Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 8 (2026) 3293

Journal of the Nigerian Society of Physical Sciences

Even vertex odd edge root square mean labeling of some cycle-related graphs

K. N. Babu [a,b], S. Meenakshi [c,*]

[a] Department of Mathematics, Vels Institute of Science, Technology and Advanced Studies (VISTAS), Chennai – 600 117, Tamil Nadu, India
[b] Sri Malolan College of Arts and Science, Madurantakam – 603 306, Tamil Nadu, India
[c] Department of Mathematics, Vels Institute of Science, Technology and Advanced Studies (VISTAS), Chennai – 600 117, Tamil Nadu, India

Abstract

An injective mapping $f : V(G) \to \{0, 2, 4, \ldots, 2|E|\}$ is said to satisfy the even vertex odd edge root square mean labeling (EVO-ERSML) condition when vertices are assigned distinct even values and each edge $uv \in E(G)$ receives a distinct odd label determined by $\left\lceil \sqrt{\frac{f(u)^2+f(v)^2}{2}} \right\rceil$ or $\left\lfloor \sqrt{\frac{f(u)^2+f(v)^2}{2}} \right\rfloor$. Graphs that admit such assignments are called EVOERSML graphs. The present study examines this labeling scheme on several graph structures closely related to cycles. Constructive methods are provided for families including ladder graphs, tadpole graphs, polygon-chain graphs, dumbbell graphs, and sunlet graphs. The obtained results emphasize how parity restrictions influence labeling existence and uniqueness. This work enriches the theory of root square mean labeling and opens further scope for studying EVOERSML-type labelings in more complex graph families and applied network models.

DOI: 10.46481/jnsps.2026.3293

Keywords: EVOERSML, Graph, Cycle, Ladder, Sunlet

Article History:
Received: 05 February 2026
Received in revised form: 13 May 2026
Accepted for publication: 3 June 2026
Available online: 07 July 2026

Communicated by: P. Thakur

1. Introduction

The present work is restricted to the study of simple graphs. The notions of root square mean labeling (RSML) and super root square mean labeling (SRSML) were originally introduced and developed in Refs. [1–4], and a comprehensive overview of developments in this area has been provided by Gallian in Ref. [5]. The concept of odd vertex even edge root square mean labeling (OVEERSML) was later examined in Ref. [6]. Furthermore, Babu and Meenakshi explored the notion of even vertex odd edge root square mean labeling (EVOERSML), with particular emphasis on path-related graphs, in Ref. [7]. Motivated by these earlier studies, the current work aims to advance and broaden the theory of EVOERSML graphs.

This study primarily concentrates on graphs associated with paths and cycles, for which several new properties, labeling criteria, and existence results are derived. Explicit construction methods are developed for selected families of graphs. Furthermore, essential conditions and deficiency measures corresponding to these labelings are investigated, offering a clearer understanding of the structural characteristics of EVOERSML graphs under different graph transformations.

The findings presented here add to the existing body of graph labeling theory by expanding the collection of graph classes known to support EVOERSML labelings. The work also reveals connections among various labeling schemes, thereby reinforcing the theoretical framework underlying

*Corresponding author Tel. No.: +91 944-578-4530.
Email address: meenakshikarthikeyan@yahoo.in (S. Meenakshi)

vertex–edge labeling techniques. In addition, the approaches employed in this study can be adapted to examine similar labeling problems and to formulate more generalized labeling methodologies.

In conclusion, this research suggests several potential directions for future work. Extending the concept of EVOERSML to broader categories of graphs, including trees, ladder graphs, grid structures, and graph products, represents a fruitful area for further exploration. The development of algorithmic procedures for constructing EVOERSML labelings, along with possible applications in areas such as network analysis, communication models, and combinatorial designs, also merits further attention. Collectively, the results of this study lay a strong groundwork for continued research in graph labeling theory.

1.1. Definitions

According to Refs. [5, 6], a one-to-one relation f on $V(G)$ with integers between 1 and twice the size of the graph increased by one is said to be an OVEERSML if the edge labels can be determined by the ceiling or floor value of

$$\sqrt{\frac{f(u)^2 + f(v)^2}{2}}, \tag{1}$$

in which all vertices and edges are distinct odd and even numbers, respectively.

According to Refs. [5, 7], a one-to-one relation f on $V(G)$ with integers between 0 and twice the size of the graph is said to be an EVOERSML if the edge labels can be determined by the ceiling or floor value of

$$\sqrt{\frac{f(u)^2 + f(v)^2}{2}}, \tag{2}$$

in which all vertices and edges are distinct even and odd numbers, respectively.

A ladder graph is a simple connected graph obtained as the Cartesian product of a path graph with a complete graph on two vertices. Equivalently, it consists of two parallel paths of equal length, where corresponding vertices of the paths are joined by edges, resembling the rungs of a ladder. The ladder graph with n rungs is commonly denoted by L_n.

A polygon-chain graph is obtained by joining m cycles consecutively such that consecutive cycles share exactly one common edge.

A dumbbell graph is a simple connected graph obtained by joining two vertex-disjoint cycles by a path. Let C_m and C_n be cycles of lengths m and n, respectively. The dumbbell graph, denoted by $D(m, n, k)$, is formed by connecting a vertex of C_m to a vertex of C_n using a path of length $k \geq 1$. When $k = 1$, the two cycles are connected by a single edge; for $k > 1$, the connecting structure is a path with k edges.

The $T(m, n)$-tadpole graph is uniquely constructed by connecting a cycle of m vertices to a path of n vertices using a single edge.

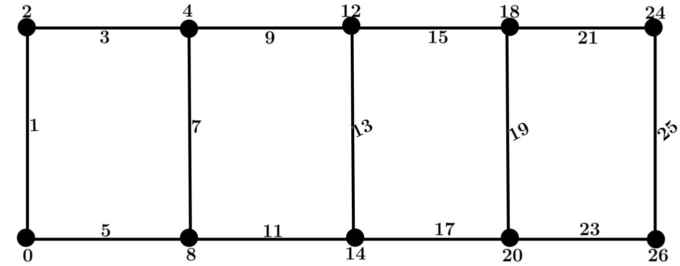

Figure 1: Ladder graph L_5.

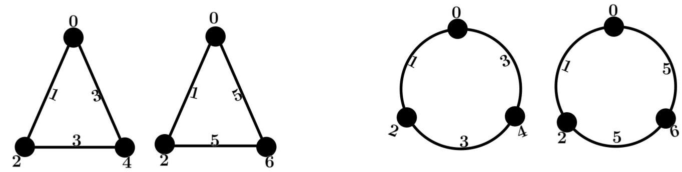

Figure 2: Graphs of K_3 and C_3.

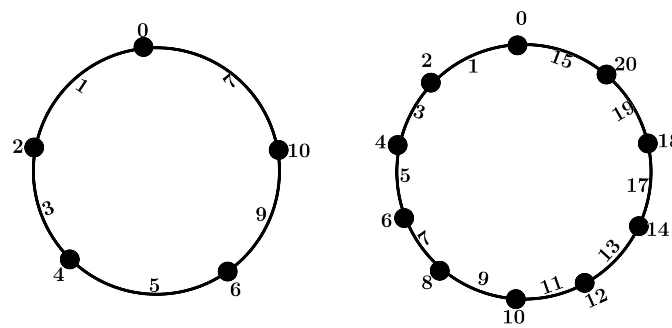

Figure 3: Graphs of C_5 and C_{10}.

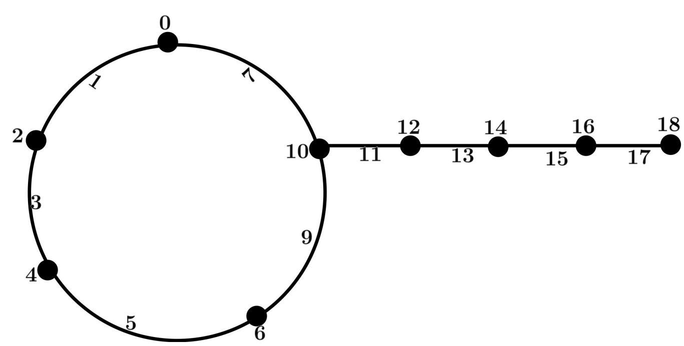

Figure 4: Graph of $T(5, 4)$.

2. Main results

2.1. Ladder graphs

Theorem 2.1. The ladder graph L_p admits an EVOERSML.

Proof. Let G denote the ladder graph L_p. Let

$$V(G) = \{u_i, v_i : i = 1, 2, \ldots, p\}, \tag{3}$$

and

$$E(G) = \{(u_i, u_{i+1}), (v_i, v_{i+1}) : i \in [1, p-1]\} \cup \{(u_i, v_i) : i \in [1, p]\}. \tag{4}$$

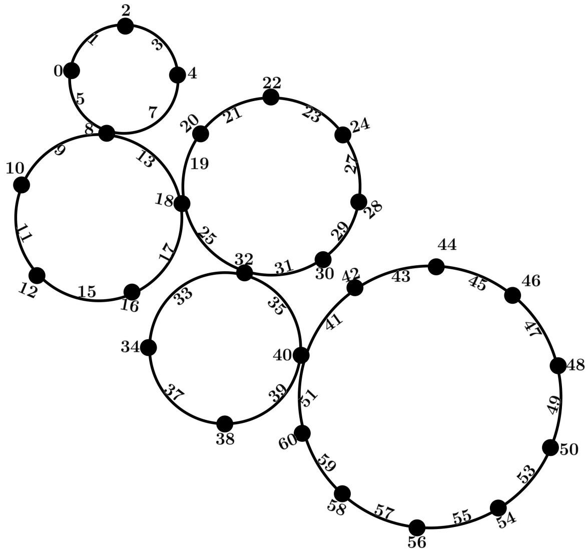

Figure 5: Polygon-chain graph with cycles of orders 4, 5, 7, 4, and 10 admitting an EVOERSML.

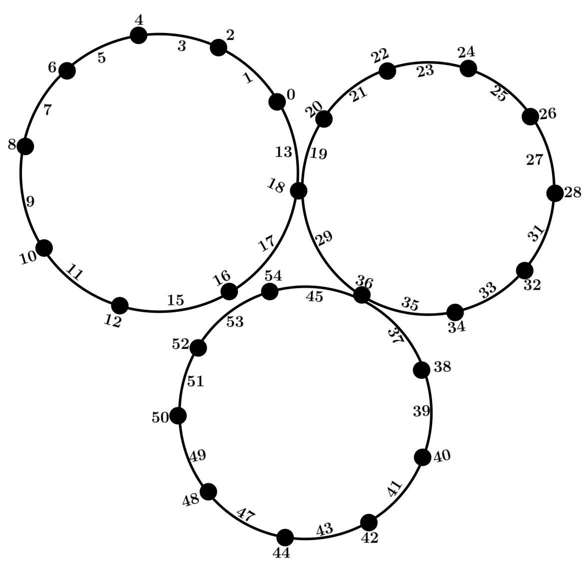

Figure 6: EVOERSML of the polygon chain formed by three copies of C_8.

Define a relation f from $V(G)$ to $\{0, 1, 2, \ldots, 2(3p-2)\}$ by

$$\begin{aligned}
&f(v_1) = 0, \\
&f(u_i) = 2i, \qquad i = 1, 2, \\
&f(u_i) = 6i - 6, \qquad i = 3, \ldots, p, \\
&f(v_i) = 6i - 4, \qquad i = 2, \ldots, p, \\
&f^*(u_i u_{i+1}) = 6i - 3, \qquad i = 1, \ldots, p-1, \\
&f^*(v_i v_{i+1}) = 6i - 1, \qquad i = 1, \ldots, p-1, \\
&f^*(u_i v_i) = 6i - 5, \qquad i = 1, \ldots, p.
\end{aligned} \tag{5}$$

Hence, L_p is an EVOERSML graph. □

Example 1. The ladder graph L_5 can be labeled as an EVOERSML, as presented in Figure 1.

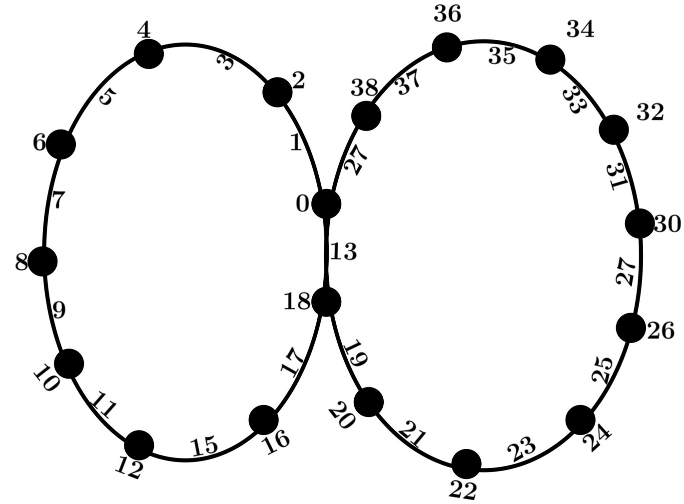

Figure 7: Graph $C_9 \circ C_{11}$.

2.2. 2-regular graphs

Theorem 2.2. A 2-regular graph is an EVOERSML graph, except when $n = 3$.

Proof. The graphs K_3 and C_3 are both 2-regular. In this case, the edge labels 3 and 5 are repeated in Figure 2, which violates the conditions required for an EVOERSML labeling.

For $n > 3$, let f be a labeling function that maps $V(G)$ to the range from 0 to $2q$, assigning labels to the vertices of the graph. Let

$$k = \sqrt{\frac{f(u_1)^2 + f(u_n)^2}{2}}. \tag{6}$$

The set of vertex labels is given by

$$\{0, 2, 4, \ldots, k-1, k+3, k+5, \ldots, 2q\}. \tag{7}$$

Let k be the pivotal value determined by the above condition. The remaining edge labels are assigned as follows: the labels $\{1, 3, 5, \ldots, k-2\}$ are allocated to a subset of the edges, while the labels $\{k+2, k+4, \ldots, 2q-1\}$ are assigned to another subset. The label k is designated to a specific edge. Since each vertex and edge receives a unique label under this scheme, there is no duplication of edge labels, and the graph adheres to the EVOERSML condition. Thus, the theorem holds. □

Example 2. The graphs C_5 and C_{10} are EVOERSML graphs, as illustrated in Figure 3.

2.3. Tadpole graphs

Theorem 2.3. Joining the first vertex of the path P_n with the last vertex of the cycle C_m yields the tadpole graph $T(m, n)$, which is an EVOERSML graph.

Proof. Let $T(m, n) = G$. Define

$$V(G) = \{x_1, x_2, \ldots, x_m\} \cup \{x'_1, x'_2, \ldots, x'_n\}, \tag{8}$$

and

$$\begin{aligned} E(G) =&\{x_i x_{i+1} : 1 \le i \le m-1\} \cup \{x_m x_1\} \\ &\cup \{x'_j x'_{j+1} : 1 \le j \le n-1\} \cup \{x_m x'_1\}. \end{aligned} \tag{9}$$

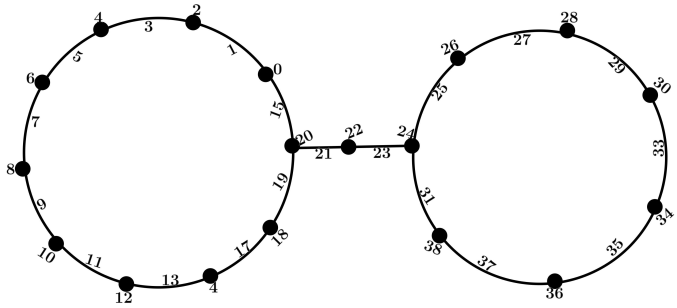

Figure 8: $DB(C(10,7))$ with P_3.

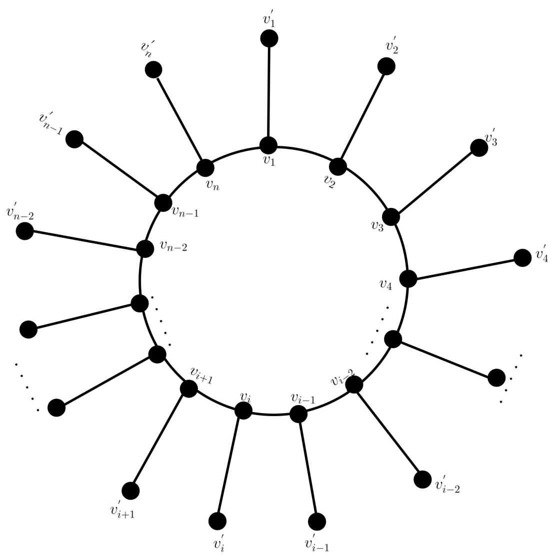

Figure 9: Structure of a sunlet graph.

Assign labels to the vertices with even integers from 0 to $2(m+n)$. The labeling scheme follows. Let

$$k = \sqrt{\frac{f(x_1)^2 + f(x_m)^2}{2}}, \tag{10}$$

where k is an odd number, since the edge $x_m x_1'$ connects the identified vertices. The labels of the vertices in G are defined as

$$\begin{aligned} &\{0, 2, \dots, k-1\} \cup \{k+3, k+5, \dots, 2m\} \\ &\cup \{2m+2, 2m+4, \dots, 2(m+n)\}. \end{aligned} \tag{11}$$

The resulting edge labels are

$$\begin{aligned} &\{1, 3, \dots, k-2\} \cup \{k+2, k+4, \dots, 2m-1, k\} \\ &\cup \{2m+1, 2m+3, \dots, 2(m+n)-1\}. \end{aligned} \tag{12}$$

Then the vertex and edge labels are distinct. Therefore, G satisfies the EVOERSML condition, as both the vertices and edges are labeled without repetition. Hence, the theorem follows. □

Example 3. Figure 4 illustrates that $T(5,4)$ is an EVOERSML graph.

2.4. Polygon-chain graphs

Theorem 2.4. A polygon-chain graph is an EVOERSML graph.

Proof. **Case 1.** If the order of the cycles is the same, let

$$X(G) = \{x_i^j : i \in [1,n] \text{ and } j \in [1,m]\}, \tag{13}$$

and

$$Y(G) = \{y_i^j : i \in [1,n] \text{ and } j \in [1,m]\}. \tag{14}$$

Assign labels to the vertices from 0 to $2mn$ under f. The labeling scheme of G proceeds as follows.

For the first copy, the vertices are assigned labels based on

$$k_1 = \sqrt{\frac{f(x_1^1)^2 + f(x_n^1)^2}{2}}, \tag{15}$$

where $f(x_1^1) = 0$ and $f(x_n^1) = 2n$. The set of vertex labels is

$$f(x_i^1) \in \{0, 2, 4, \dots, k_1 - 1, k_1 + 3, k_1 + 5, \dots, 2n\}, \quad i \in [1,n]. \tag{16}$$

For the second copy of the graph, the vertex labels are determined using

$$k_2 = \sqrt{\frac{f(x_1^2)^2 + f(x_n^2)^2}{2}}, \tag{17}$$

with $f(x_1^2) = 2n$ and $f(x_n^2) = 4n$. The vertex-labeling set becomes

$$\begin{aligned} f(x_i^2) \in \{&2(n+1), 2(n+2), \dots, k_2 - 1, \\ &k_2 + 3, k_2 + 5, \dots, 4n\}, \quad i \in [2,n]. \end{aligned} \tag{18}$$

For the m-th copy, the labeling follows

$$k_m = \sqrt{\frac{f(x_1^m)^2 + f(x_n^m)^2}{2}}, \tag{19}$$

where $f(x_1^m) = 2(m-1)n$ and $f(x_n^m) = 2mn$. The corresponding vertex-label set is

$$\begin{aligned} f(x_i^m) \in \{&2n(m-1) + 2, 2n(m-1) + 4, \dots, k_m - 1, \\ &k_m + 3, k_m + 5, \dots, 2mn\}, \quad i \in [2,n]. \end{aligned} \tag{20}$$

It is evident from the construction that the edge labels obtained through this scheme are all distinct.

Case 2. If the order of the cycles is distinct, the order of each cycle is $q_1, q_2, \dots, q_n$. The function f is defined on $V(G)$ with 0 to $2(q_1 + q_2 + \cdots + q_n)$. The set of labeled vertices in the first copy can be identified by

$$k_1 = \sqrt{\frac{f(x_1^1)^2 + f(x_n^1)^2}{2}}, \quad f(x_1^1) = 0, \quad f(x_n^1) = 2q_1, \tag{21}$$

and

$$f(x_i^1) = \{0, 2, 4, \dots, k_1 - 1, k_1 + 3, k_1 + 5, \dots, 2q_1;\ i \in [1,n]\}. \tag{22}$$

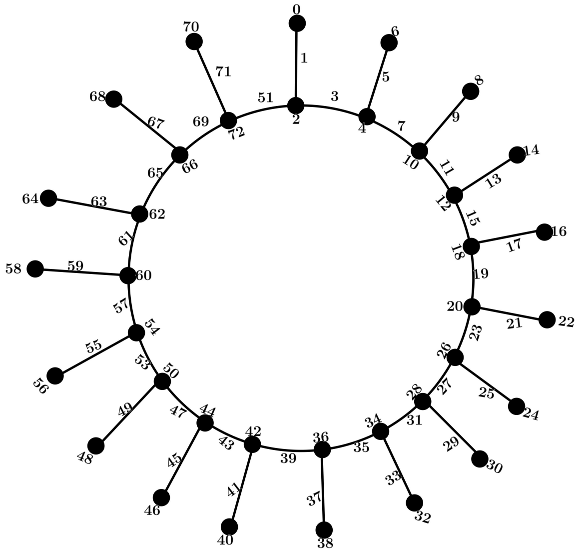

Figure 10: Sunlet graph S_{18}.

The vertices in the second copy are labeled as follows:

$$k_2 = \sqrt{\frac{f(x_1^2)^2 + f(x_n^2)^2}{2}}, \quad f(x_1^2) = 2q_1, \quad f(x_n^2) = 2(q_1 + q_2), \tag{23}$$

and

$$f(x_i^2) \in \{2q_1, 2(q_1+1), \ldots, k_2 - 1, k_2 + 3, k_2 + 5, \ldots, 2(q_1 + q_2)\}, \quad i \in [2, n]. \tag{24}$$

In general, for the j-th copy, the vertices are labeled as follows:

$$k_j = \sqrt{\frac{f(x_1^j)^2 + f(x_n^j)^2}{2}}, \tag{25}$$

where

$$f(x_1^j) = 2\left(\sum_{i=1}^{j-1} q_i\right), \quad f(x_n^j) = 2\left(\sum_{i=1}^{j} q_i\right), \tag{26}$$

and

$$f(x_i^j) = \left\{2\left(\sum_{i=1}^{j-1} q_i\right), \ldots, k_j - 1, k_j + 3, k_j + 5, \ldots, 2\left(\sum_{i=1}^{j} q_i\right)\right\}. \tag{27}$$

Hence, f is an EVOERSML. □

Example 4. The polygon-chain graph consisting of five cycles of orders 4, 5, 7, 4, and 10, where both equal and distinct cycle orders occur, admits an EVOERSML, as shown in Figure 5.

Example 5. The polygon chain formed by three copies of C_8 is an EVOERSML graph, as illustrated in Figure 6.

2.5. Cycle graphs combined by an edge

Theorem 2.5. The graph $C_n \circ C_m$, formed by combining two copies of a cycle with an edge, is an EVOERSML graph.

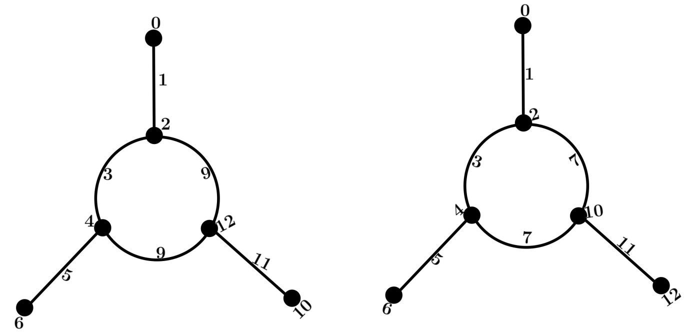

Figure 11: Sunlet graph S_3.

Proof. Let $G = C_n \circ C_m$, where

$$V(C_n) = \{u_1, u_2, \ldots, u_n\}, \quad V(C_m) = \{v_1, v_2, \ldots, v_m\}, \tag{28}$$

and

$$E(C_n) = \{e_i = u_i u_{i+1} : 1 \le i \le n-1\} \cup \{k = u_n u_1\}, \\ E(C_m) = \{e_j = v_j v_{j+1} : 1 \le j \le m-1\} \cup \{k_1 = v_m v_1\}. \tag{29}$$

Define a map $f : V(G) \to \{0, 2, \ldots, 2(n + m - 1)\}$ as follows. Fix the labels 0 and $2n$ for u_1 and u_n, respectively. Then

$$k = \sqrt{2}\, n, \tag{30}$$

where k is either the ceiling value or the floor value, and k is an odd integer. Fix the labels 0 and $2(m + n - 1)$ for v_1 and v_m, respectively. Then

$$k_1 = \sqrt{2}\,(m + n - 1), \tag{31}$$

where k_1 is either the ceiling value or the floor value, and k_1 is an odd integer. The labeling schemes are defined as follows:

$$f(u_i) = \{0, 2, \ldots, k-1\} \cup \{k+3, k+5, \ldots, 2n-2, 2n\}, \tag{32}$$

$$f(v_i) = \{0, 2n, 2n+2, \ldots, k_1-1\} \cup \{k_1+3, k_1+5, \ldots, 2(m+n-1)\}, \tag{33}$$

$$f^*(e_i) = \{1, 3, \ldots, k-2\} \cup \{k+2, k+4, \ldots, 2n-1\} \cup \{k\}, \tag{34}$$

and

$$f^*(e_i') = \{2n+1, 2n+3, \ldots, k_1 - 2\} \\ \cup \{k_1 + 2, k_1 + 4, \ldots, 2(m + n - 1) - 1\} \cup \{k_1\}. \tag{35}$$

Hence, the graph $C_n \circ C_m$ is an EVOERSML graph. □

Example 6. The graph $C_9 \circ C_{11}$, formed by combining two copies of a cycle with an edge, is EVOERSML, as presented in Figure 7.

2.6. Dumbbell graphs

Theorem 2.6. A dumbbell graph created by connecting two cycles to the end points of a path satisfies the EVOERSML condition.

Proof. Let $G = DB(C(m, n))$. The vertex set consists of

$$\{u_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\} \cup \{w_1, w_2, \ldots, w_l\}, \quad (36)$$

and the edge set consists of

$$\{(u_i, u_{i+1}) : i \in [1, m]\} \cup \{(v_i, v_{i+1}) : i \in [1, n]\} \cup \{(w_i, w_{i+1}) : i \in [1, l]\}. \quad (37)$$

Let

$$f : V(G) \to \{0, 2, \ldots, 2(m + n + l - 1)\}. \quad (38)$$

The labels of the vertices in C_m, w_l, and C_n are given as follows:

$$\{0, 2, \ldots, k-1\} \cup \{k+3, k+5, \ldots, 2m\} \cup \{2m+2, 2m+4, \ldots, k_1 - 1\} \cup \{k_1 + 3, \ldots, 2(m+n+l-1)\}. \quad (39)$$

where

$$k = \sqrt{\frac{f(u_1)^2 + f(u_m)^2}{2}}, \quad k_1 = \sqrt{\frac{f(v_1)^2 + f(v_m)^2}{2}}, \quad (40)$$

with $u_1 = 0$, $u_m = 2m$, $v_1 = 2(m+l-1)$, and $v_m = 2(m+n+l-1)$, respectively.

The edge labels are

$$\{1, 3, 5, \ldots, k-2\} \cup \{k+2, k+4, \ldots, 2m-1, k\} \cup \{2m+1, 2m+3, \ldots, k_1 - 2, k_1 + 2, \ldots, 2(m+n+l) - 3, k_1\}. \quad (41)$$

Hence, all vertex labels are even integers and all induced edge labels are distinct odd integers. Therefore, $G = DB(C_m, C_n)$ admits an EVOERSML. Thus, $DB(C_m, C_n)$ is an EVOERSML graph. □

Example 7. A dumbbell graph of C_{10} and C_7 with P_3 is an EVOERSML graph, as given in Figure 8.

2.7. Sunlet graphs

Theorem 2.7. The sunlet graph S_n admits an EVOERSML whenever $n > 3$.

Proof. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of S_n, respectively:

$$V(G) = \{v_i, v'_i : 1 \le i \le n\}, \quad (42)$$

and

$$E(G) = \{(v_i v_{i+1}) : 1 \le i \le n-1\} \cup \{v_n v_1\} \cup \{(v_i v'_i) : 1 \le i \le n\}. \quad (43)$$

Let

$$f : V(G) \to \{0, 2, 4, \ldots, 4n\}. \quad (44)$$

The vertex labeling is defined by

$$f(v_1) = 2 \quad \text{and} \quad f(v_n) = 4n. \quad (45)$$

Define

$$k = \sqrt{2(1 + 4n^2)}, \quad (46)$$

where k is either the ceiling value or the floor value, and k is an odd integer. Assign the value 0 to the first pendant vertex, say v'_1, 2 to the first vertex of the cycle, say v_1, 4 to v_2, and 6 to v'_2. Thus, the vertices can be ordered as

$$\{v'_1, v_1, v_2, v'_2\}, \quad (47)$$

with

$$\{f(v'_1), f(v_1), f(v_2), f(v'_2)\} = \{0, 2, 4, 6\}. \quad (48)$$

Continuing in this fashion, the vertices are labeled in the following order:

$$\{v'_1, v_1, v_2, v'_2, v'_3, v_3, v_4, v'_4, \ldots, v_{i-1}, v'_{i-1}, v'_i, v_i, \ldots, v_n\}, \quad (49)$$

with labels under f as

$$\{0, 2, 4, 6, \ldots, k-1\} \cup \{k+3, k+5, \ldots, 2n\}. \quad (50)$$

The ordinary graph of a sunlet is depicted in Figure 9. The graph admits several EVOERSML patterns, which are given below case-wise.

Case 1. If $f(v_{i-2}) = k - 3$, then the labels of the vertices are defined as follows:

$$\{f(v'_1), f(v_1), f(v_2), f(v'_2), \ldots, f(v_{i-1}), f(v'_{i-1}), f(v'_i), f(v_i), \ldots, f(v_{n-1}), f(v'_{n-1}), f(v'_n), f(v_n)\} = \{0, 2, 4, 6, 8, \ldots, k-1, k+3, k+5, k+7, \ldots, 4n-6, 4n-4, 4n-2, 4n\}. \quad (51)$$

Similarly, the induced edge labels are defined as follows:

$$\{f^*(v'_1 v_1), f^*(v_1 v_2), f^*(v_2 v'_2), f^*(v_2 v_3), f^*(v_3 v'_3), \ldots, f^*(v_{i-2} v_{i-1}), f^*(v_{i-1} v'_{i-1}), f^*(v_{i-1} v_i), \ldots, f^*(v_{n-2} v_{n-1}), f^*(v'_{n-1} v_{n-1}), f^*(v_{n-1} v_n), f^*(v_n v'_n), f^*(v_n v_1)\} = \{1, 3, 5, 7, \ldots, k-2, k+2, k+4, \ldots, 4n-7, 4n-5, 4n-3, 4n-1, k\}. \quad (52)$$

Case 2. If $f(v_{i-1}) = k - 3$, then the labels of the vertices are defined as follows:

$$\{f(v'_1), f(v_1), f(v_2), f(v'_2), \ldots, f(v'_{i-1}), f(v'_i), f(v_i), f(v_{i+1}), \ldots, f(v_{n-1}), f(v'_{n-1}), f(v'_n), f(v_n)\} = \{0, 2, 4, 6, 8, \ldots, k-1, k+3, k+5, k+7, \ldots, 4n-10, 4n-8, 4n-6, 4n-4, 4n-2, 4n\}. \quad (53)$$

Similarly, the induced edge labels are defined as follows:

$$\{f^*(v'_1 v_1), f^*(v_1 v_2), f^*(v_2 v'_2), f^*(v_2 v_3), f^*(v_3 v'_3), \ldots, f^*(v_{i-1} v'_{i-1}), f^*(v_{i-1} v_i), f^*(v_i v'_i), \ldots, f^*(v_{n-2} v_{n-1}), f^*(v'_{n-1} v_{n-1}), f^*(v_{n-1} v_n), f^*(v_n v'_n), f^*(v_n v_1)\} = \{1, 3, 5, 7, \ldots, k-2, k+2, k+4, \ldots, 4n-7, 4n-5, 4n-3, 4n-1, k\}. \quad (54)$$

Case 3. If $f(v'_{i-1}) = k-3$, then the labels of the vertices are defined as follows:

$$\begin{aligned}&\{f(v'_1), f(v_1), f(v_2), f(v'_2), \ldots, f(v_i), f(v'_i), f(v_{i+1}), f(v'_{i+1}), \ldots,\\ &\quad f(v_{n-1}), f(v'_{n-1}), f(v'_n), f(v_n)\}\\ &= \{0, 2, 4, 6, 8, \ldots, k-1, k+3, k+5, k+7, \ldots,\\ &\quad 4n-10, 4n-8, 4n-6, 4n-4, 4n-2, 4n\}.\end{aligned} \tag{55}$$

Similarly, the induced edge labels are defined as follows:

$$\begin{aligned}&\{f^*(v'_1v_1), f^*(v_1v_2), f^*(v_2v'_2), f^*(v_2v_3), f^*(v_3v'_3), \ldots,\\ &\quad f^*(v_iv'_i), f^*(v_iv_{i+1}), f^*(v_{i+1}v'_{i+1}), \ldots,\\ &\quad f^*(v_{n-2}v_{n-1}), f^*(v'_{n-1}v_{n-1}), f^*(v_{n-1}v_n),\\ &\quad f^*(v_nv'_n), f^*(v_nv_1)\}\\ &= \{1, 3, 5, 7, \ldots, k-2, k+2, k+4, \ldots,\\ &\quad 4n-7, 4n-5, 4n-3, 4n-1, k\}.\end{aligned} \tag{56}$$

Case 4. If $f(v'_i) = k-3$, then the labels of the vertices are defined as follows:

$$\begin{aligned}&\{f(v'_1), f(v_1), f(v_2), f(v'_2), \ldots, f(v_i), f(v_{i+1}), f(v'_{i+1}), f(v'_{i+2}), \ldots,\\ &\quad f(v_{n-1}), f(v'_{n-1}), f(v'_n), f(v_n)\}\\ &= \{0, 2, 4, 6, 8, \ldots, k-1, k+3, k+5, k+7, \ldots,\\ &\quad 4n-10, 4n-8, 4n-6, 4n-4, 4n-2, 4n\}.\end{aligned} \tag{57}$$

Similarly, the induced edge labels are defined as follows:

$$\begin{aligned}&\{f^*(v'_1v_1), f^*(v_1v_2), f^*(v_2v'_2), f^*(v_2v_3), f^*(v_3v'_3), \ldots, f^*(v'_iv_i),\\ &\quad f^*(v_iv_{i+1}), f^*(v_{i+1}v'_{i+1}), \ldots, f^*(v_{n-2}v_{n-1}),\\ &\quad f^*(v'_{n-1}v_{n-1}),\\ &\quad f^*(v_{n-1}v_n), f^*(v_nv'_n), f^*(v_nv_1)\}\\ &= \{1, 3, 5, 7, \ldots, k-2, k+2, k+4, \ldots,\\ &\quad 4n-7, 4n-5, 4n-3, 4n-1, k\}.\end{aligned} \tag{58}$$

For each of the above four cases, all vertex labels and induced edge labels are distinct for every value of n. Therefore, the sunlet graph is an EVOERSML graph. □

Example 8. Figure 10 illustrates the sunlet graph S_{18} which admits an EVOERSML..

Remark 1. The sunlet graph S_3 does not satisfy the EVOERSML condition. The diagram is depicted in Figure 11.

3. Conclusion

In this work, several families of graphs were analyzed with respect to the EVOERSML property. In particular, cycle graphs, $T(m,n)$ graphs, ladder graphs, dumbbell graphs, polygon-chain graphs, $C_n \circ C_m$ chain graphs, and sunlet graphs were investigated. Explicit constructions were provided to establish the existence of EVOERSML labelings for these classes.

Data availability

No data were used in the research described in this article.

Declaration of competing interest

The authors have no known competing financial interests or personal relationships that could have appeared to influence the work.

Funding

The authors received no external funding for this article.

References

[1] S. S. Sandhya, S. Somasundaram & S. Anusa, "Root square mean labeling of graphs", International Journal of Contemporary Mathematical Sciences **9** (2014) 667. https://doi.org/10.12988/ijcms.2014.410105.
[2] K. Thirugnanasambandam & K. Venkatesan, "Super root square mean labeling of graph", International Journal of Mathematics and Soft Computing **5** (2015) 189. https://www.researchgate.net/publication/321164979_Super_Root_Square_Mean_Labeling_of_Graphs.
[3] M. J. M. Orias & A. C. Pedrano, "On super root square mean labeling of some cycle related graphs", Acta Scientific Computer Sciences **5** (2023) 40. https://actascientific.com/ASCS/pdf/ASCS-05-0376.pdf.
[4] K. Venkatesan & K. Thirugnanasambandam, "Super root square mean labeling of some cycle related graphs", Global Journal of Engineering Science and Researches **5** (2018) 221. https://doi.org/10.5281/zenodo.1495607.
[5] J. A. Gallian, "A dynamic survey of graph labeling", The Electronic Journal of Combinatorics **18** (2015) DS6. https://www.combinatorics.org/files/Surveys/ds6/ds6v18-2015.pdf.
[6] V. Senthilkumar & K. Venkatesan, "Odd vertex even edge root square mean labelling graphs", Journal of Interdisciplinary Mathematics **27** (2024) 1001. https://doi.org/10.47974/JIM-1923.
[7] K. N. Babu & S. Meenakshi, "Even vertex odd edge root square mean labelling of path related graphs", Communications on Applied Nonlinear Analysis **32** (2025) 296. https://doi.org/10.52783/cana.v32.2401.