464,651.13
15	TLKM	19.36%	Rp193,615,909.35
16	TOWR	5.63%	Rp56,265,474.07
17	UNTR	4.27%	Rp42,719,069.83
18	UNVR	8.11%	Rp81,112,578.99

Next, using the strong Wolfe search line (5), (16), and (13) for $k-1$ then,

$$\frac{g_k^T d_k}{\|g_k\|^2} \leq -1 + \sigma \frac{|g_k^T d_{k-1}|}{\|g_{k-1}\|^2} \leq -1 - \sigma^2 \frac{g_{k-1}^T d_{k-1}}{\|g_{k-1}\|^2} \leq -1 + \frac{\sigma^2}{1-\sigma^2}. \quad (17)$$

Thus, it is proven that (13) is satisfied $\forall k \geq 1$. □

Next, the second theorem establishes the descent condition for the IDY conjugate gradient method. As before, we restate the proof following the structure of the original article and provide additional clarification where appropriate.

Theorem 2. *Suppose Assumptions 1 and 2 are satisfied and* $0 < \sigma < \frac{\sqrt{2}}{2}$*, and* d_k *is generated by the IDY conjugate gradient method, then the following descent condition is satisfied*

$$g_k^T d_k \leq -(1-\sigma)\|g_k\|^2, \ \forall k \geq 1. \quad (18)$$

Proof. The proof is done by mathematical induction. Recall the search direction of the IDY method shown by (3). For $k = 1$, by multiplying both sides by g_1^T, we get the following result

$$g_1^T d_1 = g_1^T(-g_1) = -\|g_1\|^2 \leq 0.$$

Therefore, for $k = 1$, (13) is satisfied. Furthermore, assuming that (13) is satisfied for $k-1$ ($k > 2$), we prove that (13) is also satisfied for k.

Next, for $k \geq 2$, by multiplying both sides of (3) by g_k^T and substituting the form β_k^{IDY} we get the following result,

$$\begin{aligned} g_k^T d_k &= g_k^T(-g_k + \beta_k^{IDY} d_{k-1}) \\ &= -\|g_k\|^2 + \beta_k^{IDY} g_k^T d_{k-1} \\ &= -\|g_k\|^2 + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \cdot \frac{\|g_k\|^2}{d_{k-1}^T(g_k - g_{k-1})} \cdot g_k^T d_{k-1}. \end{aligned} \quad (19)$$

By algebraic manipulation the form (19) is obtained,

$$\begin{aligned} g_k^T d_k &= -\|g_k\|^2 + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \cdot \frac{\|g_k\|^2}{d_{k-1}^T(g_k - g_{k-1})} \cdot d_{k-1}^T g_k \\ &= -\|g_k\|^2 + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \cdot \frac{\|g_k\|^2}{d_{k-1}^T(g_k - g_{k-1})} \\ &\quad \cdot (d_{k-1}^T(g_k - g_{k-1}) + g_{k-1}^T d_{k-1}) \\ &= -\|g_k\|^2 + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \cdot \|g_k\|^2 + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \cdot \frac{\|g_k\|^2 g_{k-1}^T d_{k-1}}{d_{k-1}^T(g_k - g_{k-1})} \\ &= -(1 - \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}})\|g_k\|^2 + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \cdot \frac{\|g_k\|^2 g_{k-1}^T d_{k-1}}{d_{k-1}^T(g_k - g_{k-1})}. \end{aligned} \quad (20)$$

Before further analyzing the form of (20), several conditions are required.

By describing the strong Wolfe search line procedure (5) we get,

$$\begin{aligned} & |g(x_k + \alpha_k d_k)^T d_k| \leq -\sigma g_k^T d_k \\ \Longleftrightarrow\ & |g_{k+1}^T d_k| \leq -\sigma g_k^T d_k \\ \Longleftrightarrow\ & |g_k^T d_{k-1}| \leq -\sigma g_{k-1}^T d_{k-1} \\ \Longleftrightarrow\ & \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \leq \sigma \\ \Longleftrightarrow\ & \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \|g_k\|^2 \leq \sigma \|g_k\|^2 \\ \Longleftrightarrow\ & -\|g_k\|^2 + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \|g_k\|^2 \leq -\|g_k\|^2 + \sigma \|g_k\|^2 \\ \Longleftrightarrow\ & -(1 - \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}})\|g_k\|^2 \leq -(1-\sigma)\|g_k\|^2. \end{aligned} \quad (21)$$

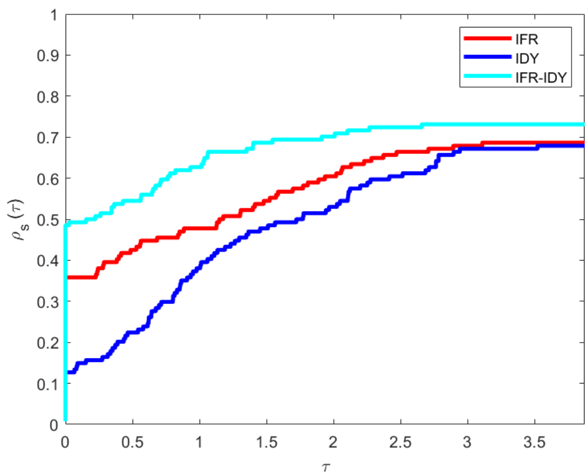

Figure 2. Performance profile curves based on CPU time.

Furthermore, based on the strong Wolfe search line procedure (5) the following inequality is also obtained,

$$\begin{aligned} &|g(x_k+\alpha_k d_k)^T d_k| \leq -\sigma g_k^T d_k \\ \iff &|g_{k+1}^T d_k| \leq -\sigma g_k^T d_k \\ \iff &|g_k^T d_{k-1}| \leq -\sigma g_{k-1}^T d_{k-1} \\ \iff &\sigma g_{k-1}^T d_{k-1} \leq g_k^T d_{k-1} \leq -\sigma g_{k-1}^T d_{k-1}. \end{aligned}$$

Based on Wolfe's strong search line inequality above, then

$$\begin{aligned} d_{k-1}^T(g_k - g_{k-1}) &= g_k^T d_{k-1} - d_{k-1}^T g_{k-1} \\ &\geq \sigma g_{k-1}^T d_{k-1} - d_{k-1}^T g_{k-1} \\ &= \sigma d_{k-1}^T g_{k-1} - d_{k-1}^T g_{k-1} \\ &= (\sigma-1) d_{k-1}^T g_{k-1} \\ &= (\sigma-1) g_{k-1}^T d_{k-1}. \end{aligned} \quad (22)$$

Since $0 < \sigma < \frac{\sqrt{2}}{2}$, and assuming that (18) holds for $k-1$, that is, $g_{k-1}^T d_{k-1} < 0$, then from (22) we get

$$d_{k-1}^T(g_k - g_{k-1}) > 0.$$

Then assuming that $g_{k-1}^T d_{k-1} < 0$ and with algebraic manipulations we obtain

$$\begin{aligned} g_{k-1}^T d_{k-1} < 0 &\iff \|g_k\|^2 g_{k-1}^T d_{k-1} < 0 \\ &\iff \frac{\|g_k\|^2 g_{k-1}^T d_{k-1}}{d_{k-1}^T(g_k - g_{k-1})} < 0 \\ &\iff |g_k^T d_{k-1}| \cdot \frac{\|g_k\|^2 g_{k-1}^T d_{k-1}}{d_{k-1}^T(g_k - g_{k-1})} < 0 \\ &\iff \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \cdot \frac{\|g_k\|^2 g_{k-1}^T d_{k-1}}{d_{k-1}^T(g_k - g_{k-1})} < 0. \end{aligned} \quad (23)$$

Then a further analysis of the form (20) is carried out using (21) and (23),

$$\begin{aligned} g_k^T d_k &= -\left(1 - \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}}\right)\|g_k\|^2 + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \cdot \frac{\|g_k\|^2 \cdot g_{k-1}^T d_{k-1}}{d_{k-1}^T(g_k - g_{k-1})} \\ &\leq -(1-\sigma)\|g_k\|^2 + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \cdot \frac{\|g_k\|^2 \cdot g_{k-1}^T d_{k-1}}{d_{k-1}^T(g_k - g_{k-1})} \\ &\leq -(1-\sigma)\|g_k\|^2. \end{aligned} \quad (24)$$

Thus, the form (18) is satisfied for $\forall k \geq 1$. □

Using theorems 1 and 2 below, we give a theorem that shows that the hybrid IFR-IDY conjugate gradient method satisfies the descent condition.

Theorem 3. *Suppose Assumptions 1 and 2 are satisfied and d_k is constructed by Algorithm 1. If the parameter σ is $0 < \sigma < \frac{\sqrt{2}}{2}$, then the following descent condition is satisfied*

$$g_k^T d_k < 0. \quad (25)$$

Proof. First, consider (3) again. For $k = 1$ and by multiplying both sides by g_k^T, we get

$$g_k^T d_k = g_k^T(-g_k) = -\|g_k\|^2 < 0.$$

So, for $k = 1$, the form (25) is satisfied. Then, for $k \geq 2$, by multiplying both sides (3) by g_k^T we get

$$\begin{aligned} g_k^T d_k &= g_k^T(-g_k + \beta_k^{IFR-IDY} d_{k-1}) \\ &= -\|g_k\|^2 + g_k^T(\beta_k^{IFR-IDY} d_{k-1}). \end{aligned} \quad (26)$$

Then, note that $\beta_k^{IFR-IDY} = \min(\beta_k^{IFR}, \beta_k^{IDY})$. This implies, $\beta_k^{IFR-IDY}$ has two possible values, namely $\beta_k^{IFR-IDY} = \beta_k^{IFR}$ or $\beta_k^{IFR-IDY} = \beta_k^{IDY}$. Thus, the proof is divided into two cases.

- Case 1: $\beta_k^{IFR-IDY} = \beta_k^{IFR}$
  By substituting the value of $\beta_k^{IFR-IDY} = \beta_k^{IFR}$ into (26), the following results are obtained,

  $$\begin{aligned} g_k^T d_k &= -\|g_k\|^2 + g_k^T(\beta_k^{IFR-IDY} d_{k-1}) \\ &= -\|g_k\|^2 + g_k^T(\beta_k^{IFR} \cdot d_{k-1}). \end{aligned} \quad (27)$$

  Based on Theorem 1 and the form of (27), we obtain

  $$g_k^T d_k = -\|g_k\|^2 + g_k^T(\beta_k^{IFR} \cdot d_{k-1}) < 0.$$

  Thus, for the case $\beta_k^{IFR-IDY} = \beta_k^{IFR}$, (25) is satisfied.

- Case 2: $\beta_k^{IFR-IDY} = \beta_k^{IDY}$ By substituting the value of $\beta_k^{IFR-IDY} = \beta_k^{IDY}$ into (26), the following results are obtained

  $$\begin{aligned} g_k^T d_k &= -\|g_k\|^2 + g_k^T(\beta_k^{IFR-IDY} d_{k-1}) \\ &= -\|g_k\|^2 + g_k^T(\beta_k^{IDY} \cdot d_{k-1}). \end{aligned} \quad (28)$$

  Based on Theorem 2 and the form (28), we obtain

  $$g_k^T d_k = -\|g_k\|^2 + g_k^T(\beta_k^{IDY} \cdot d_{k-1}) < 0.$$

  Thus, for the case $\beta_k^{IFR-IDY} = \beta_k^{IDY}$, (25) is satisfied.

Based on the results of the description for the two cases above, namely $\beta_k^{IFR-IDY} = \beta_k^{IFR}$ and $\beta_k^{IFR-IDY} = \beta_k^{IDY}$ it has been shown that the hybrid IFR-IDY gradient conjugate method satisfies the descent condition. □

Before proving the global convergence property of the hybrid IFR-IDY conjugate gradient method, the following Lemma is given first.

Lemma 1. *Suppose Assumptions 1 and 2 are satisfied. The sequence $\{x_k\}$ is generated based on the equation with d_k satisfying the descent condition, and α_k is generated using the strong Wolfe search line procedure then the following inequality is satisfied,*

$$\sum_{k=0}^{\infty} \frac{(g_k^T d_k)^2}{||g_k||^2} < +\infty. \tag{29}$$

Proof. Consider again the inequality of the two strong Wolfe search line procedures (5) as follows

$$\begin{aligned} & |g_{k+1}^T d_k| \leq -\sigma g_k^T d_k \\ \iff & \sigma g_k^T d_k \leq g_{k+1}^T d_k \leq -\sigma g_k^T d_k \\ \iff & \sigma g_k^T d_k \leq g_{k+1}^T d_k. \end{aligned} \tag{30}$$

By adding both sides (30) with $-g_k^T d_k$ we get

$$\begin{aligned} & \sigma g_k^T d_k \leq g_{k+1}^T d_k \\ \iff & \sigma g_k^T d_k - g_k^T d_k \leq g_{k+1}^T d_k - g_k^T d_k \\ \iff & (\sigma - 1) g_k^T d_k \leq (g_{k+1}^T - g_k^T) d_k. \end{aligned} \tag{31}$$

Based on Assumptions 2, (2), and the Cauchy-Scwarz inequality, the following results are obtained,

$$\begin{aligned} & ||g(x_{k+1}) - g(x_k)|| \leq L||x_{k+1} - x_k|| \\ \iff & ||g_{k+1} - g_k|| \leq L||x_{k+1} - x_k|| \\ \iff & ||g_{k+1} - g_k||||d_k|| \leq L||x_{k+1} - x_k||||d_k|| \\ \iff & ||g_{k+1} - g_k||||d_k|| \leq L||\alpha_k d_k||||d_k|| \\ \iff & ||g_{k+1} - g_k||||d_k|| \leq L\alpha_k||d_k||^2. \end{aligned} \tag{32}$$

Based on the Cauchy-Scwarz inequality and (32) note that,

$$\begin{aligned} (g_{k+1}^T - g_k^T) d_k &= (g_{k+1} - g_k)^T d_k \\ &\leq ||g_{k+1} - g_k||||d_k|| \\ &\leq L||\alpha_k||d_k||^2. \end{aligned} \tag{33}$$

Based on (31) and (33) the following results are obtained,

$$\begin{aligned} & (\sigma - 1) g_k^T d_k \leq L\alpha_k||d_k||^2 \\ \iff & \alpha_k \geq \frac{(\sigma - 1) g_k^T d_k}{L||d_k||^2}. \end{aligned} \tag{34}$$

Based on the strong Wolfe search line procedure (4) the following results are obtained,

$$\begin{aligned} & f(x_k + \alpha_k d_k) \leq f(x_k) + \delta\alpha_k g_k^T d_k \\ \iff & f(x_k) - f(x_k + \alpha_k d_k) \geq -\delta\alpha_k g_k^T d_k \\ & \geq -\delta\left(\frac{(\sigma - 1) g_k^T d_k}{L||d_k||^2}\right) g_k^T d_k \\ & \geq Z \frac{(g_k^T d_k)^2}{||d_k||^2}. \end{aligned} \tag{35}$$

with $Z = \frac{-\delta(\sigma-1)}{L} > 0$.

By substituting some values of k into (35), the following results are obtained.

For $k = 0$,

$$f(x_0) - f(x_0 + \alpha_0 d_0) \geq Z \frac{(g_0^T d_0)^2}{||d_0||^2}.$$

Based on (2), we obtain

$$f(x_0) - f(x_1) \geq Z \frac{(g_0^T d_0)^2}{||d_0||^2}.$$

For $k = 1$,

$$f(x_1) - f(x_2) \geq Z \frac{(g_1^T d_1)^2}{||d_1||^2}.$$

For $k = 2$,

$$f(x_2) - f(x_3) \geq Z \frac{(g_2^T d_2)^2}{||d_2||^2}.$$

So on until $k = n - 1$ is obtained,

$$f(x_{n-1}) - f(x_n) \geq Z \frac{(g_{n-1}^T d_{n-1})^2}{||d_{n-1}||^2}.$$

So the following results are obtained,

$$f(x_{n-1}) - f(x_n) \geq Z \sum_{i=0}^{\infty} \frac{(g_k^T d_k)^2}{||d_k||^2}.$$

Then, for $n \to \infty$

$$\begin{aligned} & f(x_{n-1}) - \lim_{n\to\infty} f(x_n) \geq Z \sum_{i=0}^{\infty} \frac{(g_k^T d_k)^2}{||d_k||^2} \\ \iff & \frac{f(x_{n-1}) - \lim_{n\to\infty} f(x_n)}{Z} \geq \sum_{i=0}^{\infty} \frac{(g_k^T d_k)^2}{||d_k||^2}. \end{aligned}$$

Since $f(x)$ is a bounded function, $\lim_{n\to\infty} f(x_n)$ has a finite value, and so do $f(x_0)$ and Z. Since it has a finite value, this means that $\sum_{i=0}^{\infty} \frac{(g_k^T d_k)^2}{||d_k||^2} < \infty$. □

The following is a theorem stating that the hybrid IFR-IDY conjugate gradient method satisfies the global convergence property.

Theorem 4. *Suppose Assumptions 1 and 2 hold and the sequence $\{x_k\}$ generated by Algorithm 1 with α_k is determined using the strong Wolfe search procedure (4) and (5), then*

$$\lim_{k\to\infty} \inf||g_k|| = 0. \tag{36}$$

Proof. The proof of the above theorem is done by the method of contradiction. Suppose (36) is not true, there exists a constant $\gamma > 0$ such that $||g_k||^2 \geq \gamma, \forall k$.

Consider again the search direction equation (3),

$$d_k = -g_k + \beta_k^{IFR-IDY} d_{k+1} \iff d_k + g_k = \beta_k^{IFR-IDY} d_{k+1}. \quad (37)$$

By squaring both sides of (37) we get the following result,

$$\begin{aligned}(d_k + g_k)^2 &= (\beta_k^{IFR-IDY} d_{k-1})^2 \\ \iff ||d_k||^2 + 2g_k^T d_k + ||g_k||^2 &= (\beta_k^{IFR-IDY})^2 ||d_{k-1}||^2 \\ \iff ||d_k||^2 &= (\beta_k^{IFR-IDY})^2 ||d_{k-1}||^2 - 2g_k^T d_k - ||g_k||^2.\end{aligned}$$

Next, by dividing both sides by $(g_k^T d_k)^2$ we get the following result,

$$\begin{aligned}\frac{||d_k||^2}{(g_k^T d_k)^2} &= \frac{(\beta_k^{IFR-IDY})^2 ||d_{k-1}||^2 - 2g_k^T d_k - ||g_k||^2}{(g_k^T d_k)^2} \\ &= \frac{(\beta_k^{IFR-IDY})^2 ||d_{k-1}||^2}{(g_k^T d_k)^2} - \frac{2g_k^T d_k}{(g_k^T d_k)^2} - \frac{||g_k||^2}{(g_k^T d_k)^2} \\ &= \frac{(\beta_k^{IFR-IDY})^2 ||d_{k-1}||^2}{(g_k^T d_k)^2} - \frac{2}{g_k^T d_k} - \frac{||g_k||^2}{(g_k^T d_k)^2} \\ &= \frac{(\beta_k^{IFR-IDY})^2 ||d_{k-1}||^2}{(g_k^T d_k)^2} - \left(\frac{1}{||g_k||} + \frac{||g_k||}{g_k^T d_k}\right)^2 + \frac{1}{||g_k||^2} \\ &\leq \frac{(\beta_k^{IFR-IDY})^2 ||d_{k-1}||^2}{(g_k^T d_k)^2} + \frac{1}{||g_k||^2}. \end{aligned} \quad (38)$$

Note that $\beta_k^{IFR-IDY} = \min(\beta_k^{IFR}, \beta_k^{IDY})$. This implies that $\beta_k^{IFR-IDY}$ has two possible values, namely $\beta_k^{IFR-IDY} = \beta_k^{IFR}$ or $\beta_k^{IFR-IDY} = \beta_k^{IDY}$. Thus, the proof is divided into two cases.

Case 1: $\beta_k^{IDY} < \beta_k^{IFR}$

That is, in this case $\beta_k^{IFR-IDY} = \min(\beta_k^{IFR}, \beta_k^{IDY}) = \beta_k^{IDY}$. Look again at (22) and (24). Since $-(1-\sigma)||g_k||^2 < 0$ it can be concluded that,

$$\begin{aligned}g_k^T d_k &\leq \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \cdot \frac{||g_k||^2 g_{k-1}^T d_{k-1}}{d_{k-1}^T (g_k - g_{k-1})} \\ &= \beta_k^{IDY} (g_{k-1}^T d_{k-1}).\end{aligned} \quad (39)$$

Next, by multiplying both sides of (39) by $\frac{1}{(g_{k-1}^T d_{k-1})}$, the following result is obtained,

$$\frac{g_k^T d_k}{(g_{k-1}^T d_{k-1})} \geq \beta_k^{IDY}. \quad (40)$$

Next, by substituting (40) into (38) we get the following results,

$$\begin{aligned}\frac{||d_k||^2}{(g_k^T d_k)^2} &\leq \frac{(\beta_k^{IFR-IDY})^2 ||d_{k-1}||^2}{(g_k^T d_k)^2} + \frac{1}{||g_k||^2} \\ &= \frac{(\beta_k^{IDY})^2 ||d_{k-1}||^2}{(g_k^T d_k)^2} + \frac{1}{||g_k||^2} \\ &\leq \frac{\left(\frac{g_k^T d_k}{g_{k-1}^T d_{k-1}}\right)^2 ||d_{k-1}||^2}{(g_k^T d_k)^2} + \frac{1}{||g_k||^2} \\ &= \frac{||d_{k-1}||^2}{(g_{k-1}^T d_{k-1})^2} + \frac{1}{||g_k||^2}.\end{aligned} \quad (41)$$

By substituting some values of k into (41) we get the following result: following.

For $k = 2$,

$$\begin{aligned}\frac{||d_2||^2}{(g_2^T d_2)^2} &\leq \frac{||d_1||^2}{(g_1^T d_1)^2} + \frac{1}{||g_2||^2} \\ &= \frac{||g_1||^2}{(||g_1||^2)^2} + \frac{1}{||g_2||^2} \\ &= \frac{1}{||g_1||^2} + \frac{1}{||g_2||^2} \\ &= \sum_{k=1}^{2} \frac{1}{||g_k||^2}.\end{aligned}$$

For $k = 3$,

$$\begin{aligned}\frac{||d_k||^2}{(g_k^T d_k)^2} &\leq \frac{||d_2||^2}{(g_2^T d_2)^2} + \frac{1}{||g_3||^2} \\ &\leq \frac{1}{||g_1||^2} + \frac{1}{||g_2||^2} + \frac{1}{||g_3||^2} \\ &= \sum_{k=1}^{3} \frac{1}{||g_k||^2}.\end{aligned}$$

So on until for $k = n$,

$$\frac{||d_n||^2}{(g_n^T d_n)^2} \leq \frac{||d_{n-1}||^2}{(g_{n-1}^T d_{n-1})^2} + \frac{1}{||g_n||^2} = \sum_{k=1}^{n} \frac{1}{||g_k||^2}. \quad (42)$$

Since $||g_k||^2 > \gamma \iff \frac{1}{||g_k||^2} \leq \frac{1}{\gamma}$, the following results are obtained,

$$\frac{1}{||g_k||^2} \leq \frac{1}{\gamma} \iff \sum_{k=1}^{n} \frac{1}{||g_k||^2} \leq \frac{n}{\gamma}. \quad (43)$$

Based on (42) and (43), the following results are obtained,

$$\begin{aligned}\frac{||d_n||^2}{(g_n^T d_n)^2} &\leq \frac{n}{\gamma} \\ \iff \frac{(g_n^T d_n)^2}{||d_n||^2} &\geq \frac{\gamma}{n}.\end{aligned} \quad (44)$$

Thus, by taking the sum of both sides of (44), we obtain the following result,

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{||d_k||^2} \geq \sum_{k=1}^{\infty} \frac{\gamma}{k} = \infty.$$

Note that $\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{||d_k||^2} \geq \infty$, which contradicts (1). Thus, for the case $\beta_k^{IFR-IDY} = \beta_k^{IDY}$ (36) holds.

Case 2: $\beta_k^{IFR} < \beta_k^{IDY}$

That is, in this case, $\beta_k^{IFR-IDY} = \min(\beta_k^{IFR}, \beta_k^{IDY}) = \beta_k^{IFR}$.

Look again at the formula β_k^{IFR} (6). Since $-g_{k-1}^T d_{k-1} > 0$, it is clear that $0 < \beta_k^{IFR} < \beta_k^{FR}$. So by substituting β_k^{IFR} into (38), the following results are obtained,

$$\frac{||d_k||^2}{(g_k^T d_k)^2} \leq \frac{(\beta_k^{IFR-IDY})^2 ||d_{k-1}||^2}{(g_k^T d_k)^2} + \frac{1}{||g_k||^2}$$
$$= \frac{(\beta_k^{IFR})^2 ||d_{k-1}||^2}{(g_k^T d_k)^2} + \frac{1}{||g_k||^2} \quad (45)$$
$$\leq \frac{(\beta_k^{FR})^2 ||d_{k-1}||^2}{(g_k^T d_k)^2} + \frac{1}{||g_k||^2}$$
$$= \frac{(\frac{||g_k||^2}{||g_{k-1}||^2})^2 ||d_{k-1}||^2}{(g_k^T d_k)^2} + \frac{1}{||g_k||^2}. \quad (46)$$

By substituting some values of k into (46) we get the following results,

For $k = 2$,

$$\frac{||d_2||^2}{(g_2^T d_2)^2} \leq \frac{\left(\frac{||g_2||^2}{||g_1||^2}\right)^2 ||d_1||^2}{(g_2^T d_2)^2} + \frac{1}{||g_2||^2}$$
$$= \frac{(\frac{||g_2||^2}{||g_1||^2})^2 ||g_1||^2}{(||g_2||^2)^2} + \frac{1}{||g_2||^2}$$
$$= \frac{1}{||g_1||^2} + \frac{1}{||g_2||^2} = \sum_{k=1}^{2} \frac{1}{||g_k||^2}.$$

For $k = 3$,

$$\frac{||d_3||^2}{(g_3^T d_3)^2} \leq \frac{(\frac{||g_3||^2}{||g_2||^2})^2 ||d_2||^2}{(g_3^T d_3)^2} + \frac{1}{||g_3||^2}$$
$$= \frac{(\frac{||g_3||^2}{||g_2||^2})^2 ||g_2||^2}{(||g_3||^2)^2} + \frac{1}{||g_3||^2}$$
$$= \frac{1}{||g_2||^2} + \frac{1}{||g_3||^2}$$
$$\leq \frac{1}{||g_1||^1} + \frac{1}{||g_2||^2} + \frac{1}{||g_3||^2} = \sum_{k=1}^{3} \frac{1}{||g_k||^2}.$$

So on until for $k = n$,

$$\frac{||d_n||^2}{(g_n^T d_n)^2} \leq \frac{(\frac{||g_n||^2}{||g_n - 1||^2})^2 ||d_{n-1}||^2}{(g_n^T d_n)^2} + \frac{1}{||g_n||^2}$$
$$\leq \frac{1}{||g_1||^2} + \frac{1}{||g_2||^2} + \ldots + \frac{1}{||g_n||^2} = \sum_{k=1}^{n} \frac{1}{||g_k||^2}. \quad (47)$$

Since $||g_k||^2 > \gamma \iff \frac{1}{||g_k||^2} \leq \frac{1}{\gamma}$, the following results are obtained,

$$\frac{1}{||g_k||^2} \leq \frac{1}{\gamma} \iff \sum_{k=1}^{n} \frac{1}{||g_k||^2} \leq \frac{n}{\gamma}. \quad (48)$$

Based on (47) and (48), the following results are obtained,

$$\frac{||d_n||^2}{(g_n^T d_n)^2} \leq \frac{n}{\gamma} \iff \frac{(g_n^T d_n)^2}{||d_n||^2} \geq \frac{\gamma}{n}. \quad (49)$$

Thus, by taking the sum of both sides of (49) we get the following result,

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{||d_k||^2} \geq \sum_{k=1}^{\infty} \frac{\gamma}{k} = \infty.$$

Note that $\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{||d_k||^2} \geq \infty$, this contradicts (1). Thus, for the case $\beta_k^{IFR-IDY} = \beta_k^{IFR}$ (36) holds.

Since for both possible values of β_k (36) always holds, it is proven that the hybrid conjugate gradient method IFR-IDY converges globally under strong Wolfe search. □

4. Numerical experiments

In this section, numerical experiments are conducted on the hybrid IFR-IDY conjugate gradient method using 134 test functions recommended by Andrei [1]. The list of test functions used is presented in Table 1 in columns one and two. When proposing a new algorithm, numerical experiments are essential to evaluate and compare its numerical performance with its predecessors. The key aspects considered in this evaluation are the number of iterations and computation time required to find the solution for the given unconstrained optimization problems. The benchmark methods for comparison with the hybrid IFR–IDY conjugate gradient method are its foundational methods, namely the IFR and IDY conjugate gradient methods.

The numerical experiments in this study were conducted using Matlab R2022a on a personal laptop running Microsoft Windows 11, equipped with an Intel Core i5-1235U processor and 16.0 GB of RAM. In these experiments, several parameters were set, including a stopping tolerance of $\epsilon = 10^{-6}$ strong Wolfe line search constants $\delta = 0.01$ and $\sigma = 0.1$, and a maximum of 2000 iterations. The output results, including the number of iterations and computation time required to solve the unconstrained optimization problems for the 134 test functions, are presented in the following Table 1 in column one and two. Based on Table 1, the number of functions successfully and unsuccessfully solved by the hybrid IFR-IDY conjugate gradient method, the IFR conjugate gradient method, and the IDY conjugate gradient method is shown as follows.

From Table 2, it is evident that the IFR-IDY hybrid conjugate gradient method successfully solved more test functions compared to the IFR and IDY conjugate gradient methods. However, the results presented in Table 2 alone are not sufficient to draw definitive conclusions regarding which method is superior. Therefore, the following subsection will introduce a performance profile curve as an analytical tool. When comparing the numerical performance of different methods, tabular formats can be difficult to analyze and interpret, especially when a large number of test functions are involved.

An alternative approach, such as computing the total or average number of iterations and computation time, may introduce bias and disadvantage methods that successfully solve a larger number of test functions (i.e., the most robust methods). Dolan and More [21] introduced performance profiles as a tool for evaluating and comparing the numerical performance of optimization software. The performance profile of a method represents the cumulative distribution function of a performance metric, which is defined as the ratio of the computation time of a given method to that of the best-performing method.

Suppose there are n_S methods and n_p test functions, and the primary aspect of interest is computation time. Let $t_{p,s}$ be the computation time required by method s to solve problem p. The performance ratio, which compares the computation time of method s for a test function with that of the best-performing method, is defined mathematically as:

$$r_{ps} = \frac{t_{p,s}}{\min\{t_{p,s} : s \in S\}}.$$

Let $r_M \geq r_{ps}$ for all cases, where $r_M = r_{ps}$ if and only if method s fails to solve test function p. Dolan and Moré then define ρ_s as the probability that the performance ratio $r_{p,s}$ of method $s \in S$ falls within a factor $\tau \in \mathbb{R}$ of the best possible performance ratio. $\rho_s(\tau)$ is given by:

$$\rho_s(\tau) = \frac{1}{n_p} \text{size}\{p \in P : r_{p,s} \leq \tau\}.$$

The function ρ_s represents the cumulative distribution function for the performance ratio. The performance profile $\rho_s : \mathbb{R} \to [0, 1]$ of a method is a piecewise constant, non-decreasing, and right-continuous function. The value of $\rho_s(1)$ indicates the probability that a method outperforms all others. In general, the method with the highest $\rho_s(\tau)$ value is considered the best. Figure 1 and Figure 2 illustrate the performance profile curves for computation time (CPU Time in second) and the number of iterations of the IFR-IDY, IFR, and IDY conjugate gradient methods, respectively.

5. Application in portfolio selection

In this section, the IFR-IDY hybrid gradient conjugate method is applied to the IDX30 stock portfolio selection problem. The purpose of this application is to determine the investment weights for each stock of interest in order to obtain minimum portfolio risk. Several studies on the application of the conjugate gradient method to stock portfolio selection can be found in Sabi'u *et al.* [22], Malik *et al.* [23], Deepho *et al.* [24], and Malik *et al.* [25].

The steps in this section are shown as follows.

1. Selecting stocks that are consistent constituents of the IDX30 stock index in the period February 2022-January 2024.
2. Retrieving closing price data of selected stocks.
3. Calculating the return and expexted return values using the following formulas, respectively.

$$r_{ij} = \frac{p_{ij} - p_{ij-1}}{p_{ij-1}}, \tag{50}$$

$$\bar{r}_i = \frac{\sum_{j=1}^{m} r_{ij}}{m}, \tag{51}$$

where p_{ij} is closing price of asset i at time j, m is number of observation periods, r_{ij} $(i = 1, \ldots, n, j = 1, \ldots, m)$ denotes the return of asset i at time j, and $\bar{r}_i$ is average return of the ith asset over m periods [26].

4. Selecting stocks with positive expexted returns and calculating the covariance between stocks using the following formula.

$$\sigma_{ik} = \frac{\sum_{j=1}^{m} (r_{ij} - \bar{r}_i)(r_{kj} - \bar{r}_k)}{m}, \tag{52}$$

where σ_{ik} is covariance between the ith asset and the kth asset, r_{kj} is rate of return of the kth asset in the jth period, and $\bar{r}_k$ is average return of the kth asset over m periods.

5. Forming a portfolio optimization model.
6. Optimizing the model that has been formed using the hybrid conjugate gradient method IFR-IDY.

The IDX30 stock index was first launched on April 23, 2012. This index measures the price performance of 30 selected constituents that meet several criteria: (1) having a large market capitalization, (2) having high liquidity, and (3) having strong company fundamentals.

Stock liquidity is assessed based on transaction value, transaction frequency, the number of days transactions occur in the regular market, and free-float market capitalization. Meanwhile, fundamentals are evaluated based on financial performance, compliance, and other factors.

Below is a list of stocks that have been constituents of the IDX30 index during the period from February 2022 to January 2024.

The list of stocks that constitute the IDX30 stock index is updated every semester. In this study, only stocks that consistently remained constituents of the IDX30 stock index for four consecutive semesters during the period of February 2022 - January 2024 are considered. The stocks that consistently remained constituents of the IDX30 stock index during the period of February 2022 - January 2024 are as follows: ADRO, ANTM, ASII, BBCA, BBNI, BBRI, BMRI, BRPT, BUKA, CPIN, EMTK, INCO, INDF, KLBF, MDKA, PGAS, PTBA, SMGR, TLKM, TOWR, UNTR, and UNVR. Therefore, only these 22 stocks will be the focus, and their daily closing price data will be collected from the Yahoo Finance website.

Next, the returns of the 22 selected stocks, as shown in Table 3, are calculated by substituting the stock closing price values into formula (50). The calculations are performed using Microsoft Excel.After obtaining the stock return values, the expected return and the return covariance for each stock are calculated using formulas (51) and (52), respectively. Stocks included in the portfolio are those with positive expected returns, as shown in Table 4.

Next, a stock portfolio model is constructed with the objective of minimizing risk. Let $y_1, y_2, y_3, ..., y_{18}$ sequentially represent the weights of the stocks ADRO, ANTM, ASII, BBCA, BBNI, BBRI, BMRI, BRPT, BUKA, CPIN, EMTK, INCO, INDF, KLBF, MDKA, PGAS, PTBA, SMGR, TLKM, TOWR, UNTR, and UNVR.Since the goal is to minimize risk, the objective function of the optimization for the 18 selected stocks is as follows.

$$\begin{cases} \text{Minimize}: & V = y^T Qy \\ \text{Subject to}: & \sum_{i=1}^{18} y_i = 1, \end{cases} \tag{53}$$

where V is the value of the portfolio risk to be minimized, y is vector of fund allocation weights for each stock in the portfolio, where y_i is the weight of the i-th stock, Q is covariance matrix of returns between stocks (size 18×18 in this case), y^T is transpose of vector y, and $\sum_{i=1}^{18} y_i = 1$ is the constraint that the total investment weight must be equal to 100% of the available capital [26].

By applying the IFR-IDY hybrid conjugate gradient method to solve problem (53), we obtained the following weights for the selected stocks:

$y_1 = -0.0282;\ y_2 = 0.0394;\ y_3 = 0.0686;\ y_4 = 0.0084;$

$y_5 = 0.0201;\ y_6 = 0.0395;\ y_7 = 0.0197;\ y_8 = 0.0574;$

$y_9 = 0.2895; y_{10} = 0.0142;\ y_{11} = -0.0317;\ y_{12} = 0.0629;$

$y_{13} = 0.0490;\ y_{14} = 0.0174;\ y_{15} = 0.1936;\ y_{16} = 0.0562;$

$y_{17} = 0.0428;\ y_{18} = 0.0811$

By substituting these weight values into formula

$$R = \sum_{i=1}^{n} \bar{r}_i y_i$$

and into the objective function (53), the expected return and risk of the portfolio are obtained as 0.00042 and 0.000050545, respectively.

Based on the obtained weights, if an investor has a capital of Rp1,000,000,000 and intends to invest in the stocks ADRO, ANTM, ASII, BRPT, BUKA, CPIN, EMTK, INCO, INDF, KLBF, MDKA, PGAS, PTBA, SMGR, TLKM, TOWR, UNTR, and UNVR, the allocation of funds for each stock to minimize risk is shown in Table 5.

The negative value of the proportion of ADRO and MDKA shares means that investors are doing short selling. Short selling is a transaction of buying and selling shares carried out by investors who do not yet own the shares with the speculation that there will be a decrease in the share price.

6. Conclusion

This study has demonstrated that the proposed hybrid IFR–IDY conjugate gradient method possesses strong theoretical and practical merits. Under appropriate assumptions, the method satisfies the descent condition and exhibits global convergence, ensuring its robustness for solving unconstrained optimization problems. These theoretical guarantees provide a solid foundation for its reliable application to large-scale numerical problems.

Extensive numerical experiments involving 134 benchmark test functions confirm the computational efficiency of the hybrid IFR–IDY conjugate gradient method. The results indicate that the proposed method consistently outperforms the classical IFR and IDY conjugate gradient methods in terms of computation time and number of iterations, highlighting its superior convergence behavior and numerical stability.

Furthermore, the applicability of the method to real-world problems has been illustrated through its implementation in an IDX30 stock portfolio optimization problem. The obtained optimal asset allocation demonstrates the method's capability in effectively minimizing portfolio risk while maintaining a reasonable expected return. This practical application emphasizes the potential of the hybrid IFR–IDY conjugate gradient method as a valuable tool in financial optimization.

Future research may extend this work by applying the proposed method to other complex optimization problems, such as constrained optimization, multi-objective portfolio models, or risk measures beyond variance. In addition, further investigation into adaptive parameter strategies and large-scale real financial datasets may enhance the method's performance and broaden its applicability.

Acknowledgment

This research is funded by Directorate of Research Funding and Ecosystem, Universitas Indonesia under Hibah PUTI 2025 (Grant No. PKS-356/UN2.RST/HKP.05.00/2025).

References

[1] N. Andrei, *Nonlinear Conjugate Gradient Methods for Unconstrained Optimization*, Springer International Publishing, Cham, Switzerland, 2020. https://doi.org/10.1007/978-3-030-42950-8.

[2] S. Wright & J. Nocedal, *Numerical Optimization*, Springer, New York, USA, 1999. https://doi.org/10.1007/978-0-387-40065-5.

[3] S. K. Mishra & B. Ram, *Introduction to Unconstrained Optimization with R*, Springer Nature, Singapore, 2019. https://doi.org/10.1007/978-981-13-7438-8.

[4] M. R. Hestenes & E. Stiefel, "Methods of conjugate gradients for solving linear systems", Journal of Research of the National Bureau of Standards **49** (1952) 409–436. https://doi.org/10.6028/JRES.049.044.

[5] R. Fletcher & C. M. Reeves, "Function minimization by conjugate gradients", The Computer Journal **7** (1964) 149. https://doi.org/10.1093/comjnl/7.2.149.

[6] E. Polak & G. Ribière, "Note sur la convergence de méthodes de directions conjuguées", Revue Française d'Informatique et de Recherche Opérationnelle, Série Rouge **3** (1969) 16. https://doi.org/10.1051/m2an/196903R100351.

[7] R. Fletcher, *Practical Methods of Optimization*, John Wiley & Sons, Chichester, United Kingdom, 2000. https://doi.org/10.1002/9781118723203.

[8] Y.-H. Dai & Y. Yuan, "A nonlinear conjugate gradient method with a strong global convergence property", SIAM Journal on Optimization **10** (1999) 177. https://doi.org/10.1137/S1052623497318992.

[9] Y. Liu & C. Storey, "Efficient generalized conjugate gradient algorithms, part 1: theory", Journal of Optimization Theory and Applications **69** (1991) 129. https://doi.org/10.1007/BF00940064.

[10] X. Jiang & J. Jian, "Improved Fletcher–Reeves and Dai–Yuan conjugate gradient methods with the strong Wolfe line search", Journal of Computational and Applied Mathematics **348** (2019) 525. https://doi.org/10.1016/j.cam.2018.09.034.

[11] M. Malik, M. Mamat, S. S. Abas, I. M. Sulaiman & F. Sukono, "A new coefficient of the conjugate gradient method with the sufficient descent condition and global convergence properties", Engineering Letters **28** (2020) 704. https://doi.org/10.3176/proc.2020.3.04.

[12] Maulana Malik, Mustafa Mamat, Siti Sabariah Abas, Ibrahim Mohammed Sulaiman & S. Sukono, "Performance analysis of new spectral and hybrid conjugate gradient methods for solving unconstrained optimization problems", IAENG International Journal of Computer Science **48** (2021) 66–79. https://doi.org/10.1142/S0218196721500031.

[13] M. Malik, A. B. Abubakar, I. M. Sulaiman, M. Mamat & S. S. Abas, "A new three-term conjugate gradient method for unconstrained optimization with applications in portfolio selection and robotic motion control", IAENG International Journal of Applied Mathematics **51** (2021). https://doi.org/10.1142/S0218196721500031.

[14] S. M. Ibrahim, A. M. Awwal, M. Malik, R. Khalid, A. M. Benjamin, M. K. M. Nawawi & E. N. Madi, "An efficient gradient-based algorithm with descent direction for unconstrained optimization with applications to image restoration and robotic motion control", PeerJ Computer Science **11** (2025) e2783. https://doi.org/10.7717/peerj-cs.2783.

[15] F. Novkaniza, M. Malik, I. M. Sulaiman & D. Aldila, "Modified spectral conjugate gradient iterative scheme for unconstrained optimization problems with application on COVID-19 model", Frontiers in Applied Mathematics and Statistics **8** (2022) 1014956. https://doi.org/10.3389/fams.2022.1014956.

[16] P. Kumam, A. B. Abubakar, M. Malik, A. H. Ibrahim, N. Pakkaranang & B. Panyanak, "A hybrid HS–LS conjugate gradient algorithm for unconstrained optimization problems with application in motion control and image recovery", Journal of Computational and Applied Mathematics **433** (2023) 115304. https://doi.org/10.1016/j.cam.2023.115304.

[17] A. M. Awwal & N. Pakkaranang, "New structured spectral gradient methods for nonlinear least squares optimization problems with application in robotic motion control", Journal of Computational and Applied Mathematics **469** (2025) 116671. https://doi.org/116671.

[18] M. Al-Baali, "Descent property and global convergence of the Fletcher–Reeves method with inexact line search", IMA Journal of Numerical Analysis **5** (1985) 121. https://doi.org/10.1093/imanum/5.1.121.

[19] Y.-H. Dai & Y. Yuan, "An efficient hybrid conjugate gradient method for unconstrained optimization", Annals of Operations Research **103** (2001) 33. https://doi.org/10.1023/A:1012912516239.

[20] X. Yang, Z. Luo & X. Dai, "A global convergence of LS–CD hybrid conjugate gradient method", Advances in Numerical Analysis **2013** (2013) 517452. https://doi.org/10.1155/2013/517452.

[21] E. D. Dolan & J. J. Moré, "Benchmarking optimization software with performance profiles", Mathematical Programming **91** (2002) 201. https://doi.org/10.1007/s101070100263.

[22] J. Sabi'u, I. M. Sulaiman, P. Kaelo, M. Malik & S. A. Kamaruddin, "An optimal choice Dai–Liao conjugate gradient algorithm for unconstrained optimization and portfolio selection", AIMS Mathematics **9** (2024) 642. https://doi.org/10.3934/math.2024032.

[23] Maulana Malik, Ibrahim Mohammed Sulaiman, Auwal Bala Abubakar, Gianinna Ardaneswari & Sukono, "A new family of hybrid three-term conjugate gradient method for unconstrained optimization with application to image restoration and portfolio selection", AIMS Mathematics **8** (2023) 1–28. https://doi.org/10.3934/math.2023001.

[24] J. Deepho, A. B. Abubakar, M. Malik & I. K. Argyros, "Solving unconstrained optimization problems via hybrid CD–DY conjugate gradient methods with applications", Journal of Computational and Applied Mathematics **405** (2022) 113823. https://doi.org/10.1016/j.cam.2021.113823.

[25] M. Malik, I. M. Sulaiman, M. Mamat & S. S. Abas, "A new class of nonlinear conjugate gradient method for unconstrained optimization models and its application in portfolio selection", Nonlinear Functional Analysis and Applications (2021) 811. https://doi.org/10.1007/978-3-030-73839-6-28.

[26] M. Bartholomew-Biggs, *Nonlinear Optimization with Financial Applications*, Springer, New York, USA, 2005. https://doi.org/10.1007/0-387-28240-6.