Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 8 (2026) 3107

Journal of the Nigerian Society of Physical Sciences

On graded J_{gr}-2-absorbing primary submodule

Shatha Alghueiri, Khaldoun Al-Zoubi*

Department of Mathematics and Statistics, Jordan University of Science and Technology, P.O.Box 3030, Irbid 22110, Jordan.

Abstract

This paper introduces the concept of *graded J_{gr}-2-absorbing primary submodules*, a new intermediate structure in graded module theory. Motivated by the need to extend and unify classical notions such as graded prime and graded primary submodules, the study develops a comprehensive framework describing their defining characteristics and relationships. Through a sequence of rigorous theorems and illustrative examples, we establish fundamental properties, equivalence conditions, and inclusion relations that clarify the behavior of these submodules within graded modules. The findings show that graded J_{gr}-2-absorbing primary submodules generalize several known structures while maintaining distinctive algebraic flexibility. Moreover, the results concerning graded homomorphisms and multiplication modules demonstrate the robustness of the concept and its potential applications in graded algebra. Overall, this work deepens the theoretical understanding of graded algebraic systems and provides a foundation for further research in module and ring theory.

DOI:10.46481/jnsps.2026.3107

Keywords: Graded J_{gr}-2-absorbing primary submodule, Graded J_{gr}-2-absorbing submodule, Graded 2-absorbing submodule

Article History :
Received: 25 August 2025
Received in revised form: 14 October 2025
Accepted for publication: 24 October 2025
Available online: 25 November 2025

Communicated by: Pankaj Thakur

1. Introduction and preliminaries

Graded algebraic structures, particularly graded modules and their submodules, have gained significant attention due to their ability to generalize classical algebraic concepts and provide insight into symmetry and modularity within algebraic systems. The study of graded prime submodules has been an important area of exploration, with extensive work done by researchers such as Atani and others [1–10]. A proper graded submodule N of a graded A-module D is said to be *a graded prime submodule* if whenever $a_g \in h(A)$ and $d_\lambda \in h(D)$ with $a_g d_\lambda \in N$, then either $d_\lambda \in N$ or $a_g \in (N :_R D) = \{a \in A : aD \subseteq N\}$. Building on this foundation, in Ref. [10] the notion of graded primary submodules was introduced, extending the concept of graded primes. A proper graded submodule K of a graded A-module D is said to be *a graded primary submodule* if whenever $a_g \in h(A)$ and $d_\lambda \in h(D)$ with $a_g d_\lambda \in K$, then either $d_\lambda \in K$ or $a_g \in Gr((K :_R D))$. The notion of a graded 2-absorbing submodule, which extends the idea of a graded prime submodule, was first introduced by Al-Zoubi and Abu-Dawwas [4] and further investigated in Refs. [3, 7, 11]. A proper graded submodule W of D is called a graded 2-absorbing submodule of D, if whenever $a_g, b_h \in h(A)$ and $d_\alpha \in h(D)$ with $a_g b_h d_\alpha \in W$, then either $a_g b_h \in (W :_A D)$ or $a_g d_\alpha \in W$ or $b_h d_\alpha \in W$. In this context, the concept of graded J_{gr}-2-absorbing primary submodules was developed to provide a refined perspective. These submodules satisfy a modified absorption property, offering greater flexibil-

*Corresponding author: Tel.: 00962-78-598-3953.
Email address: kfzoubi@just.edu.jo (Khaldoun Al-Zoubi)

ity in analyzing the product elements of graded modules.

Through new definitions, theorems, and illustrative examples, this paper investigates the properties of graded J_{gr}-2-absorbing primary submodules, expanding upon prior research and offering deeper insight into their structural roles within graded modules. By establishing equivalence conditions, inclusion relationships, and construction methods, the paper enriches the theoretical landscape of graded algebra. The inclusion of concrete examples and proofs ensures a practical understanding of these submodules, thus advancing the field of graded algebra and its applications in ring theory, module theory, and beyond. This work assumes that G is a group with identity element e, A is a commutative G-graded ring with identity, and D is a unitary graded A-module. The overarching goal of this paper is to refine and extend the theoretical framework of graded algebraic structures, thereby advancing the study of algebraic systems with graded components.

We begin by revisiting some foundational concepts about graded rings and modules, which will be used later. For a more detailed discussion on these topics, please refer to Refs. [12, 13].

Consider a group G with an identity element e. A G-graded ring is a ring A that can be expressed as a direct sum of abelian groups:

$$A = \bigoplus_{g\in G} A_g,$$

such that $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$. The non-zero elements of A_h are called homogeneous elements of degree h, and the set of all homogeneous elements is denoted as $h(A)$, where

$$h(A) = \bigcup_{h\in G} A_h.$$

Any element $t \in A$ can be uniquely expressed as a sum $\sum_{g\in G} t_g$, where t_g is the homogeneous component of t in A_g.

An ideal W in A is called a graded ideal if it can also be decomposed as:

$$W = \bigoplus_{g\in G}(W \cap A_g) := \bigoplus_{g\in G} W_g.$$

For further details, see Ref. [13].

Now, let A be a G-graded ring and D an A-module. D is referred to as a G-graded A-module if it can be written as a direct sum of additive subgroups:

$$D = \bigoplus_{g\in G} D_g,$$

and it satisfies $A_g D_h \subseteq D_{gh}$ for all $g, h \in G$. Any element of D that belongs to $\bigcup_{g\in G} D_g = h(D)$ is called a homogeneous element.

If A is a G-graded ring and D is a graded A-module, a submodule K of D is considered a graded submodule if it can be expressed as:

$$K = \bigoplus_{g\in G}(K \cap D_g) := \bigoplus_{g\in G} K_g.$$

Here, K_g is referred to as the g-component of K.

Recently, various generalizations and applications of algebraic and graph-theoretical concepts have been studied in different frameworks [9].

2. Results

Definition 2.1. Let $W = \oplus_{g\in G} W_g$ be a proper graded submodule of D and $g \in G$.

(i) We say that W_g is a g-J_{gr}-2-absorbing primary (or simply, g-J_{gr}-2A-primary) submodule of the A_e-module D_g if $W_g \neq D_g$; and whenever $a_e b_e d_g \in W_g$ where $a_e, b_e \in A_e$ and $d_g \in D_g$, then either $a_e d_g \in Gr_{D_g}(W_g) + J_{gr}(D_g)$ or $b_e d_g \in Gr_{D_g}(W_g) + J_{gr}(D_g)$ or $a_e b_e D_g \subseteq W_g + J_{gr}(D_g)$.

(ii) We say that W is a graded J_{gr}-2-absorbing primary (or simply gr-J_{gr}-2A-primary) submodule if $W \neq D$; and whenever $a_g b_h d_\alpha \in W$ where $a_g, b_h \in h(A)$ and $d_\alpha \in h(D)$, then either $a_g d_\alpha \in Gr_D(W) + J_{gr}(D)$ or $b_h d_\alpha \in Gr_D(W) + J_{gr}(D)$ or $a_g b_h D \subseteq W + J_{gr}(D)$.

Definition 2.2. A proper graded ideal I of A is called a graded J_{gr}-2-absorbing primary (or simply gr-J_{gr}-2A-primary) ideal of A if I behaves like a gr-J_{gr}-2A-primary submodule when viewed as a part of the graded A-module A.

A proper graded submodule W of D is called a graded 2-absorbing (or simply, gr-2A) submodule of D, if whenever $a_g, b_h \in h(A)$ and $d_\alpha \in h(D)$ with $a_g b_h d_\alpha \in W$, then either $a_g b_h \in (W :_A D)$ or $a_g d_\alpha \in W$ or $b_h d_\alpha \in W$ [4].

It's clear that every gr-2A submodule is also a gr-J_{gr}-2A-primary submodule. However, the example below demonstrates that the reverse is not always true.

Example 2.3. Let's start with the group $G = \mathbb{Z}_2$ (the integers modulo 2) and the ring $A = \mathbb{Z}$ (the integers). The ring A is given a grading by G, where $A_0 = \mathbb{Z}$ and $A_1 = 0$. Next, consider the ring $D = \mathbb{Z}_{16}$ (integers modulo 16). We treat D as a graded A-module. Under this grading, $D_0 = \mathbb{Z}_{16}$ and $D_1 = 0$. Now, look at the graded submodule $W = \langle \bar{8} \rangle$. This submodule W is not a gr-2A submodule of D. Why not? Because even though the product $2 \cdot 2 \cdot \bar{2}$ is in W, neither $2 \cdot \bar{2}$ is in W, nor is $2 \cdot 2$ in $(\langle \bar{8} \rangle :_{\mathbb{Z}} \mathbb{Z}_{16}) = 8\mathbb{Z}$. However, it's easy to verify that W satisfies the conditions to be a gr-J_{gr}-2A-primary submodule of D.

A proper graded submodule W of D is called a gr-prime submodule of D, if whenever $a_g \in h(A)$ and $d_h \in h(D)$ with $a_g d_h \in W$, then either $a_g \in (W :_A D)$ or $d_h \in W$ [10].

It's clear that every gr-prime submodule is also a gr-J_{gr}-2A-primary submodule. However, the next example demonstrates that the reverse is not always true.

Example 2.4. Let $G = \mathbb{Z}_2$ (the group with two elements) and $A = \mathbb{Z}$ (the integers). Here, A is structured as a G-graded ring where the component $A_0 = \mathbb{Z}$ and the component $A_1 = 0$. Now, consider $D = \mathbb{Z}_8$ (integers modulo 8). This D is a graded module over A, with $D_0 = \mathbb{Z}_8$ and $D_1 = 0$. Let us focus on a graded submodule $W = \langle \bar{4} \rangle$, which is generated by the element $\bar{4}$. This submodule W is not a gr-prime submodule of D. The reason is that $2 \cdot \bar{2} \in \langle \bar{4} \rangle$, but neither $\bar{2} \in \langle \bar{4} \rangle$ nor $2 \in (\langle \bar{4} \rangle :_{\mathbb{Z}} \mathbb{Z}_8) = 4\mathbb{Z}$.

However, straightforward calculations demonstrate that W is a gr-J_{gr}-2A-primary submodule of D.

A proper graded submodule W of D is called a gr-primary submodule of D, if whenever $a_g \in h(A)$ and $d_h \in h(D)$ with $a_g d_h \in W$, then either $a_g^k \in (W :_A D)$ for some $k \in \mathbb{Z}^+$ or $d_h \in W$ [10].

It's clear that every gr-primary submodule is also a gr-J_{gr}-2A-primary submodule. However, the following example demonstrates that the reverse is not necessarily true.

Example 2.5. Let $G = \mathbb{Z}_2$ (the integers modulo 2), and let $A = \mathbb{Z}$ (the integers). In this case, A is a ring graded by G, where the component of A corresponding to $0 \in G$ is $A_0 = \mathbb{Z}$, and the component corresponding to $1 \in G$ is $A_1 = 0$. Next, let $D = \mathbb{Z}_6$ (the integers modulo 6). Here, D is a graded module over A, where the component of D corresponding to $0 \in G$ is $D_0 = \mathbb{Z}_6$, and the component corresponding to $1 \in G$ is $D_1 = 0$. Now, consider the graded submodule $W = \langle \bar{0} \rangle$. The submodule W is not a gr-primary submodule of D because the product $3 \cdot \bar{2} \in \langle \bar{0} \rangle$, but neither $\bar{2} \in \langle \bar{0} \rangle$ nor is $3^k \in (\langle \bar{0} \rangle : \mathbb{Z}_6) = 0$ for any positive integer k. However, through straightforward calculations, it can be shown that W is a gr-J_{gr}-2A-primary submodule of D.

The following theorems provide different ways to describe gr-J_{gr}-2A-primary submodules.

Theorem 2.6. *Let W be a proper graded submodule of D. Then the following statements are equivalent:*

(i) *W is a gr-J_{gr}-2A-primary submodule of D.*

(ii) *If $a_g, b_h \in h(A)$ with $a_g b_h \notin (W + J_{gr}(D) :_A D)$, then $(W :_D a_g b_h) \subseteq (Gr_D(W) + J_{gr}(D) :_D a_g) \cup (Gr_D(W) + J_{gr}(D) :_D b_h)$.*

Proof. $(i) \Rightarrow (ii)$ Let $a_g, b_h \in h(A)$ such that $a_g b_h \notin (W + J_{gr}(D) :_A D)$ and $d_\alpha \in (W :_D a_g b_h) \cap h(D)$. Hence, $a_g b_h d_\alpha \in W$ and then $a_g d_\alpha \in Gr_D(W) + J_{gr}(D)$ or $b_h d_\alpha \in Gr_D(W) + J_{gr}(D)$. So, $d_\alpha \in (Gr_D(W) + J_{gr}(D) :_D a_g)$ or $d_\alpha \in (Gr_D(W) + J_{gr}(D) :_D b_h)$ which yields $d_\alpha \in (Gr_D(W) + J_{gr}(D) :_D a_g) \cup (Gr_D(W) + J_{gr}(D) :_D b_h)$. Thus, $(W :_D a_g b_h) \subseteq (Gr_D(W) + J_{gr}(D) :_D a_g) \cup (Gr_D(W) + J_{gr}(D) :_D b_h)$.

$(ii) \Rightarrow (i)$ Let $a_g, b_h \in h(A)$ and $d_\alpha \in h(D)$ with $a_g b_h d_\alpha \in W$ and $a_g b_h \notin (W + J_{gr}(D) :_A D)$. Hence, $d_\alpha \in (W :_D a_g b_h)$ and then $d_\alpha \in (Gr_D(W) + J_{gr}(D) :_D a_g) \cup (Gr_D(W) + J_{gr}(D) :_D b_h)$. Thus, $a_g d_\alpha \in Gr_D(W) + J_{gr}(D)$ or $b_h d_\alpha \in Gr_D(W) + J_{gr}(D)$. Therefore, W is a gr-J_{gr}-2A-primary submodule of D.

Theorem 2.7. *Let W and $L = \oplus_{g \in G} L_g$ be two graded submodules of D. If W is a gr-J_{gr}-2A-primary submodule of D, then for each $a_g, b_h \in h(A)$ and $\alpha \in G$ with $a_g b_h L_\alpha \subseteq W$ and $a_g b_h \notin (W + J_{gr}(D) :_A D)$, either $a_g L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$ or $b_h L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$.*

Proof. Let W be a gr-J_{gr}-2A-primary submodule of D. Let $a_g, b_h \in h(A)$ and $\alpha \in G$ with $a_g b_h L_\alpha \subseteq W$ and $a_g b_h \notin (W + J_{gr}(D) :_A D)$. Assume that neither $a_g L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$ nor $b_h L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$, which follows that there exist $l_{1_\alpha}, l_{2_\alpha} \in L_\alpha$ with $a_g l_{1_\alpha} \notin Gr_D(W) + J_{gr}(D)$ and $b_h l_{2_\alpha} \notin Gr_D(W) + J_{gr}(D)$. Now, we have $a_g b_h l_{1_\alpha} \in W$ which yields that $b_h l_{1_\alpha} \in Gr_D(W) + J_{gr}(D)$ as $a_g l_{1_\alpha} \notin Gr_D(W) + J_{gr}(D)$ and $a_g b_h \notin (W + J_{gr}(D) :_A D)$. Also, since $a_g b_h \notin (W + J_{gr}(D) :_A D)$, $b_h l_{2_\alpha} \notin Gr_D(W) + J_{gr}(D)$ and $a_g b_h l_{2_\alpha} \in W$, $a_g l_{2_\alpha} \in Gr_D(W) + J_{gr}(D)$. Let $l_{1_\alpha} + l_{2_\alpha} \in L_\alpha$. Now, $a_g b_h (l_{1_\alpha} + l_{2_\alpha}) \in W$ so we get either $a_g (l_{1_\alpha} + l_{2_\alpha}) \in Gr_D(W) + J_{gr}(D)$ or $b_h (l_{1_\alpha} + l_{2_\alpha}) \in Gr_D(W) + J_{gr}(D)$ as W is a gr-J_{gr}-2A-primary submodule of D and $a_g b_h \notin (W + J_{gr}(D) :_A D)$. If $a_g (l_{1_\alpha} + l_{2_\alpha}) = a_g l_{1_\alpha} + a_g l_{2_\alpha} \in Gr_D(W) + J_{gr}(D)$, then $a_g l_{1_\alpha} \in Gr_D(W) + J_{gr}(D)$ since $a_g l_{2_\alpha} \in Gr_D(W) + J_{gr}(D)$, a contradiction. Similarly, if $b_h (l_{1_\alpha} + l_{2_\alpha}) = b_h l_{1_\alpha} + b_h l_{2_\alpha} \in Gr_D(W) + J_{gr}(D)$, then $b_h l_{2_\alpha} \in Gr_D(W) + J_{gr}(D)$ since $b_h l_{1_\alpha} \in Gr_D(W) + J_{gr}(D)$, a contradiction. Thus, either $a_g L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$ or $b_h L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$.

Theorem 2.8. *Let W be a proper graded submodule of D and let $I = \oplus_{g \in G} I_g$ and $J = \oplus_{g \in G} J_g$ be two graded ideals of A and $L = \oplus_{g \in G} L_g$ a graded submodule of D. Then the following are equivalent:*

(i) *W is a gr-J_{gr}-2A-primary submodule of D.*

(ii) *If $g, h, \alpha \in G$ with $I_g J_h L_\alpha \subseteq W$, then either $I_g J_h \subseteq (W + J_{gr}(D) :_A D)$ or $I_g L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$ or $J_h L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$.*

Proof. $(i) \Rightarrow (ii)$ Suppose that W is a gr-J_{gr}-2A-primary submodule of D. Let $g, h, \alpha \in G$ such that $I_g J_h L_\alpha \subseteq W$ and $I_g J_h \nsubseteq (W + J_{gr}(D) :_A D)$. We need to prove that either $I_g L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$ or $J_h L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$. So, Assume that neither $I_g L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$ nor $J_h L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$. Then there exist $i_{1_g} \in I_g$ and $j_{1_h} \in J_h$ with $i_{1_g} L_\alpha \nsubseteq Gr_D(W) + J_{gr}(D)$ and $j_{1_h} L_\alpha \nsubseteq Gr_D(W) + J_{gr}(D)$. But $i_{1_g} j_{1_h} L_\alpha \subseteq W$, so by Theorem 2.7, we get $i_{1_g} j_{1_h} \in (W + J_{gr}(D) :_A D)$ as $i_{1_g} L_\alpha \nsubseteq Gr_D(W) + J_{gr}(D)$, $j_{1_h} L_\alpha \nsubseteq Gr_D(W) + J_{gr}(D)$ and W is a gr-J_{gr}-2A-primary submodule of D. Since $I_g J_h \nsubseteq (W + J_{gr}(D) :_A D)$, there exist $i_{2_g} \in I_g$ and $j_{2_h} \in J_h$ such that $i_{2_g} j_{2_h} \notin (W + J_{gr}(D) :_A D)$. But $i_{2_g} j_{2_h} L_\alpha \subseteq W$, so either $i_{2_g} L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$ or $j_{2_h} L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$. Then we have three cases.

Case 1: Assume that $i_{2_g} L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$ and $j_{2_h} L_\alpha \nsubseteq Gr_D(W) + J_{gr}(D)$. Since $i_{1_g} j_{2_h} L_\alpha \subseteq W$, $i_{1_g} j_{2_h} \in (W + J_{gr}(D) :_A D)$ by Theorem 2.7. Now, $i_{1_g} L_\alpha \nsubseteq Gr_D(W) + J_{gr}(D)$ and $i_{2_g} L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$ follow that $(i_{1_g} + i_{2_g}) L_\alpha \nsubseteq Gr_D(W) + J_{gr}(D)$ where $(i_{1_g} + i_{2_g}) \in I_g$. Then $(i_{1_g} + i_{2_g}) j_{2_h} L_\alpha \subseteq W$ implies that $(i_{1_g} + i_{2_g}) j_{2_h} = (i_{1_g} j_{2_h} + i_{2_g} j_{2_h}) \in (W + J_{gr}(D) :_A D)$. But $i_{1_g} j_{2_h} \in (W + J_{gr}(D) :_A D)$, so $i_{2_g} j_{2_h} \in (W + J_{gr}(D) :_A D)$, a contradiction.

Case 2 : Assume that $i_{2_g} L_\alpha \nsubseteq Gr_D(W) + J_{gr}(D)$ and $j_{2_h} L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$. Similar to case 1 we get a contradiction.

Case 3: Assume that $i_{2_g} L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$ and $j_{2_h} L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$. Since $i_{1_g} L_\alpha \nsubseteq Gr_D(W) + J_{gr}(D)$ and $i_{2_g} L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$, $(i_{1_g} + i_{2_g}) L_\alpha \nsubseteq Gr_D(W) + J_{gr}(D)$ where $(i_{1_g} + i_{2_g}) \in I_g$. Now, $(i_{1_g} + i_{2_g}) j_{1_h} L_\alpha \subseteq W$ follows that $(i_{1_g} + i_{2_g}) j_{1_h} = (i_{1_g} j_{1_h} + i_{2_g} j_{1_h}) \in (W + J_{gr}(D) :_A D)$ and then $i_{2_g} j_{1_h} \in (W + J_{gr}(D) :_A D)$ as $i_{1_g} j_{1_h} \in (W + J_{gr}(D) :_A D)$. Similarly, since $j_{1_h} L_\alpha \nsubseteq Gr_D(W) + J_{gr}(D)$ and $j_{2_h} L_\alpha \subseteq Gr_D(W) + J_{gr}(D)$, $(j_{1_h} + j_{2_h}) L_\alpha \nsubseteq Gr_D(W) + J_{gr}(D)$ where $(j_{1_h} + j_{2_h}) \in J_h$. Now, $i_{1_g} (j_{1_h} + j_{2_h}) L_\alpha \in W$ yields that $i_{1_g} (j_{1_h} + j_{2_h}) = (i_{1_g} j_{1_h} + i_{1_g} j_{2_h}) \in (W + J_{gr}(D) :_A D)$ and then $i_{1_g} j_{2_h} \in (W + J_{gr}(D) :_A D)$ as $i_{1_g} j_{1_h} \in$

$(W + J_{gr}(D) :_A D)$. Hence, since $(i_{1_g} + i_{2_g})(j_{1_h} + j_{2_h})L_\alpha \in W$, $(i_{1_g} + i_{2_g})(j_{1_h} + j_{2_h}) = i_{1_g}j_{1_h} + i_{1_g}j_{2_h} + i_{2_g}j_{1_h} + i_{2_g}j_{2_h} \in (W + J_{gr}(D) :_A D)$. but $i_{1_g}j_{1_h},\ i_{1_g}j_{2_h},\ i_{2_g}j_{1_h} \in (W + J_{gr}(D) :_A D)$, so $i_{2_g}j_{2_h} \in (W + J_{gr}(D) :_A D)$, a contradiction. Therefore, either $I_gL_\alpha \subseteq Gr_D(W) + J_{gr}(D)$ or $J_hL_\alpha \subseteq Gr_D(W) + J_{gr}(D)$.

$(ii) \Rightarrow (i)$ Assume that (ii) holds. Let $a, b \in h(A)$ and $d \in h(D)$ with $abd \subseteq W$. Let I and J be ideals of A generated by a and b, respectively. That is $I = aA$ and $J = bA$. So, $I = \oplus_{g\in G}aA_g$ and $J = \oplus_{g\in G}bA_g$. Also, let L be a graded submodule of D generated by d so $L = Ad$. Moreover, for each $g \in G$, $I_g = aA_g$, $J_g = bA_g$ and $L_g = A_gd$. Now, by our assumption $I_eJ_eL_e \subseteq W$. So, we obtain either $I_eL_e \subseteq Gr(W) + J_{gr}(D)$ or $J_eL_e \subseteq Gr(W) + J_{gr}(D)$ or $I_eJ_e \subseteq (W + J_{gr}(D) :_A D)$. Thus, either $ad \in Gr(W) + J_{gr}(D)$ or $bd \in Gr(W) + J_{gr}(D)$ or $ab \in (W + J_{gr}(D) :_A D)$. Therefore, W is a gr-J_{gr}-2A-primary submodule of D.

Theorem 2.9. *Let W be a proper graded submodule of D. If $Gr_D(W)$ is a gr-prime submodule of D, then W is a gr-J_{gr}-2A-primary submodule of D.*

Proof. Let $a_g, b_h \in h(A)$ and $d_\alpha \in h(D)$ such that $a_gb_hd_\alpha \in W$ and $b_hd_\alpha \notin Gr_D(W) + J_{gr}(D)$. Now, since $W \subseteq Gr_D(W)$, $a_gb_hd_\alpha \in Gr_D(W)$. So, as $Gr_D(W)$ is a graded prime submodule of D, we get $a_gD \subseteq Gr_D(W) \subseteq Gr_D(W) + J_{gr}(D)$ and then $a_gd_\alpha \in Gr_D(W) + J_{gr}(D)$. Thus, W is a gr-J_{gr}-2A-primary submodule of D.

Theorem 2.10. *Let W and K be two proper graded submodules of D with $W \subsetneq K$, $J_{gr}(D) \subseteq K$ and $J_{gr}(D) \cap K = J_{gr}(K)$. If W is a gr-J_{gr}-2A-primary submodule of D, then W is a gr-J_{gr}-2A-primary submodule of K.*

Proof. Let $a_g, b_h \in h(A)$ and $k_\alpha \in K \cap h(D)$ with $a_gb_hk_\alpha \in W$. Then we get either $a_gk_\alpha \in Gr_D(W)+J_{gr}(D)$ or $b_hk_\alpha \in Gr_D(W)+J_{gr}(D)$ or $a_gb_hD \subseteq W+J_{gr}(D)$. Hence, either $a_gk_\alpha \in (Gr_D(W)+J_{gr}(D)) \cap K$ or $b_hk_\alpha \in (Gr_D(W) + J_{gr}(D)) \cap K$ or $a_gb_hK \subseteq a_gb_hD \subseteq (W + J_{gr}(D)) \cap K$. Now, by the modular law we have $(Gr_D(W) + J_{gr}(D)) \cap K = (Gr_D(W) \cap K) + (J_{gr}(D) \cap K) = Gr_K(W) + J_{gr}(K)$ and $(W + J_{gr}(D)) \cap K = (W \cap K) + (J_{gr}(D) \cap K) = W + J_{gr}(K)$. So, either $a_gk_\alpha \in Gr_K(W) + J_{gr}(K)$ or $b_hk_\alpha \in Gr_K(W) + J_{gr}(K)$ or $a_gb_hK \subseteq W + J_{gr}(K)$. Thus, W is a gr-J_{gr}-2A-primary submodule of K.

In simpler terms, a graded multiplication module D is a type of module where, for every graded submodule L of D, there is a corresponding graded ideal I in A such that $L = ID$. If L is a graded submodule of a graded multiplication module D, then it can be written as $L = (L :_A D)D$, where $(L :_A D)$ is the graded ideal associated with L. Now, let's consider two graded submodules L and K of the graded multiplication module D, where $L = I_1D$ and $K = I_2D$, for some graded ideals I_1 and I_2 of A. The product of L and K, denoted LK, is defined as $LK = I_1I_2D$ where I_1I_2 is the product of the two graded ideals, [8].

Theorem 2.11. *Let W be a proper graded submodule of D. Let $g \in G$ with D_g is a multiplication A_e-module. If W is a gr-J_{gr}-2A-primary submodule of D, then for any graded submodules $K_1 = \oplus_{h\in G}K_{1_h}$, $K_2 = \oplus_{h\in G}K_{2_h}$, $K_3 = \oplus_{h\in G}K_{3_h}$ of D with $K_{1_g}K_{2_g}K_{3_g} \subseteq W$, then either $K_{1_g}K_{3_g} \subseteq Gr_D(W) + J_{gr}(D)$ or $K_{2_g}K_{3_g} \subseteq Gr_D(W) + J_{gr}(D)$ or $K_{1_g}K_{2_g}D \subseteq W + J_{gr}(D)$.*

Proof. Assume that W is a gr-J_{gr}-2A-primary submodule of D. Let $g \in G$ such that $K_{1_g}K_{2_g}K_{3_g} \subseteq W$ and $K_{1_g}K_{2_g}D \nsubseteq W+J_{gr}(D)$. Now, since D_g is a multiplication A_e-module, $K_{1_g} = (K_{1_g} :_{A_e} D_g)D_g$ and $K_{2_g} = (K_{1_g} :_{A_e} D_g)D_g$. Thus, $(K_{1_g} :_{A_e} D_g)(K_{2_g} :_{A_e} D_g)K_{3_g} \subseteq W$ and $(K_{1_g} :_{A_e} D_g)(K_{2_g} :_{A_e} D_g)D \nsubseteq W+J_{gr}(D)$. Now, by Theorem 2.8 we get either $(K_{1_g} :_{A_e} D_g)K_{3_g} \subseteq Gr_D(W) + J_{gr}(D)$ or $(K_{2_g} :_{A_e} D_g)K_{3_g} \subseteq Gr_D(W) + J_{gr}(D)$. Thus, either $K_{1_g}K_{3_g} \subseteq Gr_D(W) + J_{gr}(D)$ or $K_{2_g}K_{3_g} \subseteq Gr_D(W) + J_{gr}(D)$.

Let A be a G-graded ring and let D and D' be two graded modules over A, each graded by G. Suppose there exists a map $\digamma : D \to D'$ that preserves the module structure (i.e., a homomorphism). We define $\digamma$ to be a graded homomorphism if it maps each component D_g of D, corresponding to a particular element $g \in G$, into the component D'_g of D'. In other words, for each $g \in G$, $\digamma(D_g) \subseteq D'_g$, ensuring that $\digamma$ respects the grading [13].

Recall that a proper graded submodule W of a graded A-module D is said to be a gr-small submodule of D (for short $W <<_g D$) if for every proper graded submodule K of D, we have $W + K \neq D$ [5].

Theorem 2.12. *Let A be a G-graded ring and D, D' be two graded A-modules. Let $\digamma : D \to D'$ be a graded epimorphism with $ker(\digamma) <<_g D$. If W' is a gr-J_{gr}-2A-primary submodule of D', then $\digamma^{-1}(W')$ is a gr-J_{gr}-2A-primary submodule of D.*

Proof. It is easy to see that $\digamma^{-1}(W')$ is a proper graded submodule of D. Now, let $a_g, b_h \in h(A)$ and $d_\alpha \in h(D)$ such that $a_gb_hd_\alpha \in \digamma^{-1}(W')$. This follows that $a_gb_h\digamma(d_\alpha) \in W'$. Hence, either $a_g\digamma(d_\alpha) \in Gr_{D'}(W') + J_{gr}(D')$ or $b_h\digamma(d_\alpha) \in Gr_{D'}(W')+J_{gr}(D')$ or $a_gb_hD' \subseteq W'+J_{gr}(D')$ as W' is a gr-J_{gr}-2A-primary submodule of D'. Now, by [6, Theorem 2.12] and [2, Lemma 5.3], we get $\digamma(J_{gr}(D)) = J_{gr}(D')$ and $Gr_{D'}(W') = Gr_D(\digamma^{-1}(W'))$. So, either $a_gd_\alpha \in Gr_D(\digamma^{-1}(W')) + J_{gr}(D)$ or $b_hd_\alpha \in Gr_D(\digamma^{-1}(W')) + J_{gr}(D)$ or $a_gb_hD \subseteq \digamma^{-1}(W') + J_{gr}(D)$. Therefore, $\digamma^{-1}(W')$ is a graded J_{gr}-primary-2-absorbing submodule of D.

Theorem 2.13. *Let A be a G-graded ring, D and D' be two graded A-modules. Let $\digamma : D \to D'$ be a graded epimorphism. If W is a gr-J_{gr}-2A-primary submodule of D with $kea(\digamma) \subseteq W$, then $\digamma(W)$ is a gr-J_{gr}-2A-primary submodule of D'.*

Proof. By [2, Lemma 4.8], $\digamma(W)$ is a proper graded submodule of D'. Now, let $a_gb_hd'_\alpha \in \digamma(W)$ where $a_g, b_h \in h(A)$ and $d'_\alpha \in h(D')$. As $\digamma$ is a graded epimorphism there exists $d_\alpha \in h(D)$ with $\digamma(d_\alpha) = d'_\alpha$. So, $a_gb_h\digamma(d_\alpha) \in \digamma(W)$ and then $a_gb_h\digamma(d_\alpha) = \digamma(n_\lambda)$ for some $n_\lambda \in W \cap h(D)$. Hence, $a_gb_hd_\alpha - n_\lambda \in kea(\digamma) \subseteq W$ and this follows that $a_gb_hd_\alpha \in W$. Thus, we get either $a_gd_\alpha \in Gr_D(W) + J_{gr}(D)$

or $b_h d_\alpha \in Gr_D(W) + J_{gr}(D)$ or $a_g b_h D \subseteq W + J_{gr}(D)$. But by [6, Theorem 2.12] and [2, Lemma 5.3], we conclude that $\digamma(Gr_D(W)) + \digamma(J_{gr}(D)) \subseteq Gr_{D'}(\digamma(W)) + J_{gr}(D')$. Hence, either $a_g \digamma(d_\alpha) = a_g d'_\alpha \in Gr_D(\digamma(W)) + J_{gr}(D')$ or $b_h \digamma(d_\alpha) = b_h d'_\alpha \in Gr_D(\digamma(W)) + J_{gr}(D')$ or $a_g b_h D' \subseteq \digamma(W) + J_{gr}(D')$. Therefore, $\digamma(W)$ is a gr-J_{gr}-2A-primary submodule of D'.

Theorem 2.14. *Let D be a graded multiplication A-module and $W = \oplus_{h\in G} W_h$ a proper graded submodule of D. Let $g \in G$ with D_g is a multiplication A_e -module and $J_{gr}(A_e)D_g = J_{gr}(D_g)$. Then the following statements are equivalent.*

(i) W_g is a g–J_{gr}-2A-primary submodule of D_g.

(ii) $(W_g :_{A_e} D_g)$ is a g–J_{gr}-2A-primary ideal of A_e.

Proof. $(i) \Rightarrow (ii)$ Let $a_e, b_e \in A_e$ and $J = \oplus_{h\in G} J_h$ be a graded ideal of A such that $a_e b_e J_e \subseteq (W_g :_{A_e} D_g)$ and $a_e b_e \notin ((W_g :_{A_e} D_g) + J_{gr}(A_e) :_{A_e} A_e) = (W_g :_{A_e} D_g) + J_{gr}(A_e)$. This yields that $a_e b_e D_g \nsubseteq (W_g :_{A_e} D_g)D_g + J_{gr}(A_e)D_g = W_g + J_{gr}(D_g)$ and then $a_e b_e \notin (W_g + J_{gr}(D_g) :_{A_e} D_g)$. Now, $a_e b_e J_e \subseteq (W_g :_{A_e} D_g)$ follows that $a_e b_e J_e D_g \subseteq W_g$ and then by Theorem 2.7 we get either $a_e J_e D_g \subseteq Gr_{D_g}(W_g) + J_{gr}(D_g)$ or $b_e J_e D_g \subseteq Gr_{D_g}(W_g) + J_{gr}(D_g)$ as W_g is a g-J_{gr}-2A-primary submodule of D_g. Hence, by [10, Theorem 9], we have either $a_e J_e D_g \subseteq Gr((W_g :_{A_e} D_g))D_g + J_{gr}(A_e)D_g$ or $b_e J_e D_g \subseteq Gr((W_g :_{A_e} D_g))D_g + J_{gr}(A_e)D_g$. Thus, either $a_e J_e \subseteq Gr((W_g :_{A_e} D_g)) + J_{gr}(A_e)$ or $b_e J_e \subseteq Gr((W_g :_{A_e} D_g)) + J_{gr}(A_e)$. Therefore, $(W_g :_{A_e} D_g)$ is a g-J_{gr}-2A-primary ideal of A_e.

$(ii) \Rightarrow (i)$ Let $a_e, b_e \in A_e$ and $K = \oplus_{h\in G} K_h$ be a graded submodule of D with $a_e b_e K_g \subseteq W_g$ and $a_e b_e \notin (W_g + J_{gr}(D_g) :_{A_e} D_g)$. This follows that $a_e b_e D_g \nsubseteq W_g + J_{gr}(D_g) = (W_g :_{A_e} D_g)D_g + J_{gr}(A_e)D_g$ and then $a_e b_e \notin (W_g :_{A_e} D_g) + J_{gr}(A_e) = ((W_g :_{A_e} D_g) + J_{gr}(A_e) :_{A_e} A_e)$. Now, since D_g is a multiplication A_e-module, there exists an ideal I_e of A_e such that $K_g = I_e D_g$. Hence, $a_e b_e I_e D_g = a_e b_e K_g \subseteq W_g$ which yields that $a_e b_e I_e \subseteq (W_g :_{A_e} D_g)$. Thus, either $a_e I_e \subseteq Gr((W_g :_{A_e} D_g)) + J_{gr}(A_g)$ or $b_e I_e \subseteq Gr((W_g :_{A_e} D_g)) + J_{gr}(A_e)$. Then either $a_e I_e D_g \subseteq Gr((W_g :_{A_e} D_g))D_g + J_{gr}(A_e)D_g$ or $b_e I_e D_g \subseteq Gr((W_g :_{A_e} D_g))D_g + J_{gr}(A_e)D_g$ and then $a_e K_g \subseteq Gr_{D_g}(W_g) + J_{gr}(D_g)$ or $b_e K_g \subseteq Gr_{D_g}(W_g) + J_{gr}(D_g)$ by [10, Theorem 9]. Therefore, W_g is a g-J_{gr}-2A-primary submodule of D_g.

3. Conclusion

The theorems presented in this study provide a comprehensive framework for understanding graded J_{gr}-2-absorbing primary submodules. Key results demonstrate their equivalence to specific conditions involving graded ideals, submodules, and algebraic operations, highlighting their structural uniqueness compared to other submodules like gr-prime and gr-primary submodules. Additionally, theorems establish the behavior of these submodules under graded homomorphisms and in graded multiplication modules, further solidifying their role in graded algebra. These findings underscore the flexibility and depth of graded J_{gr}-2-absorbing primary submodules, contributing significantly to the theoretical advancement of graded module theory. Although our study is mainly theoretical, the concept of graded Jgr-2-absorbing primary submodules may serve as a useful tool in further investigations. For example, these submodules could play a role in the structural study of graded rings and modules, in particular when analyzing decomposition theorems or extending results about graded prime and graded primary submodules. They may also provide insights into homological aspects of graded modules and inspire applications in related areas of commutative algebra.

Data availability

No data was used for the research described in the article

Acknowledgment

The authors would like to thank the reviewers and editors for their valuable comments and constructive suggestions, which have greatly improved the quality of this paper.

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