Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 3 (2021) 459–468

Journal of the Nigerian Society of Physical Sciences

Computational study of some three-step hybrid integrators for solution of third order ordinary differential equations

Mark I. Modebei[a], Olumide O. Olaiya[a,], Ignatius P. Ngwongwo[b,]

[a]Department of Mathematics Programme, National Mathematical Centre, Abuja, Nigeria
[b]Department of Mathematics, Federal University of Technology, Owerri, Owerri, Nigeria.

Abstract

Block of hybrid methods with three off-step points based on collocation technique are presented in this work for direct approximation of solution of third-order Initial and Boundary Value Problems (IVPs and BVPs). These off-step points are formulated such that they exist only on a single step at a time. Hence, these points are shifted to three positions respectively in order to obtain three block different integrators for computational analysis. These analysis includes; order of the methods, consistency, stability and convergence, global error, number of functions evaluation and CPU time. The superiority of these methods over existing methods is established numerically on different test problems in literature.

DOI:10.46481/jnsps.2021.323

Keywords: Third Order Ordinary Diferential Equation, Initial Value Problem, Boundary Value Problems, Block Method, Finite Diferences, Linear Multistep Hybrid Method

Article History :
Received: 29 July 2021
Received in revised form: 01 September 2021
Accepted for publication: 13 September 2021
Published: 29 November 2021

Communicated by: T. Latunde

1. Introduction

Third-order problems considered in this work are of the form:

$$y''' = f(x, y, y', y''), \quad x \in [a, b] \tag{1}$$

with appropriate initial conditions and boundary conditions respetively.
It is assumed that; the function f is continuous in $[a, b] \times \mathbb{R}^3$. Also, as discussed in [1, 2], the existence and uniqueness of the solution of (1) is assumed. The problem in (1) is assumed well posed and the numerical solution is the interest in this work. Numerical methods for the solution of (1) with initial conditions or boundary conditions and for third order singularly perturbed boundary value problems are numerous. Some include, Collocation method [3, 4, 5, 6], Non-polynomial splines [7], Quartic Splines [8], Adomian Decomposition Method [9], Exponential Quartic Spline [10], Quartic B-spline method [11], and many others.
In this work, the collocation technique is employed. This method is found to be flexible and more efficient in that since it approximates the solution of (1) at several intra-points, its block formulations consists of several linear multistep methods which are required for direct solution of (1), such that it overcomes the overlapping of pieces of solutions and it is self starting.

In this work we seek the numerical solution of (1) directly with three different three-step continuous hybrid block methods with three off-step points. These methods are developed using the collocation approach.

Email address: gmarc.ify@gmail.com, (Olumide O. Olaiya)

2. Derivation of the Methods

Here, the derivation of a continuous implicit three mid intra-step hybrid block method is described, for the solution of (1) over the integration interval $[a, b]$,

$$\pi_N \equiv \{a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b\}$$

with h the constant step-size, $h = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$.
For the solution of (1) with initial condition, the method proposed in [12], three off-step points in the interval $0 < x_r < x_s < x_t < 1$ are given such that $(r, s, t) = (\frac{3}{8}, \frac{5}{8}, \frac{7}{8})$. Also, an optimized two-step method with three off-step points proposed in [13] is such that r, s, v are in the interval $0 < r, s, t < 2$.
Here, on the interval $[x_n, x_{n+3}]$, we consider the following subintervals; $[x_n, x_{n+1}]$, $[x_{n+1}, x_{n+2}]$ and $[x_{n+2}, x_{n+3}]$ where the points (r, s, u), (r', s', u') and (r'', s'', u'') are the assigned off-set points in each of the subinterval respectively as $x_n < x_{n+r} < x_{n+s} < x_{n+u} < x_{n+1} < x_{n+2} < x_{n+3}$, $< x_n < x_{n+1} < x_{n+r'} < x_{n+s'} < x_{n+u'} < x_{n+2} < x_{n+3}$ and $< x_n < x_{n+1} < x_{n+2} < x_{n+r''} < x_{n+s''} < x_{n+u''} < x_{n+3}$. where $(r, s, u) = (\frac{1}{4}, \frac{1}{2}, \frac{3}{4})$, $(r', s', u') = (\frac{5}{4}, \frac{3}{2}, \frac{7}{4})$ and $(r'', s'', u'') = (\frac{9}{4}, \frac{5}{2}, \frac{11}{4})$.
Consider the approximation $w(x)$ of $y(x)$ given by the polynomial

$$y(x) \simeq w(x) = \sum_{i=0}^{9} a_i x^i \,. \tag{2}$$

and the third derivative

$$y'''(x) \simeq w'''(x) = \sum_{i=3}^{9} \rho_i i(i-1)(i-2)x^{i-3} \tag{3}$$

where a_i are coefficients to be determined.

2.1. Specifications 1

To obtain the method in the interval $x_n < x_{n+r} < x_{n+s} < x_{n+u} < x_{n+1} < x_{n+2} < x_{n+3}$, we Interpolate (2) at $x = x_{n+i}$, $i = 0, 1, 2$ and collocating (3) at the points $x = x_{n+\frac{i}{4}}$, $i = 0, 1, \ldots, 4, 8, 12$, a system of 10 equations with 10 unknown a_i, $i = 0, 1, \ldots, 9$. This system can be written in matrix form as

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & \cdots & x_n^9 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & \cdots & x_{n+1}^9 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & \cdots & x_{n+2}^9 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & \cdots & 504x_n^6 \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{1}{4}} & 60x_{n+\frac{1}{4}}^2 & 120x_{n+\frac{1}{4}}^3 & \cdots & 504x_{n+\frac{1}{4}}^6 \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{1}{2}} & 60x_{n+\frac{1}{2}}^2 & 120x_{n+\frac{1}{2}}^3 & \cdots & 504x_{n+\frac{1}{2}}^6 \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{3}{4}} & 60x_{n+\frac{3}{4}}^2 & 120x_{n+\frac{3}{4}}^3 & \cdots & 504x_{n+\frac{3}{4}}^6 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & \cdots & 504x_{n+1}^6 \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & \cdots & 504x_{n+2}^6 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & \cdots & 504x_{n+3}^6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ h^3 f_n \\ h^3 f_{n+\frac{1}{4}} \\ h^3 f_{n+\frac{1}{2}} \\ h^3 f_{n+\frac{3}{4}} \\ h^3 f_{n+1} \\ h^3 f_{n+2} \\ h^3 f_{n+3} \end{pmatrix}$$

where $y_{n+i}^{(j)} \simeq y^{(j)}(x_{n+i})$, $f_{n+i} \simeq f(x_{n+i}, y_{n+i}, y'_{n+i}, y''_{n+i})$, On solving the above system and obtaining the coefficients a_i's (not included here) which are then substituted into the polynomial (2), the following formula is obtained:

$$w(x) = \sum_{i=0}^{2} \alpha_i(x) y_{n+i} + h^3 \left(\sum_{i=0}^{3} \beta_i(x) f_{n+i} + \sum_{i=1}^{3} \beta_{\frac{i}{4}}(x) f_{n+\frac{i}{4}} \right) \tag{4}$$

where the α and β are continuous coefficients (which are large expressions and are not included here, but can be easily obtained with the help of Maple software).
Evaluating $w(x)$ in (4) at the points $x = x_{n+\frac{1}{4}}, x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}$ and x_{n+3} and after some simplifications, we obtain the following methods:

$$\left.\begin{aligned} y_{n+\frac{1}{4}} &= \frac{21y_n}{32} + \frac{7y_{n+1}}{16} - \frac{3y_{n+2}}{32} + h^3\left(\frac{34621f_n}{8847360} - \frac{18589f_{n+\frac{1}{4}}}{1013760} + \frac{25783f_{n+\frac{1}{2}}}{368640} - \frac{46109f_{n+\frac{3}{4}}}{829440} + \frac{79271f_{n+1}}{1474560} + \frac{3019f_{n+2}}{2949120} - \frac{403f_{n+3}}{18247680}\right) \\ y_{n+\frac{1}{2}} &= \frac{3y_n}{8} + \frac{3y_{n+1}}{4} - \frac{y_{n+2}}{8} + h^3\left(\frac{4997f_n}{967680} - \frac{5459f_{n+\frac{1}{4}}}{194040} + \frac{221f_{n+\frac{1}{2}}}{2520} - \frac{341f_{n+\frac{3}{4}}}{4536} + \frac{23099f_{n+1}}{322560} + \frac{3083f_{n+2}}{2257920} - \frac{941f_{n+3}}{31933440}\right) \\ y_{n+\frac{3}{2}} &= \frac{5y_n}{32} + \frac{15y_{n+1}}{16} - \frac{3y_{n+2}}{32} + h^3\left(\frac{23831f_n}{6193152} - \frac{218389f_{n+\frac{1}{4}}}{9934848} + \frac{159311f_{n+\frac{1}{2}}}{2580480} - \frac{343927f_{n+\frac{3}{4}}}{5806080} + \frac{221689f_{n+1}}{4128768} + \frac{36983f_{n+2}}{36126720} - \frac{45137f_{n+3}}{2043740160}\right) \\ y_{n+3} &= y_n - 3y_{n+1} + 3y_{n+2} - h^3\left(\frac{149f_n}{1890} - \frac{8192f_{n+\frac{1}{4}}}{24255} + \frac{104f_{n+\frac{1}{2}}}{315} + \frac{512f_{n+\frac{3}{4}}}{2835} - \frac{251f_{n+1}}{315} - \frac{983f_{n+2}}{2205} - \frac{115f_{n+3}}{12474}\right) \end{aligned}\right\}$$

If $w'(x)$ of (4) is evaluated at $x = x_{\frac{i}{4}}$, $i = 0(1)4, 8, 12$, following formulae for approximating the first derivatives are obtained:

$$\left.\begin{aligned}
y'_n &= -\tfrac{3y_n}{2h} + \tfrac{2y_{n+1}}{h} - \tfrac{y_{n+2}}{2h} + h^2\left(\frac{139f_n}{5670} - \frac{5056f_{n+\frac{1}{4}}}{72765} + \frac{1784f_{n+\frac{1}{2}}}{4725} - \frac{12352f_{n+\frac{3}{4}}}{42525} + \frac{2161f_{n+1}}{7560} + \frac{181f_{n+2}}{33075} - \frac{443f_{n+3}}{3742200}\right)\\
y'_{n+\frac{1}{4}} &= -\tfrac{5y_n}{4h} + \tfrac{3y_{n+1}}{2h} - \tfrac{y_{n+2}}{4h} + h^2\left(\frac{916957f_n}{92897280} - \frac{175423f_{n+\frac{1}{4}}}{2661120} + \frac{1696777f_{n+\frac{1}{2}}}{9676800} - \frac{234391f_{n+\frac{3}{4}}}{1555200} + \frac{887681f_{n+1}}{6193152} + \frac{422549f_{n+2}}{154828800} - \frac{902123f_{n+3}}{15328051200}\right)\\
y'_{n+\frac{1}{2}} &= -\tfrac{y_n}{h} + \tfrac{y_{n+1}}{h} - h^2\left(\frac{73f_n}{725760} + \frac{577f_{n+\frac{1}{4}}}{72765} + \frac{121f_{n+\frac{1}{2}}}{4725} + \frac{337f_{n+\frac{3}{4}}}{42525} + \frac{5f_{n+1}}{48384} - \frac{f_{n+2}}{8467200} + \frac{f_{n+3}}{119750400}\right)\\
y'_{n+\frac{3}{2}} &= -\tfrac{3y_n}{4h} + \tfrac{y_{n+1}}{2h} + \tfrac{y_{n+2}}{4h} - h^2\left(\frac{958907f_n}{92897280} - \frac{1048031f_{n+\frac{1}{4}}}{18627840} + \frac{1702583f_{n+\frac{1}{2}}}{9676800} - \frac{219329f_{n+\frac{3}{4}}}{1555200} + \frac{4451939f_{n+1}}{30965760} + \frac{2958517f_{n+2}}{1083801600} - \frac{128971f_{n+3}}{2189721600}\right)\\
y'_{n+1} &= -\tfrac{y_n}{2h} + \tfrac{y_{n+2}}{2h} - h^2\left(\frac{467f_n}{22680} - \frac{8768f_{n+\frac{1}{4}}}{72765} + \frac{1516f_{n+\frac{1}{2}}}{4725} + \frac{14528f_{n+\frac{3}{4}}}{42525} + \frac{533f_{n+1}}{1890} + \frac{1447f_{n+2}}{264600} - \frac{221f_{n+3}}{1871100}\right)\\
y'_{n+2} &= \tfrac{y_n}{2h} - \tfrac{2y_{n+1}}{h} + \tfrac{3y_{n+2}}{2h} + h^2\left(\frac{607f_n}{5670} - \frac{6592f_{n+\frac{1}{4}}}{10395} + \frac{7304f_{n+\frac{1}{2}}}{4725} - \frac{10816f_{n+\frac{3}{4}}}{6075} + \frac{7951f_{n+1}}{7560} + \frac{208f_{n+2}}{4725} - \frac{2693f_{n+3}}{3742200}\right)\\
y'_{n+3} &= \tfrac{3y_n}{2h} - \tfrac{4y_{n+1}}{h} + \tfrac{5y_{n+2}}{2h} - h^2\left(\frac{22889f_n}{22680} - \frac{56384f_{n+\frac{1}{4}}}{10395} + \frac{52396f_{n+\frac{1}{2}}}{4725} - \frac{442688f_{n+\frac{3}{4}}}{42525} + \frac{169f_{n+1}}{54} - \frac{44249f_{n+2}}{37800} - \frac{105641f_{n+3}}{1871100}\right)
\end{aligned}\right\}$$

Similarly, evaluating $w''(x)$ of (4), at the points $x = x_{\frac{i}{4}}$, $i = 0(1)4, 8, 12$, we obtain the formulae which approximates the second derivatives:

$$\left.\begin{aligned}
y''_n &= \tfrac{y_n}{h^2} - \tfrac{2y_{n+1}}{h^2} + \tfrac{y_{n+2}}{h^2} - h\left(\frac{113f_n}{945} + \frac{64f_{n+\frac{1}{4}}}{693} + \frac{232f_{n+\frac{1}{2}}}{315} - \frac{1472f_{n+\frac{3}{4}}}{2835} + \frac{1411f_{n+1}}{2520} - \frac{f_{n+2}}{90} - \frac{61f_{n+3}}{249480}\right)\\
y''_{n+\frac{1}{4}} &= \tfrac{y_n}{h^2} - \tfrac{2y_{n+1}}{h^2} + \tfrac{y_{n+2}}{h^2} - h\left(\frac{23999f_n}{645120} - \frac{59981f_{n+\frac{1}{4}}}{388080} + \frac{2921f_{n+\frac{1}{2}}}{3360} - \frac{8929f_{n+\frac{3}{4}}}{15120} + \frac{53261f_{n+1}}{92160} + \frac{16361f_{n+2}}{1505280} - \frac{4967f_{n+3}}{21288960}\right)\\
y''_{n+\frac{1}{2}} &= \tfrac{y_n}{h^2} - \tfrac{2y_{n+1}}{h^2} + \tfrac{y_{n+2}}{h^2} - h\left(\frac{145f_n}{3456} - \frac{6452f_{n+\frac{1}{4}}}{24255} + \frac{73f_{n+\frac{1}{2}}}{105} - \frac{1564f_{n+\frac{3}{4}}}{2835} + \frac{22973f_{n+1}}{40320} + \frac{1031f_{n+2}}{94080} - \frac{947f_{n+3}}{3991680}\right)\\
y''_{n+\frac{3}{2}} &= \tfrac{y_n}{h^2} - \tfrac{2y_{n+1}}{h^2} + \tfrac{y_{n+2}}{h^2} - h\left(\frac{77173hf_n}{1935360} - \frac{96433hf_{n+\frac{1}{4}}}{388080} + \frac{5561hf_{n+\frac{1}{2}}}{10080} - \frac{4433hf_{n+\frac{3}{4}}}{6480} + \frac{374419hf_{n+1}}{645120} + \frac{49123hf_{n+2}}{4515840} - \frac{14941hf_{n+3}}{63866880}\right)\\
y''_{n+1} &= \tfrac{y_n}{h^2} - \tfrac{2y_{n+1}}{h^2} + \tfrac{y_{n+2}}{h^2} - h\left(\frac{107f_n}{2520} - \frac{6464f_{n+\frac{1}{4}}}{24255} + \frac{64f_{n+\frac{1}{2}}}{105} - \frac{832f_{n+\frac{3}{4}}}{945} - \frac{61f_{n+1}}{126} + \frac{13f_{n+2}}{1176} - \frac{f_{n+3}}{4158}\right)\\
y''_{n+2} &= \tfrac{y_n}{h^2} - \tfrac{2y_{n+1}}{h^2} + \tfrac{y_{n+2}}{h^2} + h\left(\frac{374f_n}{945} - \frac{56512f_{n+\frac{1}{4}}}{24255} + \frac{584f_{n+\frac{1}{2}}}{105} - \frac{17984f_{n+\frac{3}{4}}}{2835} + \frac{1243f_{n+1}}{360} + \frac{391f_{n+2}}{1470} - \frac{739f_{n+3}}{249480}\right)\\
y''_{n+3} &= \tfrac{y_n}{h^2} - \tfrac{2y_{n+1}}{h^2} + \tfrac{y_{n+2}}{h^2} - h\left(\frac{5165f_n}{1512} - \frac{453952f_{n+\frac{1}{4}}}{24255} + \frac{1792f_{n+\frac{1}{2}}}{45} - \frac{112064f_{n+\frac{3}{4}}}{2835} - \frac{9607f_{n+1}}{630} - \frac{34877f_{n+2}}{17640} - \frac{16573f_{n+3}}{62370}\right)
\end{aligned}\right\}$$

2.2. *Specification 2*

To obtain the method in the interval $< x_n < x_{n+1} < x_{n+r'} < x_{n+s'} < x_{n+u'} < x_{n+2} < x_{n+3}$, we Interpolate (2) at $x = x_{n+i}$, $i = 0, 1, 2$ and collocating (3) at the points $x = x_{n+\frac{i}{4}}$, $i = 0, 4(1)8, 12$, a system of 10 equations with 10 unknown a_i, $i = 0, 1, \ldots, 9$ in matrix form is given as

$$\begin{pmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & \cdots & x_n^9\\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & \cdots & x_{n+1}^9\\
1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & \cdots & x_{n+2}^9\\
0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & \cdots & 504x_n^6\\
0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & \cdots & 504x_{n+1}^6\\
0 & 0 & 0 & 6 & 24x_{n+\frac{5}{4}} & 60x_{n+\frac{5}{4}}^2 & 120x_{n+\frac{5}{4}}^3 & \cdots & 504x_{n+\frac{5}{4}}^6\\
0 & 0 & 0 & 6 & 24x_{n+\frac{3}{2}} & 60x_{n+\frac{3}{2}}^2 & 120x_{n+\frac{3}{2}}^3 & \cdots & 504x_{n+\frac{3}{2}}^6\\
0 & 0 & 0 & 6 & 24x_{n+\frac{7}{4}} & 60x_{n+\frac{7}{4}}^2 & 120x_{n+\frac{7}{4}}^3 & \cdots & 504x_{n+\frac{7}{4}}^6\\
0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & \cdots & 504x_{n+2}^6\\
0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & \cdots & 504x_{n+3}^6
\end{pmatrix}
\begin{pmatrix} a_0\\ a_1\\ a_2\\ a_3\\ a_4\\ a_5\\ a_6\\ a_7\\ a_8\\ a_9 \end{pmatrix}
=
\begin{pmatrix} y_n\\ y_{n+1}\\ y_{n+2}\\ h^3 f_n\\ h^3 f_{n+1}\\ h^3 f_{n+\frac{5}{4}}\\ h^3 f_{n+\frac{3}{2}}\\ h^3 f_{n+\frac{7}{4}}\\ h^3 f_{n+2}\\ h^3 f_{n+3} \end{pmatrix}$$

On solving the above system and obtaining the coefficients a_i's (not included here), are then substituted into the polynomial (2), the following formula is obtained:

$$w(x) = \sum_{i=0}^{2} \alpha_i(x) y_{n+i} + h^3 \left(\sum_{i=0}^{3} \beta_i(x) f_{n+i} + \sum_{i=5}^{7} \beta_{\frac{i}{4}}(x) f_{n+\frac{i}{4}} \right) \tag{5}$$

where the α and β are continuous coefficients (which are large expressions and are not included here, but can be easily obtained with the help of Maple software). Evaluating $w(x)$ in (5) at the points $x = x_{n+\frac{5}{4}}, x_{n+\frac{3}{2}}, x_{n+\frac{7}{4}}$ and x_{n+3} and after some simplifications, we obtain the following methods:

$$\left.\begin{aligned}
y_{n+\frac{5}{4}} &= -\frac{3y_n}{32}+\frac{15y_{n+1}}{16}+\frac{5y_{n+2}}{32}-h^3\left(\frac{16453f_n}{27095040}+\frac{311963f_{n+1}}{4128768}-\frac{588473f_{n+\frac{5}{4}}}{4515840}+\frac{240025f_{n+\frac{3}{2}}}{1548288}-\frac{361883f_{n+\frac{7}{4}}}{4515840}+\frac{38035f_{n+2}}{2064384}-\frac{45137f_{n+3}}{433520640}\right)\\
y_{n+\frac{3}{2}} &= -\frac{y_n}{8}+\frac{3y_{n+1}}{4}+\frac{3y_{n+2}}{8}-h^3\left(\frac{1097f_n}{1354752}+\frac{32509f_{n+1}}{322560}-\frac{2999f_{n+\frac{5}{4}}}{17640}+\frac{401f_{n+\frac{3}{2}}}{1890}-\frac{373f_{n+\frac{7}{4}}}{3528}+\frac{7939f_{n+2}}{322560}-\frac{941f_{n+3}}{6773760}\right)\\
y_{n+\frac{7}{4}} &= -\frac{3y_n}{32}+\frac{7y_{n+1}}{16}+\frac{21y_{n+2}}{32}-h^3\left(\frac{37607f_n}{61931520}+\frac{111511f_{n+1}}{1474560}-\frac{16343f_{n+\frac{5}{4}}}{129024}+\frac{180517f_{n+\frac{3}{2}}}{1105920}-\frac{9869f_{n+\frac{7}{4}}}{129024}+\frac{54527f_{n+2}}{2949120}-\frac{403f_{n+3}}{3870720}\right)\\
y_{n+3} &= y_n-3y_{n+1}+3y_{n+2}+h^3\left(\frac{71f_n}{13230}+\frac{316f_{n+1}}{315}-\frac{4864f_{n+\frac{5}{4}}}{2205}+\frac{3208f_{n+\frac{3}{2}}}{945}-\frac{4864f_{n+\frac{7}{4}}}{2205}+\frac{316f_{n+2}}{315}+\frac{71f_{n+3}}{13230}\right)
\end{aligned}\right\}$$

If $w'(x)$ of (4) is evaluated at $x = x_{\frac{i}{4}}$, $i = 0, 4(1)8, 12$, the following formulae for approximating the first derivatives are obtained:

$$\left.\begin{aligned}
y'_n &= -\frac{3y_n}{2h}+\frac{2y_{n+1}}{h}-\frac{y_{n+2}}{2h}+h^2\left(\frac{6043f_n}{198450}+\frac{13337f_{n+1}}{7560}-\frac{135488f_{n+\frac{5}{4}}}{33075}+\frac{2600f_{n+\frac{3}{2}}}{567}-\frac{83648f_{n+\frac{7}{4}}}{33075}+\frac{110f_{n+2}}{189}-\frac{2693f_{n+3}}{793800}\right)\\
y'_{n+1} &= -\frac{y_n}{2h}+\frac{y_{n+2}}{2h}-h^2\left(\frac{2573f_n}{793800}+\frac{377f_{n+1}}{945}-\frac{23872f_{n+\frac{5}{4}}}{33075}+\frac{332f_{n+\frac{3}{2}}}{405}-\frac{14272f_{n+\frac{7}{4}}}{33075}+\frac{149f_{n+2}}{1512}-\frac{221f_{n+3}}{396900}\right)\\
y'_{n+\frac{5}{4}} &= -\frac{y_n}{4h}-\frac{y_{n+1}}{2h}+\frac{3y_{n+2}}{4h}-h^2\left(\frac{5264363f_n}{3251404800}+\frac{6257533f_{n+1}}{30965760}-\frac{399871f_{n+\frac{5}{4}}}{1209600}+\frac{493205f_{n+\frac{3}{2}}}{1161216}-\frac{1789537f_{n+\frac{7}{4}}}{8467200}+\frac{304673f_{n+2}}{6193152}-\frac{128971f_{n+3}}{464486400}\right)\\
y'_{n+\frac{3}{2}} &= -\frac{y_{n+1}}{h}+\frac{y_{n+2}}{h}-h^2\left(\frac{f_n}{25401600}+\frac{23f_{n+1}}{241920}+\frac{263f_{n+\frac{5}{4}}}{33075}+\frac{29f_{n+\frac{3}{2}}}{1134}+\frac{263f_{n+\frac{7}{4}}}{33075}+\frac{23f_{n+2}}{241920}+\frac{f_{n+3}}{25401600}\right)\\
y'_{n+\frac{7}{4}} &= \frac{y_n}{4h}-\frac{3y_{n+1}}{2h}+\frac{5y_{n+2}}{4h}+h^2\left(\frac{5265037f_n}{3251404800}+\frac{6242651f_{n+1}}{30965760}-\frac{2879903f_{n+\frac{5}{4}}}{8467200}+\frac{2461463f_{n+\frac{3}{2}}}{5806080}-\frac{1870343f_{n+\frac{7}{4}}}{8467200}+\frac{1508483f_{n+2}}{30965760}-\frac{902123f_{n+3}}{3251404800}\right)\\
y'_{n+2} &= \frac{y_n}{2h}-\frac{2y_{n+1}}{h}+\frac{3y_{n+2}}{2h}+h^2\left(\frac{643f_n}{198450}+\frac{3047f_{n+1}}{7560}-\frac{22208f_{n+\frac{5}{4}}}{33075}+\frac{2488f_{n+\frac{3}{2}}}{2835}-\frac{12608f_{n+\frac{7}{4}}}{33075}+\frac{97f_{n+2}}{945}-\frac{443f_{n+3}}{793800}\right)\\
y'_{n+3} &= \frac{3y_n}{2h}-\frac{4y_{n+1}}{h}+\frac{5y_{n+2}}{2h}+h^2\left(\frac{3697f_n}{793800}+\frac{1972f_{n+1}}{945}-\frac{27584f_{n+\frac{5}{4}}}{4725}+\frac{27436f_{n+\frac{3}{2}}}{2835}-\frac{244928f_{n+\frac{7}{4}}}{33075}+\frac{24713f_{n+2}}{7560}+\frac{2183f_{n+3}}{56700}\right),
\end{aligned}\right\}$$

Similarly, evaluating $w''(x)$ of (4), at the points $x = x_{\frac{i}{4}}$, $i = 0, 4(1)8, 12$, we obtain the formulae which approximates the second derivatives:

$$\left.\begin{aligned}
y''_n &= \frac{y_n}{h^2}-\frac{2y_{n+1}}{h^2}+\frac{y_{n+2}}{h^2}-h\left(\frac{278f_n}{1323}+\frac{16091f_{n+1}}{2520}-\frac{35008f_{n+\frac{5}{4}}}{2205}+\frac{488f_{n+\frac{3}{2}}}{27}-\frac{22336f_{n+\frac{7}{4}}}{2205}+\frac{1481f_{n+2}}{630}-\frac{739f_{n+3}}{52920}\right)\\
y''_{n+1} &= \frac{y_n}{h^2}-\frac{2y_{n+1}}{h^2}+\frac{y_{n+2}}{h^2}+h\left(\frac{23f_n}{3528}+\frac{13f_{n+1}}{18}-\frac{1216f_{n+\frac{5}{4}}}{735}+\frac{512f_{n+\frac{3}{2}}}{315}-\frac{1984f_{n+\frac{7}{4}}}{2205}+\frac{169f_{n+2}}{840}-\frac{f_{n+3}}{882}\right)\\
y''_{n+\frac{5}{4}} &= \frac{y_n}{h^2}-\frac{2y_{n+1}}{h^2}+\frac{y_{n+2}}{h^2}+h\left(\frac{2503f_n}{387072}+\frac{523829f_{n+1}}{645120}-\frac{5633f_{n+\frac{5}{4}}}{3920}+\frac{9313f_{n+\frac{3}{2}}}{6048}-\frac{4357f_{n+\frac{7}{4}}}{5040}+\frac{41777f_{n+2}}{215040}-\frac{14941f_{n+3}}{13547520}\right)\\
y''_{n+\frac{3}{2}} &= \frac{y_n}{h^2}-\frac{2y_{n+1}}{h^2}+\frac{y_{n+2}}{h^2}+h\left(\frac{5491f_n}{846720}+\frac{32443f_{n+1}}{40320}-\frac{580f_{n+\frac{5}{4}}}{441}+\frac{1604f_{n+\frac{3}{2}}}{945}-\frac{1964f_{n+\frac{7}{4}}}{2205}+\frac{1601f_{n+2}}{8064}-\frac{947f_{n+3}}{846720}\right)\\
y''_{n+\frac{7}{4}} &= \frac{y_n}{h^2}-\frac{2y_{n+1}}{h^2}+\frac{y_{n+2}}{h^2}+h\left(\frac{5843f_n}{903168}+\frac{521837f_{n+1}}{645120}-\frac{3155f_{n+\frac{5}{4}}}{2352}+\frac{2671f_{n+\frac{3}{2}}}{1440}-\frac{27127f_{n+\frac{7}{4}}}{35280}+\frac{41113f_{n+2}}{215040}-\frac{4967f_{n+3}}{4515840}\right)\\
y''_{n+2} &= \frac{y_n}{h^2}-\frac{2y_{n+1}}{h^2}+\frac{y_{n+2}}{h^2}+h\left(\frac{43f_n}{6615}+\frac{2021f_{n+1}}{2520}-\frac{64f_{n+\frac{5}{4}}}{49}+\frac{1672f_{n+\frac{3}{2}}}{945}-\frac{1216f_{n+\frac{7}{4}}}{2205}+\frac{59f_{n+2}}{210}-\frac{61f_{n+3}}{52920}\right)\\
y''_{n+3} &= \frac{y_n}{h^2}-\frac{2y_{n+1}}{h^2}+\frac{y_{n+2}}{h^2}-h\left(\frac{13f_n}{1512}-\frac{2113f_{n+1}}{630}+\frac{5440f_{n+\frac{5}{4}}}{441}-\frac{20288f_{n+\frac{3}{2}}}{945}+\frac{5696f_{n+\frac{7}{4}}}{315}-\frac{18619f_{n+2}}{2520}-\frac{2851f_{n+3}}{13230}\right)
\end{aligned}\right\}$$

2.3. Specification 3

Finally, obtaining the method in the interval $x_n < x_{n+1} < x_{n+2} < x_{n+r''} < x_{n+s''} < x_{n+u''} < x_{n+3}$, we Interpolate (2) at $x = x_{n+i}$, $i = 0, 1, 2$ and collocating (3) at the points $x = x_{n+\frac{i}{4}}$, $i = 0, 4, 8(1)12$, a system of 10 equations with 10 unknown a_i, $i = 0, 1, \ldots, 9$ in matrix form is given as

$$\begin{pmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & \cdots & x_n^9\\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & \cdots & x_{n+1}^9\\
1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & \cdots & x_{n+2}^9\\
0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & \cdots & 504x_n^6\\
0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & \cdots & 504x_{n+1}^6\\
0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & \cdots & 504x_{n+2}^6\\
0 & 0 & 0 & 6 & 24x_{n+\frac{9}{4}} & 60x_{n+\frac{9}{4}}^2 & 120x_{n+\frac{9}{4}}^3 & \cdots & 504x_{n+\frac{9}{4}}^6\\
0 & 0 & 0 & 6 & 24x_{n+\frac{5}{2}} & 60x_{n+\frac{5}{2}}^2 & 120x_{n+\frac{5}{2}}^3 & \cdots & 504x_{n+\frac{5}{2}}^6\\
0 & 0 & 0 & 6 & 24x_{n+\frac{11}{4}} & 60x_{n+\frac{11}{4}}^2 & 120x_{n+\frac{11}{4}}^3 & \cdots & 504x_{n+\frac{11}{4}}^6\\
0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & \cdots & 504x_{n+3}^6
\end{pmatrix}
\begin{pmatrix} a_0\\ a_1\\ a_2\\ a_3\\ a_4\\ a_5\\ a_6\\ a_7\\ a_8\\ a_9 \end{pmatrix}
=
\begin{pmatrix} y_n\\ y_{n+1}\\ y_{n+2}\\ h^3 f_n\\ h^3 f_{n+1}\\ h^3 f_{n+2}\\ h^3 f_{n+\frac{9}{4}}\\ h^3 f_{n+\frac{5}{2}}\\ h^3 f_{n+\frac{11}{4}}\\ h^3 f_{n+3} \end{pmatrix}$$

Thus, solving the above system and obtaining the coefficients a_i's, then substituted into the polynomial (2), the following formula is obtained:

$$w(x) = \sum_{i=0}^{2} \alpha_i(x) y_{n+i} + h^3 \left(\sum_{i=0}^{3} \beta_i(x) f_{n+i} + \sum_{i=9}^{11} \beta_{\frac{i}{4}}(x) f_{n+\frac{i}{4}} \right) \qquad (6)$$

where the α and β are continuous coefficients (which are large expressions and are not included here, but can be easily obtained with the help of Maple software). Evaluating $w(x)$ in (24) at the points $x = x_{n+\frac{9}{4}}, x_{n+\frac{5}{2}}, x_{n+\frac{11}{4}}$ and x_{n+3} and after some simplifications, we obtain the following methods:

$$\left.\begin{aligned}
y_{n+\frac{9}{4}} &= \tfrac{5y_n}{32} - \tfrac{9y_{n+1}}{16} + \tfrac{45y_{n+2}}{32} + h^3\left(\tfrac{996379f_n}{681246720} + \tfrac{826499f_{n+1}}{12042240} + \tfrac{97453f_{n+2}}{1376256} + \tfrac{60029f_{n+\frac{9}{4}}}{1935360} - \tfrac{97477f_{n+\frac{5}{2}}}{860160} + \tfrac{247559f_{n+\frac{11}{4}}}{3311616} - \tfrac{33373f_{n+3}}{2064384}\right)\\
y_{n+\frac{5}{2}} &= \tfrac{3y_n}{8} - \tfrac{5y_{n+1}}{4} + \tfrac{15y_{n+2}}{8} + h^3\left(\tfrac{111341f_n}{31933440} + \tfrac{374389f_{n+1}}{2257920} + \tfrac{14657f_{n+2}}{64512} + \tfrac{169f_{n+\frac{9}{4}}}{22680} - \tfrac{533f_{n+\frac{5}{2}}}{2520} + \tfrac{6007f_{n+\frac{11}{4}}}{38808} - \tfrac{6721f_{n+3}}{193536}\right)\\
y_{n+\frac{11}{4}} &= \tfrac{21y_n}{32} - \tfrac{33y_{n+1}}{16} + \tfrac{77y_{n+2}}{32} + h^3\left(\tfrac{10073f_n}{1658880} + \tfrac{6018419f_{n+1}}{20643840} + \tfrac{691801f_{n+2}}{1474560} - \tfrac{10439f_{n+\frac{9}{4}}}{165888} - \tfrac{21131f_{n+\frac{5}{2}}}{73728} + \tfrac{154817f_{n+\frac{11}{4}}}{645120} - \tfrac{492349f_{n+3}}{8847360}\right)\\
y_{n+3} &= y_n - 3y_{n+1} + 3y_{n+2} + h^3\left(\tfrac{115f_n}{12474} + \tfrac{983f_{n+1}}{2205} + \tfrac{251f_{n+2}}{315} - \tfrac{512f_{n+\frac{9}{4}}}{2835} - \tfrac{104f_{n+\frac{5}{2}}}{315} + \tfrac{8192f_{n+\frac{11}{4}}}{24255} - \tfrac{149f_{n+3}}{1890}\right)
\end{aligned}\right\}$$

If $w'(x)$ of (4) is evaluated at $x = x_{\frac{i}{4}}$, $i = 0, 4(1)8, 12$, following methods for approximating the first derivatives are obtained:

$$\left.\begin{aligned}
y'_n &= -\tfrac{3y_n}{2h} + \tfrac{2y_{n+1}}{h} - \tfrac{y_{n+2}}{2h} + h^2\left(\tfrac{39883f_n}{935550} + \tfrac{132803f_{n+1}}{264600} - \tfrac{4087f_{n+2}}{945} + \tfrac{454208f_{n+\frac{9}{4}}}{42525} - \tfrac{50056f_{n+\frac{5}{2}}}{4725} + \tfrac{357824f_{n+\frac{11}{4}}}{72765} - \tfrac{20207f_{n+3}}{22680}\right)\\
y'_{n+1} &= -\tfrac{y_n}{2h} + \tfrac{y_{n+2}}{2h} - h^2\left(\tfrac{259f_n}{48600} + \tfrac{11833f_{n+1}}{66150} - \tfrac{4939f_{n+2}}{7560} + \tfrac{71872f_{n+\frac{9}{4}}}{42525} - \tfrac{8084f_{n+\frac{5}{2}}}{4725} + \tfrac{5312f_{n+\frac{11}{4}}}{6615} - \tfrac{1661f_{n+3}}{11340}\right)\\
y'_{n+2} &= \tfrac{y_n}{2h} - \tfrac{2y_{n+1}}{h} + \tfrac{3y_{n+2}}{2h} + h^2\left(\tfrac{4423f_n}{935550} + \tfrac{8219f_{n+1}}{37800} + \tfrac{22f_{n+2}}{189} + \tfrac{10688f_{n+\frac{9}{4}}}{42525} - \tfrac{328f_{n+\frac{5}{2}}}{675} + \tfrac{3008f_{n+\frac{11}{4}}}{10395} - \tfrac{1361f_{n+3}}{22680}\right)\\
y'_{n+\frac{9}{4}} &= \tfrac{3y_n}{4h} - \tfrac{5y_{n+1}}{2h} + \tfrac{7y_{n+2}}{4h} + h^2\left(\tfrac{106886797f_n}{15328051200} + \tfrac{359414603f_{n+1}}{1083801600} + \tfrac{14053789f_{n+2}}{30965760} + \tfrac{60743f_{n+\frac{9}{4}}}{10886400} - \tfrac{4098743f_{n+\frac{5}{2}}}{9676800} + \tfrac{5766623f_{n+\frac{11}{4}}}{18627840} - \tfrac{6451643f_{n+3}}{92897280}\right)\\
y'_{n+\frac{5}{2}} &= \tfrac{y_n}{h} - \tfrac{3y_{n+1}}{h} + \tfrac{2y_{n+2}}{h} + h^2\left(\tfrac{1103999f_n}{119750400} + \tfrac{3774721f_{n+1}}{8467200} + \tfrac{192743f_{n+2}}{241920} - \tfrac{8017f_{n+\frac{9}{4}}}{42525} - \tfrac{1681f_{n+\frac{5}{2}}}{4725} + \tfrac{23999f_{n+\frac{11}{4}}}{72765} - \tfrac{57289f_{n+3}}{725760}\right)\\
y'_{n+\frac{11}{4}} &= \tfrac{5y_n}{4h} - \tfrac{7y_{n+1}}{2h} + \tfrac{9y_{n+2}}{4h} + h^2\left(\tfrac{25105411f_n}{2189721600} + \tfrac{606913043f_{n+1}}{1083801600} + \tfrac{7056257f_{n+2}}{6193152} - \tfrac{4098337f_{n+\frac{9}{4}}}{10886400} - \tfrac{2296823f_{n+\frac{5}{2}}}{9676800} + \tfrac{6636359f_{n+\frac{11}{4}}}{18627840} - \tfrac{8237603f_{n+3}}{92897280}\right)\\
y'_{n+3} &= \tfrac{3y_n}{2h} - \tfrac{4y_{n+1}}{h} + \tfrac{5y_{n+2}}{2h} + h^2\left(\tfrac{51307f_n}{3742200} + \tfrac{6371f_{n+1}}{9450} + \tfrac{11197f_{n+2}}{7560} - \tfrac{23872f_{n+\frac{9}{4}}}{42525} - \tfrac{556f_{n+\frac{5}{2}}}{4725} + \tfrac{4544f_{n+\frac{11}{4}}}{10395} - \tfrac{1063f_{n+3}}{11340}\right)
\end{aligned}\right\}$$

Similarly, evaluating $w''(x)$ of (4), at the points $x = x_{\frac{i}{4}}$, $i = 0, 4(1)8, 12$, we obtain the formulae which approximates the second derivatives:

$$\left.\begin{aligned}
y''_n &= \tfrac{y_n}{h^2} - \tfrac{2y_{n+1}}{h^2} + \tfrac{y_{n+2}}{h^2} - h\left(\tfrac{7999f_n}{31185} + \tfrac{3859f_{n+1}}{2520} - \tfrac{10109f_{n+2}}{630} + \tfrac{112576f_{n+\frac{9}{4}}}{2835} - \tfrac{2488f_{n+\frac{5}{2}}}{63} + \tfrac{12736f_{n+\frac{11}{4}}}{693} - \tfrac{25229f_{n+3}}{7560}\right)\\
y''_{n+1} &= \tfrac{y_n}{h^2} - \tfrac{2y_{n+1}}{h^2} + \tfrac{y_{n+2}}{h^2} + h\left(\tfrac{1013f_n}{83160} + \tfrac{793f_{n+1}}{4410} - \tfrac{2231f_{n+2}}{840} + \tfrac{832f_{n+\frac{9}{4}}}{135} - \tfrac{1856f_{n+\frac{5}{2}}}{315} + \tfrac{21568f_{n+\frac{11}{4}}}{8085} - \tfrac{299f_{n+3}}{630}\right)\\
y''_{n+2} &= \tfrac{y_n}{h^2} - \tfrac{2y_{n+1}}{h^2} + \tfrac{y_{n+2}}{h^2} + h\left(\tfrac{8f_n}{891} + \tfrac{8059f_{n+1}}{17640} + \tfrac{269f_{n+2}}{210} - \tfrac{3008f_{n+\frac{9}{4}}}{2835} + \tfrac{88f_{n+\frac{5}{2}}}{315} + \tfrac{192f_{n+\frac{11}{4}}}{2695} - \tfrac{55f_{n+3}}{1512}\right)\\
y''_{n+\frac{9}{4}} &= \tfrac{y_n}{h^2} - \tfrac{2y_{n+1}}{h^2} + \tfrac{y_{n+2}}{h^2} + h\left(\tfrac{52169f_n}{5806080} + \tfrac{2062307f_{n+1}}{4515840} + \tfrac{888467f_{n+2}}{645120} - \tfrac{39223f_{n+\frac{9}{4}}}{45360} + \tfrac{319f_{n+\frac{5}{2}}}{1440} + \tfrac{3149f_{n+\frac{11}{4}}}{35280} - \tfrac{75403f_{n+3}}{1935360}\right)\\
y''_{n+\frac{5}{2}} &= \tfrac{y_n}{h^2} - \tfrac{2y_{n+1}}{h^2} + \tfrac{y_{n+2}}{h^2} + h\left(\tfrac{11951f_n}{1330560} + \tfrac{128917f_{n+1}}{282240} + \tfrac{18367f_{n+2}}{13440} - \tfrac{692f_{n+\frac{9}{4}}}{945} + \tfrac{23f_{n+\frac{5}{2}}}{63} + \tfrac{116f_{n+\frac{11}{4}}}{1617} - \tfrac{1487f_{n+3}}{40320}\right)\\
y''_{n+\frac{11}{4}} &= \tfrac{y_n}{h^2} - \tfrac{2y_{n+1}}{h^2} + \tfrac{y_{n+2}}{h^2} + h\left(\tfrac{573899f_n}{63866880} + \tfrac{2062267f_{n+1}}{4515840} + \tfrac{59125f_{n+2}}{43008} - \tfrac{4997f_{n+\frac{9}{4}}}{6480} + \tfrac{1087f_{n+\frac{5}{2}}}{2016} + \tfrac{7899f_{n+\frac{11}{4}}}{43120} - \tfrac{80579f_{n+3}}{1935360}\right)\\
y''_{n+3} &= \tfrac{y_n}{h^2} - \tfrac{2y_{n+1}}{h^2} + \tfrac{y_{n+2}}{h^2} + h\left(\tfrac{2239f_n}{249480} + \tfrac{403f_{n+1}}{882} + \tfrac{3419f_{n+2}}{2520} - \tfrac{1984f_{n+\frac{9}{4}}}{2835} + \tfrac{128f_{n+\frac{5}{2}}}{315} + \tfrac{10432f_{n+\frac{11}{4}}}{24255} + \tfrac{11f_{n+3}}{270}\right)
\end{aligned}\right\}$$

3. Analysis of the Method

3.1. Local truncation error and order

The linear difference operators associated with the formula in (4) is of the form

$$\mathcal{L}[z(x);\ h] \equiv w(x) - \sum_{j=0}^{2} \alpha_j z_{n+j} + h^3\left(\sum_{j=0}^{3} \beta_i(x) z'''_{n+j} + \sum_{j=1}^{3} \beta_{\frac{j}{4}} z'''_{n+\frac{j}{4}}\right) \tag{7}$$

Same can be formulated for (5) and (24). The Taylor series expansion of (7) around x yields

$$\mathcal{L}[z(x);\ h] = C_0 z(x) + C_1 h z'(x) + C_2 h^2 z''(x) + \cdots + C_p h^p z^{(p)}(x) + O(h^{(p+1)}). \tag{8}$$

where the C_i are constants. Suppose the first $p+3$ terms vanishes, that is

$$C_0 = C_1 = C_2 = \cdots = C_{p+2} = 0 \ \text{ and } \ C_{p+3} \neq 0, \ \text{ then}$$

$$\mathcal{L}[z(x);\ h] = C_{p+3} h^{p+3} y^{(p+3)}(x) + O(h^{p+4}) \tag{9}$$

Here (9) is the local Truncation Error (LTE) for (7) and equivalently for (4) and p is the order. The LTE is the amount by which the exact solution of the ODE fails to satisfy the corresponding difference operator. The method in (4) (and similarly for (5) and (24)) is said to be consistent if $p > 1$ (see [14]).
The LTEs of (2.1) are

$$C_{y_{n+1/4}} = \frac{2351}{332943851520} y^{(10)}(x_n)h^{10} + O(h^{11}) \quad (10)$$
$$C_{y_{n+1/2}} = \frac{941}{20808990720} y^{(10)}(x_n)h^{10} + O(h^{11}) \quad (11)$$
$$C_{y_{n+3/4}} = \frac{459}{4110417920} y^{(10)}(x_n)h^{10} + O(h^{11}) \quad (12)$$
$$C_{y_{n+3}} = \frac{-27}{1003520} y^{(10)}(x_n)h^{10} + O(h^{11}) \quad (13)$$

The method that approximates the first and second derivative in (2.1) and (2.1) respectively can be computed in a similar fashion.
The LTEs of (2.2) are

$$C_{y_{n+5/4}} = \frac{4667875}{66588770304} y^{(10)}(x_n)h^{10} + O(h^{11}) \quad (14)$$
$$C_{y_{n+3/2}} = \frac{20439}{183500800} y^{(10)}(x_n)h^{10} + O(h^{11}) \quad (15)$$
$$C_{y_{n+7/4}} = \frac{5519899}{33973862400} y^{(10)}(x_n)h^{10} + O(h^{11}) \quad (16)$$
$$C_{y_{n+3}} = \frac{2781}{5017600} y^{(10)}(x_n)h^{10} + O(h^{11}) \quad (17)$$

The method that approximates the first and second derivative in (2.2) and (2.2) respectively can be computed in a similar fashion.
Finally, The LTEs of (2.3) are

$$C_{y_{n+9/4}} = \frac{39621879}{20552089600} y^{(10)}(x_n)h^{10} + O(h^{11}) \quad (18)$$
$$C_{y_{n+5/2}} = \frac{10239625}{4161798144} y^{(10)}(x_n)h^{10} + O(h^{11}) \quad (19)$$
$$C_{y_{n+11/4}} = \frac{5089372651}{1664719257600} y^{(10)}(x_n)h^{10} + O(h^{11}) \quad (20)$$
$$C_{y_{n+3}} = \frac{18657}{5017600} + O[h] y^{(10)}(x_n)h^{10} + O(h^{11}) \quad (21)$$
$$(22)$$

The method that approximates the first and second derivative in (2.3) and (7) respectively can be computed in a similar fashion. The methods have order $p = 7$.

3.2. Zero-stability and convergence

A numerical method is zero-stable if the solutions remain bounded as $h \to 0$. Following the procedure in [6, 15], to show the zero-stability the block method (2.1)-(2.1) (similarly for (2.2)-(2.2) and (2.3)-(7) respectively) may be rewritten in a form such that $y^{(k)}_{n+\frac{j}{4}}$, are on the left hand side. so that the method in matrix form becomes

$$A_0^j Y_{\tau+1}^{(j)} = A_1^j Y_{\tau-1}^{(j)} + h^3 (B_1^j F_{\tau+1}^{(j)} + B_0^j F_{\tau-1}^{(j)}), \quad j = 0, 1, 2 \quad (23)$$

as $h \to 0$, (8) becomes

$$A_0^j Y_{\tau+1}^{(j)} = A_1^j Y_{\tau-1}^{(j)}, \quad j = 0, 1, 2 \quad (24)$$

where

$$Y_{\tau+1}^{(j)} = \left(y_{1/4}^{(j)}, y_{1/2}^{(j)}, y_{3/4}^{(j)}, y_1^{(j)}, y_2^{(j)}, y_3^{(j)}\right)^T,$$
$$Y_{\mu-1}^{j} = \left(y_{-3/4}^{(j)}, y_{-1/2}^{(j)}, y_{-1/4}^{(j)}, y_1^{(j)}, y_2^{(j)}, y_{-1/4}^{(j)}\right)^T,$$

A_0^j is an 18×18 identity matrix given by

$$A_0^j = \begin{pmatrix} A_0^1 & 0 & 0 \\ 0 & A_0^2 & 0 \\ 0 & 0 & A_0^3 \end{pmatrix}$$

with $A_0^1 = A_0^1 = A_0^2 = I_{6\times6}$;

$$A_1^j = \begin{pmatrix} A_1^1 & 0 & 0 \\ 0 & A_1^2 & 0 \\ 0 & 0 & A_1^3 \end{pmatrix}$$

with $A_1^1 = A_1^2 = A_1^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

The characteristic polynomial of each of the matrix A_1^1 (similarly for A_1^2 and A_1^3) is given as $\lambda^5(\lambda - 1) = 0$. The roots of the characteristic polynomial are $\lambda_j = 0$, for $j = 1, \ldots, 5$ and $\lambda_6 = 1$. Since the roots of the characteristic polynomial are all zero but one, whose modulus is 1 (see [14]), then the method ((2.1)-(2.1)) is zero stable. The method is thus convergent by the following theorem.

Theorem 3.1. *Henrici [16]. A linear multistep method is said to be convergent if it is consistent (with order $p \geq 1$) and it is zero-stable.*

3.3. Computational procedure

The methods in (2.1)-(2.1) (equivalently for the methods in (2.2)-(2.2) and (2.3)-(7) respectively) are executed in single block on the interval $[x_n, x_{n+3}]$, where $n = 0, 3, \ldots, N-3$, N, the number of subintervals must be divisible by 3 in order to obtain the last points on the integration interval $b = x_N$. Thus, the methods in (2.1)-(2.1) are put in the form $Z(y) = 0$ such that the system is solved by the Newtons method of the form

$$Y^{j+1} = Y^j - (J^j)^{-1} Z^j$$

where

$$Y = (y_0', y_0'', y_{1/4}, y_{1/4}', y_{1/4}'', y_{1/2}, y_{1/2}', y_{1/2}'', y_{3/4}, y_{3/4}', y_{3/4}'', y_1, \ldots, \ldots, y_N'')$$

J is the Jacobian matrix of Z. The starting values used in the Newtons method are the approximations given by the Taylor series

$$\begin{aligned} y_{n+\frac{j}{4}} &= y_n + j\tfrac{h}{4} y_n' + \tfrac{1}{2}\left(j\tfrac{h}{4}\right)^2 y_n'' + \tfrac{1}{6}\left(j\tfrac{h}{4}\right)^3 f_n \\ y_{n+\frac{j}{4}}' &= y_n' + j\tfrac{h}{4} y_n'' + \tfrac{1}{2}\left(j\tfrac{h}{4}\right)^2 f_n \\ y_{n+\frac{j}{4}}'' &= y_n'' + j\tfrac{h}{4} f_n \end{aligned} \quad (26)$$

for $j = 0(1)4, 8, 12$

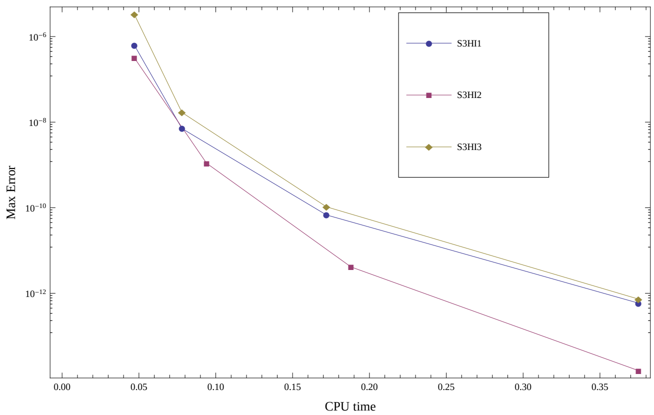

Figure 1. Efficiency plot for S3HI1, S3HI2 and S3HI3 showing the maximum error against the CPU time.

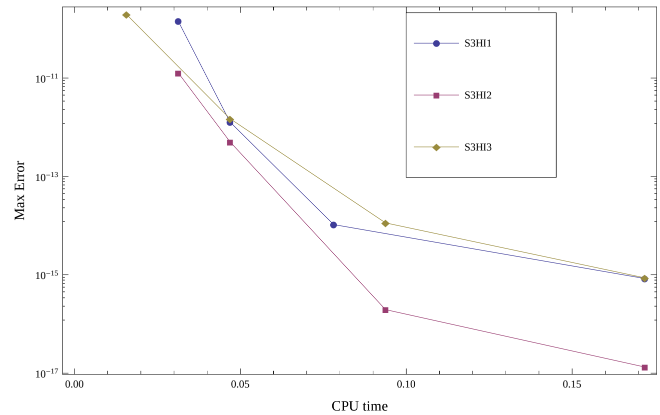

Figure 2. Efficiency plot for S3HI1, S3HI2 and S3HI3 showing the maximum error against the CPU time.

4. Numerical test Problems and results

Numerical problems are presented to show the accuracy of the three specification of methods proposed. This comparison is done based on the maximum error against the machine cost (CPU time). This is to ascertain which of the proposed methods performed favourably and in comparison to other methods in literature.
The methods used are denoted as follows:

- S3HI1: Proposed method described in specification 1;
- S3HI2: Proposed method described in specification 2;
- S3HI3: Proposed method described in specification 3.
- MAE: Maximum Absolute Error obtained in [10];
- S3HBM: 3-Step Hybrid Block Method described in [6];
- OSTBM: Step Block Method described in [3];
- TS: Total nummber of sub intervals TS on [a,b];
- N: Number of steps.

Problem 1.. Consider the third-order singularly perturbed boundary value problem discussed in [10]

$$\begin{aligned} &-\epsilon y''' + y = 81\epsilon^2 \cos(3x) + 3\epsilon \sin(3t), \quad x \in [0,1] \\ &y(0) = 0, \quad y(1) = 3\epsilon \sin(3), \quad y'(0) = 9\epsilon \end{aligned} \tag{27}$$

whose exact solution is $y(x) = 3\epsilon \sin(3x)$.

Figure 1 shows the efficiency curve of Maximum error against the CPU time for the methods S3HI1, S3HI2 and S3HI3. The maximum error are determined for $N = 6, 12, 24$ and 48 and the CPU time for each N, for S3HI1, S3HI2 and S3HI3 respectively. The curve reveals that S3HI2 performed better in terms of error than S3HI1 and S3HI3 but slightly costly in terms of its CPU time. Table 1 confirms the output of the efficiency curve for this problem. The S3HI2 performed better compared to other methods.

Problem 2.. Consider the IVP discussed in [12]

$$\begin{aligned} &y''' = 3\sin(x), \quad x \in [0,3] \\ &y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 2 \end{aligned} \tag{28}$$

whose exact solution is $y(x) = 3\cos(x) + \frac{x^2}{2}$.

The Figure 2 shows the efficiency curve of S3HI1, S3HI2 and S3HI3 where the maximum error obtained in each care is plotted against the CPU time. The maximum error are determined for N=6,12,24 and 48 and the CPU time for each N, for S3HI1, S3HI2 and S3HI3 respectively.
The efficiency curve shows that S3HI3 performed better in terms of error than S3HI1 and S3HI3 which have slightly better CPU time than S3HI2.
Table 2 shows errors obtain for Problem 2 with $h = 0.1$ using the methods S3HI1, S3HI2, S3HI3 and a method of order 8 in [12]. The table agrees with the efficiency curve which indicate that S3HI1 and S3HI3 have the same performance while S3HI2 performed better when compared to its counterparts and the method of order 7 in [12].

Problem 3.. Consider the IVP discussed in [5]

$$\begin{aligned} &y''' + 4y' = x, \quad x \in [0,1] \\ &y(0) = y'(0) = 0, \quad y''(0) = 1 \end{aligned} \tag{29}$$

whose exact solution is $y(x) = \frac{3}{16}(1\cos(2x)) + \frac{x^2}{8}$.

Table 3 shows the Maximum Error obtained for different values of N at the point $x_N = b$. The results shows the superiority of the proposed methods when the number of steps (subinterval) considered are less than those used in [5]. Figure 3 is the efficiency curve showing the different methods and their individual performance in the absolute errors obtained against the CPU time in each case. The methods are relatively the same in terms of errors but S3HI2 exhibits more CPU time for $N = 48$.

Problem 4.. Consider the nonlinear BVP discussed in [3].

$$\begin{aligned} &y''' + 2e^{-3y} = 4(1+x)^{-3}, \quad x \in [0,1] \\ &y(0) = 0, \quad y'(0) = 1, \quad y'(1) = 0.5 \end{aligned} \tag{30}$$

Table 1. Comparison of maximum Errors (ME) for Problem 1

Methods	ϵ	$N = 10$	$N = 20$	$N = 40$
S3HI1	1/16	1.49×10^{-8}	1.53×10^{-10}	1.87×10^{-12}
	1/32	7.78×10^{-9}	8.02×10^{-11}	1.01×10^{-12}
	1/64	4.22×10^{-9}	4.41×10^{-11}	5.76×10^{-13}
S3HI2	1/16	3.65×10^{-9}	1.35×10^{-11}	6.47×10^{-14}
	1/32	2.08×10^{-9}	7.59×10^{-12}	3.73×10^{-14}
	1/64	1.31×10^{-9}	4.70×10^{-12}	2.44×10^{-14}
S3HI3	1/16	4.70×10^{-8}	2.74×10^{-10}	2.46×10^{-12}
	1/32	2.60×10^{-8}	1.48×10^{-10}	1.34×10^{-12}
	1/64	1.57×10^{-8}	8.62×10^{-11}	7.93×10^{-13}
MAE in [**?**]	1/16	2.32×10^{-4}	6.12×10^{-5}	1.52×10^{-5}
	1/32	9.77×10^{-5}	2.59×10^{-5}	6.45×10^{-6}
	1/64	3.78×10^{-5}	1.00×10^{-6}	2.50×10^{-6}

Table 2. Comparison of errors for Problem 2

x	Error in S3HI1	Error in S3HI2	Error in S3HI3	Error in [12]
0.1	1.2458×10^{-16}	2.1548×10^{-17}	5.1274×10^{-16}	4.1078×10^{-15}
0.2	2.2580×10^{-15}	1.7852×10^{-17}	4.4770×10^{-15}	1.6875×10^{-14}
0.3	1.2358×10^{-15}	4.2878×10^{-17}	1.2258×10^{-15}	5.0848×10^{-14}
0.4	5.2145×10^{-15}	3.2154×10^{-17}	1.5551×10^{-15}	1.1779×10^{-13}
0.5	3.0125×10^{-15}	1.3358×10^{-15}	1.6712×10^{-15}	2.4081×10^{-13}
0.6	2.1258×10^{-15}	1.2015×10^{-15}	7.1255×10^{-15}	4.3709×10^{-13}
0.7	1.1125×10^{-14}	1.3287×10^{-15}	1.0021×10^{-14}	7.3708×10^{-13}
0.8	7.1258×10^{-14}	8.2148×10^{-15}	4.2014×10^{-14}	1.1662×10^{-12}
0.9	2.2158×10^{-14}	3.2358×10^{-15}	2.1584×10^{-14}	1.7587×10^{-12}
1.0	1.0012×10^{-14}	1.9985×10^{-15}	1.8877×10^{-14}	2.5466×10^{-12}

Table 3. Comparison of maximum errors for Problem 3

	S3HI1		S3HI2	S3HI3	Method of order 7 in [5]	
b	TS	ME	ME	ME	TS	ME
5	30	1.25×10^{-13}	5.58×10^{-13}	5.54×10^{-13}	46	1.20×10^{-10}
	45	5.21×10^{-13}	1.12×10^{-13}	5.02×10^{-13}	56	3.69×10^{-11}
	60	2.35×10^{-13}	3.87×10^{-13}	1.11×10^{-12}	88	2.44×10^{-12}
10	60	4.25×10^{-13}	1.57×10^{-13}	8.41×10^{-12}	61	5.54×10^{-09}
	75	1.22×10^{-13}	1.01×10^{-13}	6.63×10^{-12}	91	5.04×10^{-10}
	90	8.14×10^{-13}	1.87×10^{-13}	6.74×10^{-12}	136	4.53×10^{-11}
15	75	1.66×10^{-12}	9.81×10^{-13}	3.42×10^{-12}	76	2.67×10^{-08}
	90	2.25×10^{-12}	1.77×10^{-13}	1.08×10^{-12}	110	2.91×10^{-09}
	105	1.11×10^{-12}	2.31×10^{-13}	9.52×10^{-11}	180	1.52×10^{-10}
20	90	2.58×10^{-11}	3.27×10^{-13}	2.57×10^{-11}	91	5.29×10^{-08}
	105	2.78×10^{-10}	1.14×10^{-13}	1.21×10^{-10}	129	6.54×10^{-09}
	120	2.36×10^{-11}	2.48×10^{-13}	1.18×10^{-10}	204	4.19×10^{-10}

Table 4. Comparison of maximum Errors (ME) for Example 5

h	ME in S3HI1	ME in S3HI2	ME in S3HI3	ME in [12]
$\frac{1}{16}$	5.11541×10^{-14}	2.23457×10^{-14}	1.00124×10^{-15}	1.03179×10^{-11}
$\frac{1}{32}$	6.55230×10^{-15}	1.20048×10^{-15}	5.47895×10^{-15}	3.24907×10^{-13}
$\frac{1}{64}$	3.75411×10^{-16}	4.78951×10^{-17}	3.47785×10^{-16}	1.02789×10^{-14}

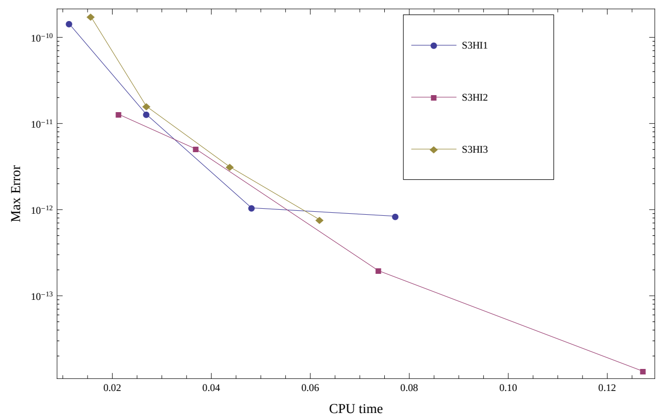

Figure 3. Efficiency plot for S3HI1, S3HI2 and S3HI3 showing the maximum error against the CPU time for Problem 3 for $N = 6, 12, 24, 48$.

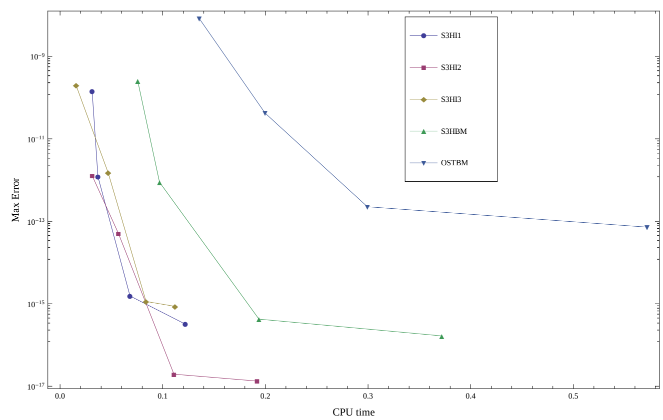

Figure 4. Efficiency plot for S3HI1, S3HI2, S3HI3, S3BHM and OSTBM showing the maximum error against the CPU time for Problem 4 for N=6,12,24,48.

whose exact solution is $y(x) = \ln(1 + x)$.
Figure 4 is the efficiency curve showing the different methods and their individual performance in the absolute errors obtained against the CPU time in each case. From Figure 4, one observes that, S3BHM shows good error performance relative to OSTEM (where OSTEM is an order 8 method). The proposed S3HI1, S3HI2, S3HI3 performed well in terms of error and time as such, they out performed S3BHM and OSTEM in terms of accuracy and time cost.

Example 5.. Consider the BVP discussed in [12].

$$\begin{aligned} &y''' + y = (x-4)\sin(x) + (1-x)\cos(x), \quad x \in [0,1] \\ &y(0) = 0, \quad y'(0) = 1, \quad y'(1) = -\sin 1 \end{aligned} \tag{31}$$

whose exact solution is $y(x) = (x - 1)\sin(x)$.

Table 4 shows the maximum error obtained using different step-sizes and compared with the an order 7 method in [12]. It clearly shows that S3HI1, S3HI2 and S3HI3 are almost equivalent in performance but more superior to the method cited.

5. Conclusion

Block of 3 points hybrid method are proposed and applied to solve third-order linear and non linear IVPs and BVPs in ordinary differential equations. The method are such that the off-step points are shifted between two step points in order to investigate which position is more efficient. All three methods possess the same characteristics, viz-a-viz, order of accuracy and number of functions evaluations. They are found to be consistent, zero stable and convergent as well. They are less ambiguous to derive. It should also be noted that this paper proposes computational comparison of 3 classes of a family of 3 steps linear multistep method. However, less emphases are placed on these methods outperforming existing methods in literature. Albeit, they performed favourably well when compared to other methods in literature.

References

[1] J. Henderson & and K. R. Prasad, ”Existence and uniqueness of solutions of three-point boundary value problems on time scales”, Journal Nonlinear Science **8** (2001) 1.

[2] C. P. Gupta & V. Lakshmikantham, ”Existence and uniqueness theorems for a third-order three point boundary value problem”, Journal of Nonlinear Analysis: Theory and Methods in Application **15** (1991) 949.

[3] R. K. Sahi, S. N. Jator & N. A. Khan, ”Continuous Fourth Derivative Method For Third-order Boundary Value Problems”, International Journal of Pure and Applied Mathematics **85** (2013) 907.

[4] S. N. Jator, ”Novel Finite Difference Schemes For Third Order Boundary Value Problems” International Journal of Pure and Applied Mathematics **53** (2009) 37.

[5] O. Adeyeye & Z. Omar, ”Solving Third Order Ordinary Differential Equations Using One-Step Block Method with Four Equidistant Generalized Hybrid Points”, IAENG International Journal of Applied Mathematics (IJAM) **49** (2019) 1.

[6] M. I. Modebei, O. O. Olaiya & A. C. Onyekonwu, ”A 3-step fourth derivatives method for numerical integrationof third order ordinary differential equations”, Int. J. Math. Ana Opt.; Theory and Applications **7** (2021) 32.

[7] S. Islam, M. A. Khan, I. A. Tirmizi & E. H. Twizell, ”Non-polynomial splines approach to the solution of a system of third-order boundary-value problems”, Applied Mathematics and Computations **168** (2005) 152.

[8] P. K. Pandey, ”Solving third-order Boundary Value Problems with Quartic Splines”, SpringerPlus, **5** (2016) 1.

[9] Y. Q. Hasan & S. A. Alaqel, ”Application of Adomian Decomposition Method to Solving Higher Order Singular Value Problems for Ordinary Differential Equations”, Asian Journal of Probability and Statistics **9** (2020) 28.

[10] A. Khan & P. Khandelwal, ”Numerical Solution of Third Order Singularly Perturbed Boundary Value Problems Using Exponential Quartic Spline”, Thai Journal of Mathematics **17** (2019) 663.

[11] H. K. Mishra & S. Saini, ”Quartic B-Spline Method for Solving a Singular Singularly Perturbed Third-Order Boundary Value Problems”, American Journal of Numerical Analysis **3** (2015) 18.

[12] Ra'ft Abdelrahim, "Numerical solution of third order boundary value problems using one-step hybrid block method", Ain Shams Engineering Journal **10** (2019) 179.

[13] B. S. H. Kashkari & S. Alqarni, "Optimization of two-step block method with three hybrid points for solving third order initial value problems", Journal of Nonlinear Science and Application **12** (2019) 450.

[14] J. D. Lambert, "Computational Methods in Ordinary Differential Equations", John Wiley, New York (1973).

[15] M. I. Modebei, R. B. Adeniyi, S. N. Jator & H. C. Ramos, "A block hybrid integrator for numerically solving fourth-order Initial Value Problems", Applied Mathematics and Computation **346** (2019) 680.

[16] P. Henrici, "Discrete Variable Methods in Ordinary Differential Equations", John Wiley & Sons, New York, USA, (1962).