Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 8 (2026) 3337

Check for updates

Journal of the Nigerian Society of Physical Sciences

Bipolar soft neighborhoods with application in decision-making

Saif Z. Hameed[a,*], Heba I. Mustafa[b], Abdelaziz E. Radwan[c]

[a]Department of Mathematics, College of Basic Education, Mustansiriyah University, Baghdad, 10001, Iraq
[b]Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, 44511, Egypt
[c]Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, 11311, Egypt

Abstract

This paper introduces bipolar soft (BS) neighborhood structures within the framework of soft set theory. First, several properties associated with lower and upper approximations based on a binary BS relation are examined, and the proposed approximation operators are shown to satisfy several fundamental properties. Next, an additional structure formulated through BS neighborhoods is explored. The concept of bipolar soft N_j^{BS}-neighborhood spaces is investigated, and their main characteristics are analyzed. Two types of topologies, $\overline{T}_\zeta$ and $\underline{T}_\zeta$, induced by bipolar soft reflexive relations are then characterized. These topological structures are used to approximate rough sets, and a direct method is proposed to generate such topologies from the underlying relations. Finally, a numerical example is provided to illustrate the applicability of the proposed approach to decision-making.

DOI: 10.46481/jnsps.2026.3337

Keywords: Bipolar soft set, Bipolar soft approximation, Bipolar soft neighborhoods, Decision-making

Article History:
Received: 22 February 2026
Received in revised form: 27 May 2026
Accepted for publication: 08 June 2026
Available online: 04 August 2026

Communicated by: J. Ndam

1. Introduction

In 1999, the novel concept of soft set was offered by Molodtsov [1] as an alternative mathematical strategy to handle inaccuracies and uncertainties. There has been heightened interest in soft set theory. Maji *et al.* [2] launched algebraic operations of soft set theory. Shabir and Naz [3] offered the idea of soft topological spaces. Akdag and Ozkan [4, 5] defined soft α-open and introduced the soft b-open and continuous functions of them. The strongly soft b^*-separation axioms, strongly soft b^*-connectedness, N-bipolar soft connectedness, disconnectedness and strongly soft b^*-ideal were studied [6–9]. The notion of a bipolar soft set was proposed by Shabir and Naz [10]. In Ref. [11], Fatimah *et al.* introduced the concept of the N-soft set as an extended model of the soft set. Fadel and Dzul-Kifli [12] defined the bipolar soft topological spaces and some applications. Binary bipolar soft sets have also been studied [13].

In Ref. [14], Pawlak defined lower and upper approximations based on equivalence relation. Later, lower and upper approximations were generalized using arbitrary binary relations. Abd ElMonsef *et al.* [15] defined j-neighborhoods and j-NS, then constructed new approximations based on these neighborhoods. In 2022, Abu-Gdairi *et al.* [16] defined the j-adhesion neighborhoods and formed other approximations through them. In 2020, Shabir *et al.* [17] defined a soft binary relation, introduced a new type of neighborhood based on this relation, constructed lower and upper approximations based on this neighborhood, and defined a generalized soft approximation space. According to Dubois and Prada [18], decision-making was structured around two aspects: the positive and the negative.

*Corresponding author Tel. No.: +00964774577108
Email address: saif.zuhar.edbs@uomustansiriyah.edu.iq (Saif Z. Hameed)

Ref. [19] developed the idea of bipolar soft minimal structures. Then, Shbair *et al.* [20] also offered the notions of rough sets through minimal neighborhoods. Fuzzy soft set-valued maps have been introduced and applied to homotopy [21]. Some information measures have been discussed with applications to decision-making problems and diagnostic analysis [22]. Varol *et al.* [23] investigated the neighborhood structures in bipolar fuzzy settings. Hosny *et al.* [24] introduced new definitions for generalized rough set models based on the concepts of initial neighborhoods and ideals. In Ref. [25] Mustafa *et al.* introduced eight soft neighborhood types built from parameterized relations, generalizing existing models, enabling flexible approximations. Also, they developed refined lower and upper approximation operators by constructing topologies from these neighborhoods.

In the present work, we introduce lower and upper approximations based on binary bipolar soft relations, study the properties of these approximations, and generate new topologies from them. In addition, we explore their characterizations and properties. The paper is organized as follows. Section 2 introduces the definitions needed for the study. Section 3 introduces a new type of bipolar soft lower and upper approximations and studies some of their properties. Section 4 discusses bipolar soft neighborhoods. Section 5 studies the topologies generated from bipolar soft N_j-neighborhoods. In the final section, we apply the concept of BS-sets to model a real-world decision-making problem. We propose a structured algorithm designed to facilitate the selection of the optimal object based on the available information set.

2. Preliminaries

In this section, we review the fundamental definitions and terminology essential for the subsequent sections of this paper.

Definition 2.1. (Shabir and Naz [10]) Let $\mathfrak{X}$ be a universal set, $\check{E}$ be a set of attributes, and $\wp \subseteq \check{E}$. We say that $(£, \wp)$ is a soft set (S-set) over $\mathfrak{X}$ if $£ : \wp \rightarrow P(\mathfrak{X})$.

Definition 2.2. (Shabir and Naz [10]) Let $\mathfrak{X}$ and $\mathfrak{D}$ be universal sets, $(£, \wp)$ be an S-set over $\mathfrak{X} \times \mathfrak{D}$. then $(£, \wp)$ is called a binary soft relation (BS-relation) from $\mathfrak{X}$ to $\mathfrak{D}$.

Definition 2.3. (Shabir and Naz [10]) Let $\mathfrak{X}$ be a universal set, $\check{E}$ be a set of attributes, and $\wp \subseteq \check{E}$. We say that a triple $(£, Ł, \wp)$ is a bipolar soft set (BS-set) on $\mathfrak{X}$ if $£ : \wp \rightarrow P(\mathfrak{X})$ and $Ł : \neg\wp \rightarrow P(\mathfrak{X})$ with the property that for each $\nabla \in \wp$, $£(\nabla) \cap Ł(\neg\nabla) = \phi$.

Definition 2.4. (Shabir and Naz [10]) Let $(£_1, Ł_1, \wp_1)$,$(£_2, Ł_2, \wp_2)$ be BS- sets on $\mathfrak{X}$, then:

1. $(£_1, Ł_1, \wp_1)$ is said to be a BS-subset of $(£_2, Ł_2, \wp_2)$ denoted by $(£_1, Ł_1, \wp_1) \subseteq (£_2, Ł_2, \wp_2)$ if $\wp_1 \subseteq \wp_2$, $£_1(\nabla) \subseteq £_2(\nabla)$ and $Ł_2(\neg\nabla) \subseteq Ł_1(\neg\nabla)$, $\forall\ \nabla \in \wp$ and $\forall\ \neg\nabla \in \neg\wp$.
2. $(£_1, Ł_1, \wp_1)$ and $(£_2, Ł_2, \wp_2)$ are equal if $(£_1, Ł_1, \wp_1) \subseteq (£_2, Ł_2, \wp_2)$ and $(£_2, Ł_2, \wp_2) \subseteq (£_1, Ł_1, \wp_1)$.
3. The complement of $(£_1, Ł_1, \wp_1)$ is defined by $(£_1, Ł_1, \wp_1)^c = (£_1^c, Ł_1^c, \wp_1)$ where the two mappings $£_1^c$ and $Ł_1^c$ are defined by $£_1^c(\nabla) = Ł_1(\neg\nabla)$ and $Ł_1^c(\neg\nabla) = £_1(\nabla)$, $\forall\ \nabla \in \wp$ and $\forall\ \neg\nabla \in \neg\wp$.
4. $(£_1, Ł_1, \wp_1)$ is called a null bipolar soft set if $\forall\ \nabla \in \wp$, $£_1(\nabla) = \phi$ and $\neg\nabla \in \neg\wp$, $Ł_1^c(\neg\nabla) = \mathfrak{X}$, we denote it by $(\Phi, \mathfrak{X}, \wp)$.
5. $(£_1, Ł_1, \wp_1)$ is called an absolute bipolar soft set if $\forall\ \nabla \in \wp$, $£_1(\nabla) = \mathfrak{X}$ and $\neg\nabla \in \neg\wp$, $Ł_1^c(\neg\nabla) = \phi$, we denote it by $(\mathfrak{X}, \Phi, \wp)$.
6. The union (intersection) of $(£_1, Ł_1, \wp_1)$ and $(£_2, Ł_2, \wp_2)$ is the BS-set (ψ, ξ, ϱ) on $\mathfrak{X}$ where $\varrho = \wp_1 \cup \wp_2$, denoted by $(£_1, Ł_1, \wp_1) \cup (\cap)(£_2, Ł_2, \wp_2) = \psi, \xi, \varrho)$, is defined as:

$$\psi(\nabla) = \begin{cases} £_1(\nabla), & \text{if } \nabla \in \wp_1 \setminus \wp_2, \\ £_2(\nabla), & \text{if } \nabla \in \wp_2 \setminus \wp_1, \\ £_1(\nabla)\ \cup\ (\cap)£_2(\nabla), & \text{if } \nabla \in \wp_1 \cap \wp_2. \end{cases} \tag{1}$$

$$\xi(\neg\nabla) = \begin{cases} Ł_1(\neg\nabla), & \text{if } \neg\nabla \in \neg\nabla\wp_1 \setminus \neg\nabla\wp_2, \\ Ł_1(\neg\nabla), & \text{if } \neg\nabla \in \neg\nabla\wp_2 \setminus \neg\nabla\wp_1, \\ Ł_1(\neg\nabla)\ \cup\ (\cap)Ł_1(\neg\nabla), & \text{if } \neg\nabla \in \neg\nabla\wp_1 \cap \neg\nabla\wp_2. \end{cases} \tag{2}$$

Definition 2.5. (Fadel and Dzul-Kifli [12]) Let $(£, Ł, \wp)$ be BS-set and Γ be a family of BS-subsets from $(£, Ł, \wp)$. Γ is called a bipolar soft topology (BST) on $(£, Ł, \wp)$ if:

1. $(£, Ł, \wp), (\Phi, \mathfrak{X}, \wp) \in \Gamma$,
2. If $\{(£_i, Ł_i, \wp_i) \subseteq (£, Ł, \wp), i \in I\} \subseteq \Gamma$, then $\bigcup_{i\in I}(£_i, Ł_i, \wp_i) \in \Gamma$,
3. If $\{(£_i, Ł_i, \wp) \subseteq (£, Ł, \wp), 1 \leq i \leq \lambda, \lambda \in \mathbb{N}\} \subseteq \Gamma$, then $\bigcap_{i=1}^{\lambda}(£_i, Ł_i, \wp) \in \Gamma$.

Then, $(£, \Gamma, \wp, \neg\wp)$ is said to be bipolar soft topological spaces (BST_s).

Definition 2.6. (Mustafa and Wassefa [25]) Let $(£, \wp)$ be a BS-relation from $\mathfrak{X}$ to $\mathfrak{X}$, $\nabla \in \wp$, then the soft N_j neighborhoods (N_j^S-neighborhoods) of $\omega \in \mathfrak{X}$ are defined for each $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$ as follows:

1. $N_l^{£(\nabla)}(\omega) = \{\wp \in \mathfrak{X} : (\omega, \wp) \in £(\nabla)\}$,
2. $N_r^{£(\nabla)}(\omega) = \{\wp \in \mathfrak{X} : (\wp, \omega) \in £(\nabla)\}$,
3. $N_i^{£(\nabla)}(\omega) = N_r^{£(\nabla)}(\omega) \cap N_l^{£(\nabla)}(\omega)$,
4. $N_u^{£(\nabla)}(\omega) = N_r^{£(\nabla)}(\omega) \cup N_l^{£(\nabla)}(\omega)$,
5.

$$N_{\langle r\rangle}^{£(\nabla)}(\omega) = \begin{cases} \bigcap_{\omega \in N_r^{£(\nabla)}(\wp)} N_r^{£(\nabla)}(\wp), & \text{if } \exists\ \omega \in N_r^{£(\nabla)}(\wp), \\ \phi, & \text{otherwise.} \end{cases} \tag{3}$$

6.
$$N^{\pounds(\nabla)}_{\langle l\rangle}(\omega)=\begin{cases}\bigcap_{\omega\in N_l^{\pounds(\nabla)}(\wp)} N_l^{\pounds(\nabla)}(\wp), & \text{if } \exists\ \omega\in N_l^{\pounds(\nabla)}(\wp),\\ \phi, & \text{otherwise},\end{cases} \tag{4}$$

7. $N^{\pounds(\nabla)}_{\langle i\rangle}(\omega)=N^{\pounds(\nabla)}_{\langle r\rangle}(\omega)\cap N^{\pounds(\nabla)}_{\langle l\rangle}(\omega)$,

8. $N^{\pounds(\nabla)}_{\langle u\rangle}(\omega)=N^{\pounds(\nabla)}_{\langle r\rangle}(\omega)\cup N^{\pounds(\nabla)}_{\langle l\rangle}(\omega)$.

3. Generalized bipolar soft approximations

In this section, we introduce lower and upper approximations based on binary bipolar soft relations and study their properties.

Definition 3.1. If $(\pounds,\L,\wp)$ is BS-set over $\mathfrak{X}\times\mathfrak{D}$, then $(\pounds,\L,\wp)$ is called a binary bipolar soft relation (BBS-relation) from $\mathfrak{X}$ to $\mathfrak{D}$.

Definition 3.2. Let $\zeta\subseteq\mathfrak{D}$ and $\vartheta=(\pounds,\L,\wp)$ be a BBS-relation, then we can define lower approximation $\underline{\vartheta}^{\zeta}=(\underline{\pounds}^{\zeta},(\underline{\L}^{\zeta})$ and upper approximation $\overline{\vartheta}^{\zeta}=(\overline{\pounds}^{\zeta},\overline{\L}^{\zeta})$ by:

1. $\underline{\pounds}^{\zeta}(\nabla)=\{\omega\in\mathfrak{X}:\phi\neq\omega\pounds(\nabla)\subseteq\zeta\}$,

   $\underline{\L}^{\zeta}(\neg\nabla)=\{\omega\in\mathfrak{X}:\omega\L(\neg\nabla)\cap\zeta'\neq\phi\}$.

2. $\overline{\pounds}^{\zeta}(\nabla)=\{\omega\in\mathfrak{X}:\omega\pounds(\nabla)\cap\zeta\neq\phi\}$,

   $\overline{\L}^{\zeta}(\neg\nabla)=\{\omega\in\mathfrak{X}:\phi\neq\omega\L(\neg\nabla)\subseteq\zeta'\}$.

Where $\omega\pounds(\nabla)=\{\rho\in\mathfrak{D}:(\omega,\rho)\in\pounds(\nabla)\}$ and $\omega\L(\neg\nabla)=\{\rho\in\mathfrak{D}:(\rho,\omega)\in\L(\neg\nabla)\}$ for all $\nabla\in\wp$ and $\neg\nabla\in\neg\wp$.
Moreover, $\underline{\pounds}^{\zeta}(\nabla):\wp\to P(\mathfrak{X}),\overline{\pounds}^{\zeta}(\nabla):\wp\to P(\mathfrak{X}),\underline{\L}^{\zeta}(\neg\nabla):\neg\wp\to P(\mathfrak{X})$ and $\overline{\L}^{\zeta}(\neg\nabla):\neg\wp\to P(\mathfrak{X})$. We say $(\mathfrak{X},\mathfrak{D},(\pounds,\L,\wp))$ a generalized bipolar soft approximation space (GBS_{AS}).

Example 3.3. Let $\mathfrak{X}=\{\varpi_1,\varpi_2,\varpi_3,\varpi_4,\varpi_5\}$ and $\wp=\{\nabla_1,\nabla_2\}$, $\neg\wp=\{\neg\nabla_1,\neg\nabla_2\}$. Let $\mathfrak{D}=\{o,n,w,b\}$. Suppose $\zeta\subseteq\mathfrak{D}$ defined as:
$\zeta=\{o,n\}$, $\zeta'=\{w,b\}$ and define a BBS-relation $(\pounds,\L,\wp)$ by
$\pounds(\nabla_1)=\{(\varpi_1,o),(\varpi_1,b),(\varpi_2,n)\},\L(\neg\nabla_1)=\{(\varpi_4,b),\ (\varpi_5,n)\}$.

$\pounds(\nabla_2)=\{(\varpi_2,o),(\varpi_2,n),(\varpi_3,o)\}$, $\L(\neg\nabla_2)=\{(\varpi_1,w),(\varpi_1,b),(\varpi_4,w)\}$.

Now, $\varpi_1\pounds(\nabla_1)=\{o,b\}$, $\varpi_2\pounds(\nabla_1)=\{n\}$, $\varpi_3\pounds(\nabla_1)=\phi$, $\varpi_4\pounds(\nabla_1)=\phi$ and $\varpi_5\pounds(\nabla_1)=\phi$.

$\varpi_1\pounds(\nabla_2)=\phi$, $\varpi_2\pounds(\nabla_2)=\{o,n\}$, $\varpi_3\pounds(\nabla_2)=\{o\}$, $\varpi_4\pounds(\nabla_2)=\phi$ and $\varpi_5\pounds(\nabla_2)=\phi$.

and
$\varpi_1\L(\neg\nabla_1)=\phi$, $\varpi_2\L(\neg\nabla_1)=\phi$, $\varpi_3\L(\neg\nabla_1)=\phi$, $\varpi_4\L(\neg\nabla_1)=\{b\}$ and $\varpi_5\ \L(\neg\nabla_1)=\{n\}$.

$\varpi_1\L(\neg\nabla_2)=\{w,b\}$, $\varpi_2\L(\neg\nabla_2)=\phi$, $\varpi_3\L(\neg\nabla_2)=\phi$, $\varpi_4\L(\neg\nabla_2)=\{w\}$ and $\varpi_5\L(\neg\nabla_2)=\phi$.

Then, $\underline{\pounds}^{\zeta}(\nabla_1)=\{\varpi_2\},\underline{\pounds}^{\zeta}(\nabla_2)=\{\varpi_2\}$ and $\underline{\L}^{\zeta}(\neg\nabla_1)=\{\varpi_4\},\underline{\L}^{\zeta}(\neg\nabla_1)=\{\varpi_1,\varpi_4\}$.

Also, $\overline{\pounds}^{\zeta}(\nabla_1)=\{\varpi_1,\varpi_2\}$, $\overline{\pounds}^{\zeta}(\nabla_2)=\{\varpi_2,\varpi_3\}$ and $\overline{\L}^{\zeta}(\neg\nabla_1)=\{\varpi_4\}$, $\overline{\L}^{\zeta}(\neg\nabla_2)=\{\varpi_1,\varpi_4\}$.

Definition 3.4. Let $(\mathfrak{X},\mathfrak{D},(\pounds,\L,\wp))$ be a GBS_{AS} over $\mathfrak{X}$ and $\zeta,\delta\subseteq\mathfrak{X}$ and $\underline{\vartheta}^{\zeta}=(\underline{\pounds}^{\zeta},\underline{\L}^{\zeta})$, then

1. $\underline{\vartheta}^{\zeta}\subseteq\underline{\vartheta}^{\delta}$ iff $\underline{\pounds}^{\zeta}\subseteq\underline{\pounds}^{\delta}$ and $\underline{\L}^{\delta}\subseteq\underline{\L}^{\zeta}$.
2. $\underline{\vartheta}^{\zeta}\cap\underline{\vartheta}^{\delta}=(\underline{\pounds}^{\zeta}\cap\underline{\pounds}^{\delta},\underline{\L}^{\zeta}\cup\underline{\L}^{\delta})$.
3. $\underline{\vartheta}^{\zeta}\cup\underline{\vartheta}^{\delta}=(\underline{\pounds}^{\zeta}\cup\underline{\pounds}^{\delta},\underline{\L}^{\zeta}\cap\underline{\L}^{\delta})$.

Proposition 3.5. *Let $(\mathfrak{X},\mathfrak{D},(\pounds,\L,\wp))$ be a GBS_{AS} over $\mathfrak{X}$ and $\zeta,\delta\subseteq\mathfrak{X}$ and $\underline{\vartheta}^{\zeta}=(\underline{\pounds}^{\zeta},\underline{\L}^{\zeta})$, then*

1. *If $\zeta\subseteq\delta$ then $\underline{\vartheta}^{\zeta}\subseteq\underline{\vartheta}^{\delta}$*
2. *$\underline{\vartheta}^{\zeta}\cup\underline{\vartheta}^{\delta}\subseteq\underline{\vartheta}^{\zeta\cup\delta}$.*
3. *$\underline{\vartheta}^{\zeta}\cap\underline{\vartheta}^{\delta}=\underline{\vartheta}^{\zeta\cap\delta}$.*

Proof.

1. We show that $\underline{\pounds}^{\zeta}(\nabla)\subseteq\underline{\pounds}^{\delta}$ and $\underline{\L}^{\delta}(\neg\nabla)\subseteq\underline{\L}^{\zeta}(\neg\nabla)$. By Ref. [17], $\underline{\pounds}^{\zeta}(\nabla)\subseteq\underline{\pounds}^{\delta}$.
   Now, we show that if $\delta'\subseteq\zeta'$ then $\underline{\L}^{\delta}(\neg\nabla)\subseteq\underline{\L}^{\zeta}(\neg\nabla)$ for all $\neg\nabla\in\neg\wp$. Let $\omega\in\underline{\L}^{\delta}(\neg\nabla)$, then, $\omega\underline{\L}(\neg\nabla)\cap\delta'\neq\phi$. As $\delta'\subseteq\zeta'$, we have $\omega\underline{\L}(\neg\nabla)\cap\zeta'\neq\phi$. Thus, $\omega\in\underline{\L}^{\zeta}(\neg\nabla)$. Hence, $\underline{\L}^{\delta}(\neg\nabla)\subseteq\underline{\L}^{\zeta}(\neg\nabla)$. Therefore, $\underline{\vartheta}^{\zeta}\subseteq\underline{\vartheta}^{\delta}$.

2. By Ref. [17], $\underline{\pounds}^{\zeta}(\nabla)\cup\underline{\pounds}^{\delta}(\nabla)\subseteq\underline{\pounds}^{\zeta\cup\delta}(\nabla)$. Now, we show that $\underline{\L}^{\zeta\cup\delta}(\neg\nabla)\subseteq\underline{\L}^{\zeta}(\neg\nabla)\cup\underline{\L}^{\delta}(\neg\nabla)$.
   Let $\omega\in\underline{\L}^{\zeta\cup\delta}(\neg\nabla)$, this implies that $\omega\underline{\L}(\neg\nabla)\cap(\zeta\cup\delta)'=\omega\underline{\L}(\neg\nabla)\cap(\zeta'\cup\delta')\neq\phi$. So, $\omega\underline{\L}(\neg\nabla)\cap\zeta'\neq\phi$ and $\omega\underline{\L}(\neg\nabla)\cap\delta'\neq\phi$. Hence, $\omega\in\underline{\L}^{\zeta}(\neg\nabla)$ and $\omega\in\underline{\L}^{\delta}(\neg\nabla)$. Therefore, $\omega\in\underline{\L}^{\zeta}(\neg\nabla)\cap\underline{\L}^{\delta}(\neg\nabla)$.
   Now, $\underline{\vartheta}^{\zeta}\cup\underline{\vartheta}^{\delta}=(\underline{\pounds}^{\zeta}\cup\underline{\pounds}^{\delta},\underline{\L}^{\zeta}\cap\underline{\L}^{\delta})\subseteq(\underline{\pounds}^{\zeta\cup\delta},\underline{\L}^{\zeta\cup\delta})=\underline{\vartheta}^{\zeta\cup\delta}$.

3. By Ref. [17], $\underline{\pounds}^{\zeta}(\nabla)\cap\underline{\pounds}^{\delta}(\nabla)=\underline{\pounds}^{\zeta\cap\delta}(\nabla)$. Now, we show that $\underline{\L}^{\zeta\cap\delta}(\neg\nabla)=\underline{\L}^{\zeta}(\neg\nabla)\cup\underline{\L}^{\delta}(\neg\nabla)$.
   Let $\omega\in\underline{\L}^{\zeta\cap\delta}(\neg\nabla)\Rightarrow\omega\underline{\L}(\neg\nabla)\cap(\zeta\cap\delta)'=\phi$
   $\Longleftrightarrow\omega\underline{\L}(\neg\nabla)\cap[\zeta'\cup\delta']\neq\phi$
   $\Longleftrightarrow\omega\underline{\L}(\neg\nabla)\cap\zeta'\neq\phi$ or $\omega\underline{\L}(\neg\nabla)\cap\delta'\neq\phi$.
   $\Longleftrightarrow\omega\in\underline{\L}^{\zeta}(\neg\nabla)$ or $\omega\in\underline{\L}^{\delta}(\neg\nabla)$
   $\Longleftrightarrow\omega\in\underline{\L}^{\zeta}(\neg\nabla)\cup\underline{\L}^{\delta}(\neg\nabla)$.
   Hence, $\underline{\L}^{(\zeta\cup\delta)}(\neg\nabla)=\underline{\L}^{\zeta}(\neg\nabla)\cup\underline{\L}^{\delta}(\neg\nabla)$ Now, $\underline{\vartheta}^{\zeta}\cap\underline{\vartheta}^{\delta}=(\underline{\pounds}^{\zeta}\cap\underline{\pounds}^{\delta},\underline{\L}^{\zeta}\cup\underline{\L}^{\delta})\subseteq(\underline{\pounds}^{\zeta\cap\delta},\underline{\L}^{\zeta\cap\delta})=\underline{\vartheta}^{\zeta\cap\delta}$.

□

Definition 3.6. Let $(\mathfrak{X},\mathfrak{D},(\pounds,\L,\wp))$ be a GBS_{AS} over $\mathfrak{X}$ and $\zeta,\delta\subseteq\mathfrak{X}$ and $\overline{\vartheta}^{\zeta}=(\overline{\pounds}^{\zeta},\overline{\L}^{\zeta})$, then:

1. $\overline{\vartheta}^{\zeta}\subseteq\overline{\vartheta}^{\delta}$ iff $\overline{\pounds}^{\zeta}\subseteq\overline{\pounds}^{\delta}$ and $\overline{\L}^{\delta}\subseteq\overline{\L}^{\zeta}$.
2. $\overline{\vartheta}^{\zeta}\cap\overline{\vartheta}^{\delta}=(\overline{\pounds}^{\zeta}\cap\overline{\pounds}^{\delta},\overline{\L}^{\zeta}\cup\overline{\L}^{\delta})$.

3. $\overline{\vartheta}^{\zeta} \cup \overline{\vartheta}^{\delta} = (\overline{£}^{\zeta} \cup \overline{£}^{\delta}, \overline{Ł}^{\zeta} \cap \overline{Ł}^{\delta})$.

Proposition 3.7. *Let $(\mathfrak{X}, \mathfrak{D}, (£, Ł, \wp))$ be a GBS_{AS} over $\mathfrak{X}$ and $\zeta, \delta \subseteq \mathfrak{X}$ and $\overline{\vartheta}^{\zeta} = (\overline{£}^{\zeta}, \overline{Ł}^{\zeta})$, then*

1. *If $\zeta \subseteq \delta$ then $\overline{\vartheta}^{\zeta} \subseteq \overline{\vartheta}^{\delta}$*
2. *$\overline{\vartheta}^{\zeta\cap\delta} = \overline{\vartheta}^{\zeta} \cap \overline{\vartheta}^{\delta}$.*
3. *$\overline{\vartheta}^{\zeta\cup\delta} \subseteq \overline{\vartheta}^{\zeta} \cup \overline{\vartheta}^{\delta}$.*

Proof. 1. The part: $\overline{£}^{\zeta}(\nabla) \subseteq \overline{£}^{\delta}(\nabla)$ is already proved in Ref. [17].
Now, we show that if $\delta' \subseteq \zeta'$ then $\overline{Ł}^{\delta}(\neg\nabla) \subseteq \overline{Ł}^{\zeta}(\neg\nabla)$ for all $\neg\nabla \in \neg\wp$. Let $\omega \in \overline{Ł}^{\delta}(\neg\nabla)$, then $\phi \neq \omega\overline{Ł}(\neg\nabla) \subseteq \delta'$. As $\delta' \subseteq \zeta'$, we have $\phi \neq \omega\overline{Ł}(\neg\nabla) \subseteq \zeta'$. Thus, $\omega \in \overline{Ł}^{\zeta}(\neg\nabla))$. and, hence $\overline{Ł}^{\delta}(\neg\nabla) \subseteq \overline{Ł}^{\zeta}(\neg\nabla)$. Therefore, $\overline{\vartheta}^{\zeta} \subseteq \overline{\vartheta}^{\delta}$.

2. The part: $\overline{£}^{\zeta\cap\delta}(\nabla) \subseteq \overline{£}^{\zeta}(\nabla) \cap \overline{£}^{\delta}(\nabla)$ is already proved in Ref. [17].
Now, we show that $\overline{Ł}^{\zeta\cap\delta}(\neg\nabla) \subseteq \overline{Ł}^{\zeta}(\neg\nabla) \cup \overline{Ł}^{\delta}(\neg\nabla)$.
Let $\omega \in \overline{Ł}^{\zeta\cap\delta}(\neg\nabla)$ and so $\phi \neq \omega\overline{Ł}(\neg\nabla) \subseteq (\zeta \cap \delta)'$
$\Longrightarrow \phi \neq \omega\overline{Ł}(\neg\nabla) \subseteq \zeta' \cup \delta'$
$\Longrightarrow \phi \neq \omega\overline{Ł}(\neg\nabla) \subseteq \zeta'$ or $\phi \neq \omega\overline{Ł}(\neg\nabla) \subseteq \delta'$.
$\Longrightarrow \omega \in \overline{Ł}^{\zeta}(\neg\nabla)$ or $\omega \in \overline{Ł}^{\delta}(\neg\nabla)$
$\Longrightarrow \omega \in \overline{Ł}^{\zeta}(\neg\nabla) \cup \overline{Ł}^{\delta}(\neg\nabla)$.
Now, $\overline{\vartheta}^{\zeta\cap\delta} = (\overline{£}^{\zeta\cap\delta}, \overline{Ł}^{\zeta\cap\delta})$ then,
$\overline{\vartheta}^{\zeta\cap\delta} = (\overline{£}^{\zeta\cap\delta}, \overline{Ł}^{\zeta\cap\delta}) \subseteq (\overline{£}^{\zeta} \cap \overline{£}^{\delta}, \overline{Ł}^{\zeta} \cup \overline{Ł}^{\delta}) = \overline{\vartheta}^{\zeta} \cap \overline{\vartheta}^{\delta}$.

3. The part: $\overline{£}^{\zeta\cup\delta}(\nabla) \subseteq \overline{£}^{\zeta}(\nabla) \cup \overline{£}^{\delta}(\nabla)$ is already proved in Ref. [17],. Now, we show that $\overline{Ł}^{\zeta\cup\delta}(\neg\nabla) = \overline{Ł}^{\zeta}(\neg\nabla) \cap \overline{Ł}^{\delta}(\neg\nabla)$.
Then $\omega \in \overline{Ł}^{\zeta\cup\delta}(\neg\nabla)$, this implies that $\phi \neq \omega\overline{Ł}(\neg\nabla) \subseteq (\zeta \cup \delta)' = \zeta' \cup \delta'$.
$\Longleftrightarrow \phi \neq \omega\overline{Ł}(\neg\nabla) \subseteq \zeta'$ and $\phi \neq \omega\overline{Ł}(\neg\nabla) \subseteq \delta'$.
$\Longleftrightarrow \omega \in \overline{Ł}^{\zeta}(\neg\nabla)$ and $\omega \in \overline{Ł}^{\delta}(\neg\nabla)$.
$\Longleftrightarrow \omega \in \overline{Ł}^{\zeta}(\neg\nabla) \cap \overline{Ł}^{\delta}(\neg\nabla)$.
Now, $\overline{\vartheta}^{\zeta\cup\delta} = (\overline{£}^{\zeta\cup\delta}, \overline{Ł}^{\zeta\cup\delta}) \subseteq (\overline{£}^{\zeta} \cup \overline{£}^{\delta}, \overline{Ł}^{\zeta} \cap \overline{Ł}^{\delta}) = \overline{\vartheta}^{\zeta} \cup \overline{\vartheta}^{\delta}$.

□

The following example shows that the inclusions in part (2) of Proposition 3.5 and Proposition 3.7 may be proper.

Example 3.8. Let $\mathfrak{X} = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$ and $\wp = \{\nabla_1, \nabla_2\}$, $\neg\wp = \{\neg\nabla_1, \neg\nabla_2\}$. Let $\mathfrak{D} = \{o, n, w, b\}$.
Define a BBS-relation $(£, Ł, \wp)$ by
$£(\nabla_1) = \{(o, \varpi_1), (o, \varpi_3), (n, \varpi_2)\}$, $Ł(\neg\nabla_1) = \{(w, \varpi_3), (w, \varpi_1), (b, \varpi_4)\}$.

$£(\nabla_2) = \{(w, \varpi_2), (w, \varpi_2), (b, \varpi_3)\}$, $Ł(\neg\nabla_2) = \{(o, \varpi_1), (o, \varpi_3), (n, \varpi_4)\}$.

Now, $o£(\nabla_1) = \{\varpi_1, \varpi_3\}$, $n£(\nabla_1) = \{\varpi_2\}$, $w£(\nabla_1) = \phi$, $b£(\nabla_1) = \phi$ and
$oŁ(\neg\nabla_1) = \phi$, $nŁ(\neg\nabla_1) = \phi$, $wŁ(\neg\nabla_1) = \{\varpi_1, \varpi_3\}$, $bŁ(\neg\nabla_1) = \{\varpi_4\}$
Suppose $\zeta, \delta \subseteq \mathfrak{D}$, defined as: $\zeta = \{\varpi_1, \varpi_2\}, \zeta' = \{\varpi_3, \varpi_4\}$, $\delta = \{\varpi_2, \varpi_3\}, \delta' = \{\varpi_1, \varpi_4\}$.

Then, $\zeta \cap \delta = \{\varpi_2\}$, $\zeta' \cap \delta' = \{\varpi_4\}$ and $\zeta \cup \delta = \{\varpi_1, \varpi_2, \varpi_3\}$, $\zeta' \cup \delta' = \{\varpi_1, \varpi_3, \varpi_4\}$
Now, $\overline{£}^{\zeta\cap\delta}(\nabla_1) = \{n\}$, $\underline{£}^{\zeta\cap\delta}(\nabla_1) = \{o, n\}$, $\overline{£}^{\zeta}(\nabla_1) = \{o, n\}$, $\overline{£}^{\delta}(\nabla_1) = \{o, n\}$, $\underline{£}^{\zeta}(\nabla_1) = \{n\}$, $\underline{£}^{\delta}(\nabla_1) = \{n\}$, and $\overline{Ł}^{\zeta\cup\delta}(\neg\nabla_1) = \{w, b\}$, $\underline{Ł}^{\zeta\cap\delta}(\neg\nabla_1) = \{b\}$, $\underline{Ł}^{\zeta}(\neg\nabla) = \{w, b\}$, $\underline{Ł}^{\delta}(\neg\nabla) = \{w, b\}$, $\overline{Ł}^{\zeta}(\neg\nabla) = \{b\}$, $\overline{Ł}^{\delta}(\neg\nabla) = \{b\}$.
Hence, $\overline{£}^{\delta}(\nabla_1) \cap \overline{£}^{\zeta}(\nabla_1) = \{o, n\} \nsubseteq \{n\} = \overline{£}^{\zeta\cap\delta}(\nabla_1)$ and $\overline{Ł}^{\zeta\cup\delta}(\neg\nabla_1) = \{w, b\} \nsubseteq \{b\} = \overline{Ł}^{\zeta}(\neg\nabla_1) \cup \overline{Ł}^{\delta}(\neg\nabla_1)$, $\underline{£}^{\zeta\cap\delta}(\nabla_1) = \{o, n\} \nsubseteq \{n\} = \underline{£}^{\zeta}(\nabla_1) \cup \underline{£}^{\delta}(\nabla_1)$ and $\underline{Ł}^{\zeta}(\neg\nabla_1) \cap \underline{Ł}^{\delta}(\neg\nabla_1) = \{w, b\} \nsubseteq \{b\} = \underline{Ł}^{\zeta\cap\delta}(\neg\nabla_1)$.

Proposition 3.9. *Let $(\mathfrak{X}, \mathfrak{D}, (£, Ł, \wp))$ be a GBS_{AS}. Let ζ_i be any family of subsets of $\mathfrak{D}$ and $\underline{\vartheta}^{\zeta} = (\underline{£}^{\zeta}, \underline{Ł}^{\zeta}, \wp)$ and $\overline{\vartheta}^{\zeta} = (\overline{£}^{\zeta}, \overline{Ł}^{\zeta}, \wp)$. Then:*

1. *$\underline{\vartheta}^{\bigcap_{i\in\mathfrak{I}}\zeta_i} = \bigcap_{i\in\mathfrak{I}} \underline{\vartheta}^{\zeta_i}$*
2. *$\overline{\vartheta}^{\bigcap_{i\in\mathfrak{I}}\zeta_i} = \bigcap_{i\in\mathfrak{I}} \overline{\vartheta}^{\zeta_i}$*

Proof. 1. The part: $\underline{£}^{\bigcap_{i\in\mathfrak{I}}\zeta_i}(\nabla) = \bigcap_{i\in\mathfrak{I}} \underline{£}^{\zeta_i}(\nabla)$ is already proved in Ref. [17]. Now, we show that $\underline{Ł}^{\bigcap_{i\in\mathfrak{I}}\zeta_i}(\neg\nabla) = \bigcup_{i\in\mathfrak{I}} \underline{Ł}^{\zeta_i}(\neg\nabla)$.
Let $\omega \in \underline{Ł}^{\bigcap_{i\in\mathfrak{I}}\zeta_i}(\neg\nabla) \Longleftrightarrow \omega Ł(\neg\nabla) \bigcap (\bigcup_{i\in\mathfrak{I}} \zeta_i)' \neq \phi$.
$\Longleftrightarrow \omega Ł(\neg\nabla) \bigcap (\bigcup_{i\in\mathfrak{I}} \zeta_i') \neq \phi$.
$\Longleftrightarrow \omega Ł(\neg\nabla) \bigcap \zeta_i' \neq \phi$ for all $i \in \mathfrak{I} \Rightarrow \omega \in \underline{Ł}^{\zeta_i}(\neg\nabla)$ for all $i \in \mathfrak{I}$.
$\Longleftrightarrow \omega \in \bigcup_{i\in\mathfrak{I}} \underline{Ł}^{\zeta_i}(\neg\nabla)$. Hence, $\underline{Ł}^{\bigcap_{i\in\mathfrak{I}}\zeta_i}(\neg\nabla) = \bigcup_{i\in\mathfrak{I}} \underline{Ł}^{\zeta_i}(\neg\nabla)$. Thus, $\underline{\vartheta}^{\bigcap_{i\in\mathfrak{I}}\zeta_i} = (\underline{£}^{\bigcap_{i\in\mathfrak{I}}\zeta_i}, \underline{Ł}^{\bigcap_{i\in\mathfrak{I}}\zeta_i}) = (\bigcap_{i\in\mathfrak{I}} \underline{£}^{\zeta_i}, \bigcup_{i\in\mathfrak{I}} \underline{Ł}^{\zeta_i}) = \bigcap_{i\in\mathfrak{I}} \underline{\vartheta}^{\zeta_i}$.

2. The part: $\overline{£}^{\bigcap_{i\in\mathfrak{I}}\zeta_i}(\nabla) \subseteq \bigcap_{i\in\mathfrak{I}} \overline{£}^{\zeta_i}(\nabla)$ is already proved in Ref. [17]. Now, we show that $\overline{Ł}^{\bigcap_{i\in\mathfrak{I}}\zeta_i}(\neg\nabla) \subseteq \bigcup_{i\in\mathfrak{I}} \overline{Ł}^{\zeta_i}(\neg\nabla)$.
Let $\omega \in \overline{Ł}^{\bigcap_{i\in\mathfrak{I}}\zeta_i}(\neg\nabla) \Longleftrightarrow \omega Ł(\neg\nabla) \subseteq (\bigcup_{i\in\mathfrak{I}} \zeta_i)' \neq \phi$.
$\Longleftrightarrow \phi \neq \omega Ł(\neg\nabla) \subseteq (\bigcup_{i\in\mathfrak{I}} \zeta_i')$.
$\Longleftrightarrow \phi \neg \omega Ł(\neg\nabla) \bigcap \zeta_i'$ for all $i \in \mathfrak{I} \Rightarrow \omega \in \overline{Ł}^{\zeta_i}(\neg\nabla)$ for all $i \in \mathfrak{I}$.
$\Longleftrightarrow \omega \in \bigcup_{i\in\mathfrak{I}} \overline{Ł}^{\zeta_i}(\neg\nabla)$.
Hence, $\overline{\vartheta}^{\bigcap_{i\in\mathfrak{I}}\zeta_i} = (\overline{£}^{\bigcap_{i\in\mathfrak{I}}\zeta_i}, \overline{Ł}^{\bigcap_{i\in\mathfrak{I}}\zeta_i}) \subseteq (\bigcap_{i\in\mathfrak{I}} \overline{£}^{\zeta_i}, \bigcup_{i\in\mathfrak{I}} \overline{Ł}^{\zeta_i}) = \bigcap_{i\in\mathfrak{I}} \overline{\vartheta}^{\zeta_i}$.

□

Definition 3.10. If $(£, Ł, \wp)$ is a BS- set over $\mathfrak{X}\times\mathfrak{X}$ then $(£, Ł, \wp)$ is called a binary BS- relation (BBS-relation) from $\mathfrak{X}$ to $\mathfrak{X}$. The set of all smooth soft binary relations from $\mathfrak{X}$ to $\mathfrak{X}$ will be known as $\varrho B_B S\, Br(\mathfrak{X})$.

Definition 3.11. A BBS-relation $(£, Ł, \wp)$ from $\mathfrak{X}$ to $\mathfrak{X}$ is said to be BS-reflexive relation from $\mathfrak{X}$ to $\mathfrak{X}$ if $£(\nabla)$ and $Ł(\neg\nabla)$ are reflexive relation from $\mathfrak{X}$ to $\mathfrak{X}$ for all $\nabla \in \wp$ and $\neg\nabla \in \neg\wp$. In this case, each $\omega\pounds(\nabla)$ and $\omega\text{Ł}(\neg\nabla)$ are non-empty with $\omega \in \omega\pounds(\nabla)$ and $\omega \in \omega\text{Ł}(\neg\nabla)$.

Definition 3.12. Define a map $\pi^{\zeta}_{\vartheta} = (\pi^{\zeta}_{\pounds}, \pi^{\zeta}_{\text{Ł}})$ where $\pi^{\zeta}_{\pounds} : \wp \to P(\mathfrak{X})$ and $\pi^{\zeta}_{\text{Ł}} : \neg\wp \to P(\mathfrak{X})$ such that$\pi^{\zeta}_{\pounds} = \zeta$ and $\pi^{\zeta}_{\text{Ł}} = \zeta'$ for all $\nabla \in \wp$ and $\neg\nabla \in \neg\wp$, where $\zeta \subseteq \mathfrak{X}$.

Theorem 3.13. *Let $\pounds : \wp \to P(\mathfrak{X} \times \mathfrak{X})$ and $\text{Ł} : \neg\wp \to P(\mathfrak{X} \times \mathfrak{X})$ be a BS-reflexive relation from $\mathfrak{X}$ to $\mathfrak{X}$. Let $\phi \neq \zeta \subseteq \mathfrak{X}$, then:*

1. $\underline{\vartheta}^{\zeta} \subseteq \pi^{\zeta}_{\vartheta}$.
2. $\pi^{\zeta}_{\vartheta} \subseteq \overline{\vartheta}^{\zeta}$.
3. $\underline{\vartheta}^{(\mathfrak{X},\Phi,\wp)} = (\mathfrak{X}, \Phi, \wp) = \overline{\vartheta}^{(\mathfrak{X},\Phi,\wp)}$

Proof. 1. The part: $\underline{\pounds}^{\zeta}(\nabla) \subseteq \pi^{\zeta}_{\pounds}(\nabla)$ is proved already in Ref. [17]. Now, we show that $\pi^{\zeta}_{\text{Ł}}(\neg\nabla) \subseteq \underline{\text{Ł}}^{\zeta}(\neg\nabla)$. Let $\omega \in \pi^{\zeta}_{\text{Ł}}(\neg\nabla)$. This implies $\omega \in \zeta'$ and so $\omega \in \text{Ł}(\neg\nabla) \cap \zeta'$. Then $\omega\text{Ł}(\neg\nabla) \cap \zeta' \neq \phi$ and therefore $\omega \in \underline{\text{Ł}}^{\zeta}(\neg\nabla)$. Thus, $\zeta' \subseteq \underline{\text{Ł}}^{\zeta}(\neg\nabla)$. That is $\pi^{\zeta}_{\text{Ł}}(\neg\nabla) \subseteq \underline{\text{Ł}}^{\zeta}(\neg\nabla)$.
Hence, $\underline{\vartheta}^{\zeta} = (\underline{\pounds}^{\zeta}, \underline{\text{Ł}}^{\zeta}) \subseteq (\zeta, \zeta') = (\pi^{\zeta}_{\pounds}, \pi^{\zeta}_{\text{Ł}}) = \pi^{\zeta}_{\vartheta}$.

2. The part: $\pi^{\zeta}_{\pounds}(\nabla) \subseteq \overline{\pounds}^{\zeta}(\nabla)$ is proved already in Ref. [17]. Now, we show that $\overline{\text{Ł}}^{\zeta}(\neg\nabla) \subseteq \pi^{\zeta}_{\text{Ł}}(\neg\nabla)$.
Let $\omega \in \overline{\text{Ł}}^{\zeta}(\neg\nabla)$. Then $\omega\text{Ł}(\neg\nabla) \subseteq \zeta'$. But $\omega \in \omega\text{Ł}(\neg\nabla)$, so $\omega \in \zeta'$. Thus, $\overline{\text{Ł}}^{\zeta}(\neg\nabla) \subseteq \zeta'$, that is $\overline{\text{Ł}}^{\zeta}(\neg\nabla) \subseteq \pi^{\zeta}_{\text{Ł}}(\neg\nabla)$.
Hence, $\pi^{\zeta}_{\vartheta} = (\pi^{\zeta}_{\pounds}, \pi^{\zeta}_{\text{Ł}}) = (\zeta, \zeta') \subseteq (\overline{\pounds}^{\zeta}, \overline{\text{Ł}}^{\zeta}) \subseteq \overline{\vartheta}^{\zeta}$.

3. It is direct.

□

Definition 3.14. A BBS-relation $(\pounds, \text{Ł}, \wp)$ from $\mathfrak{X}$ to $\mathfrak{X}$ is said to be a *BS*-symmetric relation from $\mathfrak{X}$ to $\mathfrak{X}$ if $\pounds(\nabla)$ and $\text{Ł}(\neg\nabla)$ are symmetric relations from $\mathfrak{X}$ to $\mathfrak{X}$ for all $\nabla \in \wp$ and $\neg\nabla \in \neg\wp$.

Lemma 3.15. *If $(\pounds, \text{Ł}, \wp)$ is a BS-symmetric relation from $\mathfrak{X}$ to $\mathfrak{X}$, then $\varpi \in \omega\pounds(\nabla)$ implies $\omega \in \varpi\pounds(\nabla)$ and $\omega \in \varpi\text{Ł}(\neg\nabla)$ implies $\omega \in \varpi\text{Ł}(\neg\nabla)$.*

Theorem 3.16. *$\pounds : \wp \to P(\mathfrak{X} \times \mathfrak{X})$ and $\text{Ł} : \neg\wp \to P(\mathfrak{X} \times \mathfrak{X})$ be a BS-symmetric relation from $\mathfrak{X}$ to $\mathfrak{X}$. For $\phi \neq \zeta \subseteq \mathfrak{X}$, the following properties hold:*

1. $\pi^{\zeta}_{\pounds} \subseteq \overline{\text{Ł}}^{(\underline{\text{Ł}}^{\zeta(\neg\nabla)})'}$.
2. $\underline{\text{Ł}}^{(\overline{\text{Ł}}^{\zeta(\neg\nabla)})'} \subseteq \pi^{\zeta}_{\text{Ł}}$

Proof. 1. Let $\omega \notin \overline{\text{Ł}}^{(\underline{\text{Ł}}\zeta(\neg(\nabla))'}(\neg\nabla)$. This implies that $\omega\text{Ł}(\neg\nabla) \notin \overline{\text{Ł}}^{\zeta}(\neg\nabla)$. So there exists at least one $\omega_1 \in \text{Ł}(\neg\nabla)$ such that $\omega_1 \notin \underline{\text{Ł}}^{\zeta}(\neg\nabla)$ and hence $\omega_1\text{Ł}(\neg\nabla) \cap \zeta' = \phi$. There are three possibilities:

(i) If $\zeta' = \phi$ then $\omega \notin \zeta'$.

(ii) If $\omega_1\text{Ł}(\neg\nabla) = \phi$ then this is no possible since $\omega_1 \in \omega\text{Ł}(\neg\nabla)$ implies $\omega \in \omega_1\text{Ł}(\neg\nabla)$.

(iii) If $\omega_1\text{Ł}(\neg\nabla) \subseteq \zeta$ while $\omega \in \omega_1\text{Ł}(\neg\nabla)$, then $\omega \in \zeta$ which implies $\omega \notin \zeta'$. Thus, $\zeta' \subseteq \overline{\text{Ł}}^{(\underline{\text{Ł}}^{\zeta(\neg(\nabla))})'}(\neg\nabla)$.
Hence, $\pi^{\zeta}_{\pounds}(\neg\nabla) \subseteq \overline{\text{Ł}}^{(\underline{\text{Ł}}^{\zeta(\neg(\nabla))})'}(\neg\nabla)$.

2. Let $\omega \in \underline{\text{Ł}}^{(\overline{\text{Ł}}^{\zeta(\neg\nabla)})'}(\neg\nabla)$. This implies that $\omega\text{Ł}(\neg\nabla) \cap \overline{\text{Ł}}^{\zeta}(\neg\nabla) \neq \phi$ and so there exists at least one $\omega_1 \in \omega\text{Ł}(\neg\nabla) \cap \overline{\text{Ł}}^{\zeta}(\neg\nabla)$. Hence, $\omega_1 \in \omega\text{Ł}(\neg\nabla)$ and $\omega_1 \in \overline{\text{Ł}}^{\zeta}(\neg\nabla)$. Now, $\omega_1 \in \overline{\text{Ł}}^{\zeta}(\neg\nabla)$ implies $\phi \neq \omega_1\text{Ł}(\neg\nabla) \subseteq \zeta'$. Since $\omega_1 \in \omega\text{Ł}(\neg\nabla)$ and the relation is *BS*- symmetric, then $\omega \in \omega_1\text{Ł}(\neg\nabla)$. Thus, $\omega \in \zeta'$. Therefore, $\underline{\text{Ł}}^{(\overline{\text{Ł}}^{\zeta(\neg\nabla)})'}(\neg\nabla) \subseteq \zeta'$ and hence, $\underline{\text{Ł}}^{(\overline{\text{Ł}}^{\zeta(\neg\nabla)})'}(\neg\nabla) \subseteq \pi^{\zeta}_{\text{Ł}}(\neg\nabla)$.

□

Definition 3.17. A *BBS*-relation $(\pounds, \text{Ł}, \wp)$ from $\mathfrak{X}$ to $\mathfrak{X}$ is said to be a *BS*-transitive relation from $\mathfrak{X}$ to $\mathfrak{X}$ if $\pounds(\nabla)$ and $\text{Ł}(\neg\nabla)$ are transitive relation from $\mathfrak{X}$ to $\mathfrak{X}$ for all $\nabla \in \wp$ and $\neg\nabla \in \neg\wp$.

Theorem 3.18. *Let $\pounds : \wp \to P(\mathfrak{X} \times \mathfrak{X})$ and $\text{Ł} : \neg\wp \to P(\mathfrak{X} \times \mathfrak{X})$ be a BS-transitive relation from $\mathfrak{X}$ to $\mathfrak{X}$. For $\zeta \subseteq \mathfrak{X}$, $\overline{\text{Ł}}^{\zeta}(\neg\nabla) \subseteq \overline{\text{Ł}}^{(\overline{\text{Ł}}^{\zeta(\neg\nabla)})'}(\neg\nabla)$.*

Proof. Assume that $\omega \notin \overline{\text{Ł}}^{(\overline{\text{Ł}}^{\zeta(\neg\nabla)})'}(\neg\nabla)$. This implies that $\phi \neq \omega\text{Ł}(\neg\nabla) \nsubseteq \overline{\text{Ł}}^{\zeta}(\neg\nabla)$, so there exists at least one $\omega_1 \in \omega\text{Ł}(\neg\nabla)$ such that $\omega_1 \in \overline{\text{Ł}}^{\zeta}(\neg\nabla)$. Hence, $\omega_1\text{Ł}(\neg\nabla) \subseteq \zeta'$ and so there exists at least one $\varpi \in \omega_1\text{Ł}(\neg\nabla)$ such that $\varpi \notin \zeta'$. But $\varpi \in \omega_1\text{Ł}(\neg\nabla)$ and $\omega_1 \in \omega\text{Ł}(\neg\nabla)$ implies $\varpi \in \omega\text{Ł}(\neg\nabla)$, since the relation is *BS*- transitive. This implies that $\phi \notin \omega\text{Ł}(\neg\nabla) \subseteq \zeta'$. So, $\omega \notin \overline{\text{Ł}}^{\zeta}(\neg\nabla)$. Thus, $\overline{\text{Ł}}^{\zeta}(\neg\nabla) \subseteq \overline{\text{Ł}}^{(\overline{\text{Ł}}^{\zeta(\neg\nabla)})'}(\neg\nabla)$. □

Theorem 3.19. *If a BS B-relation $\pounds : \wp \to P(\mathfrak{X} \times \mathfrak{X})$ and $\text{Ł} : \neg\wp \to P(\mathfrak{X} \times \mathfrak{X})$ from $\mathfrak{X}$ to $\mathfrak{X}$ be a BS-reflexive relation. Then for any subset $\zeta \subseteq \mathfrak{X}$, $\overline{\text{Ł}}^{\zeta}(\neg\nabla) = \overline{\text{Ł}}^{(\overline{\text{Ł}}^{\zeta(\neg\nabla)})'}(\neg\nabla)$.*

Proof. Since the relation is *BS*-reflexive, then by Theorem 3.13(2) $\text{Ł}^{\zeta}(\neg\nabla) \subseteq \zeta'$. So, $\zeta \subseteq (\overline{\text{Ł}}^{\zeta}(\neg\nabla))'$. Hence, $\overline{\text{Ł}}^{\zeta}(\neg\nabla) \subseteq \overline{\text{Ł}}^{(\overline{\text{Ł}}^{\zeta(\neg\nabla)})'}(\neg\nabla)$.
Now, we show that $\overline{\text{Ł}}^{(\overline{\text{Ł}}^{\zeta(\neg\nabla)})'}(\neg\nabla) \subseteq \overline{\text{Ł}}^{\zeta}(\neg\nabla)$. Let $\omega \in \overline{\text{Ł}}^{(\overline{\text{Ł}}^{\zeta(\neg\nabla)})'}(\neg\nabla)$, then $\omega \in \omega\text{Ł}(\neg\nabla) \subseteq \overline{\text{Ł}}^{\zeta}(\neg\nabla)$. Since the relation is *BS*-reflexive, then $\omega \in \omega\text{Ł}(\neg\nabla)$ and hence $\omega \in \overline{\text{Ł}}^{\zeta}(\neg\nabla)$. So, $\overline{\text{Ł}}^{\zeta}(\neg\nabla) = \overline{\text{Ł}}^{(\overline{\text{Ł}}^{\zeta(\neg\nabla)})'}(\neg\nabla)$. □

4. Bipolar soft neighborhoods

In this section, we discuss and explore one of the most important properties of neighborhoods, called bipolar soft neighborhoods.

Definition 4.1. Let $(£, Ł, \wp)$ be a binary BS-relation over $\mathfrak{X}$, $\nabla \in \wp$, then the bipolar soft N_j neighborhoods (N_j^{BS}-neighborhoods) of $\omega \in \mathfrak{X}$ are defined for each $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$ as follows: let $\vartheta = (£, Ł, \wp)$ and $N_j^{\vartheta} = (N_j^{£}, N_j^{Ł})$. Then:

1. $N_r^{£(\nabla)}(\omega) = \{\wp \in \mathfrak{X} : (\omega, \wp) \in £(\nabla)\}$,
   $N_r^{Ł(\neg\nabla)}(\omega) = \{\wp \in \mathfrak{X} : (\wp, \omega) \in Ł(\neg\nabla)\}$,
2. $N_l^{£(\nabla)}(\omega) = \{\wp \in \mathfrak{X} : (\wp, \omega) \in £(\nabla)\}$,
   $N_l^{Ł(\neg\nabla)}(\omega) = \{\wp \in \mathfrak{X} : (\omega, \wp) \in Ł(\neg\nabla)\}$,
3. $N_i^{£(\nabla)}(\omega) = N_r^{£(\nabla)}(\omega) \cap N_l^{£(\nabla)}(\omega)$,
   $N_i^{Ł(\neg\nabla)}(\omega) = N_r^{Ł(\neg\nabla)}(\omega) \cup N_l^{Ł(\neg\nabla)}(\omega)$,
4. $N_u^{£(\nabla)}(\omega) = N_r^{£(\nabla)}(\omega) \cup N_l^{£(\nabla)}(\omega)$,
   $N_i^{Ł(\neg\nabla)}(\omega) = N_r^{Ł(\neg\nabla)}(\omega) \cap N_l^{Ł(\neg\nabla)}(\omega)$,
5. $$N_{\langle r\rangle}^{£(\nabla)}(\omega) = \begin{cases} \bigcap_{\omega \in N_r^{£(\nabla)}(\wp)} N_r^{£(\nabla)}(\wp), & \text{if } \exists\, \omega \in N_r^{£(\nabla)}(\wp), \\ \phi, & \text{otherwise}, \end{cases}$$
   $$N_{\langle r\rangle}^{Ł(\neg\nabla)}(\omega) = \begin{cases} \bigcup_{\omega \in N_r^{Ł(\neg\nabla)}(\wp)} N_r^{Ł(\neg\nabla)}(\wp), & \text{if } \exists\, \omega \in N_r^{Ł(\neg\nabla)}(\wp), \\ \phi, & \text{otherwise}, \end{cases}$$
6. $$N_{\langle l\rangle}^{£(\nabla)}(\omega) = \begin{cases} \bigcap_{\omega \in N_l^{£(\nabla)}(\wp)} N_l^{£(\nabla)}(\wp), & \text{if } \exists\, \omega \in N_l^{£(\nabla)}(\wp), \\ \phi, & \text{otherwise}, \end{cases} \tag{5}$$
   $$N_{\langle l\rangle}^{Ł(\neg\nabla)}(\omega) = \begin{cases} \bigcup_{\omega \in N_l^{Ł(\neg\nabla)}(\wp)} N_l^{Ł(\neg\nabla)}(\wp), & \text{if } \exists\, \omega \in N_l^{Ł(\neg\nabla)}(\wp), \\ \phi, & \text{otherwise}, \end{cases} \tag{6}$$
7. $N_{\langle i\rangle}^{£(\nabla)}(\omega) = N_{\langle r\rangle}^{£(\nabla)}(\omega) \cap N_{\langle l\rangle}^{£(\nabla)}(\omega)$,
   $N_{\langle i\rangle}^{Ł(\neg\nabla)}(\omega) = N_{\langle r\rangle}^{Ł(\neg\nabla)}(\omega) \cup N_{\langle l\rangle}^{Ł(\neg\nabla)}(\omega)$,
8. $N_{\langle u\rangle}^{£(\nabla)}(\omega) = N_{\langle r\rangle}^{£(\nabla)}(\omega) \cup N_{\langle l\rangle}^{£(\nabla)}(\omega)$,
   $N_{\langle u\rangle}^{Ł(\neg\nabla)}(\omega) = N_{\langle r\rangle}^{Ł(\neg\nabla)}(\omega) \cap N_{\langle l\rangle}^{Ł(\neg\nabla)}(\omega)$.

Definition 4.2. Let $(£, Ł, \wp)$ be a BBS-relation over $\mathfrak{X}$, and π_j be a mapping from $\mathfrak{X}$ to $P(\mathfrak{X})$ which associates each $\omega \in \mathfrak{X}$ with its N_j^{BS}-neighborhood in $P(\mathfrak{X})$, then $(\mathfrak{X}, (£, Ł, \wp), \Pi_j)$ is called an N_j^{BS}-neighborhood space (briefly BS_jN_s).

Proposition 4.3. *Let $(\mathfrak{X}, (£, Ł, \wp), \Pi_j)$ be an N_j^{BS}-neighborhood space, $N_j^{\vartheta} = (N_j^{£}, N_j^{Ł})$. Then*

1. $N_i^{\vartheta}(\omega) \subseteq N_r^{\vartheta}(\omega) \subseteq N_u^{\vartheta}(\omega)$,
2. $N_i^{\vartheta}(\omega) \subseteq N_l^{\vartheta}(\omega) \subseteq N_u^{\vartheta}(\omega)$,
3. $N_{\langle i\rangle}^{\vartheta}(\omega) \subseteq N_{\langle r\rangle}^{\vartheta}(\omega) \subseteq N_{\langle u\rangle}^{\vartheta}(\omega)$,
4. $N_{\langle i\rangle}^{\vartheta}(\omega) \subseteq N_{\langle l\rangle}^{\vartheta}(\omega) \subseteq N_{\langle u\rangle}^{\vartheta}(\omega)$,
5. *If Π_j is symmetric, then $N_i^{\vartheta}(\omega) = N_l^{\vartheta}(\omega) = N_u^{\vartheta}(\omega)$.*

Table 1: BS-$N_j^{£(\nabla_i)}(\omega)$-neighborhoods.

ω	$N_r^{£(\nabla_1)}(\omega)$	$N_l^{£(\nabla_1)}(\omega)$	$N_i^{£(\nabla_1)}(\omega)$	$N_u^{£(\nabla_1)}(\omega)$
ϖ_1	$\{\varpi_2\}$	$\{\varpi_2\}$	$\{\varpi_2\}$	$\{\varpi_2\}$
ϖ_2	$\{\varpi_1, \varpi_3\}$	$\{\varpi_1\}$	$\{\varpi_1\}$	$\{\varpi_1, \varpi_3\}$
ϖ_3	ϕ	$\{\varpi_2\}$	ϕ	$\{\varpi_2\}$
ϖ_4	ϕ	ϕ	ϕ	ϕ
ω	$N_{\langle r\rangle}^{£(\nabla_1)}(\omega)$	$N_{\langle l\rangle}^{£(\nabla_1)}(\omega)$	$N_{\langle i\rangle}^{£(\nabla_1)}(\omega)$	$N_{\langle u\rangle}^{£(\nabla_1)}(\omega)$
ϖ_1	$\{\varpi_1, \varpi_3\}$	$\{\varpi_1, \varpi_3\}$	$\{\varpi_1, \varpi_3\}$	$\{\varpi_1, \varpi_3\}$
ϖ_2	$\{\varpi_2\}$	$\{\varpi_2\}$	$\{\varpi_2\}$	$\{\varpi_2\}$
ϖ_3	ϕ	$\{\varpi_1, \varpi_3\}$	ϕ	$\{\varpi_1, \varpi_3\}$
ϖ_4	ϕ	ϕ	ϕ	ϕ
ω	$N_r^{£(\nabla_2)}(\omega)$	$N_l^{£(\nabla_2)}(\omega)$	$N_i^{£(\nabla_2)}(\omega)$	$N_u^{£(\nabla_2)}(\omega)$
ϖ_1	$\{\varpi_3\}$	ϕ	ϕ	$\{\varpi_3\}$
ϖ_2	$\{\varpi_2, \varpi_3\}$	$\{\varpi_2\}$	$\{\varpi_2\}$	$\{\varpi_2, \varpi_3\}$
ϖ_3	ϕ	$\{\varpi_1, \varpi_2\}$	ϕ	$\{\varpi_1, \varpi_2\}$
ϖ_4	ϕ	ϕ	ϕ	ϕ
ω	$N_{\langle r\rangle}^{£(\nabla_2)}(\omega)$	$N_{\langle l\rangle}^{£(\nabla_2)}(\omega)$	$N_{\langle i\rangle}^{£(\nabla_2)}(\omega)$	$N_{\langle u\rangle}^{£(\nabla_2)}(\omega)$
ϖ_1	$\{\varpi_1, \varpi_2\}$	ϕ	$\{\varpi_1, \varpi_3\}$	$\{\varpi_1, \varpi_3\}$
ϖ_2	$\{\varpi_1, \varpi_2\}$	$\{\varpi_2, \varpi_3\}$	$\{\varpi_2, \varpi_3\}$	$\{\varpi_1, \varpi_2, \varpi_3\}$
ϖ_3	ϕ	$\{\varpi_2, \varpi_3\}$	ϕ	$\{\varpi_2, \varpi_3\}$
ϖ_4	ϕ	ϕ	ϕ	ϕ

Proof. 1. By Ref. [23], $N_i^{£(\nabla)}(\omega) \subseteq N_r^{£(\nabla)}(\omega) \subseteq N_u^{£(\nabla)}(\omega)$. We prove the other part. $N_i^{Ł(\neg\nabla)}(\omega) \subseteq N_r^{Ł(\neg\nabla)}(\omega) \subseteq N_u^{Ł(\neg\nabla)}(\omega)$. Now,
$N_u^{Ł(\neg\nabla)}(\omega) = N_r^{Ł(\neg\nabla)}(\omega) \cap N_l^{Ł(\neg\nabla)}(\omega) \subseteq N_r^{Ł(\neg\nabla)}(\omega) \cap N_r^{Ł(\neg\nabla)}(\omega) \cup N_l^{Ł(\neg\nabla)}(\omega) = N_i^{Ł(\neg\nabla)}(\omega)$.

The other parts can be proved similarly. □

Example 4.4. Let $\mathfrak{X} = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$ and $\wp = \{\nabla_1, \nabla_2\}$, $\neg\wp = \{\neg\nabla_1, \neg\nabla_2\}$. Define $£ : \wp \to 2^{\mathfrak{X}\times\mathfrak{X}}$ and $Ł : \neg\wp \to 2^{\mathfrak{X}\times\mathfrak{X}}$ by

$£(\nabla_1) = \{(\{\varpi_1, \varpi_2\}), (\{\varpi_2, \varpi_1\}), (\{\varpi_2, \varpi_3\})\}$,
$Ł(\neg\nabla_1) = \{(\{\varpi_3, \varpi_4\}), (\{\varpi_4, \varpi_3\}), (\{\varpi_1, \varpi_4\})\}$,

$£(\nabla_2) = \{(\{\varpi_1, \varpi_3\}), (\{\varpi_2, \varpi_3\}), (\{\varpi_2, \varpi_2\})\}$,
$Ł(\neg\nabla_2) = \{(\{\varpi_2, \varpi_4\}), (\{\varpi_1, \varpi_4\}), (\{\varpi_3, \varpi_3\})\}$.

Then the BS-N_j^{BS}-neighborhoods of each of them, where $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$ shown in Table 1 and Table 2.

5. Topologies generated by bipolar soft N_j neighborhoods

In the following theorem, we will study the topologies generated from bipolar soft N_j neighborhoods.

Theorem 5.1. *Let $(\mathfrak{X}, £, Ł, \wp, \Pi_j)$ be a N_j^{BS}-neighborhood space on $\mathfrak{X}$. Let $\vartheta = (£, Ł, \wp)$. Then the collection $\mathrm{T}_{N_j^{-S}} = \{\delta \subseteq \mathfrak{X} \mid \forall\omega \in \delta, \delta' \subseteq N_j^{Ł(\neg\nabla)}(\omega), \text{for all } \neg\nabla \in \wp\}$ for all $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$ is a topological space on $\mathfrak{X}$.*

Table 2: BS-$N_j^{Ł(\neg\nabla_i)}(\omega)$-neighborhoods.

ω	$N_r^{Ł(\neg\nabla_1)}(\omega)$	$N_l^{Ł(\neg\nabla_1)}(\omega)$	$N_i^{Ł(\neg\nabla_1)}(\omega)$	$N_u^{Ł(\neg\nabla_1)}(\omega)$
ϖ_1	ϕ	$\{\varpi_4\}$	$\{\varpi_4\}$	ϕ
ϖ_2	ϕ	ϕ	ϕ	ϕ
ϖ_3	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$
ϖ_4	$\{\varpi_1,\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_1,\varpi_3\}$	$\{\varpi_3\}$
ω	$N_{\langle r\rangle}^{Ł(\neg\nabla_1)}(\omega)$	$N_{\langle l\rangle}^{Ł(\neg\nabla_1)}(\omega)$	$N_{\langle i\rangle}^{Ł(\neg\nabla_1)}(\omega)$	$N_{\langle u\rangle}^{Ł(\neg\nabla_1)}(\omega)$
ϖ_1	ϕ	$\{\varpi_1,\varpi_3\}$	$\{\varpi_1,\varpi_3\}$	ϕ
ϖ_2	ϕ	ϕ	ϕ	ϕ
ϖ_3	$\{\varpi_1,\varpi_3\}$	$\{\varpi_1,\varpi_3\}$	$\{\varpi_1,\varpi_3\}$	$\{\varpi_1,\varpi_3\}$
ϖ_4	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$
ω	$N_r^{Ł(\neg\nabla_2)}(\omega)$	$N_l^{Ł(\neg\nabla_2)}(\omega)$	$N_i^{Ł(\neg\nabla_2)}(\omega)$	$N_u^{Ł(\neg\nabla_2)}(\omega)$
ϖ_1	ϕ	$\{\varpi_4\}$	$\{\varpi_4\}$	ϕ
ϖ_2	ϕ	$\{\varpi_4\}$	$\{\varpi_4\}$	ϕ
ϖ_3	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$
ϖ_4	$\{\varpi_1,\varpi_2\}$	ϕ	$\{\varpi_1,\varpi_2\}$	ϕ
ω	$N_{\langle r\rangle}^{Ł(\neg\nabla_2)}(\omega)$	$N_{\langle l\rangle}^{Ł(\neg\nabla_2)}(\omega)$	$N_{\langle i\rangle}^{Ł(\neg\nabla_2)}(\omega)$	$N_{\langle u\rangle}^{Ł(\neg\nabla_2)}(\omega)$
ϖ_1	ϕ	$\{\varpi_1,\varpi_2\}$	$\{\varpi_1,\varpi_2\}$	ϕ
ϖ_2	ϕ	$\{\varpi_1,\varpi_2\}$	$\{\varpi_1,\varpi_2\}$	ϕ
ϖ_3	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$
ϖ_4	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$

Table 3: Results of the decision algorithm with respect to after-sets.

$\vartheta(\nabla_i)$	ϖ_1	ϖ_2	ϖ_3	ϖ_4	ϖ_5	ϖ_6
$\underline{\vartheta}(\nabla_1)$	1	1	1	−1	−1	−1
$\overline{\vartheta}(\nabla_1)$	1	1	1	−1	−1	−1
$\underline{\vartheta}(\nabla_2)$	0	−1	0	0	0	−1
$\overline{\vartheta}(\nabla_2)$	0	0	0	0	0	−1
$\underline{\vartheta}(\nabla_3)$	−1	0	1	0	−1	−1
$\overline{\vartheta}(\nabla_3)$	−1	1	1	−1	1	0
α_{ij}	0	2	4	−3	−2	−5

Proof. 1. Clearly, $\mathfrak{X}, \Phi \in \mathrm{T}_{N_j^{-s}}$;

2. Let $\{\delta_i\}_{i\in\mathfrak{I}} \subseteq \mathrm{T}_{N_j^{-s}}$, we need to prove $\bigcup_{i\in\mathfrak{I}} \delta_i \in \mathrm{T}_{N_j^{-s}}$. Let $\omega \in \bigcup_{i\in\mathfrak{I}} \delta_i$, this implies that there exists $i \in \mathfrak{I}$ such that$\omega \in \delta_i$. So, $\delta_i' \subseteq N_j^{Ł(\neg\nabla)}(\omega)$ Hence, $\bigcap_{i\in\mathfrak{I}} \delta_i' \subseteq N_j^{Ł(\neg\nabla)}(\omega)$. Therefore, $(\bigcup_{i\in\mathfrak{I}} \delta_i)' = \bigcap_{i\in\mathfrak{I}} \delta_i' \subseteq N_j^{Ł(\neg\nabla)}(\omega)$. i.e. $\bigcup_{i\in\mathfrak{I}} \delta_i \in N_j^{Ł(\neg\nabla)}(\omega)$.

3. Let $\delta_1, \delta_2 \in \mathrm{T}_{N_j^{-s}}$ and $\omega \in \delta_1 \cap \delta_2$, we have $\omega \in \delta_1$ and $\omega \in \delta_2$. So, $\delta_1' \subseteq N_j^{Ł(\neg\nabla)}(\omega)$ and $\delta_2' \subseteq N_j^{Ł(\neg\nabla)}(\omega)$. Hence, $(\delta_1 \cap \delta_2)' = \delta_1' \cup \delta_2' \subseteq N_j^{Ł(\neg\nabla)}(\omega)$. Therefore, $\delta_1 \cap \delta_2 \in \mathrm{T}_{N_j^{-s}}$.

□

Theorem 5.2. *Let $(\mathfrak{X}, £, Ł, \wp, \Pi_j)$ be a N_j^{BS}-neighborhood space on $\mathfrak{X}$. Let $\vartheta = (£, Ł, \wp)$. Then the collection $\mathrm{T}_{N_j^{\vartheta}} = (\mathrm{T}_{N_j^{+s}}, \mathrm{T}'_{N_j^{-s}})$ is a topology on $(\mathfrak{X}, \mathfrak{X}')$.*

Proof. 1. Clearly, $(\mathfrak{X}, \Phi)$ and $(\Phi, \mathfrak{X}) \in \mathrm{T}_{N_j^{\vartheta}}$;

2. Let $(\delta_i, \mu_i)_{i\in\mathfrak{I}} \subseteq \mathrm{T}_{N_j^{\vartheta}}$, we need to prove $\bigcup_{i\in\mathfrak{I}} (\delta_i, \mu_i) = (\bigcup_{i\in\mathfrak{I}} \delta_i, \bigcap_{i\in\mathfrak{I}} \mu_i) \in \mathrm{T}_{N_j^{\vartheta}}$. Since $\mathrm{T}_{N_j^{+s}}$ and $\mathrm{T}_{N_j^{-s}}$ are topologies in $\mathfrak{X}$, we have $\bigcup_{i\in\mathfrak{I}} \delta_i \in \mathrm{T}_{N_j^{+s}}$ and $(\bigcap_{i\in\mathfrak{I}} \mu_i)' = \bigcup_{i\in\mathfrak{I}} \mu_i' \in \mathrm{T}_{N_j^{-s}}$. Thus $\bigcap_{i\in\mathfrak{I}} \mu_i \in \mathrm{T}'_{N_j^{-s}}$ and consequently, $\bigcup_{i\in\mathfrak{I}} (\delta_i, \mu_i) \in \mathrm{T}_{N_j^{\vartheta}}$. Similarly, we can prove that for every $(\delta_1, \mu_1), (\delta_2, \mu_2) \in \mathrm{T}_{N_j^{\vartheta}}$, $((\delta_1, \mu_1) \bigcap (\delta_2, \mu_2)) \in \mathrm{T}_{N_j^{\vartheta}}$.

□

Next, we study the relations between these different topologies.

Proposition 5.3. *$(\mathfrak{X}, £, Ł, \wp, \Pi_j)$ be a N_j^{BS}-neighborhood space over $\mathfrak{X}$. Let $\vartheta = (£, Ł, \wp)$ then*

1. $\mathrm{T}_{N_j^{\vartheta}} \subseteq \mathrm{T}_{N_i^{\vartheta}}$ *for all* $j \in \{r, l\}$;
2. $\mathrm{T}_{N_u^{\vartheta}} \subseteq \mathrm{T}_{N_j^{\vartheta}}$ *for all* $j \in \{r, l, i\}$;
3. $\mathrm{T}_{N_{\langle i\rangle}^{\vartheta}} \subseteq \mathrm{T}_{N_j^{\vartheta}}$ *for all* $j \in \{\langle r\rangle, \langle l\rangle\}$;
4. $\mathrm{T}_{N_{\langle u\rangle}^{\vartheta}} \subseteq \mathrm{T}_{N_j^{\vartheta}}$ *for all* $j \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle\}$;

Proof. 1. By Ref. [17], $\mathrm{T}_{N_{\langle r\rangle}^{+S}} \subseteq \mathrm{T}_{N_{\langle i\rangle}^{+S}}$. On the other hand, assume that $\delta \in \mathrm{T}_{N_{\langle r\rangle}^{-S}}$. So $\forall \omega \in \delta$, $\delta' \subseteq N_r^{Ł(\neg\nabla)}(\omega)$. for all $\neg\nabla \in \neg\wp$. Hence, $\delta' \subseteq N_r^{Ł(\neg\nabla)}(\omega) \subseteq N_r^{Ł(\neg\nabla)}(\omega) \bigcup N_l^{Ł(\neg\nabla)}(\omega) = N_i^{Ł(\neg\nabla)}(\omega)$. Therefore, $\delta \in \mathrm{T}_{N_r^{-S}}$ and thus $\mathrm{T}'_{N_i^{-S}} \subseteq \mathrm{T}'_{N_r^{-S}}$. Consequently, $\mathrm{T}_{N_j^{\vartheta}} \subseteq \mathrm{T}_{N_i^{\vartheta}}$. The remaining part when $j = l$ can be proved similarly.
(2), (3) and (4) are similar to (1).

□

Proposition 5.4. *$(\mathfrak{X}, £, Ł, \wp, \Pi_j)$ be a N_j^{BS}-neighborhood space over $\mathfrak{X}$. Let $\vartheta = (£, Ł, \wp)$ then $\mathrm{T}_{N_u^{\vartheta}} \subseteq \mathrm{T}_{N_r^{\vartheta}} \subseteq \mathrm{T}_{N_i^{\vartheta}}$.*

Proof. Obvious by Proposition 5.3 □

In the following theorem, we will discuss two types of topologies induced by bipolar soft reflexive relation.

Theorem 5.5. *Let $£ : \wp \to P(\mathfrak{X} \times \mathfrak{X})$ and $Ł : \neg\wp \to P(\mathfrak{X} \times \mathfrak{X})$ be a BS-symmetric relation from $\mathfrak{X}$ to $\mathfrak{X}$. For $\phi \notin \zeta \subseteq \mathfrak{X}$, If $(£, Ł, \wp)$ be a bipolar soft reflexive relation on $\mathfrak{X}$, then $\underline{T}_{\zeta} = \{\zeta \subseteq \mathfrak{X} : \underline{\vartheta}^{\zeta} = \pi_{\vartheta}^{\zeta}\}$ ($\overline{T}_{\zeta} = \{\zeta \subseteq \mathfrak{X} : \overline{\vartheta}^{\zeta} = \pi_{\vartheta}^{\zeta}\}$) is a topology on $\mathfrak{X}$.*

Proof. 1. Since $\underline{\vartheta}^{\mathfrak{X}} = (\underline{£}^{\mathfrak{X}}, \underline{Ł}^{\mathfrak{X}}) = (\mathfrak{X}, \Phi)$. Then $\mathfrak{X} \in \underline{T}_{\zeta}$. Similar $\Phi \in \underline{T}_{\zeta}$.

2. Suppose that $\{\zeta_i\}_{i\in\mathfrak{I}} \subseteq \underline{T}_{\zeta}$. So, $\underline{\vartheta}^{\zeta_i} = \pi_{\vartheta}^{\zeta_i}$, $\forall i \in \mathfrak{I}$. To show that $\bigcup_{i\in\mathfrak{I}} \zeta_i \in \underline{T}_{\zeta}$ we must prove that $\underline{\vartheta}^{\bigcup_{i\in\mathfrak{I}} \zeta_i} = \pi_{\vartheta}^{\bigcup_{i\in\mathfrak{I}} \zeta_i}$. Since $\underline{\vartheta}^{\bigcup_{i\in\mathfrak{I}} \zeta_i} \subseteq \pi_{\vartheta}^{\bigcup_{i\in\mathfrak{I}} \zeta_i}$ by Theorem 3.13, it is enough to show that $\pi_{\vartheta}^{\bigcup_{i\in\mathfrak{I}} \zeta_i} \subseteq \underline{\vartheta}^{\bigcup_{i\in\mathfrak{I}} \zeta_i}$. Since $\zeta_i \subseteq \bigcup_{i\in\mathfrak{I}} \zeta_i$, then $\underline{\vartheta}^{\zeta_i} \subseteq \underline{\vartheta}^{\bigcup_{i\in\mathfrak{I}} \zeta_i}$ for $i \in \mathfrak{I}$ by Properties 3.5. Hence $\bigcup_{i\in\mathfrak{I}} \underline{\vartheta}^{\zeta_i} \subseteq \underline{\vartheta}^{\bigcup_{i\in\mathfrak{I}} \zeta_i}$. But $\pi_{\vartheta}^{\bigcup_{i\in\mathfrak{I}} \zeta_i} = (\bigcup_{i\in\mathfrak{I}} \zeta_i, (\bigcup_{i\in\mathfrak{I}} \zeta_i)') = (\bigcup_{i\in\mathfrak{I}} \zeta_i, \bigcap_{i\in\mathfrak{I}} \zeta_i') = \bigcup_{i\in\mathfrak{I}} (\zeta_i, \zeta_i') = \bigcup_{i\in\mathfrak{I}} \underline{\vartheta}^{\zeta_i}$. Therefore, $\pi_{\vartheta}^{\bigcup_{i\in\mathfrak{I}} \zeta_i} \subseteq \underline{\vartheta}^{\bigcup_{i\in\mathfrak{I}} \zeta_i}$ and consequently $\underline{\vartheta}^{\bigcup_{i\in\mathfrak{I}} \zeta_i} = \pi_{\vartheta}^{\bigcup_{i\in\mathfrak{I}} \zeta_i}$. Thus $\bigcup_{i\in\mathfrak{I}} \zeta_i \in \underline{T}_{\zeta}$.

3. Suppose that $\zeta_1, \zeta_2 \in \underline{T}_{\zeta}$. So, $\underline{\vartheta}^{\zeta_1} = \pi_{\vartheta}^{\zeta_1}$ and $\underline{\vartheta}^{\zeta_2} = \pi_{\vartheta}^{\zeta_2}$. By Properties 3.5. Now, $\underline{\vartheta}^{(\zeta_1 \cap \zeta_2)} = \underline{\vartheta}^{\zeta_1} \bigcap \underline{\vartheta}^{\zeta_2} = \pi_{\vartheta}^{\zeta_1} \bigcap \pi_{\vartheta}^{\zeta_2} = \pi_{\vartheta}^{(\zeta_1 \cap \zeta_2)}$. So, $\zeta_1 \bigcap \zeta_2 \in \underline{T}_{\zeta}$.

Similarly, we can show that $\overline{T}_{\zeta}$ is a topology on $\mathfrak{X}$. □

6. Application in a decision-making problem

Decision-making is an important factor in all scientific careers, and experts use their knowledge of the field to make informed decisions. We apply the concept of *BS*- sets to model a given problem and then provide an algorithm for selecting the best object based on the available information set.

Example 6.1. Let $\mathfrak{X} = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5, \varpi_6\}$ is the set of t-shirts. $\wp = \{\nabla_1, \nabla_2, \nabla_3, d\}$ and $\neg\wp = \{\neg\nabla_1, \neg\nabla_2, \neg\nabla_3, d\}$ is the set of parameters, where $\nabla_1 : colorful, \neg\nabla_1 : plain, \nabla_2 : sport, \neg\nabla_2 : classic, \nabla_3 : cheap, \neg\nabla_3 : expensive$. Suppose that $C = \{\nabla_1, \nabla_2, \nabla_3\}$ and $D = \{d\}$ where C and D are the sets of condition parameters and decision parameters, respectively. Let $\vartheta = (£, Ł, \wp)$, define a *BSE*-relation $£(\nabla) : \wp \to P(\mathfrak{X} \times \mathfrak{X})$ and $Ł(\neq \nabla) : \neg\nabla \to P(\mathfrak{X} \times \mathfrak{X})$ for each parameter $\nabla \in \wp$ and $\neg\nabla \in \neg\wp$.

$$£(\nabla_1) = \{(\{\varpi_1, \varpi_2\}), (\{\varpi_2, \varpi_2\}), (\{\varpi_3, \varpi_3\}), (\{\varpi_1, \varpi_2\}), (\{\varpi_2, \varpi_3\}), (\{\varpi_1, \varpi_3\})\},$$

$$Ł(\neg\nabla_1) = \{(\{\varpi_4, \varpi_4\}), (\{\varpi_4, \varpi_5\}), (\{\varpi_5, \varpi_6\}), (\{\varpi_4, \varpi_6\}), (\{\varpi_5, \varpi_5\}), (\{\varpi_6, \varpi_6\})\}$$

$$£(\nabla_2) = \{(\{\varpi_3, \varpi_4\}), (\{\varpi_1, \varpi_4\}), (\{\varpi_4, \varpi_5\}), (\{\varpi_4, \varpi_4\}), (\{\varpi_1, \varpi_5\})\},$$

$$Ł(\neg\nabla_2) = \{(\{\varpi_6, \varpi_6\}), (\{\varpi_2, \varpi_6\}), (\{\varpi_6, \varpi_5\}), (\{\varpi_2, \varpi_5\}), (\{\varpi_2, \varpi_2\})\},$$

$$£(\nabla_3) = \{(\{\varpi_5, \varpi_5\}), (\{\varpi_2, \varpi_5\}), (\{\varpi_5, \varpi_3\}), (\{\varpi_2, \varpi_3\}), (\{\varpi_3, \varpi_3\})\},$$

$$Ł(\neg\nabla_3) = \{(\{\varpi_4, \varpi_5\}), (\{\varpi_6, \varpi_1\}), (\{\varpi_1, \varpi_4\}), (\{\varpi_6, \varpi_4\}), (\{\varpi_6, \varpi_6\})\}.$$

The following *BSE*-classes are gotten for each *BSE*-relation: $\varpi_1£(\nabla_1) = \{\varpi_1, \varpi_2, \varpi_3\}, \varpi_2£(\nabla_1) = \{\varpi_2, \varpi_3\}, \varpi_3£(\nabla_1) = \{\varpi_3\}, \varpi_4£(\nabla_1) = \phi, \varpi_5£(\nabla_1) = \phi$ and $\varpi_6£(\nabla_1) = \phi$.
$\varpi_1Ł(\neg\nabla_1) = \phi, \varpi_2Ł(\neg\nabla_1) = \phi, \varpi_3Ł(\neg\nabla_1) = \phi, \varpi_4Ł(\neg\nabla_1) = \{\varpi_4, \varpi_5, \varpi_6\}, \varpi_5Ł(\neg\nabla_1) = \{\varpi_5, \varpi_6\}$ and $\varpi_6Ł(\neg\nabla_1) = \{\varpi_6\}$.
$\varpi_1£(\nabla_2) = \{\varpi_4, \varpi_5\}, \varpi_2£(\nabla_2) = \phi, \varpi_3£(\nabla_2) = \{\varpi_4\}, \varpi_4£(\nabla_2) = \{\varpi_4, \varpi_5\}, \varpi_5£(\nabla_2) = \phi$ and $\varpi_6£(\nabla_2) = \phi$.
$\varpi_1Ł(\neg\nabla_2) = \phi, \varpi_2Ł(\neg\nabla_2) = \{\varpi_2, \varpi_5, \varpi_6\}, \varpi_3Ł(\neg\nabla_2) = \phi, \varpi_4Ł(\neg\nabla_2) = \phi, \varpi_5Ł(\neg\nabla_2) = \phi$ and $\varpi_6Ł(\neg\nabla_2) = \{\varpi_5, \varpi_6\}$.

$\varpi_1£(\nabla_3) = \phi, \varpi_2£(\nabla_3) = \{\varpi_3, \varpi_5\}, \varpi_3£(\nabla_3) = \{\varpi_3\}, \varpi_4£(\nabla_3) = \phi, \varpi_5£(\nabla_3) = \phi$ and $\varpi_6£(\nabla_3) = \phi$.
$\varpi_1Ł(\neg\nabla_3) = \{\varpi_4\}, \varpi_2Ł(\neg\nabla_3) = \phi, \varpi_3Ł(\neg\nabla_3) = \phi, \varpi_4Ł(\neg\nabla_3) = \{\varpi_5\}, \varpi_5Ł(\neg\nabla_3) = \phi$ and $\varpi_6Ł(\neg\nabla_3) = \{\varpi_1, \varpi_4, \varpi_6\}$.

Suppose $\zeta \subseteq \mathfrak{X}$, define as: $\zeta = \{\varpi_1, \varpi_2, \varpi_3\}$, $\zeta' = \{\varpi_4, \varpi_5, \varpi_6\}$.

Describe the "the special features of the t-shirts" which Mr. X is going to buy respectively.

Algorithm 1.

1. Input the lower soft approximation $\underline{\vartheta}^{\zeta}$ and upper soft approximation $\overline{\vartheta}^{\zeta}$ of a *BS*-set ζ with respect to aftersets.
2. Corresponding to each $\omega_i \in \mathfrak{X}$, we calculate $\underline{\vartheta}^{\zeta}$ which is 1 if $\omega_i \in \underline{£}^{\zeta}(\nabla_i)$, -1 if $\omega_i \in \underline{Ł}^{\zeta}(\neg\nabla_i)$ and 0 otherwise. Similarly, we calculate $\overline{\vartheta}^{\zeta}$ which is 1 if $\omega_i \in \overline{£}^{\zeta}(\nabla_i)$, -1 if $\omega_i \in \overline{Ł}^{\zeta}(\neg\nabla_i)$ and 0 otherwise.
3. Compute the choice value $\alpha_{ij} = \sum_{i=1}^{k} \underline{\vartheta}^{\zeta} + \sum_{i=1}^{k} \overline{\vartheta}^{\zeta}$ with respect to aftersets.
4. The best decision is $\omega_\epsilon \in \mathfrak{X}$ if $\alpha_\epsilon = \max_i \alpha_i, i = 1, 2, 3, \ldots, |\mathfrak{X}|$.
5. The worst decision is $\omega_\epsilon \in \mathfrak{X}$ if $\alpha_\epsilon = \min_i \alpha_i, i = 1, 2, 3, \ldots, |\mathfrak{X}|$.
6. If ϵ has more than one value, then we can equally choose anyone of ω_ϵ.

Here in Table 3 the choice value $\alpha_{ij} = \sum_{i=1}^{k} \underline{\vartheta}^{\zeta} + \sum_{i=1}^{k} \overline{\vartheta}^{\zeta}$ is calculated with respect to aftersets. The t-shirt of ϖ_3 scores the maximum choice value $\alpha_\epsilon = 4$, and decision is in favour of the t-shirt of ϖ_3 for selection. In case that ϖ_3 cannot be chosen, ϖ_2 will be selected. Moreover, the t-shirt of ϖ_6 is totally ignored. Hence, Mr. X will choose the t-shirt ϖ_3 and ϖ_2 for his personal use.

7. Conclusion

In this work, we introduced lower and upper approximations based on binary *BS* relations, studied the properties of these approximations, and generated new topologies from them. We also explored their characterizations and properties and introduced *BS*-N_j^{BS}-neighborhood spaces. Furthermore, these spaces were used to obtain methods for approximating rough sets and to provide a direct way of generating the corresponding topologies from relations without using subbases or bases. This technique may make several topological concepts easier to apply in future research. Finally, the proposed structures open the way for further topological applications in rough contexts and for applications based on real-world data. In future work, we will investigate additional topological applications in rough contexts and real-world applications using the structure proposed in this paper.

Data availability

We do not have any research data outside the submitted manuscript file.

Declaration of competing interest

The authors declare no conflict of interest.

Funding

This research received no external funding.

Acknowledgment

The authors would like to thank Mustansiriyah University (www.uomustansiriyah.edu.iq), Baghdad, Iraq, for its support in the present work. The authors also thank the referees and the editor for their thoughtful reading and helpful comments.

References

[1] D. Molodtsov, "Soft set theory–first results", Computers & Mathematics with Applications **37** (1999) 19. https://doi.org/10.1016/S0898-1221(99)00056-5.
[2] P. K. Maji, R. Biswas & A. R. Roy, "Soft set theory", Computers & Mathematics with Applications **45** (2003) 555. https://doi.org/10.1016/S0898-1221(03)00016-6.
[3] M. Shabir & M. Naz, "On soft topological spaces", Computers & Mathematics with Applications **61** (2011) 1786. https://doi.org/10.1016/j.camwa.2011.02.006.
[4] M. Akdag & A. Ozkan, "Soft α-open sets and soft α-continuous functions", Abstract and Applied Analysis **2014** (2014) 891341. https://doi.org/10.1155/2014/891341.
[5] M. Akdag & A. Ozkan, "Soft b-open sets and soft b-continuous functions", Mathematical Sciences **8** (2014) 124. https://doi.org/10.1007/s40096-014-0124-7.
[6] S. Z. Hameed, H. I. Mustafa, A. E. Radwan & E. El-Seidy, "N-bipolar soft connectedness, disconnectedness and compactness spaces with application in infinite game", Nonlinear Functional Analysis and Applications **30** (2025) 25. https://doi.org/10.22771/nfaa.2025.30.01.03.
[7] S. Z. Hameed, A. E. Radwan & E. El-Seidy, "On soft strongly b^*-compactness and soft strongly b^*-connectedness in soft topological spaces", Scalable Computing: Practice and Experience **25** (2024) 4335. https://doi.org/10.12694/scpe.v25i5.3178.
[8] S. Z. Hameed, A. E. Radwan & E. El-Seidy, "On soft strongly b^*-separation axioms", Iraqi Journal of Science **65** (2024) 3259. https://doi.org/10.24996/ijs.2024.65.6.25.
[9] A. E. Radwan, E. El-Seidy & S. Z. Hameed, "On soft strongly b^*-closed via soft ideal", Communications on Applied Nonlinear Analysis **31** (2024) 284. https://doi.org/10.52783/cana.v31.551.
[10] M. Shabir & M. Naz, "On bipolar soft sets", arXiv preprint arXiv:1303.1344 (2013). https://doi.org/10.48550/arXiv.1303.1344.
[11] F. Fatimah, D. Rosadi, R. B. F. Hakim & J. C. R. Alcantud, "N-soft sets and their decision-making algorithms", Soft Computing **22** (2018) 3829. https://doi.org/10.1007/s00500-017-2838-6.
[12] A. Fadel & S. C. Dzul-Kifli, "Bipolar soft topological spaces", European Journal of Pure and Applied Mathematics **13** (2020) 227. https://doi.org/10.29020/nybg.ejpam.v13i2.3645.
[13] O. Dalkılıç & N. Demirtaş, "Binary bipolar soft sets", Boletim da Sociedade Paranaense de Matemática **41** (2023) 1. https://doi.org/10.5269/bspm.51003.
[14] Z. Pawlak, "Rough sets", International Journal of Computer and Information Sciences **11** (1982) 341. https://doi.org/10.1007/BF01001956.
[15] M. E. Abd El-Monsef, O. A. Embaby & M. K. El-Bably, "Comparison between rough set approximations based on different topologies", International Journal of Granular Computing, Rough Sets and Intelligent Systems **3** (2015) 292. https://doi.org/10.1504/IJGCRSIS.2014.068032.
[16] R. Abu-Gdairi, M. A. El-Gayar, T. M. Al-shami, A. S. Nawar & M. K. El-Bably, "Some topological approaches for generalized rough sets and their decision-making applications", Symmetry **14** (2022) 95. https://doi.org/10.3390/sym14010095.
[17] M. Shabir, R. S. Kanwal & M. I. Ali, "Reduction of an information system", Soft Computing **24** (2020) 10801. https://doi.org/10.1007/s00500-019-04582-3.
[18] D. Dubois & H. Prade, "An introduction to bipolar representations of information and preference", International Journal of Intelligent Systems **23** (2008) 866. https://doi.org/10.1002/int.20297.
[19] A. M. Ramadhan, "Bipolar soft minimal structures", European Journal of Pure and Applied Mathematics **18** (2025) 5928. https://doi.org/10.29020/nybg.ejpam.v18i2.5928.
[20] I. T. Shbair, A. S. Salama, O. A. Embaby & A. A. El-Atik, "Some topological approaches of rough sets through minimal neighborhoods and decision making", Journal of Mathematics **2024** (2024) 2214422. https://doi.org/10.1155/2024/2214422.
[21] S. S. Mohammed, "On fuzzy soft set-valued maps with application", Journal of the Nigerian Society of Physical Sciences **2** (2020) 26. https://doi.org/10.46481/jnsps.2020.48.
[22] P. A. Ejegwa, I. C. Onyeke, B. T. Terhemen, M. P. Onoja, A. Ogiji & C. U. Opeh, "Modified Szmidt and Kacprzyk's intuitionistic fuzzy distances and their applications in decision-making", Journal of the Nigerian Society of Physical Sciences **4** (2022) 174. https://doi.org/10.46481/jnsps.2022.530.
[23] B. P. Varol & H. Malkoç, "Bipolar fuzzy supra topology via (Q-) neighborhood and its application in data mining process", Symmetry **16** (2024) 216. https://doi.org/10.3390/sym16020216.
[24] R. A. Hosny, R. Abu-Gdairi & M. K. El-Bably, "Enhancing dengue fever diagnosis with generalized rough sets: utilizing initial-neighborhoods and ideals", Alexandria Engineering Journal **94** (2024) 68. https://doi.org/10.1016/j.aej.2024.03.028.
[25] H. I. Mustafa & R. Wassef, "Improved soft rough approximations based on soft neighborhoods", Filomat **39** (2025) 4887. https://doi.org/10.2298/FIL2514887M.