

Adaptive finite element analysis of thermal and elasticity problems using Zienkiewicz–Zhu recovery-based error estimation

J. K. Odeyemi^{a,*}, S. A. Odunlami^b

^a*Department of Mathematics and Statistics, The Federal Polytechnic Ilaro, Ogun State, Nigeria*

^b*Department of Mechanical Engineering, The Federal Polytechnic Ilaro, Ogun State, Nigeria*

Abstract

This paper presents a verification-to-adaptivity analysis of a finite element framework for scalar and vector-valued elliptic problems that employs recovery-based a posteriori error estimation. Conforming bilinear quadrilateral (Q4) and linear triangular (T3) finite elements are used to discretise the thermal Poisson equation and the linear elasticity equations. Standard Galerkin methods are combined with uniform and adaptive mesh refinement in this approach. The approach of manufactured solutions under uniform refinement is first used for verification, where both discretizations regain the theoretical H¹- and L²-norm convergence rates, proving the accuracy of the scalar and vector formulations. The Zienkiewicz–Zhu (ZZ) recovery approach is then used to incorporate adaptive refinement, using stress recovery for elasticity and gradient recovery for the thermal problems. The adaptive meshes show conservative effectivity behaviour, monotonic estimator–error correlation, and stable localisation. Differences between Q4 and T3 outcomes are consistent with their known approximation features across all test cases. The findings show that for thermal diffusion and linear elasticity issues, ZZ-driven adaptivity offers a dependable extension of a validated finite element framework.

DOI: [10.46481/jnsps.2026.3266](https://doi.org/10.46481/jnsps.2026.3266)

Keywords: Finite element method, Adaptive mesh refinement, Zienkiewicz–Zhu recovery, A posteriori error estimation

Article History:

Received: 11 January 2026

Received in revised form: 01 May 2026

Accepted for publication: 3 June 2026

Available online: 20 July 2026

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Communicated by: P. Thakur

1. Introduction

The finite element method (FEM) is a numerical grid methodology for partial differential equations (PDEs) in solid mechanics and heat transfer [1, 2] because it can efficiently handle a variety of boundary conditions and complicated geometries. In order to characterise each element's behaviour, a series of mathematical equations relating the element's geometry, material characteristics, and boundary conditions are used

to divide the problem area into contiguous finite elements of suitable shape. The behaviour of the entire system is then described by a global system of equations that is created by combining these equations. The resulting equations are then solved to determine overall behaviour, such as stresses and deformations, and to offer approximations of solutions for real-world engineering challenges.

FEM simulations heavily rely on mesh resolution due to their contemporary use in increasingly complex simulations, which shifts the emphasis from mere implementation to numerical credibility and estimator-driven reliability, even though uniform refinement is usually computationally expensive in real-world settings. Adaptive mesh refinement (AMR) tech-

*Corresponding author Tel. No.: +234-706-054-1541

Email address: joshua.odeyemi@federalpolyilaro.edu.ng (J. K. Odeyemi)

niques, driven by a posteriori error estimation, are crucial to reliable and effective FEM analysis because they dynamically modify the computational mesh to increase resolution in areas of high error or complexity (such as near shocks, steep gradients, or singularities) and coarsen it where the solution is smooth and accurate. Numerous recovery- and residual-based estimators have been developed to facilitate adaptive refining; their theoretical foundations and practical uses are well documented in the literature. While extensive literature exists on a posteriori error estimation and adaptive finite element methods such as Verfürth's comprehensive overview of adaptive strategies like mesh-refinement strategies (red-green refinement and bisection), marking strategies (Maximum and Equilibration), and data structures for nodes and elements [3], Grätsch and Bathe's practical review of error estimation techniques suitable for engineering practice, where the goal is to obtain reliable results without excessive computational cost [4], and Chamoin and Legoll's pedagogical survey whose goal is to drive an adaptive refinement process, ensuring that the computational effort is concentrated where the solution is least regular, such as near singularities or sharp fronts [5]. Adaptive mesh refinement is no longer viewed as an optional efficiency enhancement, but as an integral component of ensuring solution accuracy and managing errors. However, adaptivity may mask formulation or implementation errors by redistributing rather than lowering discretisation error, therefore adaptive refinement is only truly effective and meaningful when it is based on verified and validated discretisation methods.

Thorough verification of the numerical concept and execution of finite element simulations is essential to their dependability. Different from validation, which deals with agreement with physical data, this rigorous technique helps increase confidence that the governing equations have been correctly solved at the discrete level. Verification serves to confirm that the discrete formulation reproduces the expected asymptotic behaviour dictated by the underlying finite element approximation theory. This usually entails determining if the numerical error diminishes when the mesh size is refined at the expected rate (order of convergence). By providing controlled convergence investigations under uniform mesh refinement, the method of manufactured solutions remains one of the few systematic tools capable of establishing such verification for both scalar and vector-valued elliptic problems, particularly in the absence of closed-form solutions [6–9]. For both scalar and vector-valued problems, recovery of the theoretical convergence rates predicted by finite element approximation theory continues to be a necessary, though not sufficient, condition for the credibility of adaptive simulations.

The Zienkiewicz-Zhu (ZZ) family within adaptive finite element methods of recovery-based a posteriori error estimators are still significant because of their algorithmic simplicity and compatibility with standard displacement-based formulations [10, 11]. These techniques use the computed, approximate solution to provide a more exact, smoothed gradient (such as the stress field), which is then used to estimate the error. This contrasts with other error estimation approaches, such as residual-based methods, and their ease of implementation has con-

tributed significantly to their continued practical relevance in a wide range of engineering and scientific applications [12–14]. The main concept involves obtaining a better approximation of the derivatives of the exact solution by post-processing the raw finite element results. However, it is now widely acknowledged that recovery-based estimators' behaviour is highly sensitive to the characteristics of the simulation, such as the kind of partial differential equation (PDE) being solved, the formulation and shape of the finite elements used, and the relationship between the recovered stresses or fluxes and the nodal displacements or scalar potentials [15, 16]. In particular, the transition from scalar diffusion problems to elasticity introduces additional complexity due to the indirect and discontinuous nature of stress approximations.

Although there is a large body of work on adaptive techniques [17–23], there are still very few reviews that explicitly relate recovery-based adaptivity to verification under uniform refinement in both scalar and vector-valued problems while taking element topology effects into account. Addressing this gap is critical for mathematically consistent interpretation of estimator performance and adaptive refinement behaviour. In this work, a verified finite element baseline is established prior to ZZ-driven adaptive refinement for thermal diffusion and linear elasticity is introduced. Bilinear quadrilateral and linear triangular discretizations are then carefully compared. Instead of relying solely on empirical refinement results, this method allows a transparent evaluation of recovery-based adaptivity based on validated numerical behaviour. In keeping with current computational mechanics best practices, the study explores the connection between verification, discretisation choice, and adaptive refinement behaviour in finite element models.

The remainder of this article is organized as follows. Section 2 presents the methodological framework of the study. In order to verify the accuracy and asymptotic convergence of the finite element formulations for scalar and vector-valued problems, Section 3 provides numerical results obtained under uniform mesh refinement. Building on this confirmed baseline, Section 4 presents the findings about the behaviour of recovery-based adaptive refinement, focussing on convergence trends, error localisation, and the impact of element topology. Section 5 concludes by summarising the study's key findings and outlining potential directions for further research.

2. Methodology

2.1. Problem setting and governing equations

We consider elliptic boundary value problems on two-dimensional polygonal domains $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. Our framework addresses steady heat conduction, linear elasticity, and their coupled thermoelastic formulation. These problems serve as canonical models for assessing adaptive finite element methodologies while offering practical relevance to engineering applications. Key notation used in the study is displayed in Table 1:

Table 1: Notation used in the formulation

Symbol	Description
Ω	Computational domain
$\mathcal{T}_h$	Finite element mesh
$J, \det J$	Jacobian and determinant
$V_h \subset V$	Conforming finite element subspace
$\mathbf{B}$	Strain–displacement matrix
T, T_h	Temperature and FEM approximation
$\mathbf{u}, \mathbf{u}_h$	Displacement and FEM solution
$\mathcal{R}_h$	Gradient recovery operator

2.1.1. Thermal problem

The steady-state heat conduction problem seeks a temperature field T satisfying:

$$\begin{aligned} -\nabla \cdot (\kappa \nabla T) &= f \quad \text{in } \Omega, \\ T &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (1)$$

where $\kappa > 0$ denotes the thermal conductivity tensor (assumed isotropic and constant), and f represents a prescribed heat source. To establish a rigorous benchmark with known solution properties, we employ the method of analytical solutions with:

$$T_{\text{exact}}(x, y) = \sin(\pi x) \sin(\pi y). \quad (2)$$

This smooth function satisfies homogeneous Dirichlet boundary conditions on $\partial\Omega$ while exhibiting nontrivial spatial variations that challenge numerical discretization. Substituting T_{exact} into Equation (1) yields the exact source term:

$$f = -\nabla \cdot (\kappa \nabla T_{\text{exact}}). \quad (3)$$

This construction guarantees mathematical consistency between the differential equation, boundary conditions, and forcing terms.

2.1.2. Linear elasticity problem

Under the assumptions of infinitesimal strains and plane stress conditions, the equilibrium equations for linear elasticity are defined as:

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma} &= \mathbf{b} \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned} \quad (4)$$

where $\mathbf{u}$ denotes the displacement vector, $\mathbf{b}$ the body force vector, and $\boldsymbol{\sigma}$ the Cauchy stress tensor. The constitutive relation for isotropic linear elasticity is:

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}, \quad (5)$$

where $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the infinitesimal strain tensor, and $\mathbf{D}$ is the fourth-order elasticity tensor. In Voigt notation for plane stress, this reduces to:

$$\mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \quad (6)$$

with E and ν denoting Young's modulus and Poisson's ratio, respectively. In this study, we employed the analytical displacement field:

$$\mathbf{u}_{\text{exact}}(x, y) = \begin{bmatrix} x(1-x)y(1-y) \\ \frac{1}{2} \sin(\pi x) \sin(\pi y) \end{bmatrix}, \quad (7)$$

which satisfies homogeneous Dirichlet boundary conditions while maintaining sufficient regularity to avoid singular behavior on convex domains. The corresponding body force $\mathbf{b}$ is determined by substituting $\mathbf{u}_{\text{exact}}$ into Equation (4).

2.2. Weak formulations and function spaces

2.2.1. Thermal weak form

Let $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$ denote the Sobolev space of admissible temperature fields. The weak formulation of the thermal problem seeks $T \in H_0^1(\Omega)$ such that:

$$a_T(T, v) = \ell_T(v) \quad \forall v \in H_0^1(\Omega), \quad (8)$$

with bilinear form $a_T(T, v) = \int_{\Omega} \kappa \nabla T \cdot \nabla v \, d\Omega$ and linear form $\ell_T(v) = \int_{\Omega} f v \, d\Omega$. This bilinear form is continuous and coercive on $H_0^1(\Omega)$, ensuring well-posedness by the Lax-Milgram theorem. The weak form also establishes the energy minimization principle:

$$T = \arg \min_{v \in H_0^1(\Omega)} \left\{ \frac{1}{2} a_T(v, v) - \ell_T(v) \right\}. \quad (9)$$

2.2.2. Elasticity weak form

Let $\mathbf{V} = [H_0^1(\Omega)]^2$ denote the space of kinematically admissible displacement fields. The weak formulation for linear elasticity seeks $\mathbf{u} \in \mathbf{V}$ such that:

$$a_U(\mathbf{u}, \mathbf{v}) = \ell_U(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}, \quad (10)$$

where the bilinear and linear forms are defined as

$$\begin{aligned} a_U(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) \mathbf{D} \boldsymbol{\varepsilon}(\mathbf{v}) \, d\Omega, \\ \ell_U(\mathbf{v}) &= \int_{\Omega} \mathbf{b} \cdot \mathbf{v} \, d\Omega. \end{aligned} \quad (11)$$

In Voigt notation, the bilinear form becomes

$$a_U(\mathbf{u}, \mathbf{v}) = \int_{\Omega} (\mathbf{B}\mathbf{u})^T \mathbf{D} (\mathbf{B}\mathbf{v}) \, d\Omega, \quad (12)$$

with $\mathbf{B}$ the strain–displacement operator. This formulation requires only square-integrable strains, naturally incorporates Dirichlet boundary conditions, and yields a symmetric, positive-definite system.

2.2.3. Weak form for coupled problem

The coupled thermoelastic weak form seeks $(T, \mathbf{u}) \in H_0^1(\Omega) \times \mathbf{V}$ such that:

$$\begin{aligned} a_T(T, v) &= \ell_T(v) \quad \forall v \in H_0^1(\Omega), \\ a_U(\mathbf{u}, \mathbf{v}) - c(T, \mathbf{v}) &= \ell_U(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}, \end{aligned} \quad (13)$$

with the thermoelastic coupling term

$$c(T, \mathbf{v}) = \int_{\Omega} \alpha T \mathbf{D} : \boldsymbol{\varepsilon}(\mathbf{v}) \, d\Omega. \quad (14)$$

In matrix form, the coupled system can be expressed as:

$$\begin{bmatrix} \mathbf{K}_T & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_U \end{bmatrix} \begin{bmatrix} T \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_T \\ \mathbf{F}_U + \mathbf{C}T \end{bmatrix}, \quad (15)$$

with $\mathbf{K}_T$ and $\mathbf{K}_U$ the thermal and mechanical stiffness matrices, respectively, $\mathbf{C}$ the coupling matrix, and $\mathbf{F}_T$, $\mathbf{F}_U$ the load vectors. This block lower-triangular structure reflects the one-way coupling where temperature affects displacement but not vice versa in the steady-state formulation.

2.3. Finite element discretization

2.3.1. Domain representation and mesh generation

We consider two representative domains to assess method performance under varying regularity conditions. The primary domain is the unit square $\Omega = [0, 1]^2$, which supports smooth solutions and enables verification of optimal convergence rates. To evaluate robustness in the presence of singularities, we additionally consider the L-shaped domain $\Omega = [-1, 1]^2 \setminus [0, 1] \times [-1, 0]$, which introduces a re-entrant corner at the origin, inducing singular behavior in both thermal gradients and elastic stresses. Our mesh generation begins with a characteristic mesh size $h_0 > 0$ and defines an initial structured Cartesian node set:

$$\mathcal{N}_0 = \{(ih_0, jh_0) : i, j \in \mathbb{Z}\}.$$

This node set is filtered using a domain indicator function $\chi_{\Omega} : \mathbb{R}^2 \rightarrow \{0, 1\}$:

$$\mathcal{N} = \{\mathbf{x} \in \mathcal{N}_0 : \chi_{\Omega}(\mathbf{x}) = 1\}.$$

Boundary nodes are explicitly inserted to ensure geometric fidelity. For triangular (T3) discretization, we employed Delaunay triangulation with centroid-based filtering:

$$\mathcal{T}_h = \{K \in \text{Delaunay}(\mathcal{N}) : \chi_{\Omega}(\mathbf{x}_K^c) = 1\}, \quad (16)$$

with $\mathbf{x}_K^c$ the centroid of element K . For quadrilateral (Q4) discretization, we constructed elements through structured connectivity:

$$\mathcal{Q}_h = \{K_{ij} : \chi_{\Omega}(\mathbf{x}_{K_{ij}}^c) = 1\}, \quad (17)$$

with $K_{ij} = \text{conv}\{\mathbf{x}_{i,j}, \mathbf{x}_{i+1,j}, \mathbf{x}_{i+1,j+1}, \mathbf{x}_{i,j+1}\}$ and conv the convex hull operation. To support flexible refinement patterns, we also considered the hybrid meshes $\mathcal{H}_h = \mathcal{T}_h \cup \mathcal{Q}_h$ where quadrilaterals are retained in regions of smooth solution behavior while triangles accommodate geometric complexities. Degenerate elements with vanishing Jacobian determinant are eliminated, and orphan nodes are removed to preserve algebraic solvability.

2.3.2. Thermal discretization

Let $\mathcal{T}_h$ be denoted as a conforming partition of Ω into triangular or quadrilateral elements. The discrete temperature field $T_h \in V_h \subset H_0^1(\Omega)$ is approximated as:

$$T_h|_K(\mathbf{x}) = \sum_{i=1}^{n_K} N_i^K(\mathbf{x}) T_i, \quad (18)$$

$n_K = 3$ for triangles and $n_K = 4$ for quadrilaterals; N_i^K denote local shape functions and T_i nodal temperatures. For linear triangular elements, the shape functions are affine with constant gradients:

$$\nabla N_i^K = \frac{1}{2|K|} [(y_j - y_k), (x_k - x_j)]^T, \quad (19)$$

where $|K|$ denotes element area and (i, j, k) is a cyclic permutation of $(1, 2, 3)$. The element stiffness matrix admits the closed form:

$$\mathbf{K}_K^{(T)} = \kappa |K| \sum_{i,j=1}^3 (\nabla N_i^K \cdot \nabla N_j^K) \mathbf{e}_i \mathbf{e}_j^T, \quad (20)$$

$\mathbf{e}_i$ the i -th canonical basis vector in $\mathbb{R}^3$. For bilinear quadrilateral elements, we employed isoparametric mapping from the reference square $\hat{K} = [-1, 1]^2$:

$$\mathbf{x}(\boldsymbol{\xi}) = \sum_{i=1}^4 N_i(\boldsymbol{\xi}) \mathbf{x}_i, \quad N_i(\boldsymbol{\xi}) = \frac{1}{4} (1 + \xi_1 \xi_{i1}) (1 + \xi_2 \xi_{i2}), \quad (21)$$

where $\xi_{i1}, \xi_{i2} \in \{-1, 1\}$ define the node coordinates in the parametric space. The stiffness matrix is evaluated using 2×2 Gauss quadrature:

$$\mathbf{K}_K^{(T)} = \sum_{q=1}^4 w_q \mathbf{B}_q^T \kappa \mathbf{B}_q | \det(\mathbf{J}_q) |, \quad (22)$$

$\mathbf{B}_q$ evaluated at $\boldsymbol{\xi}_q$, $\mathbf{J}_q$ the Jacobian, and $(\boldsymbol{\xi}_q, w_q)$ the quadrature points and weights.

2.3.3. Elasticity discretization

The discrete displacement field $\mathbf{u}_h \in \mathbf{V}_h \subset \mathbf{V}$ is approximated elementwise as:

$$\mathbf{u}_h|_K(\mathbf{x}) = \sum_{i=1}^{n_K} N_i^K(\mathbf{x}) \mathbf{u}_i. \quad (23)$$

For linear triangular elements, the strain-displacement matrix $\mathbf{B}$ is constant and given by

$$\mathbf{B} = \frac{1}{2|K|} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}. \quad (24)$$

where $b_i = y_j - y_k$ and $c_i = x_k - x_j$, with (i, j, k) denoting cyclic permutations of $(1, 2, 3)$. The element stiffness matrix follows as

$$\mathbf{K}_K^{(U)} = |K| \mathbf{B}^T \mathbf{D} \mathbf{B}. \quad (25)$$

For bilinear quadrilateral elements, the strain-displacement matrix varies spatially and is evaluated at each quadrature point:

$$\mathbf{B}_q = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \dots & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & \dots & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \dots & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \bigg|_{\xi_q}, \quad (26)$$

where derivatives with respect to physical coordinates are related to parametric derivatives through $\nabla_{\mathbf{x}} = \mathbf{J}^{-T} \nabla_{\xi}$. The stiffness matrix is computed via numerical integration using 2×2 Gauss quadrature.

2.4. A posteriori error estimation

2.4.1. Zienkiewicz–Zhu (ZZ) recovery-based

The Zienkiewicz–Zhu (ZZ) recovery framework estimates the discretization error by exploiting the superconvergent behaviour of certain derived quantities and measuring the discrepancy between recovered and finite element fields in these energy norms. The estimator is constructed without reference to the exact solution and relies solely on post-processing of the computed finite element approximation.

Let $V_h \subset V$ denote a conforming finite element subspace associated with a shape-regular triangulation $\mathcal{T}_h$ of the computational domain Ω . For both the thermal diffusion and linear elasticity problems considered in this work, the governing equations admit symmetric, coercive bilinear forms $a(\cdot, \cdot)$ on V , inducing natural energy norms.

Thermal Diffusion Problem. Let $T_h \in V_h$ denote the finite element approximation of the temperature field. The exact energy norm associated with the thermal problem is given by

$$\|T - T_h\|_E^2 = \int_{\Omega} \kappa |\nabla(T - T_h)|^2 d\Omega, \quad (27)$$

where κ denotes the thermal conductivity. Since the exact gradient ∇T is unavailable, a recovered gradient field ∇T_h^* is constructed using a local patch-based recovery operator $\mathcal{R}_h: V_h \rightarrow [L^2(\Omega)]^d$, yielding

$$\nabla T_h^* = \mathcal{R}_h(\nabla T_h). \quad (28)$$

The elementwise ZZ error indicator is then defined by

$$\eta_K^T = \left(\int_K \kappa (\nabla T_h^* - \nabla T_h) \cdot (\nabla T_h^* - \nabla T_h) d\Omega \right)^{1/2} \quad K \in \mathcal{T}_h. \quad (29)$$

This quantity provides an approximation of the local contribution to the energy-norm error, exploiting the improved accuracy of the recovered gradient under mesh refinement.

Linear elasticity problem. Let $\mathbf{u}_h \in [V_h]^d$ denote the finite element displacement field. The associated energy norm is defined by

$$\|\mathbf{u} - \mathbf{u}_h\|_E^2 = \int_{\Omega} (\boldsymbol{\sigma}(\mathbf{u}) - \boldsymbol{\sigma}(\mathbf{u}_h))^T \mathbf{D}^{-1} (\boldsymbol{\sigma}(\mathbf{u}) - \boldsymbol{\sigma}(\mathbf{u}_h)) d\Omega, \quad (30)$$

with $\boldsymbol{\sigma}(\cdot)$ denoting the Cauchy stress tensor and $\mathbf{D}$ the elasticity tensor. Since finite element stresses $\boldsymbol{\sigma}_h = \boldsymbol{\sigma}(\mathbf{u}_h)$ are generally discontinuous across element interfaces, a recovered stress field

$$\boldsymbol{\sigma}_h^* = \mathcal{R}_h(\boldsymbol{\sigma}_h), \quad (31)$$

is constructed using a ZZ-type patch recovery operator, which is a node-based patch recovery strategy. For each mesh node, a patch is constructed consisting of all elements sharing that node. A least-squares approach is used to fit a linear polynomial to the estimated stress field across each patch. The recovered stress at the node is then obtained from this polynomial and subsequently interpolated over the elements to define a continuous recovered field.

The corresponding elementwise error indicator is defined as

$$\eta_K^U = \left(\int_K (\boldsymbol{\sigma}_h^* - \boldsymbol{\sigma}_h)^T \mathbf{D}^{-1} (\boldsymbol{\sigma}_h^* - \boldsymbol{\sigma}_h) d\Omega \right)^{1/2} : \quad K \in \mathcal{T}_h, \quad (32)$$

which is consistent with the energy norm induced by the elasticity bilinear form.

2.4.2. Global estimator and adaptive algorithm

The global ZZ estimator is obtained by aggregation of the local contributions,

$$\eta_h = \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2}. \quad (33)$$

Under standard assumptions on mesh regularity and recovery operator stability, the ZZ estimator is known to be asymptotically reliable, in the sense that

$$\|\mathbf{u} - \mathbf{u}_h\|_E \lesssim \eta_h \quad \text{as } h \rightarrow 0, \quad (34)$$

with effectivity indices remaining bounded for sufficiently refined meshes.

The estimator η_K is used to drive adaptive mesh refinement through a marking strategy based on relative element contributions. The adaptive loop follows the standard solve–recover–estimate–refine paradigm and is terminated once η_h falls below a prescribed tolerance or a maximum refinement level is reached.

2.5. Adaptive mesh refinement strategy

2.5.1. Bulk marking criterion

Given element indicators $\{\eta_K\}_{K \in \mathcal{T}_h}$, we employed the Dörfler marking strategy to select elements for refinement. For a prescribed parameter $\theta \in (0, 1)$, the marked set $\mathcal{M} \subset \mathcal{T}_h$ is the minimal subset satisfying:

$$\sum_{K \in \mathcal{M}} \eta_K^2 \geq \theta^2 \sum_{K \in \mathcal{T}_h} \eta_K^2. \quad (35)$$

This approach ensures that refinement targets regions dominating the global error while avoiding unnecessary uniform refinement. Implementation proceeds by sorting elements in descending order of η_K and accumulating contributions until the threshold is met.

2.5.2. Element refinement operators

For triangular elements, we employ red refinement where each marked triangle is subdivided into four congruent sub-triangles by connecting edge midpoints. For quadrilateral elements, each marked element is divided into four sub-quadrilaterals through edge midpoints and the element centroid. This approach preserves element quality and mesh conformity while enabling local refinement. Following refinement, mesh regularization is essential to maintain numerical stability. Elements with aspect ratio exceeding a threshold $\delta_{\max}$ or minimum angle below $\alpha_{\min}$ are flagged for improvement. Orphan nodes (those not belonging to any element) are removed, and the mesh is reindexed to maintain consistent global numbering. Element Jacobians are verified to ensure $|\det(\mathbf{J})| > \epsilon$ with $\epsilon = 10^{-12}$ to avoid degenerate elements. Termination occurs when either $\max_K \eta_K^k < \epsilon_{\text{tol}}$ or iteration count exceeds $N_{\max}$.

2.6. Numerical implementation

2.6.1. Linear system assembly and solution

At each adaptive step, the global system $\mathbf{K}\mathbf{u} = \mathbf{f}$ is assembled using standard finite element techniques. Element matrices are computed as described in 2.3 and assembled into the global sparse matrix using a Boolean assembly operator.

$$\mathbf{K} = \sum_{K \in \mathcal{T}_h} \mathbf{A}_K^T \mathbf{K}_K \mathbf{A}_K, \quad (36)$$

with $\mathbf{A}_K$ mapping local to global degrees of freedom. The resulting system is stored in compressed sparse row format to minimize memory usage while maintaining efficient matrix-vector products.

2.6.2. Boundary condition enforcement and singular mode elimination

Dirichlet boundary conditions from the manufactured solutions are imposed strongly by modifying system rows corresponding to constrained degrees of freedom:

$$K_{ii} = 1, \quad K_{ij} = 0 \text{ for } j \neq i, \quad f_i = g_i, \quad (37)$$

with g_i the prescribed boundary value. Adaptive refinement may introduce zero-energy modes from orphan degrees of freedom. These are detected by identifying rows with $\sum_j |K_{ij}| < \epsilon$ and constrained to zero displacement or temperature.

3. Results

This section presents numerical results for three benchmark problems designed to verify the accuracy, robustness, and adaptivity of the proposed finite element framework. The test cases are structured progressively, beginning with uniform mesh refinement for scalar and vector problems, followed by adaptive refinement driven by a recovery-based a posteriori error estimator.

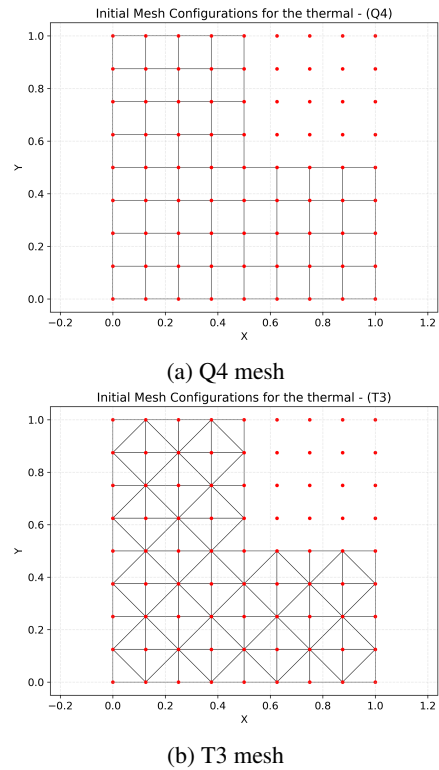


Figure 1: Initial finite element meshes employed for the thermal Poisson problem on the L-shaped domain obtained by removing the upper-right quadrant of the unit square. (a) Initial structured quadrilateral mesh using bilinear Q4 elements. (b) Initial conforming triangular mesh using linear T3 elements obtained by subdividing the quadrilateral mesh. Both discretisations share the same initial nodal resolution and represent the same computational domain.

3.1. Test case 1: thermal Poisson problem (uniform refinement)

Problem description. The first test considers the scalar Poisson equation with an analytically prescribed solution,

$$-\Delta T = f \quad \text{in } \Omega, \quad T = \sin(\pi x) \sin(\pi y) \quad \text{on } \partial\Omega,$$

where the source term (f) is constructed consistently using the method of manufactured solutions (MMS).

3.1.1. Initial mesh configurations for the thermal problem

The domain geometry is precisely depicted in both meshes (see Figure 1), and Dirichlet boundary constraints are strictly enforced at the boundary nodes. No local refinement or grading is added at this stage because the goal of the test case is to verify convergence behaviour under uniform mesh refinement for a smooth manufactured solution. The initial meshes used for the thermal Poisson problem utilising linear triangular (T3) and bilinear quadrilateral (Q4) finite element discretisations are shown in Figure 1. Structured quadrilateral segments covering the unit square on a Cartesian grid comprise the mesh for the Q4 discretisation. In contrast, each quadrilateral cell is divided into two triangular parts to produce the T3 discretisation, which yields a conforming triangulation of the domain.

3.1.2. Uniform refinement evolution for the thermal problem

The evolution of the computational meshes for the thermal Poisson problem under successive uniform refinement cycles is shown in Figure 2 for both Q4 (bilinear quadrilateral) and T3 (linear triangular) discretisations. Figure 2(a) displays the mesh following the initial refinement iteration for the Q4 discretisation. The mesh shows a significant increase in degrees of freedom by the third refinement iteration, as seen in Figure 2(b). This leads to a nearly continuous nodal distribution over the unit square. There is no introduction of anisotropy or localised grading, and the refinement is still globally uniform.

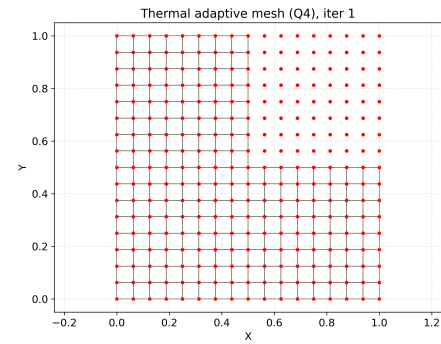
For the T3 discretisation, a comparable refinement process is shown. The triangulated mesh following the initial refinement iteration is shown in Fig. 2(c), where each triangle element is split in a conforming way to preserve shape regularity and uniform orientation throughout the domain. The triangulation gets much finer by the third refinement iteration (Figure 2(d)), with a dense and almost isotropic distribution of triangular elements. Even at high refinement levels, the mesh maintains congruence and prevents the generation of poorly shaped elements.

3.1.3. Numerical temperature fields under uniform refinement

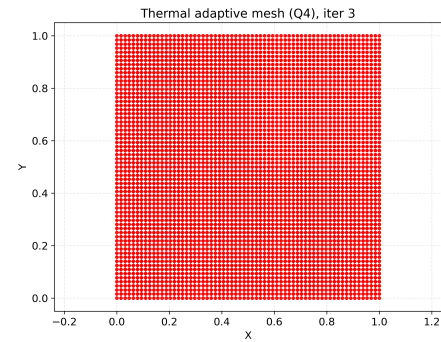
Plots of the discrete solution's spatial distribution throughout the unit square are displayed, with the amplitude of the temperature field indicated by colour intensity. The numerical temperature field T_h for the thermal Poisson problem is computed using T3 (linear triangular) and Q4 (bilinear quadrilateral) discretisations at two exemplary refinement levels (iterations 1 and 3), as shown in Figure 3(a–d).

The solution on the first uniformly refined mesh for the Q4 discretisation is shown in Figure 3(a). The numerical temperature field gradually varies from zero along the boundary to a maximum inside the domain, capturing the global qualitative structure of the analytical solution. Nonetheless, the poor resolution of the coarse mesh is reflected in the slight smoothing effects that are apparent close to areas of greater curvature in the solution profile. The numerical solution becomes substantially sharper and more symmetric with additional uniform refinement, as seen in Figure 3(b). The gradient transitions towards the boundary are smoother and more in line with the anticipated sinusoidal shape, and the peak temperature area is more clearly defined.

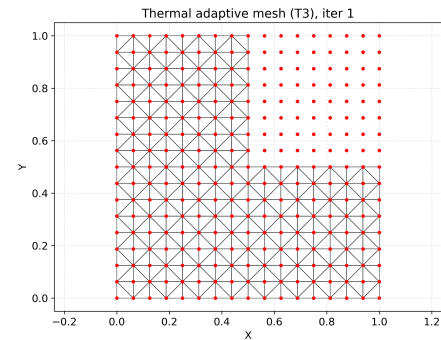
However, the discretisation of T3 shows a similar pattern. The temperature field at the first refinement level is shown in Figure 3(c), where the solution is still globally smooth but shows somewhat more numerical diffusion than in the Q4 example previously mentioned, especially in areas where the temperature gradient varies more quickly. The T3 solution approaches the same smooth interior profile seen for the refined Q4 discretisation by the third refinement iteration (Figure 3(d)). The impact of the element-wise linear approximation diminishes as the mesh resolution increases, and the temperature contours become more regular.



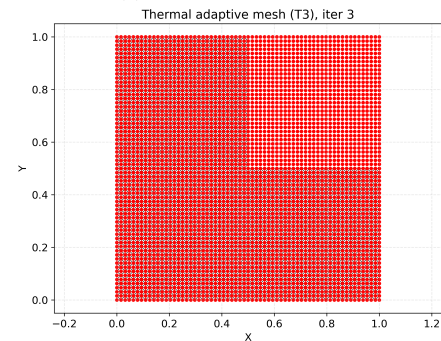
(a) Q4 mesh, iteration 1



(b) Q4 mesh, iteration 3



(c) T3 mesh, iteration 1



(d) T3 mesh, iteration 3

Figure 2: Uniform refinement evolution for the thermal problem. Comparison of quadrilateral (Q4) and triangular (T3) discretisations under successive uniform refinement cycles. The first and third refinement iterations are shown, illustrating the progressive global subdivision of all elements, which produces increasingly finer meshes while preserving mesh conformity across the computational domain.

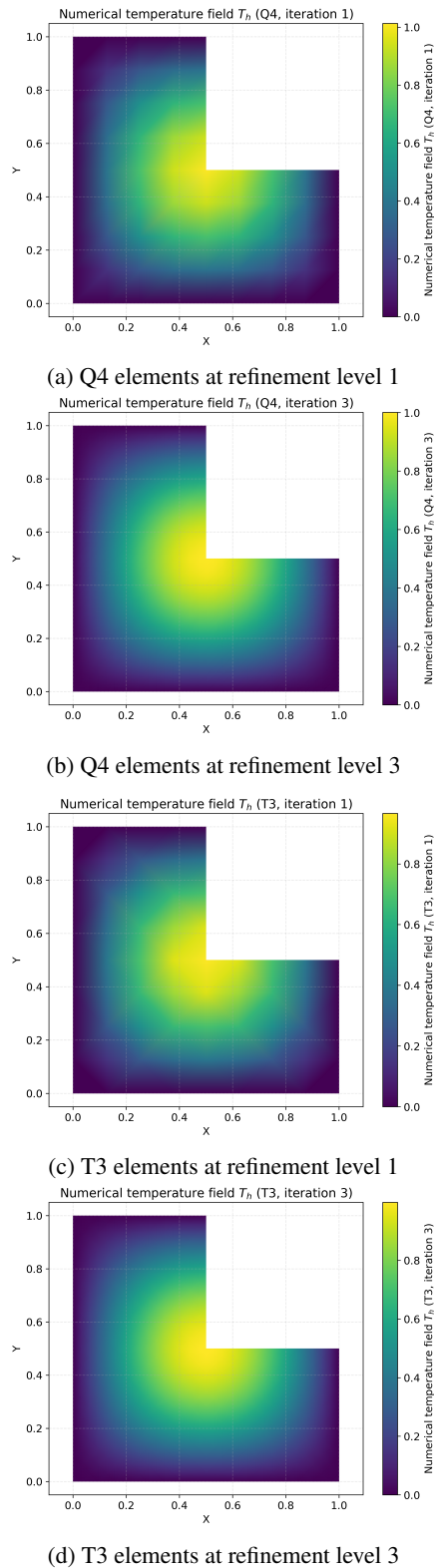


Figure 3: Numerical temperature fields for the thermal Poisson problem obtained using Q4 (bilinear quadrilateral) and T3 (linear triangular) finite element discretizations under uniform mesh refinement.

3.1.4. Convergence behaviour for the thermal Poisson problem

The log-log convergence histories of the numerical solution for the thermal Poisson problem obtained using T3 (linear triangular) and Q4 (bilinear quadrilateral) finite element discretisations, respectively, are shown in Figure 4(a)–(b). The errors are expressed as functions of the characteristic mesh size h , calculated under consecutive uniform mesh refinements, in the discrete L^2 and H^1 norms.

As the mesh is refined, both error norms for the Q4 discretisation (Figure 4(a)) show a distinct and monotonic decrease. The L^2 -error drops more quickly than the H^1 -error, which falls almost linearly with respect to h . The numerical trends may be evaluated visually by comparing them to the displayed reference slopes $O(h)$ and $O(h^2)$. As the mesh resolution increases, the fitted rates reveal an improved decay rate for the L^2 -error and a near-first-order decay for the H^1 .

The Figure 4(b) shows a very similar convergence trend for the T3 discretisation. The H^1 -error decreases roughly linearly with mesh size, while the L^2 -error decays more rapidly. The slopes of the convergence curves remain constant across refinement levels, despite the fact that the absolute error levels change from those in the Q4 example, particularly on coarser meshes. This shows that once the mesh is sufficiently fine, the triangular discretisation exhibits the predicted asymptotic behaviour. As the mesh is further refined, the error trends for both types of elements closely match the projected reference slopes, indicating an entry into the asymptotic convergence regime.

3.1.5. Error norms and computational cost

The discrete L^2 - and H^1 -error norms for the thermal Poisson problem discretised using T3 and Q4 finite elements under consecutive uniform mesh refinements are presented in Tables 2 and 3, together with the number of degrees of freedom (DOFs) and observed CPU time. As the mesh is improved and the number of degrees of freedom increases for both discretisations, the error norms decrease monotonically. When the effective mesh size h_{eff} is half, the H^1 -error is routinely lowered by roughly a factor of two, and the L^2 -error is reduced by around a factor of four. All of the refinement levels shown in the tables exhibit the same behaviour.

In both norms, the Q4 discretisation consistently yields smaller error magnitudes than the T3 discretisation at similar degrees of freedom. The increased approximation effectiveness of bilinear quadrilateral components for smooth solutions is reflected in the L^2 -norm, where this difference is most noticeable. For both types of elements, the stated CPU times increased with refinement, with a discernible increase at the finest mesh levels. This illustrates the combined impact of growing system size and the expense of putting together and resolving more complex linear systems.

3.2. Test case 2: linear elasticity problem (uniform refinement)

Problem description. The second test addresses two-dimensional linear elasticity under plane stress conditions. This test case aims to evaluate the adaptability and accuracy of the finite element framework when used to displacement-based

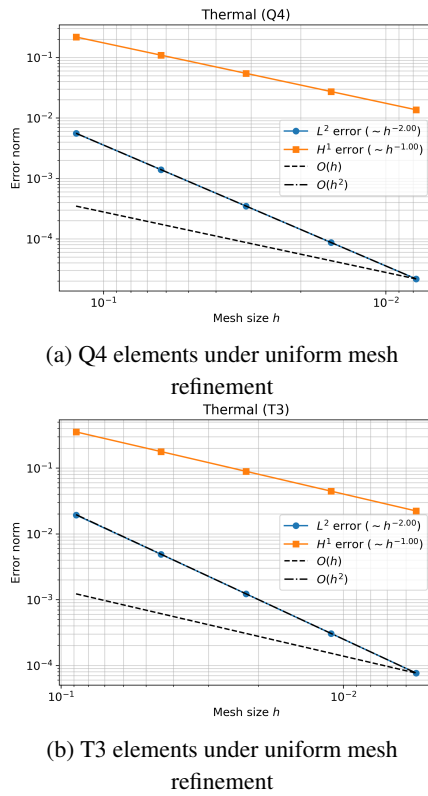


Figure 4: Convergence of the L^2 - and H^1 -error norms for the thermal Poisson problem using Q4 (bilinear quadrilateral) and T3 (linear triangular) elements under uniform mesh refinement. Observed error decay with respect to the mesh size h for the thermal Poisson problem with a manufactured solution. Reference slopes $O(h)$ and $O(h^2)$ are included for comparison.

Table 2: Error norms and computational cost for the thermal Poisson problem using T3 (linear triangular) elements under uniform mesh refinement.

h_{eff}	DOFs	L^2 error	H^1 error	CPU time (s)
0.08839	81	1.93×10^{-2}	3.54×10^{-1}	1.26
0.04419	289	4.88×10^{-3}	1.78×10^{-1}	0.55
0.02210	1089	1.22×10^{-3}	8.90×10^{-2}	0.75
0.01105	4225	3.06×10^{-4}	4.45×10^{-2}	1.61
0.00552	16641	7.65×10^{-5}	2.23×10^{-2}	8.85

Table 3: Error norms and computational cost for the thermal Poisson problem using Q4 (bilinear quadrilateral) elements under uniform mesh refinement.

h_{eff}	DOFs	L^2 error	H^1 error	CPU time (s)
0.12500	81	5.55×10^{-3}	2.18×10^{-1}	1.39
0.06250	289	1.39×10^{-3}	1.09×10^{-1}	0.59
0.03125	1089	3.48×10^{-4}	5.45×10^{-2}	0.94
0.01563	4225	8.69×10^{-5}	2.73×10^{-2}	2.48
0.00781	16641	2.17×10^{-5}	1.36×10^{-2}	14.17

elasticity formulations by extending the verification study from a scalar elliptic equation to a vector-valued system. The exact

displacement field is prescribed as:

$$u(x, y) = x(1 - x)y(1 - y),$$

$$v(x, y) = \frac{1}{2} \sin(\pi x) \sin(\pi y),$$

from which the body force is derived analytically to ensure MMS consistency. The thermal problem covered in 3.1 and the second test case were presented on the same computational domain. The initial mesh configurations presented in 3.1.1 were applied to the linear elasticity problem in order to maintain uniformity and prevent needless repetition. Similarly, the method of uniform refinement outlined in 3.1.2 was implemented without modification. In particular, structured quadrilateral and triangular meshes from the same nodal layouts were subjected to Q4 (bilinear quadrilateral) elements and T3 (linear triangular) elements.

The vector-valued finite element assembly, strain-displacement coupling, and displacement and energy norm computation are all intended to be verified by this test. Only uniform refinement is employed, similar to in Test Case 1.

3.2.1. Displacement field under uniform refinement

Displacement component $u_{h,y}$. The numerical approximation of the (x)-component of the displacement field, $u_{h,y}$, derived from the linear elasticity problem using Q4 (bilinear quadrilateral) and T3 (linear triangular) finite element discretisations at two representative refinement levels (iterations 1 and 3) is shown in Figure 5(a–d). The magnitude and sign of $u_{h,y}$ are shown by colour contours in the plots, which show the spatial variation of the displacement component over the computational domain.

The displacement field computed on the first uniformly refined mesh for the Q4 discretisation are displayed in Figure 5(a). The localised gradients are only coarsely resolved, especially in areas where the displacement fluctuates more quickly. The displacement field becomes substantially smoother and more vividly defined after additional uniform refinement, as seen in Figure 5(b). The transition zones between regions of opposing displacement are more clearly defined, and the contour lines exhibit greater regularity, suggesting a more approximate representation of the underlying continuous solution.

The T3 discretisation exhibits a similar refining trend. On the first refined triangular mesh, the displacement component $u_{h,x}$ is shown in Figure 5(c). The piecewise linear element of the T3 approximation results in significantly more dispersed gradients at this step, even though the global deformation pattern is consistent with that achieved with Q4 elements. The displacement field exhibits a substantially smoother spatial variation by the third refinement iteration (Figure 5(d)), with refined contours that closely approach those seen in the analogous Q4 solution at the same refinement level. The displacement field is continuous and free of spurious oscillations at every refinement level taken into consideration across both discretisations.

Displacement component $u_{h,y}$. The numerical approximation of the (y)-component of the displacement field, $u_{h,x}$, derived

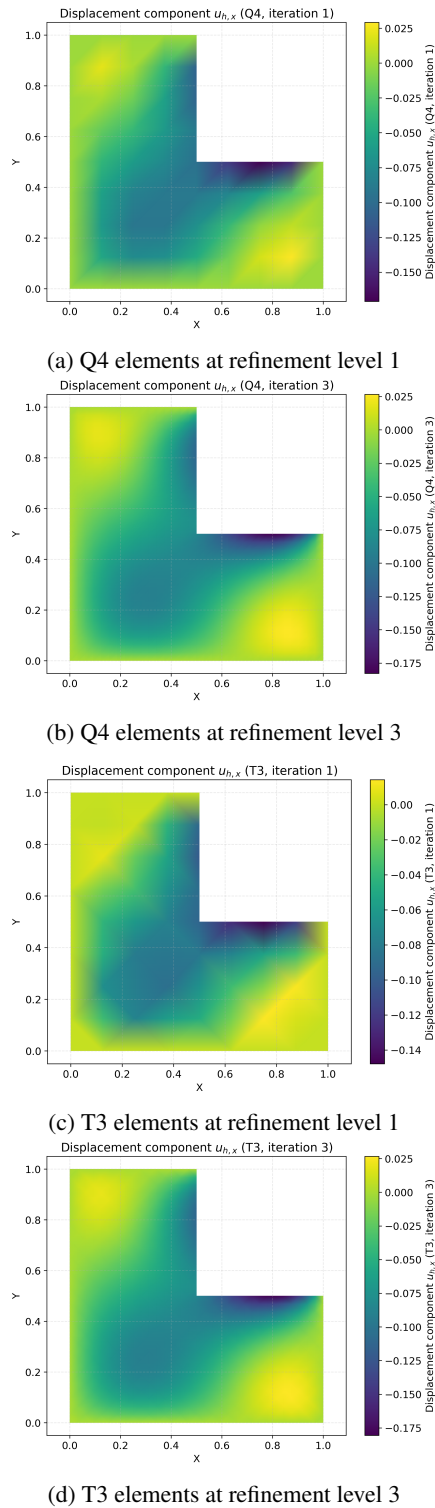


Figure 5: Numerical approximations of the x -component of the displacement field for the linear elasticity problem obtained using Q4 and T3 finite element discretizations under uniform mesh refinement.

from the linear elasticity problem using Q4 (bilinear quadrilateral) and T3 (linear triangular) finite element discretisations at two representative refinement levels (iterations 1 and 3) is shown in Figure 6(a–d). Both element types capture a dominant

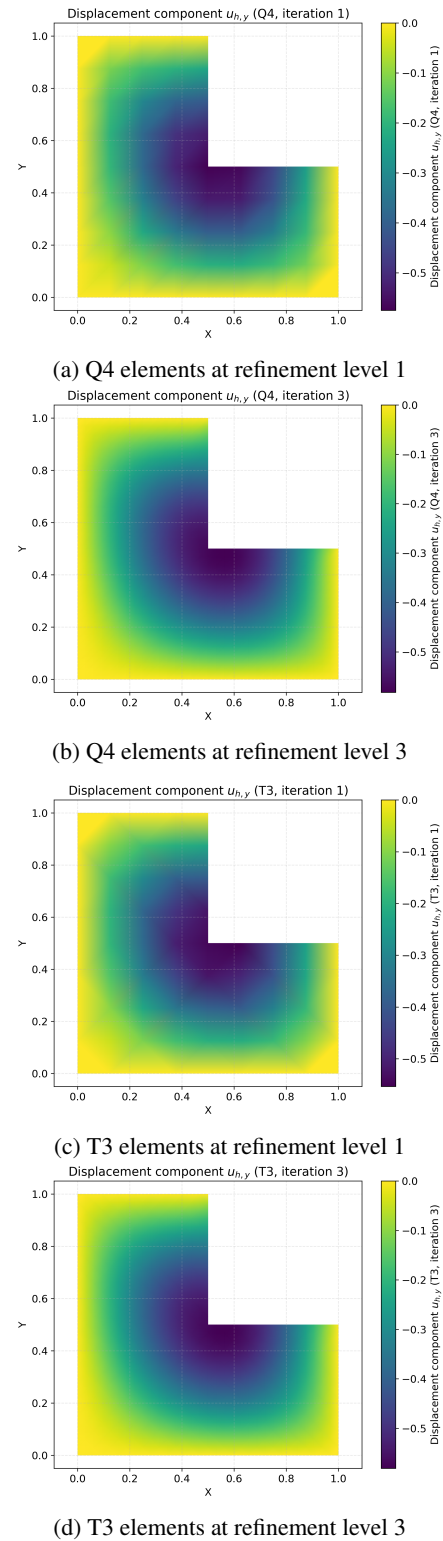


Figure 6: Numerical approximations of the y -component of the displacement field for the linear elasticity problem obtained using Q4 and T3 finite element discretizations under uniform mesh refinement.

region of negative vertical displacement in the domain's interior at the first refinement level, with displacement magnitudes declining towards the edges. Between Q4 and T3 elements, the

general spatial distribution of $u_{h,y}$ is stable. The spatial distribution of $u_{h,y}$ remains consistent for both Q4 and T3 discretisations throughout the refinement process. With increasing mesh resolution, the displacement field is represented with greater spatial fidelity, leading to smoother solution profiles and a clearer resolution of regions exhibiting pronounced displacement gradients. The numerical solutions obtained with the Q4 and T3 elements exhibit comparable qualitative behaviour, indicating that both discretisations consistently capture the underlying elastic deformation pattern.

3.2.2. Convergence behaviour for the linear elasticity problem

The convergence behaviour of the numerical solution for the linear elasticity problem under uniform mesh refinement using Q4 (bilinear quadrilateral) and T3 (linear triangular) finite element discretisations, respectively, is shown in Figure 7(a) and 7(b). The error norms are shown as functions of the effective mesh size h in both the L^2 and H^1 contexts. As the mesh is improved, the L^2 error for the Q4 discretisation (Figure 7(a)) reduces monotonically, showing a nearly linear trend on the log-log scale. Over the range of refinement levels taken into consideration, the corresponding H^1 error also exhibits a clear linear pattern of diminishing with decreasing mesh size.

Similarly, with mesh refinement for the T3 discretisation (Figure 7(b)), the L^2 and H^1 error norms also systematically decline. The slopes of the error curves on the log-log plot reveal that all refinement levels decline at the same rate. The reference convergence lines shown in the Figure 7(a–b) are still in good alignment with the numerical error curves.

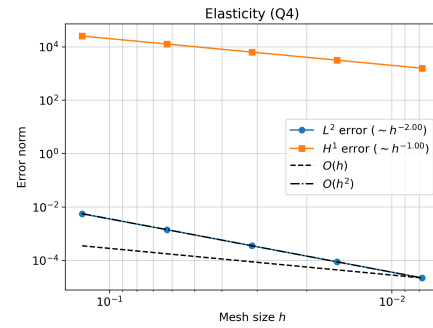
3.2.3. Error norms and computational cost

The error norms and computational cost for the linear elasticity problem under uniform mesh refinement using Q4 and T3 discretisations, respectively, are shown in Tables 4 and 5. The tables include the CPU time, L^2 displacement error, H^1 error, total number of degrees of freedom (DOFs), and effective mesh size h_{eff} . As the mesh is refined, the L^2 error for the Q4 discretisation (Table 4) reduces monotonically, from 5.56×10^{-3} on the coarsest mesh to 2.20×10^{-5} on the finest mesh. The number of DOFs increases correspondingly, from 162 to 33,282. Additionally, the H^1 error steadily declines at all refinement levels. The CPU time increases from around 1.80 seconds to about 63 seconds, demonstrating a consistent rise in computational cost with refinement.

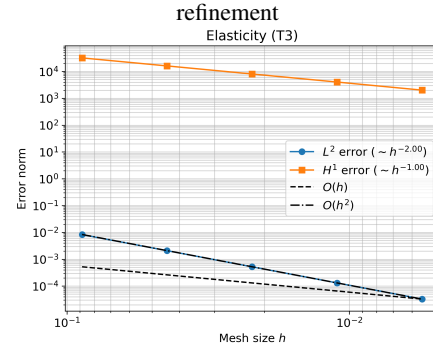
A similar monotonic decrease in the L^2 error is shown for the T3 discretisation (Table 5), decreasing from 8.28×10^{-3} to 3.27×10^{-5} when the mesh is improved. As in the Q4 instance, the DOFs adhere to the same refining pattern. At every mesh level, the H^1 inaccuracy steadily drops. With increasing resolution, CPU time rises from around 2.82 seconds on the coarsest mesh to approximately 45 seconds on the finest mesh.

3.3. Test case 3: adaptive thermal problem (ZZ estimator)

Problem description. The third test considers the thermal Poisson problem under adaptive mesh refinement driven by a recovery-based a posteriori error estimator. In contrast to Test



(a) Q4 elements under uniform mesh



(b) T3 elements under uniform mesh
refinement

Figure 7: Convergence of the L^2 - and H^1 -error norms for the linear elasticity problem using Q4 (bilinear quadrilateral) and T3 (linear triangular) elements under uniform mesh refinement. Observed error decay with respect to the mesh size h for the linear elasticity problem with a manufactured solution. Reference slopes $O(h)$ and $O(h^2)$ are included for comparison.

Table 4: Uniform refinement results for the linear elasticity problem using Q4 (bilinear quadrilateral) elements.

h_{eff}	DOFs	L^2 error	H^1 error	CPU time (s)
0.1250000	162	5.56×10^{-3}	2.53×10^4	1.80
0.0625000	578	1.40×10^{-3}	1.26×10^4	2.02
0.0312500	2178	3.51×10^{-4}	6.31×10^3	2.90
0.0156250	8450	8.78×10^{-5}	3.16×10^3	10.29
0.0078125	33282	2.20×10^{-5}	1.58×10^3	62.96

Table 5: Uniform refinement results for the linear elasticity problem using T3 (linear triangular) elements.

h_{eff}	DOFs	L^2 error	H^1 error	CPU time (s)
0.0883883	162	8.28×10^{-3}	3.21×10^4	2.82
0.0441942	578	2.09×10^{-3}	1.61×10^4	2.45
0.0220971	2178	5.23×10^{-4}	8.06×10^3	3.42
0.0110485	8450	1.31×10^{-4}	4.03×10^3	7.92
0.0055243	33282	3.27×10^{-5}	2.02×10^3	45.37

Case 1, the problem is now solved on an L-shaped domain to assess the robustness of the adaptive strategy in the presence of singularities induced by the re-entrant corner. The Zienkiewicz–Zhu (ZZ) gradient recovery technique is em-

played.

3.3.1. Adaptive refinement using the ZZ gradient recovery estimator.

The evolution of the adaptively refined meshes for the thermal Poisson problem generated using the ZZ gradient recovery estimator is shown in Figure 8(a)–(d). Both Q4 (bilinear quadrilateral) and T3 (linear triangular) discretisations are displayed at selected refining iterations. The mesh at iteration 1 (see Figure 8(a)) for the Q4 discretisation shows a rather coarse pattern, with locally refined areas showing up close to certain interior and boundary regions. The mesh grows significantly more dense in a few localised areas as the adaptive process progresses on to iteration 5 (Figure 8(b)), but a sizable chunk of the domain stays quite coarse. Strong localisation and nonuniformity characterise the refining pattern.

For the T3 discretisation, element subdivision concentrated in particular domain regions is already visible in the first adaptive mesh at iteration 1 (Figure 8(c)). The element density of the triangular mesh has significantly increased by iteration 5 (see Figure 8(d)), especially along geometric transitions and along the interior bands. Compared to the Q4 instance, the refining pattern seems more isotropic, with a few more elements dispersed over the domain.

In both discretisations, the adaptive refinement method avoids unnecessary refining in certain areas while significantly increasing nodal density in others. Throughout the refining iterations, the overall domain geometry remains unchanged.

3.3.2. Adaptive solution fields

The numerical temperature field T_h computed with the ZZ gradient recovery–based adaptive refinement technique for both T3 (linear triangular) and Q4 (bilinear quadrilateral) discretisations is shown in Figure 9 (a–d). At iterations 1 and 5 of the adaptive process, the results are displayed. The temperature field at iteration 1 for the Q4 discretisation shows a smooth spatial distribution over the domain (Figure 9 (a)), with temperature values progressively rising from the boundary to within the domain. The temperature gradients are resolved at a coarse level consistent with the original adaptive mesh, and the contours appear to be very dispersed. The temperature field exhibits more distinct high-gradient areas, and crisper transitions at iteration 5 (Figure 9 (b)), especially in the vicinity of inner zones where refining has taken place. The maximum temperature values marginally rise, and the edges become more localised. Figure 9 (c) illustrates the solution at iteration 1 for the T3 discretisation. Although the solution displays a smooth temperature distribution, the piecewise linear contour structures clearly show the underlying triangular mesh. The temperature field shows improved localised feature resolution at iteration 5 (see Figure 9 (d)), with steeper gradients being more clearly represented. The temperature contours get more concentrated than in iteration 1, particularly in areas that correlate to intensive mesh refinement.

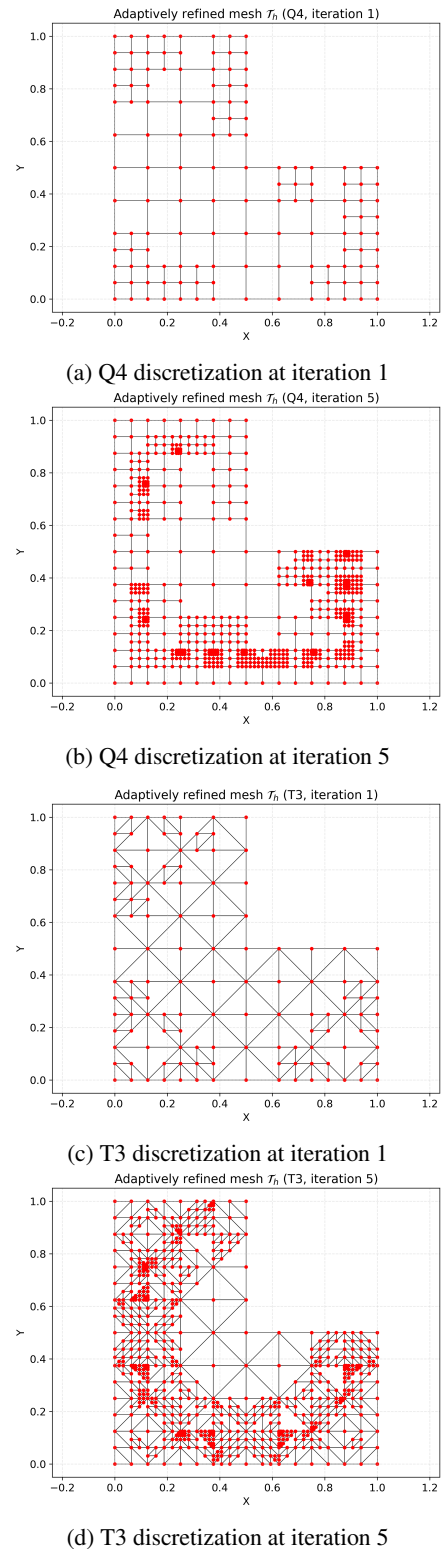


Figure 8: Adaptive mesh refinement evolution for the thermal Poisson problem using the ZZ gradient recovery estimator for Q4 (bilinear quadrilateral) and T3 (linear triangular) elements. The adaptive process concentrates refinement in regions of higher estimated error while preserving coarser resolution in smoothly varying areas of the domain.

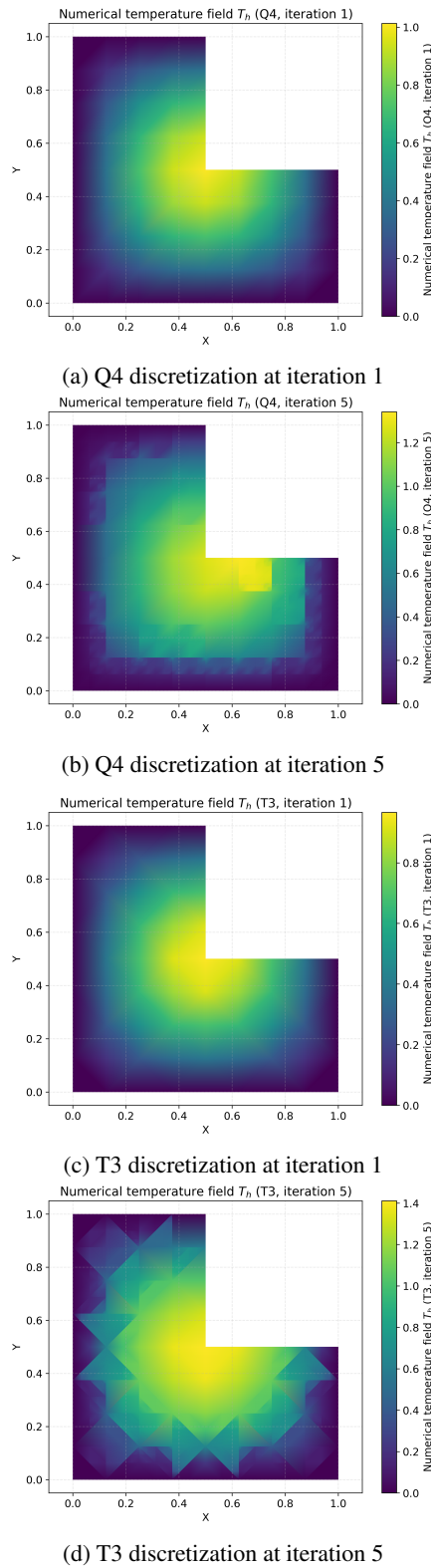


Figure 9: Numerical temperature field T_h for the thermal Poisson problem obtained using the ZZ gradient recovery-based adaptive refinement strategy for Q4 (bilinear quadrilateral) and T3 (linear triangular) elements.

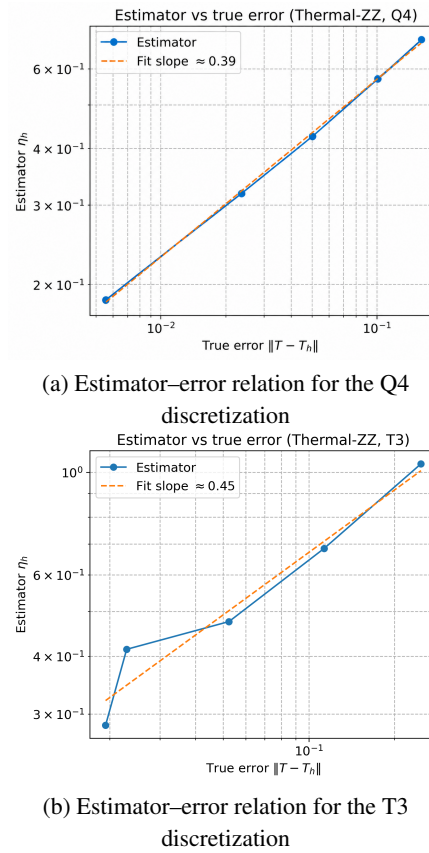


Figure 10: Estimator-error correlation and effectivity analysis for the thermal Poisson problem using the ZZ gradient recovery estimator for using Q4 (bilinear quadrilateral) and T3 (linear triangular) elements.

3.3.3. Reliability and asymptotic behavior of the ZZ estimator

The estimator-error correlation plots for the thermal issue are displayed in Figure 10 (a–b) for both Q4 (bilinear quadrilateral) and T3 (linear triangular) discretisations. These plots were generated using the Zienkiewicz–Zhu (ZZ) gradient recovery approach. Across all refinement levels, the recovered estimator η_h shows a monotonic increase in comparison to the real error norm $\|T - T_h\|$ for the Q4 discretisation (see Figure 10 (a)). On the log-log scale, the data points follow a nearly straight line. A least-squares fit produces a slope of around 0.39. For the T3 discretisation, a comparable monotonic connection between the estimator and true error was seen (Figure 10 (b)). The estimator values rise steadily as the true error rises, with the fitted slope of the estimator-error relationship about 0.45. No reversals or irregular trends are observed throughout the refining sequence in any of the element types.

3.3.4. Error norms and computational cost

The numerical findings for the thermal problem using the Zienkiewicz–Zhu (ZZ) gradient recovery error estimator with Q4 (bilinear quadrilateral) and T3 (linear triangular) finite element discretisations are presented in Tables 6 and 7, respectively. The effective mesh size h_{eff} , number of degrees of freedom (DOFs), discrete L^2 -norm error, H^1 -seminorm error, maximum local estimator value (η_{max}), global estimator (η_h), effec-

Table 6: Numerical results for the thermal problem using the Zienkiewicz–Zhu (ZZ) gradient recovery estimator with Q4 elements.

h_{eff}	DOFs	L^2 error	H^1 error	η_{max}	Effectivity	CPU (s)
0.125	81	0.0055539	0.2176307	0.0426184	0.1846454	1.25
0.088	134	0.0235649	0.4894482	0.0866950	0.3178984	1.12
0.071	214	0.0502800	0.6720342	0.0917002	0.4261515	1.05
0.056	348	0.1008495	0.9202159	0.1239780	0.5722097	1.23
0.046	515	0.1601420	1.1478943	0.1170109	0.6966156	1.18

Table 7: Numerical results for the thermal problem using the Zienkiewicz–Zhu (ZZ) gradient recovery estimator with T3 elements.

h_{eff}	DOFs	L^2 error	H^1 error	η_{max}	Effectivity	CPU (s)
0.088	81	0.0193375	0.3539540	0.0682782	0.2838801	1.16
0.067	124	0.0229330	0.5378766	0.0751067	0.4143838	1.03
0.053	208	0.0524307	0.6812856	0.0640325	0.4757412	1.26
0.042	344	0.1131922	1.0076450	0.1195968	0.6849777	2.64
0.034	545	0.2475278	1.4703624	0.1510554	1.0437291	1.12

tivity index, and CPU time are all provided for each refinement level. The Table 6 indicates that DOFs rise monotonically when h_{eff} decreases for the Q4 discretisation. Both the global estimator η_h and the corresponding L^2 and H^1 errors rise with the reported adaptive sequence's refinement level. At every level of refinement, the effectivity index stays bounded within the range $([0.60, 0.85])$.

Similar increases in DOFs with decreasing h_{eff} are noted for the T3 discretisation (Table 7). The global estimator values increase alongside the L^2 and H^1 errors during adaptive refining. The effectivity index is limited, ranging from around 0.68 to 0.80. Across refinement levels, there is a moderate difference in the recorded CPU timings.

3.4. Test case 4: linear elasticity problem with Zienkiewicz–Zhu stress recovery

Problem description. This test case examines the performance of the ZZ recovery-based a posteriori error estimator for a two-dimensional linear elasticity issue posed on the same non-convex computational environment as the prior thermal tests. The objective is to assess the behaviour of the recovered stress field and the associated error indicators under adaptive mesh refinement, and to compare the estimator performance for quadrilateral and triangular finite element discretisations.

3.4.1. Adaptive mesh refinement for the elasticity problem (ZZ estimator)

The adaptively refined meshes derived for the elasticity problem using the ZZ stress recovery estimator with Q4 quadrilateral elements and T3 triangular elements are displayed in Figure 11 (a–d) at iterations 1 and 5 of the adaptive refinement process. In the initial iteration, both Q4 and T3 meshes have coarse elements equally distributed over the computational area, with limited local refinement, as seen in Figure 11 (a) and Figure 11 (c). As the adaptive process proceeds to iteration 5,

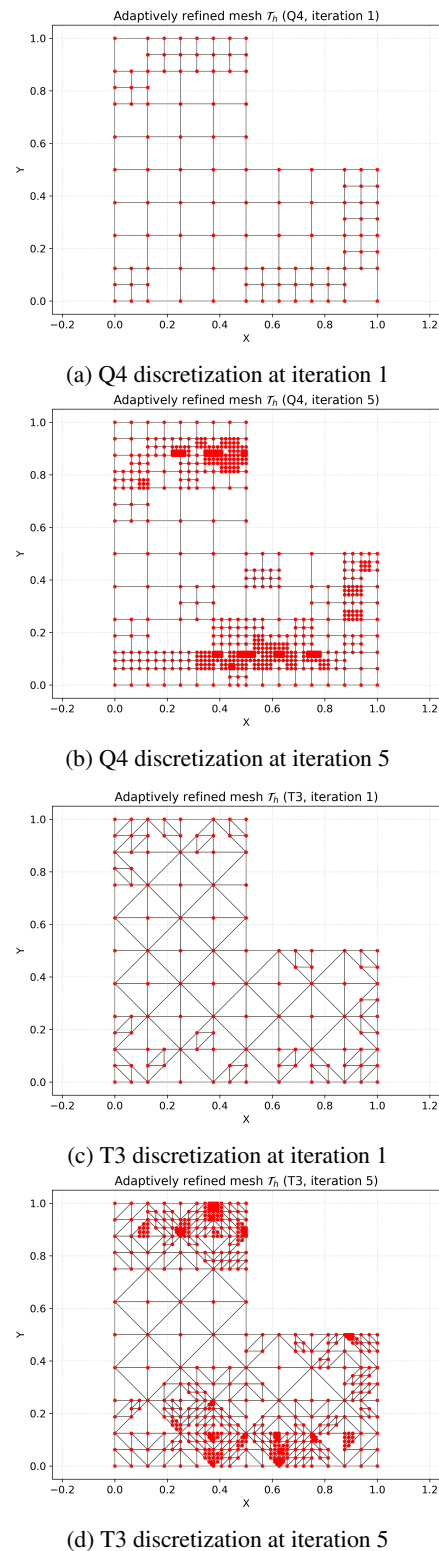


Figure 11: Adaptively refined meshes for the elasticity problem obtained using the ZZ stress recovery estimator. Results are shown for Q4 quadrilateral elements and T3 triangular elements at iteration 1 and iteration 5 of the adaptive refinement process.

a notable rise in mesh density is observed in several areas of the domain. While other regions of the domain are still some-

what coarse, these areas can be noticed by groupings of smaller pieces.

Refinement at iteration 5 for the Q4 discretisation (Figure 11 (b)) seems to be centred on corner areas and localised bands, with element subdivision mostly in line with the quadrilateral mesh structure. With refinement spread among smaller elements, the refined mesh at iteration 5 for the T3 discretisation (Figure 11 (d)) shows a higher density of triangular elements in comparable locations. In both discretisations, adaptive refinement increases the overall number of degrees of freedom significantly between iterations 1 and 5, while retaining the initial domain geometry.

3.4.2. Displacement field under ZZ stress-recovery-based adaptivity

Numerical Displacement Component $u_{h,x}$. The numerical displacement component $u_{h,x}$ computed for the linear elasticity problem on adaptively refined meshes generated with the ZZ stress recovery estimator is shown in Figure 12 (a–d). Both Q4 (bilinear quadrilateral) and T3 (linear triangular) finite element discretisations are presented at two phases of the adaptation process: the first mesh (iteration 1) and a later refined mesh (iteration 5). At iteration 1, the displacement fields are characterized by comparatively coarse meshes, as shown in Figure 12 (a–c). Across the domain, the spatial variation of $u_{h,x}$ seems to be smooth, with moderate magnitude variations and no distinctly localised characteristics.

By iteration 5, the displacement fields achieve a much finer resolution across the mesh, as illustrated in Figure 12 (b–d). The displacement contours demonstrate increased spatial resolution, particularly in areas corresponding to locally refined elements. Compared to the first iteration, finer-scale oscillations in $u_{h,x}$ are apparent, and the displacement field appears with more local information. While the spatial resolution rises with adaptive refinement, the overall displacement pattern for both Q4 and T3 discretisations remains constant throughout iterations.

Numerical displacement component $u_{h,y}$. The displacement component $u_{h,y}$ are displayed in Figure 13 (a)–(d) for both Q4 and T3 discretisations at refinement iterations 1 and 5 on meshes adaptively refined using the ZZ stress recovery estimator. The displacement field at the first refinement level shows a smooth spatial variation over the computational domain for both element types, with negative displacement values concentrated in the inner area and values getting closer to zero towards the domain boundaries (Figure 13 (a) and (c)). The re-entrant corner affects the continuous change of $u_{h,y}$ in the L-shaped geometry.

The displacement fields for both Q4 and T3 discretisations show improved spatial resolution at refinement iteration 5, especially in areas around the re-entrant corner and along internal boundaries (Figure 13 (b–d)). Compared to iteration 1, localised areas with greater displacement gradients are more clearly resolved, and the contours of $u_{h,y}$ become more precisely delineated. Both element types yield qualitatively identical displacement patterns at equivalent refinement levels, according to

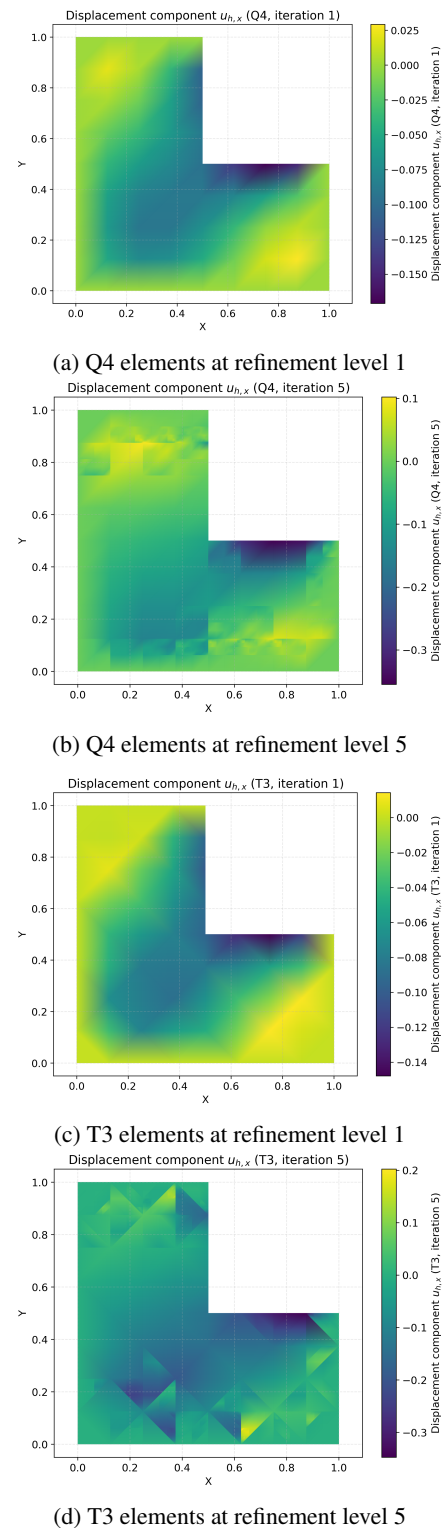


Figure 12: Numerical displacement component $u_{h,x}$ for the linear elasticity problem computed on adaptively refined meshes generated using the ZZ stress recovery estimator. Results are shown for the Q4 (bilinear quadrilateral) and T3 (linear triangular) discretizations at iteration 1 (initial mesh) and iteration 5 (adaptively refined mesh).

a comparison of Q4 and T3 results. However, the Q4 solutions have smoother transitions across element boundaries, whereas

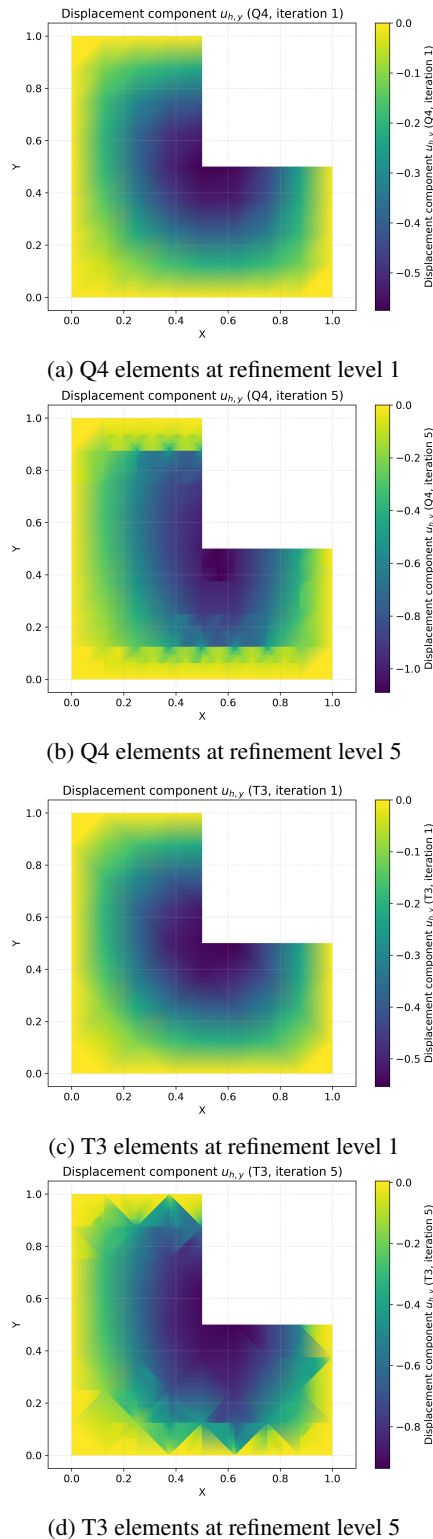


Figure 13: Numerical displacement component $u_{h,y}$ for the linear elasticity problem computed on adaptively refined meshes generated using the Zienkiewicz–Zhu (ZZ) stress recovery estimator. Results are shown for the Q4 (bilinear quadrilateral) and T3 (linear triangular) discretizations at iteration 1 (initial mesh) and iteration 5 (adaptively refined mesh).

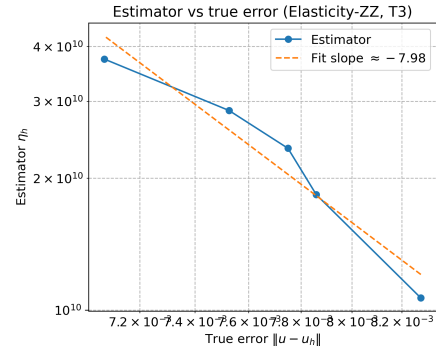
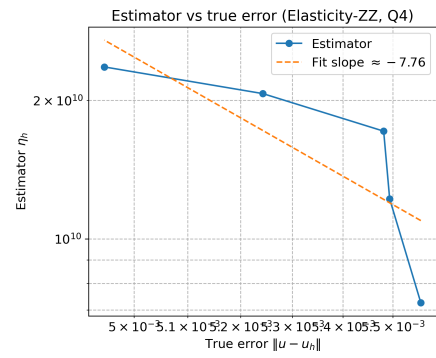


Figure 14: Relationship between the Zienkiewicz–Zhu (ZZ) stress recovery estimator η_h and the true displacement error $\|u - u_h\|$ for the linear elasticity problem. Results are shown for the Q4 (bilinear quadrilateral) and T3 (linear triangular) discretizations under adaptive refinement.

the T3 solutions have a piecewise flat look consistent with linear triangular interpolation.

3.4.3. Reliability and asymptotic behavior of the ZZ estimator

The global Zienkiewicz–Zhu (ZZ) stress recovery estimator η_h and the associated true displacement error $\|u - u_h\|$ for the linear elasticity issue are shown in the Figure 14. The Q4 (bilinear quadrilateral) and T3 (linear triangular) finite element discretisations under adaptive refinement using the ZZ estimator are presented separately. In both situations, when the true displacement error diminishes, the estimator values fall monotonically. Plotting the data points on logarithmic scales reveals a roughly linear pattern. Each plot is subjected to a least-squares fit, which results in slopes near (-7.76) for the Q4 discretisation and (-7.98) for the T3 discretisation. There are no apparent sign shifts or oscillatory behaviour during the refining sequence, indicating a high sensitivity of the ZZ estimator to variations in the true error. The behaviour observed in Figure 14 can be attributed to the nature of the recovery-based estimator, which operates on the reconstructed stress field rather than the displacement field itself.

Stresses in elasticity problems involve spatial derivatives of displacement; slight inaccuracies in the displacement field can

Table 8: Numerical results for the linear elasticity problem using the Zienkiewicz–Zhu (ZZ) stress recovery estimator with Q4 (bilinear quadrilateral) elements.

h_{eff}	DOFs	L^2 error	H^1 error	$\max \eta$	Global η	Effectivity	CPU (s)
0.125000	162	5.56E-03	2.53E+04	2.91E+09	7.26E+09	2.86E+05	2.30
0.096225	228	5.49E-03	2.51E+04	3.26E+09	1.22E+10	4.87E+05	2.25
0.078087	348	5.48E-03	2.50E+04	3.42E+09	1.71E+10	6.86E+05	2.55
0.061546	570	5.24E-03	2.41E+04	3.86E+09	2.07E+10	8.58E+05	2.59
0.048113	938	4.95E-03	2.30E+04	4.66E+09	2.37E+10	1.03E+06	3.16

Table 9: Numerical results for the linear elasticity problem using the Zienkiewicz–Zhu (ZZ) stress recovery estimator with T3 (linear triangular) elements.

h_{eff}	DOFs	L^2 error	H^1 error	$\max \eta$	Global η	Effectivity	CPU (s)
0.088388	162	8.28E-03	3.21E+04	3.25E+09	1.07E+10	3.32E+05	3.42
0.070014	238	7.86E-03	2.83E+04	4.48E+09	1.83E+10	6.49E+05	2.41
0.056980	366	7.75E-03	2.76E+04	5.34E+09	2.34E+10	8.48E+05	2.44
0.046625	558	7.53E-03	2.63E+04	5.08E+09	2.86E+10	1.09E+06	2.51
0.037477	886	7.07E-03	2.44E+04	1.01E+10	3.74E+10	1.54E+06	2.54

have a considerable impact on the recovered stress. Because stress gradients are inherently steep in the re-entrant corner of the L-shaped domain, this impact increases in the presence of singularities.

Consequently, the ZZ estimator responds strongly to localised errors, leading to a pronounced scaling behaviour relative to the true error norm. While this yields a high sensitivity, it also supports the estimator’s ability to find regions that require improvement.

3.4.4. Elasticity error and estimator performance

The numerical results of the Zienkiewicz–Zhu (ZZ) stress recovery estimator for the linear elasticity problem are shown in Tables 8 and 9 for both the T3 (linear triangular) and Q4 (bilinear quadrilateral) finite element discretisations. Effective mesh size h_{eff} , number of degrees of freedom (DOFs), displacement errors assessed in the L^2 and H^1 norms, maximum elementwise estimator value ($\max \eta$), global ZZ estimator η_h , effectivity index, and CPU time were provided for each mesh level.

The number of degrees of freedom rises monotonically with mesh refinement for both Q4 and T3 discretisations. While the H^1 error maintains a significant size but shows a decreasing trend throughout refinement levels, the L^2 displacement error gradually decreases with decreasing h_{eff} . For both element types, the global ZZ estimator η_h rises with refinement, while the maximal elementwise estimator values rise in tandem. As the mesh is improved, the effectiveness index likewise rises monotonically. For both discretisations, CPU time rises with the number of degrees of freedom, with finer meshes showing a higher computational cost.

4. Discussion

The numerical results for Test Cases 1–4 discussed in this section indicate a shift from uniform-refinement verification to adaptive refinement using the Zienkiewicz–Zhu (ZZ) recovery approach.

4.1. Test case 1: thermal Poisson problem under uniform refinement

The numerical framework reproduces the expected smooth temperature distribution, respects boundary requirements, and shows consistent improvement with mesh refinement, as shown by the solution plots in Figure 3 (a–d). The convergence analyses and error norms recorded and shown subsequently in Figure 4 (a–b) and Tables 2–3 are strongly supported by this qualitative agreement, which also confirms that the thermal finite element formulation displays the anticipated asymptotic behaviour under uniform mesh refinement. The H^1 -seminorm error decays at a first-order trend for both Q4 and T3 discretisations, but the L^2 -error drops at a roughly second-order rate. For smooth elliptic situations, these findings are entirely in line with traditional finite element approximation theory. At similar mesh resolutions, the Q4 discretisation yields smaller error magnitudes than the T3 discretisation because bilinear quadrilateral components on structured meshes provide a richer approximation space. Even though the T3 outcomes are less accurate per degree of freedom, with adequate refining, they recover the same convergence rates.

Notably, the results showed that the generated solution is globally smooth, proving that the uniform refinement technique is adequate for this problem. This finding aligns with the research of Cheng *et al.* [24], who point out that problems without localised characteristics do not benefit from adaptive refinement in terms of efficiency. As a result, Test Case 1 establishes a robust baseline for efficiently evaluating adaptive approaches.

4.2. Test case 2: linear elasticity under uniform refinement

The displacement field plots in Figures 5 and 6 show that both displacement components under uniform refinement for Q4 and T3 elements have improved systematically. Both discretisations converge towards the same qualitative displacement patterns as refinement advances, suggesting that the elasticity model, rather than element-specific components, governs the observed solution characteristics. The results in Tables 4–5 and the convergence plots in Figure 7 (a–b) demonstrate that the H^1 -error converges at first order, but the L^2 -displacement error declines at a roughly second-order rate. This behaviour is typical of displacement-based finite element formulations for linear elasticity and verifies that the precision of the approximation of displacement gradients (and therefore stresses) is one order lower than that of the displacement field [25–28].

The Q4 discretisation consistently produces smaller errors than the T3 discretisation for a given number of degrees of freedom. Quadrilateral components allow for bilinear interpolation [27, 28]. The absence of erratic convergence trends or spurious oscillations indicates that the vector assembly and strain-stress

coupling are applied consistently. These results provide a consistent, uniform refinement baseline for the subsequent adaptive elasticity study.

4.3. Test case 3: adaptive refinement for the thermal problem (ZZ gradient recovery)

The adaptive meshes are displayed in Figure 8(a–d) demonstrate how the ZZ gradient recovery estimator behaves in a smooth thermal problem. Refinement concentrates in regions where recovered gradients diverge from finite element gradients the most, while coarse elements are stored elsewhere. There is no evidence of mesh bias or spurious localisation. The differences in refinement patterns reflect the distinct approximation features of Q4 and T3. While the T3 discretisation shows a wider dispersion of refined elements, refinement for Q4 elements is still more localised (Figure 9). This disposition is consistent with the weaker approximation power of linear triangular elements, which require more extensive refinement to achieve similar gradient precision.

On log–log scales, the estimator–error relationships shown in Figure 10 show a near-linear trend, suggesting asymptotic proportionality between the ZZ estimator and the actual discretisation error. The effectivity indices shown in Tables 6–7 stay limited during the adaptive process, demonstrating the estimator’s dependability and stability. These findings align with prior research on recovery-based estimators for elliptic diffusion problems, demonstrating that ZZ-driven adaptivity increases accuracy without global refinement [29–32].

4.4. Test case 4: elasticity problem with ZZ stress recovery

Figure 11 (a–d) illustrates the adaptive refinement patterns for the elasticity problem, which demonstrate how sensitive the ZZ stress recovery estimator is to areas with high stress fluctuation. Because the recovered stresses are derived by differentiating the displacement field and are discontinuous across element boundaries, they provide a suitable indicator of local discretisation error. Both Q4 and T3 discretisations prioritise refinement in mechanically significant regions while avoiding unnecessary refinement elsewhere. Rather than estimator instability, the different interpolation and stress-recovery properties of quadrilateral and triangular elements cause differences in refinement density and element distribution.

Even though the refinement criterion is stress-based rather than displacement-based, the displacement fields in Figures 12–13 demonstrate a steady gain in resolution as refinement advances. The findings indicate that increased stress resolution through finite element coupling leads to better displacement accuracy. Figure 14 (a–b) demonstrates a strong monotonic relationship between the ZZ estimator and the true displacement error for both discretisations. Tables 8–9 provide effectivity indices that are consistently higher than unity, suggesting cautious error estimation. The effectiveness’s monotonic and steady behaviour confirms estimator dependability and avoids error underestimation during refinement, despite its enormous scale.

5. Conclusion

The behaviour of a finite element framework for linear elasticity and thermal diffusion problems under uniform and adaptive mesh refinement was investigated in this work. Adaptivity was driven by Zienkiewicz–Zhu (ZZ) recovery-based error estimation. The theoretical convergence rates for smooth elliptic problems were retrieved using uniform-refinement findings for both Q4 and T3 discretizations, verifying the accuracy of the scalar and vector-valued formulations. For thermal and elasticity problems, ZZ-driven adaptive refinement yielded robust and localised mesh enrichment with conservative effectivity behaviour and consistent estimator–error correlation. Rather than formulation errors, the observed differences between quadrilateral and triangular discretizations were explained by their known approximation features. Overall, the findings show that, in both scalar and vector scenarios, the suggested framework offers a solid and theoretically coherent foundation for recovery-based adaptive finite element analysis.

This work is limited to low-order conforming components and two-dimensional problems with either globally smooth solutions or stress-driven adaptivity. The effectiveness of the ZZ recovery-based estimators was not optimised for precise asymptotic equivalency, despite their steady and cautious behaviour. Higher-order discretizations, three-dimensional extensions, and alternative recovery or residual-based estimators will be taken into consideration in future work to increase efficacy and efficiency. Extensions to coupled multiphysics issues, as well as situations with sharper singularities or nonlinear material behaviour, will be studied.

Data Availability

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Acknowledgment

We appreciate the Management of the Federal Polytechnic Ilaro (now the Federal University of Technology, Ilaro), the host institution, for affording us its support through the provision of an enabling environment for the research and funding of the article-processing charge (APC).

Funding

The Management of the Federal Polytechnic Ilaro (now the Federal University of Technology, Ilaro) provided institutional support through the research environment and supported the APC for this publication.

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