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Existence and uniqueness results for double jump fractional uncertain differential equations

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Abstract

Fractional uncertain differential equations have been used to model random processes in economics and other fields that exhibit jumps, dependency, and nonlinearities, and which possess uncertainties due to limited data and inadequate models. In this paper, a double V-jump fractional uncertain differential equation (DV-FUDE) is presented as a θ^{th}-order Riemann-Liouville or Caputo fractional uncertain differential equation with the addition of two V-jump independent processes on different filtrations. The equation models systems possessing two sources of uncertain shocks attributed to internal and external factors, respectively. Exact solutions in the case of time-dependent coefficients are given in terms of the Mittag-Leffler function. Sample continuity, existence, and uniqueness for the general Riemann-Liouville and Caputo DV-FUDE are established using the Banach Fixed Point Theorem, under global Lipschitz and linear growth conditions on the coefficients. Some extensions and possible areas of application are highlighted for future research.

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1. Introduction

Beginning with the seminal work of Einstein in 1905 [1], stochastic differential equations (SDEs) driven by Brownian motion (BM) have been widely applied not only in the natural and social sciences but also in engineering. In finance, a fundamental assumption of the highly celebrated Black-Scholes (BS) model for option pricing is that the dynamics that describe the underlying process of the asset price are SDEs with Brownian motion [2]. Similar models have been employed to define interest rates, exchange rates, volatility, and other financial objects, see [3] and the references therein. SDE-based models can also be modified to model processes that undergo discrete random jumps. For a comprehensive review of jump-diffusion models [4].

Empirical investigations have shown that underlying asset prices exhibit dependencies and nonlinearities, as well as jumps [5, 6]. Thus these prices cannot be satisfactorily characterized by SDEs or jump-diffusion models. Instead, stochastic fractional differential equations (FDE) with jumps may be more suitable for capturing this complicated behavior. Roughly

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speaking, a FDE with jumps can be characterized as a stochastic differential equation with a fractional Brownian motion and a jump term. Fractional Brownian motion with jumps is suitable because of its self-similarity and dependency properties. The applications of SDEs and FDEs with and without jumps have extended beyond the field of finance; see Refs. [7–11] for applications of FDEs in electrochemistry, rheology, viscoelasticity, and electromagnetism.

In the nonfractional setting, uncertain differential systems with multiple V-type jumps have been studied from different perspectives, including optimal control problems, existence and uniqueness analysis for nonlinear switched systems, and hybrid stochastic differential game models with equilibrium strategies [12–14].

Uncertainty in financial modeling arises from limited data, unknown system structures, and model mis-specification. To formalize such uncertainty beyond probability theory, Liu [15] introduced uncertainty theory, defining uncertainty spaces and uncertain processes [15–18]. Uncertain differential equations (UDEs) were subsequently proposed for uncertain dynamic systems [17] and have been widely applied [16, 19].

To capture abrupt shocks, UDEs were extended via V-jump uncertain processes [20], with applications to extreme events such as epidemics and natural disasters [21]. Existence and uniqueness results for fractional uncertain differential equations with a single V-jump were established in Refs. [20–22]. This work extends these results to fractional uncertain differential equations driven by double V-jump processes using a θth-order Riemann-Liouville formulation.

The extension from single- to double-jump FUDEs is motivated by the need to model uncertain dynamics driven by dual shock sources, such as interest rates or foreign exchange rates exposed to financial and climate or catastrophic shocks with memory effects. For the proposed DV-FUDE, we establish fundamental theoretical results, including sample continuity and analytical existence and uniqueness of solutions.

The remaining sections of this paper are structured as follow. Section 2 gives definitions required for the mathematical analysis of DV-FUDE. The DV-FUDE model and the main results in this paper are presented in Section 3, in particular in Theorem 8, existence and uniqueness results for Riemann-Liouville type DV-FUDE are established using the Banach fixed point theorem. In Section 4, we conclude and propose future research directions.

2. Basic definitions and assumptions

We begin with definitions related to uncertainty. All definitions are taken from Ref. [15], and are based on earlier work by Liu ([17, 23]). In the following, Γ is a nonempty set and $\mathcal{L}$ is a σ-algebra over Γ . The elements $\Lambda \in \mathcal{L}$ are referred to as *events*.

Definition 2.1. A function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is an *uncertain measure* if it satisfies the following axioms:

- *Normality*: $\mathcal{M}(\Gamma) = 1$;
- *Duality*: $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$, for all $\Lambda \in \mathcal{L}$;
- *Subadditivity*: $\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$ for each countable sequence of events, $\Lambda_1, \Lambda_2, \cdots$.

Definition 2.2. An *uncertainty space* is a triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ where $\mathcal{L}$ is a σ-algebra defined on the set Γ, and $\mathcal{M}$ is an uncertainty measure defined on $\mathcal{L}$.

Definition 2.3. Given uncertain spaces $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$, with $k = 1, 2, 3, \cdots$, the *product uncertainty measure* $\mathfrak{M} : \prod_{k=1}^{\infty} \mathcal{L}_k \to [0, 1]$ is an uncertainty measure that satisfies the *product axiom*:

$$\mathfrak{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\left\{\Lambda_k\right\}, \tag{1}$$

where $\{\Lambda_k\}_{k=1\ldots}$ are arbitrary events in $\{\mathcal{L}\}_{k=1\ldots}$.

Definition 2.4. An *uncertain variable* is a measurable function $\xi : (\Gamma, \mathcal{L}, \mathcal{M}) \to \mathbb{R}$, from an uncertainty space to the set of real numbers. That is, for any Borel set $B \subseteq \mathbb{R}$, the set

$$\{\xi \in B\} = \{\gamma \,|\, \xi(\gamma) \in B\}$$

is an event.

Definition 2.5. An *uncertain process* is a set of uncertain variables $\{X_t\}_{t\geq 0}$ defined on the same uncertainty space. For brevity we denote $\{X_t\}_{t\geq 0}$ as X_t.

Definition 2.6. An *uncertainty distribution* $\Phi(x)$ of an uncertain variable ξ is defined by:

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}, \tag{2}$$

for any $x \in \mathbb{R}$, a real number.

Definition 2.7. An uncertain process C_t is said to be a *Liu canonical process* if it satisfies the following:

- at time $t_0 = 0$, $C_0 = 0$ and almost all its sample paths are Lipschitz continuous,
- C_t has stationary and independent increments.
- every increment $C_{s+t} - C_s$ is a normal uncertain variable with an uncertainty distribution given as:

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(E - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \; x \in \mathbb{R},$$

where E is the expected value and σ^2 is the variance.

Definition 2.8. Let C_t be a Liu canonical process and X_t be an uncertain process. Consider a closed interval $[a, b]$ with partitions such that $a = t_1 < t_2 < \cdots < t_{n+1} = b$, the mesh is given as

$$\Delta = \max_{1\leq i\leq k} |t_{i+1} - t_i| \,.$$

The *integral* of the uncertain process X_t is defined as:

$$\int_a^b X_t dt = \lim_{\Delta\mapsto 0}\sum_{i=1}^{n} X_{t_i}\cdot(t_{i+1}-t_i), \tag{3}$$

and the *Liu integral* is defined as:

$$\int_a^b X_t dC_t = \lim_{\Delta\mapsto 0}\sum_{i=1}^{n} X_{t_i}\cdot(C_{t_{i+1}}-C_{t_i}), \tag{4}$$

provided these limits exists almost surely.

Definition 2.9. An *uncertain differential equation* (UDE) is a SDE in which the Brownian motion term has been replaced by a Liu canonical process C_t. For example, if $f,g:\mathbb{R}^+\to\mathbb{R}$ are integrable functions, then

$$dZ_t = f(t,Z_t)dt + g(t,Z_t)dC_t \tag{5}$$

is a UDE. A solution Z_t of (5) is an uncertain process that satisfies

$$Z_t = Z_0 + \int_0^s f(s,Z_s)ds + \int_0^s g(s,Z_s)dC_s, \tag{6}$$

almost surely for all $t\in\mathbb{R}^+$, where Z_0 is an uncertain variable.

The existence and uniqueness of solutions for UDEs without jumps were proved by Chen and Liu [24] in which an analytical solution was obtained.

Before introducing UDEs of fractional order, we first recall several definitions related to fractional calculus.

Definition 2.10 (Fractional integral operator [25])**.** Let f be an integrable function. The fractional integral of f of order θ is defined as:

$$\mathcal{I}^{\theta}_{t+}f(\tau) = \frac{1}{\Gamma(\theta)}\int_0^{\tau}(\tau-\xi)^{\theta-1}f(\xi)d\xi,\ \tau>0,\theta>0 \tag{7}$$

whenever the right-hand side is defined pointwise on the interval $[t,\infty)$ where $\Gamma(\cdot)$ is the gamma function.

Lemma 2.1 (**Semigroup property**)**.** For any real numbers θ_1,θ_2 for which $\mathcal{I}^{\theta_2}_{t+}f(\tau)$ and $\mathcal{I}^{\theta_1+\theta_2}_{t+}f(\tau)$ are defined we have

$$\mathcal{I}^{\theta_1}_{t+}\mathcal{I}^{\theta_2}_{t+}f(\tau) = \mathcal{I}^{\theta_1+\theta_2}_{t+}f(\tau). \tag{8}$$

Lemma 2.1 follows immediately from (7) upon exchanging the order of integration.

Definition 2.11 (Riemann-Liouville fractional derivative)**.** [26] Let f be differentiable on an interval $[t,\tau]$. The θth - order Riemann-Liouville (RL) fractional derivative of f is given as

$$\begin{aligned}{}^{RL}D^{\theta}_{t+}f(\tau) &= \frac{d^n}{d\tau^n}\mathcal{I}^{n-\theta}_{t+}f(\tau)\\ &= \frac{1}{\Gamma(n-\theta)}\frac{d^n}{d\tau^n}\int_t^{\tau}(\tau-\xi)^{n-\theta-1}f(\xi)d\xi,\end{aligned} \tag{9}$$

where $n-1<\theta\le n$, $n\in\mathbb{N}$.

It follows immediately from Lemma 2.1 that:

$$\begin{aligned}{}^{RL}D^{\theta}_{t+}\mathcal{I}^{\theta}_{t+}f &= \frac{d^n}{d\tau^n}\mathcal{I}^{n-\theta}_{t+}\mathcal{I}^{\theta}_{t+}f(\tau)\\ &= \frac{d^n}{d\tau^n}\mathcal{I}^{n}_{t+}f(\tau)\\ &= f(\tau).\end{aligned} \tag{10}$$

Hence the Riemann-Liouville fractional derivative is a left inverse of the corresponding integral operator.

Definition 2.12 (Caputo fractional derivative [26])**.** Let $f:[t,\tau]\to\mathbb{R}$ be a n times differentiable function. The θth- Caputo fractional derivative of f is given as:

$${}^{c}D^{\theta}_{t+}f(\tau) = \frac{1}{\Gamma(n-\theta)}\int_t^{\tau}(\tau-\xi)^{n-\theta-1}f^{(n)}(\xi)d\xi, \tag{11}$$

where $n-1<\theta\le n$, $n\in\mathbb{N}$, and $f^{(n)}(\xi)$ is the nth derivative of f.

Lemma 2.2 ([7], Property 3 in Ref. [27])**.** Let f be a n-times differentiable function for $\tau\in[t,T]$. Then, the two operators ${}^{RL}D^{\theta}_{t+}f(\tau)$ and ${}^{c}D^{\theta}_{t+}f(\tau)$ are related as:

$${}^{RL}D^{\theta}_{t+}f(\tau) = {}^{c}D^{\theta}_{t+}f(\tau) + \sum_{j=0}^{n-1}\frac{(\tau-t)^{j-\theta}}{\Gamma(j-\theta+1)}f^{(j)}(t), \tag{12}$$

where $n-1<\theta\le n$, $n\in\mathbb{N}$ and $\tau>0$.

Henceforth in this paper, we set $t=0$ in both the Riemann-Liouville and the Caputo fractional order differential operators, and restrict $\theta\in(0,1)\subset\mathbb{R}$. We will abbreviate $\mathcal{I}^{\theta}_{0+}$, ${}^{RL}D^{\theta}_{0+}$, ${}^{c}D^{\theta}_{0+}$ as $\mathcal{I}^{\theta}$, ${}^{RL}D^{\theta}$, and ${}^{c}D^{\theta}$, respectively.

The definitions of fractional integral and fractional derivatives may be readily extended to uncertain processes, using Definition 2.8.

The fractional order counterparts of the UDE (5) for Riemann-Liouville and Caputo differential operators are:

$$\begin{aligned}{}^{RL}D^{\theta}X_{\tau} &= f(\tau,X_{\tau}) + g(\tau,X_{\tau})\frac{dC_{\tau}}{d\tau},\\ {}^{c}D^{\theta}X_{\tau} &= f(\tau,X_{\tau}) + g(\tau,X_{\tau})\frac{dC_{\tau}}{d\tau}.\end{aligned} \tag{13}$$

Existence and uniqueness results for these UDE have been investigated in Ref. [28].

Next we give definitions related to uncertain processes with jumps.

Definition 2.13 (V-jump process ([21, 29]))**.** Let V_t be an uncertain process for $t\ge 0$. Then, V_t is said to be a V-jump process endowed with parameters ϑ_1 and ϑ_2 for $0<\vartheta_1<\vartheta_2<1$ if the following holds:

- $V_0=0$,
- V_{τ} exhibits stationary and independent increments,
- Given $\tau,t>0$, the increment $\xi_{t,\tau}\equiv V_{t+\tau}-V_t$ possesses an uncertainty distribution given by:

$$\Phi(x) = \begin{cases} 0 & \text{for} \quad x < 0, \\ \dfrac{2\vartheta_1}{\tau} x & \text{for} \quad 0 \le x < \dfrac{\tau}{2}, \\ \vartheta_2 + \dfrac{2(1-\vartheta_2)}{\tau}\left(x - \dfrac{\tau}{2}\right) & \text{for} \quad \dfrac{\tau}{2} \le x < \tau, \\ 1 & \text{for} \quad x \ge \tau. \end{cases} \quad (14)$$

Ref. [29] used the notation $\xi_{t,\tau} \sim \mathcal{Z}(\vartheta_1, \vartheta_2, \tau)$ to denote uncertain variables with the distribution (14).

Definition 2.14. A fractional uncertain differential equation (FUDE) with single V-jump process using Riemann-Liouville fractional differential operator is defined as

$$^{RL}\mathcal{D}^\theta X_\tau = f(\tau, X_\tau) + g(\tau, X_\tau)\frac{dC_\tau}{d\tau} + h(\tau, X_\tau)\frac{dV_\tau}{d\tau}, \quad (15)$$

where C_τ is a Liu canonical process, f, g, h are functions and V is an uncertain jump process.

The definition of the double V-jumps fractional uncertain differential equation (DV-FUDE) follows from the single jump FUDE by adding a second jump process defined on different filtration with its own coefficient function. The two distinct jump processes are intended to model shocks due to internal and external uncertainty factors. In order to set the stage for the definition of DV-FUDEs, we first state the assumptions required for the definition.

Assumptions 1: Let $\left(\Gamma, \mathcal{L}, \{\mathcal{L}\}_{t\ge 0}, \mathcal{M}\right)$ be a filtered uncertainty space with Liu canonical process C_t. Let $\mathcal{F}_t^{int}, \mathcal{F}_t^{ext} \subset \{\mathcal{L}\}_{t\ge 0}$ be internal and external filtrations respectively, and let $V_1(t, X_t) = \left(V_1(t, X_t)\middle|\mathcal{F}_t^{int}\right)$, $V_2(t, X_t) = \left(V_2(t, X_t)\middle|\mathcal{F}_t^{ext}\right)$ be independent V-jump processes with parameters $\{\vartheta_1^{(1)}, \vartheta_2^{(1)}\}$ and $\{\vartheta_1^{(2)}, \vartheta_2^{(2)}\}$, respectively. (For simplicity we write $V_j(t, X_t) := V_{jt}$, $j = 1, 2$).

Definition 2.15. Given $\left(\Gamma, \mathcal{L}, \{\mathcal{L}\}_{t\ge 0}, \mathcal{M}\right)$, C_τ, and $V_{j\tau} j = 1, 2$ as in Assumptions 1. Let $f, g, h_1, h_2 : \mathbb{R}_0^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable functions. Then

$$\begin{aligned} ^{RL}\mathcal{D}^\theta X_\tau =& f(\tau, X_\tau) + g(\tau, X_\tau)\frac{dC_\tau}{d\tau} + h_1(\tau, X_\tau)\frac{dV_{1\tau}}{d\tau} \\ &+ h_2(\tau, X_\tau)\frac{dV_{2\tau}}{d\tau} \end{aligned} \quad (16)$$

is a Riemann - Liouville type fractional uncertain differential equation with double V-jump process (DV-FUDE). Any solution X_τ of equation (16) satisfies:

$$\begin{aligned} X_\tau =& \tau^{\theta-1}x_0 + \mathcal{I}^\theta\Big(f(\tau, X_\tau) + g(\tau, X_\tau)\frac{dC_\tau}{d\tau} \\ &+ h_1(v, X_\tau)\frac{dV_{1\tau}}{d\tau} + h_2(\tau, X_\tau)\frac{dV_{2\tau}}{d\tau}\Big) \end{aligned} \quad (17)$$

almost surely, where X_τ satisfies the initial condition $x_0 = \lim_{\tau \mapsto 0+} \tau^{1-\theta} X_\tau$. Explicitly, we have:

$$\begin{aligned} X_\tau =& \tau^{\theta-1}x_0 + \frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1} f(\xi, X_\xi)d\xi \\ &+ \frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1} g(\xi, X_\xi)dC_\xi \\ &+ \frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1} h_1(\xi, X_\xi)dV_{1\xi} \\ &+ \frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1} h_2(\xi, X_\xi)dV_{2\tau}, \end{aligned} \quad (18)$$

almost surely.

The Caputo version of Definition 2.15 may be given as follows.

Definition 2.16. Suppose the canonical Liu process C_τ and two uncertain V-jump processes V_1 and V_2 be given. Let the functions $f, g, h_1, h_2 : \mathbb{R}_0^+ \times \mathbb{R} \longrightarrow \mathbb{R}$. Then

$$\begin{aligned} ^{c}\mathcal{D}^\theta X_\tau =& f(\tau, X_\tau) + g(\tau, X_\tau)\tfrac{dC_\tau}{d\tau} + h_1(\tau, X_\tau)\tfrac{dV_{1\tau}}{d\tau} \\ &+ h_2(\tau, X_\tau)\tfrac{dV_{2\tau}}{d\tau} \end{aligned} \quad (19)$$

is a Caputo-type DV-FUDE. Any solution X_τ of equation (19) satisfies:

$$\begin{aligned} X_\tau =& x_0 + \mathcal{I}^\theta\Big(f(\tau, X_\tau) + g(\tau, X_\tau)\frac{dC_\tau}{d\tau} + h_1(\tau, X_\tau)\frac{dV_1\tau}{d\tau} \\ &+ h_2(\tau, X_\tau)\frac{dV_2\tau}{d\tau}\Big), \\ =& x_0 + \frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1} f\left(\xi, X_\xi\right)d\xi \\ &+ \frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1} g\left(\xi, X_\xi\right)dC_\xi \\ &+ \frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1} h_1\left(\xi, X_\xi\right)dV_{1\xi} \\ &+ \frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1} h_2\left(\xi, X_\xi\right)dV_{2\tau}, \end{aligned} \quad (20)$$

almost surely.

Definition 2.17. [24] The Mittag-Leffler function is defined as:

$$E_{p,q}(x) := \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(pj+q)}, \ p > 0, \ q > 0. \quad (21)$$

Remark 1. Let X_τ^{RL} and X_τ^c be solutions of equations (18) and (20), respectively that are continuous at 0. We then have:

$$\begin{aligned} ^{c}\mathcal{D}^\theta X_\tau^c &= {}^{RL}\mathcal{D}^\theta X_\tau^c + {}^{c}\mathcal{D}^\theta X_\tau^c - {}^{RL}\mathcal{D}^\theta X_\tau^c \\ &= {}^{RL}\mathcal{D}^\theta X_\tau^c + {}^{RL}\mathcal{D}^\theta X_\tau^{RL} - {}^{RL}\mathcal{D}^\theta X_\tau^c \\ &= {}^{RL}\mathcal{D}^\theta X_\tau^c + {}^{RL}\mathcal{D}^\theta (X_\tau^{RL} - X_\tau^c) \\ &= {}^{RL}\mathcal{D}^\theta X_\tau^c - {}^{RL}\mathcal{D}^\theta (x_0) \\ &= {}^{RL}\mathcal{D}^\theta X_\tau^c - \frac{1}{\Gamma(1-\theta)}\frac{d}{d\tau}\int_0^\tau (\tau-\xi)^{-\theta} x_0 d\xi \\ &= {}^{RL}\mathcal{D}^\theta X_\tau^c - \frac{x_0 \tau^{-\theta}}{\Gamma(1-\theta)}, \end{aligned} \quad (22)$$

which agrees with equation (12) with $f = X^c_\tau$.

The transition from the first to the second line follows by adding and subtracting the same Riemann–Liouville derivative, using linearity, and noting that $X^{RL}_\tau - X^c_\tau = x_0$ for solutions continuous at the origin, which yields the Caputo–Riemann–Liouville correction term. A similar relation holds for X^{RL}_τ.

3. Results

In this section, the DV-FUDE model and the main results in this paper are presented. Firstly, we give closed-form solutions for some special cases. The solution obtained shows the difference in the type of solution based on the fractional differential operator used; the jump processes are defined on a separate filtration source. Other results follow, while the main result is presented in Theorem 8.

Note that the results in this section closely follow the analogous results in Ref. [21], with some new results added to the existing literature, including double uncertain jumps and its generalization to a finite number of jumps coupled with rigorous mathematical presentations.

3.1. Closed-form solutions

Theorem 1. Let $\left(\Gamma, \mathcal{L}, \{\mathcal{L}\}_{t\geq 0}, \mathcal{M}\right)$, C_τ V_1, V_2 be as specified in Assumptions 1 over the interval $\tau \in (0, T]$. Let $\mu_\tau, \nu_\tau, \sigma_{1\tau}$ and $\sigma_{2\tau}$ be functions defined on $[0, T]$, and let $\theta \in (0, 1)$. Then:

(a) The θ-th order Riemann-Liouville DV-FUDE given by:

$$^{RL}\mathcal{D}^\theta X_\tau = \mu_\tau + \nu_\tau \frac{dC_\tau}{d\tau} + \sigma_{1\tau}\frac{dV_{1\tau}}{d\tau} + \sigma_{2\tau}\frac{dV_{2\tau}}{d\tau}, \tau > 0, \tag{23}$$

subject to initial condition $x_0 = \lim\limits_{\tau\mapsto 0+} \tau^{1-\theta} X_\tau$ has the general closed-form solution:

$$\begin{aligned}\tilde{X}_\tau &= \tau^{\theta-1}x_0 + \frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1}\mu_\xi d\xi \\ &+\frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1}\nu_\xi dC_\xi \\ &+\sum_{j=1}^{2}\left[\frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1}\sigma_{j\tau}dV_{j\xi}\right], \ \tau > 0,\end{aligned} \tag{24}$$

which leads to:

$$\tilde{X}_\tau = \tau^{\theta-1}x_0 + \mathcal{I}^\theta\left(\mu_\tau + \nu_\tau\frac{dC_\tau}{d\tau} + \sum_{j=1}^{2}\sigma_{j\xi}\frac{dV_{j\tau}}{d\tau}\right), \tag{25}$$

(b) The Caputo DV-FUDE process given by:

$$^{c}\mathcal{D}^\theta X_\tau = \mu_\tau + \nu_\tau \frac{dC_\tau}{d\tau} + \sigma_{1\tau}\frac{dV_{1\tau}}{d\tau} + \sigma_{2\tau}\frac{dV_{2\tau}}{d\tau}, \tau > 0, \tag{26}$$

has the general closed-form solution:

$$\begin{aligned}\tilde{X}_\tau &= x_0 + \frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1}\mu_\xi d\xi \\ &+\frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1}\nu_\xi dC_\xi \\ &+\sum_{j=1}^{2}\left[\frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1}\sigma_{j\tau}dV_{j\xi}\right], \ \tau > 0. \\ &= x_0 + \mathcal{I}^\theta\left(\mu_\tau + \nu_\tau\frac{dC_\tau}{d\tau} + \sum_{j=1}^{2}\sigma_{j\xi}\frac{dV_{j\tau}}{d\tau}\right), \tau > 0.\end{aligned} \tag{27}$$

Proof. Both results follow directly by replacing $f(\xi, X_\xi), g(\xi, X_\xi), h_1(\xi, X_\xi)$, and $h_2(\xi, X_\xi)$ with $\mu_\xi, \nu_\xi, \sigma_{1\xi}$, and $\sigma_{2\xi}$ respectively in equations (16) through (20).

Theorem 2. Suppose $\mu_\tau, \nu_\tau, \sigma_{1\tau}, \sigma_{2\tau}$, are integrable functions defined on the closed interval $[0, T]$, $\theta \in (0, 1)$, and $a \in \mathbb{R}$. Then the θth-order Riemann-Liouville-type fractional differential equation of the form:

$$\begin{aligned}&^{RL}\mathcal{D}^\theta X_\tau = aX_\tau + \mu_\tau + \nu_\tau \frac{dC_\tau}{d\tau} + \sigma_{1\tau}\frac{dV_{1\tau}}{d\tau} + \sigma_{2\tau}\frac{dV_{2\tau}}{d\tau}, \\ &\qquad 0 < \tau \leq T,\end{aligned} \tag{28}$$

subject to initial condition

$$x_0 = \lim_{\tau\mapsto 0+} \tau^{1-\theta} X_\tau, \tag{29}$$

has the general solution:

$$\begin{aligned}\tilde{X}_\tau &= x_0\Gamma(\theta)\,\tau^{\theta-1}E_{\theta,\theta}\left(a\tau^\theta\right) \\ &+ \int_0^\tau (\tau-\xi)^{\theta-1}\,E_{\theta,\theta}\left(a(\tau-\xi)^\theta\right)\mu_\xi d\xi \\ &+ \int_0^\tau (\tau-\xi)^{\theta-1}\,E_{\theta,\theta}\left(a(\tau-\xi)^\theta\right)\nu_\xi dC_\xi \\ &+ \sum_{i=1}^{2}\left[\int_0^\tau (\tau-\xi)^{\theta-1}\,E_{\theta,\theta}\left(a(\tau-\xi)^\theta\right)\sigma_{i\xi}dV_{i\xi}\right],\end{aligned} \tag{30}$$

which simplifies to:

$$\begin{aligned}\tilde{X}_\tau &= x_0\Gamma(\theta)\,\tau^{\theta-1}E_{\theta,\theta}\left(a\tau^\theta\right) \\ &+ \mathcal{I}^\theta\left(E_{\theta,\theta}\left(a\tau^\theta\right)\left(\mu_\tau + \nu_\tau\frac{dC_\tau}{d\tau} + \sum_{i=1}^{2}\sigma_{i\tau}\frac{dV_{i\tau}}{d\tau}\right)\right),\end{aligned} \tag{31}$$

almost surely.

Proof. The argument follows the same lines as the proof of Theorem 3 in Ref. [21]. First, we show that the initial condition in equation (29) is satisfied by equation (30). We have from equation (30):

$$\begin{aligned}
&\lim_{\tau\to 0+} \tau^{1-\theta}\tilde{X}_\tau \\
&= \lim_{\tau\to 0+} x_0\Gamma(\theta)E_{\theta,\theta}\left(a\tau^\theta\right) \\
&\quad + \lim_{\tau\mapsto 0+} \tau^{1-\theta}\int_0^\tau (\tau-\xi)^{\theta-1} E_{\theta,\theta}\left(a(\tau-\xi)^\theta\right)\mu_\xi d\xi \\
&\quad + \lim_{\tau\to 0+} \tau^{1-\theta}\int_0^\tau (\tau-\xi)^{\theta-1} E_{\theta,\theta}\left(a(\tau-\xi)^\theta\right)\nu_\xi dC_\xi \\
&\quad + \sum_{j=1}^{2}\left[\lim_{\tau\to 0+} \tau^{1-\theta}\int_0^\tau (\tau-\xi)^{\theta-1} E_{\theta,\theta}\left(a(\tau-\xi)^\theta\right)\sigma_{j\xi}dV_{j\xi}\right] \\
&= x_0.
\end{aligned} \tag{32}$$

Next, We verify the integral form of equation (28). Applying $\mathcal{I}^\theta$ to $a\tilde{X}_\tau$ in equation (30) gives:

$$\begin{aligned}
&\mathcal{I}^\theta(a\tilde{X}_\tau) \\
&= \frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1} aX_\xi d\xi \\
&= ax_0\int_0^\tau (\tau-\xi)^{\theta-1}\xi^{\theta-1}E_{\theta,\theta}\left(a\xi^\theta\right)d\xi \\
&\quad + \frac{a}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1}\int_0^\xi (\xi-\hat{\xi})^{\theta-1}E_{\theta,\theta}\left(a(\xi-\hat{\xi})^\theta\right)\mu_{\hat{\xi}}d\hat{\xi}d\xi \\
&\quad + \frac{a}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1}\int_0^\xi (\xi-\hat{\xi})^{\theta-1}E_{\theta,\theta}\left(a(\xi-\hat{\xi})^\theta\right)\nu_{\hat{\xi}}dC_{\hat{\xi}}d\xi \\
&\quad + \sum_{j=1}^{2}\left[\frac{a}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1}\int_0^\xi (\xi-\hat{\xi})^{\theta-1} E_{\theta,\theta}\left(a(\xi-\hat{\xi})^\theta\right)\times \right. \\
&\qquad\qquad \left. \sigma_{j\hat{\xi}}dV_{j\hat{\xi}}d\xi\right].
\end{aligned} \tag{33}$$

Using equation (13) in the proof of Theorem 3 in Ref. [30] and the Mittag-Leffler function in equation (21), the first term on the right-hand side of (33) becomes:

$$\begin{aligned}
&ax_0\int_0^\tau (\tau-\xi)^{\theta-1}\xi^{\theta-1}E_{\theta,\theta}(a\xi^\theta)d\xi \\
&\qquad = x_0\Gamma(\theta)\tau^{\theta-1}E_{\theta,\theta}(a\tau^\theta) - x_0\tau^{\theta-1}.
\end{aligned} \tag{34}$$

The other three terms on the right-hand side of (33) all have similar transformations; we will work out the calculation for the fourth term in detail. By interchanging the order of integration, we have:

$$\begin{aligned}
&\frac{a}{\Gamma(\theta)}\sum_{i=1}^{2}\left[\int_0^\tau (\tau-\xi)^{\theta-1}\right. \\
&\qquad\qquad \left.\times\int_0^\xi (\xi-\hat{\xi})^{\theta-1}E_{\theta,\theta}\left(a(\xi-\hat{\xi})^\theta\right)\sigma_{i\hat{\xi}}dV_{i\hat{\xi}}d\xi\right] \\
&\quad = \frac{a}{\Gamma(\theta)}\sum_{i=1}^{2}\int_0^\tau\left(\int_{\hat{\xi}}^\tau (\tau-\xi)^{\theta-1}(\xi-\hat{\xi})^{\theta-1}\right. \\
&\qquad \left.\times\ E_{\theta,\theta}\left(a(\xi-\hat{\xi})^\theta\right)d\xi\right)\sigma_{i\hat{\xi}}dV_{i\hat{\xi}}.
\end{aligned} \tag{35}$$

Making the change of variable $\beta := \dfrac{\xi-\hat{\xi}}{\tau-\hat{\xi}}$, we have:

$$\begin{aligned}
&\frac{a}{\Gamma(\theta)}\sum_{i=1}^{2}\int_0^\tau\left(\int_{\hat{\xi}}^\tau (\tau-\xi)^{\theta-1}(\xi-\hat{\xi})^{\theta-1}\ E_{\theta,\theta}\left(a(\xi-\hat{\xi})^\theta\right)d\xi\right)\sigma_{i\hat{\xi}}dV_{i\hat{\xi}} \\
&\quad = \frac{a}{\Gamma(\theta)}\sum_{i=1}^{2}\int_0^\tau\left(\int_0^1 (\tau-\hat{\xi})^{\theta-1}(1-\beta)^{\theta-1}\beta^{\theta-1}(\tau-\hat{\xi})^{\theta-1}\right. \\
&\qquad \left.\times\ E_{\theta,\theta}\left(a\beta^\theta(\tau-\hat{\xi})^\theta\right)d\beta\right)\sigma_{i\hat{\xi}}dV_{i\hat{\xi}} \\
&= \frac{a}{\Gamma(\theta)}\sum_{i=1}^{2}\int_0^\tau (\tau-\hat{\xi})^{2\theta-1}\left(\int_0^1 (1-\beta)^{\theta-1}\beta^{\theta-1}\sum_{j=0}^{\infty}\frac{a^j\beta^{\theta j}(\tau-\hat{\xi})^{\theta j}}{\Gamma(\theta(j+1))}d\beta\right)\sigma_{i\hat{\xi}}dV_{i\hat{\xi}}
\end{aligned} \tag{36}$$

$$\begin{aligned}
&= \frac{a}{\Gamma(\theta)}\sum_{i=1}^{2}\int_0^\tau (\tau-\hat{\xi})^{2\theta-1}\sum_{j=0}^{\infty}\frac{a^j(\tau-\hat{\xi})^{\theta j}}{\Gamma(\theta(j+1))} \\
&\qquad \left(\int_0^1 (1-\beta)^{\theta-1}\beta^{\theta(j+1)-1}d\beta\right)\sigma_{i\hat{\xi}}dV_{i\hat{\xi}} \\
&= a\sum_{i=1}^{2}\left[\int_0^\tau (\tau-\hat{\xi})^{2\theta-1}\sum_{j=0}^{\infty}\frac{a^j(\tau-\hat{\xi})^{\theta j}}{\Gamma(\theta(j+2))}\sigma_{i\hat{\xi}}dV_{i\hat{\xi}}\right] \\
&= a\sum_{i=1}^{2}\left[\int_0^\tau (\tau-\hat{\xi})^{\theta-1}\left(E_{\theta,\theta}\left(a(\tau-\hat{\xi})^\theta\right)-\frac{1}{\Gamma(\theta)}\right)\sigma_{i\hat{\xi}}dV_{i\hat{\xi}}\right] \\
&= a\sum_{i=1}^{2}\mathcal{I}^\theta\left(E_{\theta,\theta}\left(a\tau^\theta\right)\sigma_{i\hat{\xi}}\frac{dV_{i\tau}}{d\tau}-\sigma_{i\hat{\tau}}\frac{dV_{i\hat{\tau}}}{d\tau}\right).
\end{aligned}$$

Next, following the similar procedure as above, we have for the second term on the right-hand side of equation (33):

$$\begin{aligned}
&\frac{a}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1}\int_0^\xi (\xi-\hat{\xi})^{\theta-1}E_{\theta,\theta}\left(a(\xi-\hat{\xi})^\theta\right)\mu_{\hat{\xi}}d\hat{\xi}d\xi \\
&\qquad = \mathcal{I}^\theta\left(E_{\theta,\theta}\left(a\tau^\theta\right)\mu_{\hat{\xi}}-\mu_{\hat{\xi}}\right),
\end{aligned} \tag{37}$$

and for the third term,

$$\begin{aligned}
&\frac{a}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1}\int_0^\xi (\xi-\hat{\xi})^{\theta-1}E_{\theta,\theta}\left(a(\xi-\hat{\xi})^\theta\right)\nu_{\hat{\xi}}dC_{\hat{\xi}}d\xi \\
&\qquad = \mathcal{I}^\theta\left(E_{\theta,\theta}\left(a\tau^\theta\right)\nu_{\hat{\tau}}\frac{dC_\tau}{d\tau}-\nu_{\hat{\tau}}\frac{dC_\tau}{d\tau}\right).
\end{aligned} \tag{38}$$

Putting equations (36)-(38) in equation (33) leads to:

$$\begin{aligned}
&\mathcal{I}^\theta(a\tilde{X}_\tau) \\
&\quad = x_0\Gamma(\theta)\tau^{\theta-1}E_{\theta,\theta}(a\tau^\theta) - x_0\tau^{\theta-1} + \mathcal{I}^\theta\left(E_{\theta,\theta}\left(a\tau^\theta\right)\mu_{\hat{\xi}}-\mu_{\hat{\xi}}\right) \\
&\qquad + \mathcal{I}^\theta\left(E_{\theta,\theta}\left(a\tau^\theta\right)\nu_{\hat{\tau}}\frac{dC_\tau}{d\tau}-\nu_{\hat{\tau}}\frac{dC_\tau}{d\tau}\right) \\
&\qquad + a\sum_{i=1}^{2}\mathcal{I}^\theta\left(E_{\theta,\theta}\left(a\tau^\theta\right)\sigma_{i\hat{\xi}}\frac{dV_{i\tau}}{d\tau}-\sigma_{i\hat{\tau}}\frac{dV_{i\hat{\tau}}}{d\tau}\right),
\end{aligned} \tag{39}$$

and rearranging gives:

$$\begin{aligned}
&x_0\Gamma(\theta)\,\tau^{\theta-1}E_{\theta,\theta}\left(a\tau^\theta\right) \\
&\quad + \mathcal{I}^\theta\left(E_{\theta,\theta}\left(a\tau^\theta\right)\left(\mu_\tau+\nu_\tau\frac{dC_\tau}{d\tau}+\sum_{i=1}^{2}\sigma_{i\tau}\frac{dV_{i\tau}}{d\tau}\right)\right) \\
&\quad = x_0\tau^{\theta-1} + \mathcal{I}^\theta\left(aX_\tau+\mu_\tau+\nu_\tau\frac{dC_\tau}{d\tau}+\sum_{i=1}^{2}\sigma_{1\tau}\frac{dV_{i\tau}}{d\tau}\right),
\end{aligned} \tag{40}$$

and applying the Riemann-Liouville derivative ${}^{RL}\mathcal{D}$ to both sides gives equation (28) for $X_\tau = \tilde{X}_\tau$.

Remark 2. The theorem can be trivially extended to FUDEs with any number of jump processes, simply by adding more terms to the final sum.

3.2. Existence and uniqueness solution of the DV-FUDE

In order to prove the existence and uniqueness of solutions to the DV-FUDE, we first establish several lemmas.

Lemma 3.1 ([24])**.** Let C_t be a canonical Liu process and X_t be an integrable uncertain process on $[a, b]$ with respect to time t. Then, the inequality

$$\left|\int_a^b X_t(\gamma) dC_t(\gamma)\right| \leq K(\gamma) \int_a^b |X_t(\gamma)|\, dt, \tag{41}$$

holds, where $K(\gamma)$ is the Lipschitz constant of the sample path $X_t(\gamma)$.

The proof of Lemma 3.1 may be found in Ref. [24].

Lemma 3.2. Let V_t be an uncertain jump process on a time-bounded domain of $\mathbb{R}$ and X_t is an integrable uncertain process on $[a, b]$ with respect to time t. Then we have almost surely:

$$\left|\int_a^b X_t(\gamma) dV_t(\gamma)\right| \leq \int_a^b |X_t(\gamma)|\, dt. \tag{42}$$

Proof. Using the definition of integral (equation (3)),

$$\left|\int_a^b X_t(\gamma) dV_t(\gamma)\right| = \lim_{\Delta\to 0}\left|\sum_{i=1}^{k} X_{t_i}(\gamma)\left(V_{t_{i+1}(\gamma)} - V_{t_i(\gamma)}\right)\right|,$$

where $a = t_i < t_2 < \cdots < t_k = b$.

By Definition 2.13, $0 \leq V_{t_{i+1}}(\gamma) - V_{t_i}(\gamma) \leq t_{i+1} - t_i$ almost surely. Therefore, we have:

$$\begin{aligned}
&\lim_{\Delta\to 0}\left|\sum_{i=1}^{k} X_t(\gamma)\left(V_{t_{i+1}}(\gamma) - V_{t_i}(\gamma)\right)\right| \\
&\qquad \leq \lim_{\Delta\to 0}\sum_{i=1}^{k}\left|X_{t_i}(\gamma)\right|\left|V_{t_{i+1}}(\gamma) - V_{t_i}(\gamma)\right| \\
&\qquad \leq \lim_{\Delta\to 0}\sum_{i=1}^{k}\left|X_{t_i}(\gamma)\right|(t_{i+1} - t_i) \\
&\qquad \leq \int_a^b |X_t(\gamma)|\, dt.
\end{aligned} \tag{43}$$

□

In the following, $C_{[a,b]}$ represents the Banach space of continuous $\mathbb{R}$- valued functions defined on $[a, b]$, with norm $\|f\| \equiv \sup_{\tau\in[a,b]} |f|$ for $f \in C_{[a,b]}$.

Lemma 3.3 (Sample continuity)**.** Let X_τ be an uncertain process on $[0, \infty)$. Given differentiable functions f, g, h_1, and $h_2 : [0, \infty) \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ that satisfy the linear growth condition:

$$\begin{aligned}
&|f(\tau, x)| + |g(\tau, x)| + |h(_1\tau, x)| + |h_2(\tau, x)| \leq L(1 + |x|), \\
&\quad \forall x \in \mathbb{R},\ 0 \leq \tau < \infty,
\end{aligned} \tag{44}$$

where $L > 0$ is constant. In the following, we have $x_0, y \in \mathbb{R}$, and $0 < \theta < 1$.

(a) For each $\gamma \in \Gamma$, define a mapping $\varphi_{x_0,\gamma} : \mathbb{R}^+ \times C_{[0,\infty)} \to \mathbb{R}$ as follows:

$$\begin{aligned}
&\varphi_{x_0,\gamma}(\tau, \psi) \\
&\quad = \frac{1}{\Gamma(\theta)}\int_0^\tau (\tau - \xi)^{\theta-1} f\left(\xi, \psi(\xi) + x_0\xi^{\theta-1}\right) d\xi \\
&\quad + \frac{1}{\Gamma(\theta)}\int_0^\tau (\tau - \xi)^{\theta-1} g\left(\xi, \psi(\xi) + x_0\xi^{\theta-1}\right) dC_\xi(\gamma) \\
&\quad + \frac{1}{\Gamma(\theta)}\sum_{j=1}^{2}\int_0^\tau (\tau - \xi)^{\theta-1} h_j\left(\xi, \psi(\xi) + x_0\xi^{\theta-1}\right) dV_{j\xi}(\gamma).
\end{aligned} \tag{45}$$

Then $\varphi_{x_0,\gamma}(\tau, \psi)$ is a continuous function of τ for all $0 < \tau < \infty$.

(b) Given $a > 0$ and $y \in C[a, \tau]$. For each $\gamma \in \Gamma$, define a mapping $\varphi_{a,x_0,y,\gamma} : \mathbb{R}^+ \times C_{[a,\infty)} \to \mathbb{R}$ as follows:

$$\begin{aligned}
&\varphi_{a,x_0,y,\gamma}(\tau, \psi) \\
&\quad = \tau^{\theta-1}x_0 + y + \frac{1}{\Gamma(\theta)}\int_a^\tau (\tau - \xi)^{\theta-1} f(\xi, \psi(\xi))\, d\xi \\
&\quad + \frac{1}{\Gamma(\theta)}\int_a^\tau (\tau - \xi)^{\theta-1} g(\xi, \psi(\xi))\, dC_\xi(\gamma) \\
&\quad + \frac{1}{\Gamma(\theta)}\sum_{j=1}^{2}\int_a^\tau (\tau - \xi)^{\theta-1} h_j(\xi, \psi(\xi))\, dV_{j\xi}(\gamma).
\end{aligned} \tag{46}$$

Then $\varphi_{a,x_0,y,\gamma}(\tau, \psi)$ is continuous as a function of τ for all $a < \tau < \infty$, for any $\psi \in C_{[a,\infty)}$.

Proof. The proof resembles that of Lemma 1 in Ref. [21], but with slight modifications. We show case (b): case (a) is similar. For $\gamma \in \Gamma$ and $u \in (a, \tau)$, from equation (46) and the triangle inequality we have:

$$\begin{aligned}
&\left|\varphi_{a,x_0,y,\gamma}(\tau, \psi) - \varphi_{a,x_0,y,\gamma}(u, \psi)\right| \\
&\leq \left(u^{\theta-1} - \tau^{\theta-1}\right)|x_0| + \frac{1}{\Gamma(\theta)}\int_u^\tau (\tau - \xi)^{\theta-1} |f(\xi, \psi(\xi))|\, d\xi \\
&+ \frac{1}{\Gamma(\theta)}\left|\int_u^\tau (\tau - \xi)^{\theta-1} g(\xi, \psi(\xi))\, dC_\xi(\gamma)\right| \\
&+ \frac{1}{\Gamma(\theta)}\sum_{j=1}^{2}\left|\int_u^\tau (\tau - \xi)^{\theta-1} h_j(\xi, \psi(\xi))\, dV_{j\xi}(\gamma)\right| \\
&+ \frac{1}{\Gamma\theta)}\int_a^u \left[(\tau - \xi)^{\theta-1} - (u - \xi)^{\theta-1}\right]|f(\xi, \psi(\xi))|\, d\xi \\
&+ \frac{1}{\Gamma(\theta)}\left|\int_a^u \left[(\tau - \xi)^{\theta-1} - (u - \xi)^{\theta-1}\right] g(\xi, \psi(\xi))\, dC_\xi(\gamma)\right| \\
&+ \frac{1}{\Gamma(\theta)}\sum_{j=1}^{2}\left|\int_a^u \left[(\tau - \xi)^{\theta-1} - (u - \xi)^{\theta-1}\right] h_j(\xi, \psi(\xi))\, dV_{j\xi}(\gamma)\right|.
\end{aligned} \tag{47}$$

Using the bounds from Lemmas 3.1 and 3.2, we have:

$$\begin{aligned}
&\left|\varphi_{a,x_0,y,\gamma}(\tau,\psi)-\varphi_{a,x_0,y,\gamma}(u,\psi)\right| \\
&\le \left(u^{\theta-1}-\tau^{\theta-1}\right)|x_0| + \frac{1}{\Gamma(\theta)}\left(\int_u^\tau (\tau-\xi)^{\theta-1}|f(\xi,\psi(\xi))|\,d\xi\right. \\
&\quad + K_\gamma \int_u^\tau (\tau-\xi)^{\theta-1}|g(\xi,\psi(\xi))|\,d\xi \\
&\quad \left. + \sum_{j=1}^{2}\int_u^\tau (\tau-\xi)^{\theta-1}\left|h_j(\xi,\psi(\xi))\right| d\xi\right) \\
&\quad + \frac{1}{\Gamma(\theta)}\left(\int_a^u \left[(\tau-\xi)^{\theta-1}-(u-\xi)^{\theta-1}\right]|f(\xi,\psi(\xi))|\,d\xi\right. \\
&\quad + K_\gamma \int_a^u \left[(\tau-\xi)^{\theta-1}-(u-\xi)^{\theta-1}\right]|g(\xi,\psi(\xi))|\,d\xi \\
&\quad \left. + \sum_{j=1}^{2}\int_a^u \left[(\tau-\xi)^{\theta-1}-(u-\xi)^{\theta-1}\right]\left|h_j(\xi,\psi(\xi))\right| d\xi\right),
\end{aligned} \tag{48}$$

so that in view of the growth condition (44) we obtain:

$$\begin{aligned}
&\left|\varphi_{a,x_0,y,\gamma}(\tau,\psi)-\varphi_{a,x_0,y,\gamma}(u,\psi)\right| \\
&\le \left(u^{\theta-1}-\tau^{\theta-1}\right)|x_0| \\
&+ \frac{1+K_\gamma}{\Gamma(\theta)}\left(\int_u^\tau (\tau-\xi)^{\theta-1} L\left(1+\sup_{u\le\xi\le\tau}\|\psi(\xi)\|\right)d\xi\right) \\
&+ \frac{(1+K_\gamma}{\Gamma(\theta)}\left(\int_a^u \left[(\tau-\xi)^{\theta-1}-(u-\xi)^{\theta-1}\right]\left(1+\sup_{u\le\xi\le\tau}\|\psi(\xi)\|\right)d\xi\right).
\end{aligned}$$

Evaluating the integrals, we have finally:

$$\begin{aligned}
&\left|\varphi_{a,x_0,y,\gamma}(\tau,\psi)-\varphi_{a,x_0,y,\gamma}(u,\psi)\right| \\
&\le \left(u^{\theta-1}-\tau^{\theta-1}\right)|x_0| + \frac{L}{\Gamma(\theta+1)}\left(1+\sup_{u\le\xi\le\tau}\|\psi(\xi)\|\right)\left(1+K_\gamma\right) \\
&\qquad \times\left[(\tau-u)^\theta+(\tau-a)^\theta-(u-a)^\theta\right].
\end{aligned} \tag{49}$$

Hence $|\tau-u|\longrightarrow 0$ implies $\left|\varphi_{a,x_0,y,\gamma}(\tau,\psi)-\varphi_{a,x_0,y,\gamma}(u,\psi)\right|\longrightarrow 0$. This concluded the proof that $\varphi_{a,x_0,y,\gamma}(\tau,\psi)$ is continuous in τ for any $\psi\in C[a,\infty]$. The proof that $\varphi_{x_0,\gamma}$ is sample continuous is similar.

Now, we may establish the existence and uniqueness of solutions for DV-FUDE, given the Lipschitz and linear growth conditions on the coefficient functions.

Theorem 3 (Existence and uniqueness of solution)**.** Given a canonical Liu process C_τ and an integrable uncertain process X_τ for $0\le\tau\le T$. Then, there exists unique solutions to the RL and Caputo DV-FUDEs (16) and (19) if the coefficient terms $f(\tau,x), g(\tau,x), h_1(\tau,x), h_2(\tau,x)$ are Lipschitz continuous:

$$\begin{aligned}
&|f(\tau,x)-f(\tau,\check{x})|+|g(\tau,x)-g(\tau,\check{x})|+|h_1(\tau,x)-h_1(\tau,\check{x})| \\
&+|h_2(\tau,x)-h_2(\tau,\check{x})| \le L|x-\check{x}|,\ \forall\, x,\check{x}\in\mathbb{R}, 0\le\tau<+\infty,
\end{aligned} \tag{50}$$

and satisfy the linear growth condition:

$$\begin{aligned}
&|f(\tau,x)|+|g(\tau,x)|+|h_1(\tau,x)|+|h_2(\tau,x)| \le L(1+|x|), \\
&\forall x\in\mathbb{R},\ 0\le\tau<+\infty, \quad \text{where } L>0 \text{ is a constant.}
\end{aligned} \tag{51}$$

Proof. The proof parallels that of Theorem 1 in Ref. [28] and Theorem 5 in Ref. [21]. It appears that these proofs are incorrect in some technical details, and we supply corrections.

We begin with the RL DV-FUDE. First, for $k>0$ define $\widetilde{\varphi}_{x_0,\gamma}: C_{[0,k]}\to C_{[0,k]}$ as follows:

$$\widetilde{\varphi}_{x_0,\gamma}(\psi)(\tau)=\varphi_{x_0,\gamma}(\tau,\psi) \qquad \text{for } \tau\in[0,k]. \tag{52}$$

(Note that Lemma 3.3(a) guarantees that $\widetilde{\varphi}_{x_0}(\psi)$ is continuous for any continuous function $\psi\in C_{[0,k]}$.)

For any two functions $\psi,\check{\psi}\in C_{[0,k]}$, we then have:

$$\begin{aligned}
&\left\|\widetilde{\varphi}_{x_0,\gamma}(\psi)-\varphi_{x_0,\gamma}(\check{\psi})\right\| = \max_{\tau\in[0,k]}\left|\varphi_{x_0,\gamma}(\tau,\psi)-\varphi_{x_0,\gamma}(\tau,\check{\psi})\right| \\
&\le \max_{\tau\in[0,k]}\left|\frac{1}{\Gamma(\theta)}\int_0^\tau (t-\xi)^{\theta-1}\left[f(\xi,\psi(\xi)+x_0\xi^{\theta-1})-f(\xi,\check{\psi}(\xi)+x_0\xi^{\theta-1})\right]d\xi\right. \\
&+\frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1}\left[g\left(\xi,\psi(\xi)+x_0\xi^{\theta-1}\right)-g\left(\xi,\check{\psi}(\xi)+x_0\xi^{\theta-1}\right)\right]dC_\xi(\gamma) \\
&+\frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1}\left[h_1\left(\xi,\psi(\xi)+x_0\xi^{\theta-1}\right)-h_1\left(\xi,\check{\psi}(\xi)+x_0\xi^{\theta-1}\right)\right]dV_{1\xi}(\gamma) \\
&\left.+\frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-u)^{\theta-1}\left[h_2\left(\xi,\psi(\xi)+x_0\xi^{\theta-1}\right)-h_2\left(u,\check{\psi}(\xi)+x_0\xi^{\theta-1}\right)\right]dV_{2\xi}(\gamma)\right| \\
&\le \max_{\tau\in[0,k]}\left\{\frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1}\left|f\left(\xi,\psi(\xi)+x_0\xi^{\theta-1}\right)-f\left(\xi,\check{\psi}(\xi)+x_0\xi^{\theta-1}\right)\right|d\xi\right. \\
&+\frac{K_\gamma}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1}\left|g\left(\xi,\psi(\xi)+x_0\xi^{\theta-1}\right)-g\left(\xi,\check{\psi}(\xi)+x_0\xi^{\theta-1}\right)\right|d\xi \\
&+\frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-\xi)^{\theta-1}\left|h_1\left(\xi,\psi(\xi)+x_0\xi^{\theta-1}\right)-h_1\left(\xi,\check{\psi}(\xi)+x_0\xi^{\theta-1}\right)\right|d\xi \\
&\left.+\frac{1}{\Gamma(\theta)}\int_0^\tau (t-\xi)^{\theta-1}\left|h_2\left(\xi,\psi(\xi)+x_0\xi^{\theta-1}\right)-h_2\left(\xi,\check{\psi}(\xi)+x_0\xi^{\theta-1}\right)\right|d\xi\right\} \\
&\le \frac{\left(1+K_\gamma\right)L}{\Gamma(\theta)}\max_{\tau\in[0,k]}\int_0^\tau (\tau-\xi)^{\theta-1}\left|\psi(\xi)-\check{\psi}(\xi)\right|d\xi, \text{(by Lipschitz condition).} \\
&\le \frac{\left(1+K_\gamma\right)Lk^\theta}{\Gamma(\theta+1)}\left\|\psi-\check{\psi}\right\|.
\end{aligned} \tag{53}$$

We take $k=k(\gamma)$ sufficiently small such that:

$$\frac{\left(1+K_\gamma\right)Lk^\theta}{\Gamma(\theta+1)}<1. \tag{54}$$

With this choice of k then $\widetilde{\varphi}_{x_0,\gamma}$ is a contraction mapping on $C_{[0,k]}$. The Banach fixed point theorem implies there exists a unique fixed point $\psi^{(0)}\in C_{[0,k]}$. We then define:

$$X_\tau^{(0)}(\gamma):=\psi^{(0)}+x_0\tau^{1-\theta}. \tag{55}$$

In view of the fact that $\psi^{(0)}$ is a fixed point of $\widetilde{\phi}_{x_0,\gamma}$ which is defined in terms of $\phi_{x_0,\gamma}$, we may replace $\varphi_{x_0,\gamma}(\tau,\psi)$ and $\psi(\xi)$ in equation (46) with $X_\tau^{(0)}(\gamma)-x_0\tau^{1-\theta}$ and $X_\xi^{(0)}(\gamma)-x_0\xi^{1-\theta}$ respectively, and obtain that $X_\tau^{(0)}(\gamma)$ is the unique solution to equation (18) on $[0,k]$.

Next, define $\widetilde{\varphi}_{x_0,X_\tau^{(0)},k,\gamma}: C_{[k,2k]}\to C_{[k,2k]}$ as follows:

$$\begin{aligned}
&\widetilde{\varphi}_{x_0,X_\tau^{(0)},k,\gamma}(\psi)(\tau):=x_0\tau^{\theta-1} \\
&+\frac{1}{\Gamma(\theta)}\left(\int_0^k (\tau-\xi)^{\theta-1}f\left(\xi,X_\xi^{(0)}(\gamma)\right)d\xi+\int_0^k (\tau-\xi)^{\theta-1}g\left(\xi,X_\xi^{(0)}(\gamma)\right)dC_\xi\right. \\
&\left.+\int_0^k (\tau-\xi)^{\theta-1}h_1\left(\xi,X_\xi^{(0)}(\gamma)\right)dV_{1\xi}+\int_0^k (\tau-\xi)^{\theta-1}h_2\left(\xi,X_\xi^{(0)}(\gamma)\right)dV_{2\xi}\right) \\
&+\frac{1}{\Gamma(\theta)}\left(\int_k^{2k}(\tau-\xi)^{\theta-1}f(\xi,\psi(\xi)d\xi+\int_k^{2k}(\tau-\xi)^{\theta-1}g(\xi,\psi(\xi)dC_\xi\right. \\
&\left.+\int_k^{2k}(\tau-\xi)^{\theta-1}h_1(\xi,\psi(\xi)dV_{1\xi}\int_k^{2k}(\tau-\xi)^{\theta-1}h_2(\xi,\psi(\xi)dV_{2\xi}\right),
\end{aligned}$$

(as before, Lemma 3.3(a) guarantees that $\widetilde{\varphi}_{x_0,X_\tau^{(0)},k,\gamma}(\psi)$ is continuous for any continuous function $\psi \in C_{[k,2k]}$.

For any two functions $\psi, \breve{\psi} \in C_{[k,2k]}$, a similar chain of inequalities as in equation (53) shows that:

$$\left\|\widetilde{\varphi}_{x_0,X_\tau^{(0)},k,\gamma}(\psi) - \widetilde{\varphi}_{x_0,X_\tau^{(0)},k,\gamma}(\breve{\psi})\right\| \leq \frac{\left(3 + K_\gamma\right) Lk^\theta}{\Gamma(\theta+1)} \left\|\psi - \breve{\psi}\right\| < \left\|\psi - \breve{\psi}\right\|, \tag{56}$$

where the last inequality follows from equation (54), but the fact that the quantity in equation (54) is less than 1 does not imply that (56) must also be less than 1.

As before, the Banach fixed point theorem implies there exists a unique fixed point $X_\tau^{(1)}(\gamma) \in C_{[k,\,2k]}$. It follows that the function $X_\tau^{(0,1)}(\gamma) : C_{[0,\,2k]}$ defined by:

$$X_\tau^{(0,1)}(\gamma) = \begin{cases} X_\tau^{(0)}(\gamma) & 0 < \tau \leq k, \\ X_\tau^{(1)}(\gamma) & k \leq \tau \leq 2k, \end{cases} \tag{57}$$

is the unique solution to equation (18) on $(0, 2k]$.

We may then define $X_\tau^{(0,\ldots J)}(\gamma)$ for any integer $J \geq 2$ inductively as follows. First, define $\varphi_{x_0,X_\tau^{(0,1,\ldots J-1)},k,\gamma} : C[Jk, (J+1)k] \to C[Jk, (J+1)k]$ as follows:

$$\begin{aligned}
&\widetilde{\varphi}_{x_0,X_\tau^{(0,1,\ldots J-1)},k,\gamma}(\psi)(\tau) \\
&= x_0\tau^{\theta-1} + \frac{1}{\Gamma(\theta)}\Bigg(\int_0^{Jk} (\tau-\xi)^{\theta-1} f(\xi, X^{(0,\ldots J-1)}(\xi)d\xi \\
&\quad + \int_0^{Jk} (\tau-\xi)^{\theta-1} g(\xi, X^{(0,\ldots J-1)}(\xi)dC_\xi \\
&\quad + \int_0^{Jk} (\tau-\xi)^{\theta-1} h_1(\xi, X^{(0,\ldots J-1)}(\xi)dV_{1\xi} \\
&\quad + \int_0^{Jk} (\tau-\xi)^{\theta-1} h_2(\xi, X^{(0,\ldots J-1)}(\xi)dV_{2\xi} \Bigg) \\
&\quad + \frac{1}{\Gamma(\theta)}\Bigg(\int_{Jk}^{(J+1)k} (\tau-\xi)^{\theta-1} f(\xi, \psi(\xi)d\xi \\
&\quad + \int_{Jk}^{(J+1)k} (\tau-\xi)^{\theta-1} g(\xi, \psi(\xi)dC_\xi \\
&\quad + \int_{Jk}^{(J+1)k} (\tau-\xi)^{\theta-1} h_1(\xi, \psi(\xi)dV_{1\xi} \int_{Jk}^{(J+1)k} (\tau-\xi)^{\theta-1} h_2(\xi, \psi(\xi)dV_{2\xi} \Bigg).
\end{aligned} \tag{58}$$

Using exactly the same arguments as before, we find that $\varphi_{x_0,X^{(0,1,\ldots J-1)},k,\gamma}$ has a unique fixed point, which we denote as $X_\tau^{(J)}(\gamma)$. Then we define:

$$X_\tau^{(0,\ldots J)}(\gamma) = \begin{cases} X_\tau^{(0,\ldots J-1)}(\gamma) & 0 \leq \tau \leq Jk, \\ X_\tau^{(J)}(\gamma) & Jk \leq \tau \leq (J+1)k. \end{cases} \tag{59}$$

It follows that $X^{(0,\ldots J)}$ is the unique solution of (18) on $(0, (J+1)k]$. Since J can be made arbitrarily large, we have established the theorem.

For the Caputo-type DV-FUDE, the proof differs only in the replacement of $x_0\tau^{\theta-1}$ terms by x_0 in the definitions of $\varphi_{a,x_0,y,\gamma}(\tau,\psi)$, $X_\tau^{(0)}(\gamma)$, and $\varphi_{x_0,X_\tau^{(0,1,\ldots J-1)},k,\gamma}$. With these replacements, the proofs of both Lemma 3.3 and Theorem 3 go through with very minor changes.

Remark 3. The theorem can be trivially extended to FUDEs with any number of jump processes, simply by adding more jumps.

The summary of the contribution of this study to the existing literature are highlighted as follows:

- We developed a framework that captures dual sources of shock in the dynamics that describe uncertain processes.
- We proposed a double V-jump fractional uncertain differential equation (DV-FUDE) for modelling uncertain systems attributed to internal and external factors with different independent filtration resulting to double uncertain shocks.
- We established exact solutions in the case of time-dependent coefficients for DV-FUDE in terms of the Mittag-Leffler function.
- We introduced functionals to the solution of the uncertain process for the DV-FUDEs.
- We established continuity, existence, and uniqueness results for the DV-FUDE in terms of the general Riemann-Liouville and Caputo operators.
- Finally, we propose relevant areas of applications of this study.

4. Conclusion

This study introduces a novel double V-jump fractional uncertain differential equation (DV-FUDE) formulated via a θth-order Riemann-Liouville fractional operator, providing a rigorous framework for modeling uncertain systems driven by multiple shock sources with memory effects. Existence and uniqueness of solutions are established analytically through the Banach fixed point theorem under global Lipschitz and linear growth conditions, thereby ensuring well-posedness of the proposed model. When these conditions are relaxed, the applicability of the Schauder fixed point theorem offers a viable pathway for guaranteeing existence of solutions to equations (16) and (19). These results significantly extend the theory of fractional uncertain differential equations and lay a solid foundation for future analytical and applied investigations involving multi-jump uncertainty dynamics.

While the present study is purely analytical, future research will extend the proposed DV-FUDE framework to include stability and robustness analysis, the development of efficient numerical schemes for solution approximation, and applications to real-world uncertain systems such as financial and climate-driven dynamics. Further investigations will also address multi-dimensional and coupled DV-FUDE models, as well as parameter estimation and model calibration under uncertainty, thereby broadening both the theoretical depth and practical relevance of the framework.

Data availability

We do not have any research data outside the submitted manuscript file.

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