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On subgraph relationships between graph products

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Abstract

Graph products play an important role in graph theory by providing systematic methods for constructing complex graphs from simpler ones and by revealing structural relationships among different graph classes. In this paper, we investigate subgraph relations among several standard graph products, including the Cartesian, lexicographic, tensor, modular, co-normal, strong, homomorphic, and rooted products. By comparing their defining adjacency conditions, we establish a collection of inclusion results between these products. In particular, we prove that the Cartesian product is a subgraph of the lexicographic, co-normal, and strong products, while the lexicographic product is itself a subgraph of the co-normal product. We further show that the tensor product is a subgraph of the lexicographic, modular, strong, and co-normal products. In addition, the strong product is shown to be a subgraph of both the lexicographic and co-normal products. Finally, we establish that the rooted product forms a subgraph of the Cartesian, lexicographic, co-normal, and strong products. Illustrative examples are included to visualise these relationships. The results obtained provide a clearer understanding of the structural hierarchy among graph product operations and may support further investigations in algebraic graph theory and network modelling.

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1. Introduction

Graph theory is one of the most active areas of modern mathematics and has applications in computer science, engineering, communication systems, biology, transportation networks, and the social sciences. A graph consists of a set of vertices together with a set of edges describing relationships between pairs of vertices. Because of their ability to represent complex structures in a simple mathematical form, graphs are widely used in modelling and analysing real-world systems [1].

Among the different topics studied in graph theory, graph products occupy an important position. A graph product is a binary operation that combines two graphs to produce another graph whose structure depends on the properties of the original graphs. Graph products are useful in the study of network design, parallel processing, coding theory, and structural graph analysis. Several standard graph products have been introduced and investigated in the literature, including the Cartesian product, lexicographic product, tensor product, strong product, modular product, co-normal product, homomorphic product, and rooted product [2, 3].

Many researchers have contributed to the development of graph product theory. Bozovic and Peterin studied efficient open domination in products of graphs [4], while Changat *et al.*

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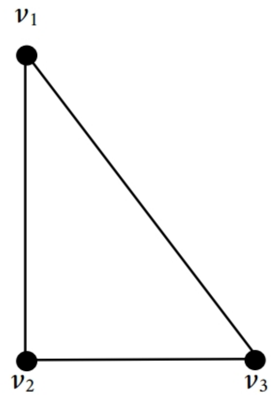

Figure 1: Graph $\Omega_1 = (\Delta_1, \Xi_1)$.

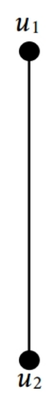

Figure 2: Graph $\Omega_2 = (\Delta_2, \Xi_2)$.

examined boundary-type sets associated with strong products [5]. Feigenbaum investigated factorisation properties of Cartesian products [6], and Manion analysed cancellation properties related to lexicographic products [7]. Potocnik and Wilson also discussed structural aspects of graph products and their variations [8]. More recently, George, Jose and Thumbakara studied vertex and edge counts together with degree-related properties of several graph products [9]. Jose *et al.* further explored product operations and their properties in broader graph-theoretic settings [10].

Although different graph products have been studied extensively, comparatively less attention has been given to the subgraph relationships among these products. Understanding such relationships helps in comparing their structural behaviour and identifying how one product can be embedded within another. Motivated by this observation, the present work investigates subgraph relations among several standard graph products. In particular, we establish inclusion relations involving Cartesian, lexicographic, tensor, modular, co-normal, strong, and rooted products. Illustrative examples are also provided to visualise these relationships and support the theoretical results obtained in this work.

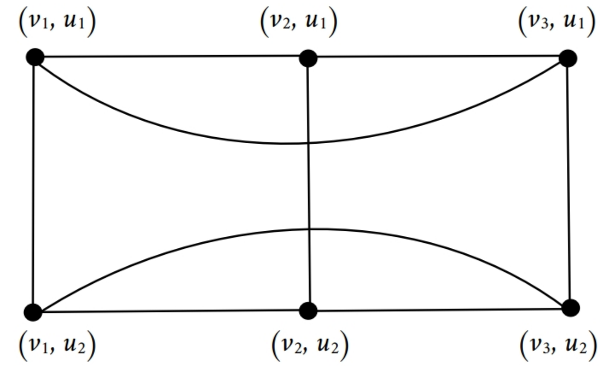

Figure 3: $\Omega_1 \Box \Omega_2$.

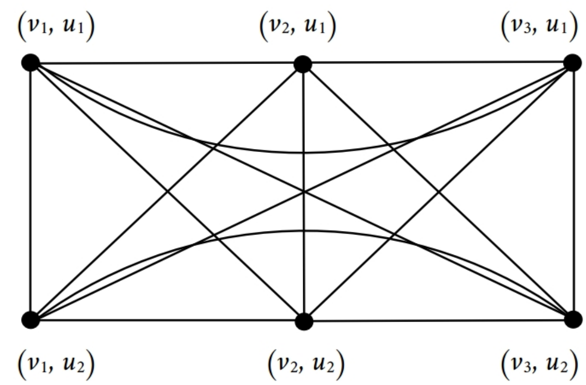

Figure 4: $\Omega_1 \circ \Omega_2$.

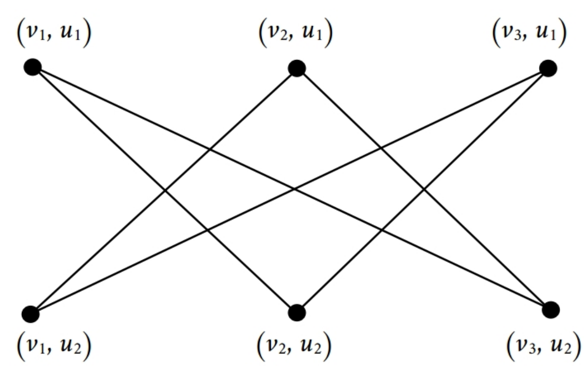

Figure 5: $\Omega_1 \times \Omega_2$.

2. Product operations in graphs

2.1. Cartesian product

Definition 2.1. Let $\Omega_1 = (\Delta_1, \Xi_1)$ and $\Omega_2 = (\Delta_2, \Xi_2)$ be two graphs. Then the *Cartesian product (or box product)* of Ω_1 and Ω_2 denoted by $\Omega_1 \Box \Omega_2$ is a graph with vertex set $\Delta(\Omega_1 \Box \Omega_2) = \Delta_1 \times \Delta_2$ and edge set $\Xi(\Omega_1 \Box \Omega_2)$, where $e \in \Xi(\Omega_1 \Box \Omega_2)$ is an edge joining the vertices $(\vartheta_1, \vartheta_1')$ and $(\vartheta_2, \vartheta_2')$ if and only if

1. $\vartheta_1 = \vartheta_2$ and the vertices ϑ_1' and ϑ_2' are adjacent in Ω_2 or
2. $\vartheta_1' = \vartheta_2'$ and the vertices ϑ_1 and ϑ_2 are adjacent in Ω_1.

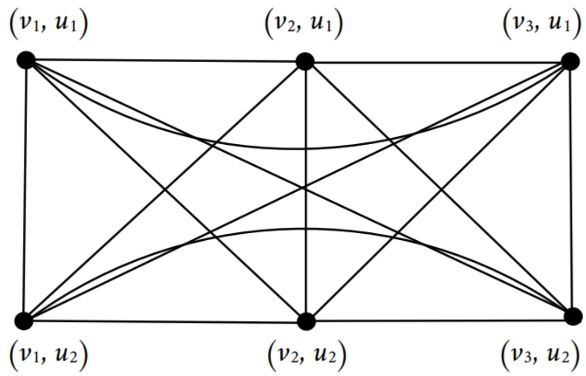

Figure 6: $\Omega_1 \boxtimes \Omega_2$.

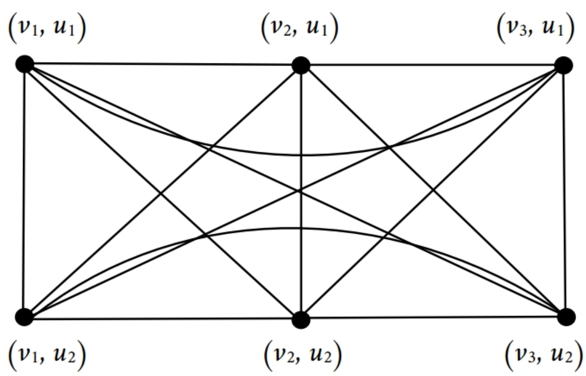

Figure 7: $\Omega_1 * \Omega_2$.

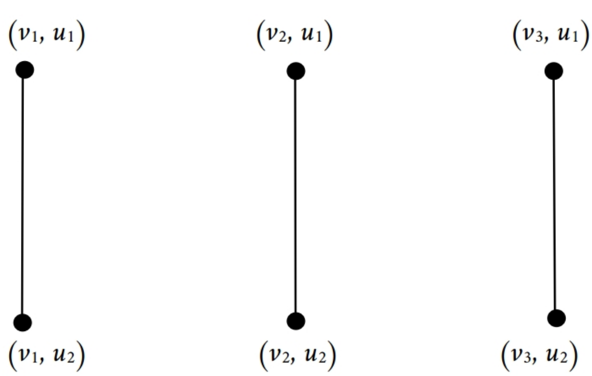

Figure 8: $\Omega_1 \ltimes \Omega_2$.

2.2. *Lexicographic product*

Definition 2.2. Let $\Omega_1 = (\Delta_1, \Xi_1)$ and $\Omega_2 = (\Delta_2, \Xi_2)$ be two graphs. Then the *lexicographic product (or composition)* of Ω_1 and Ω_2 denoted by $\Omega_1 \circ \Omega_2$ is a graph with vertex set $\Delta(\Omega_1 \circ \Omega_2) = \Delta_1 \times \Delta_2$ and edge set $\Xi(\Omega_1 \circ \Omega_2)$, where $e \in \Xi(\Omega_1 \circ \Omega_2)$ is an edge joining the vertices $(\vartheta_1, \vartheta_1')$ and $(\vartheta_2, \vartheta_2')$ if and only if

1. the vertices ϑ_1 and ϑ_2 are adjacent in Ω_1 or
2. $\vartheta_1 = \vartheta_2$ and the vertices ϑ_1' and ϑ_2' are adjacent in Ω_2.

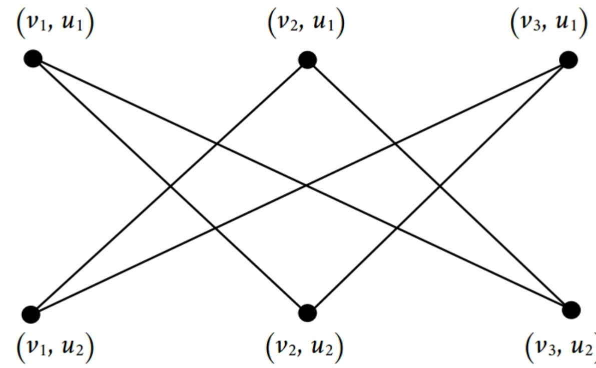

Figure 9: $\Omega_1 \bigcirc \Omega_2$.

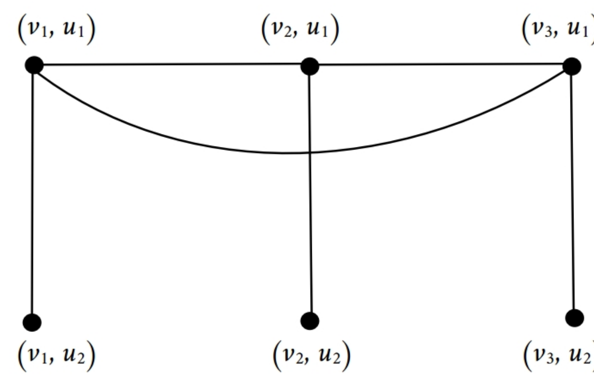

Figure 10: $\Omega_1 \circ_{u_1} \Omega_2$.

2.3. *Modular product*

Definition 2.3. Let $\Omega_1 = (\Delta_1, \Xi_1)$ and $\Omega_2 = (\Delta_2, \Xi_2)$ be two graphs. Then the *modular product* of Ω_1 and Ω_2, represented by $\Omega_1 \bigcirc \Omega_2$ is a graph with vertex set $\Delta(\Omega_1 \bigcirc \Omega_2) = \Delta_1 \times \Delta_2$ and edge set $\Xi(\Omega_1 \bigcirc \Omega_2)$, where $(\vartheta_1, \vartheta_1')$ and $(\vartheta_2, \vartheta_2')$ are adjacent in $\Omega_1 \bigcirc \Omega_2$ if and only if

1. ϑ_1 and ϑ_2 are adjacent in Ω_1 and ϑ_1' and ϑ_2' are adjacent in Ω_2 or
2. ϑ_1 and ϑ_2 are not adjacent in Ω_1 and ϑ_1' and ϑ_2' are not adjacent in Ω_2.

2.4. *Co-normal product*

Definition 2.4. Let $\Omega_1 = (\Delta_1, \Xi_1)$ and $\Omega_2 = (\Delta_2, \Xi_2)$ be two graphs. Then the *co-normal product* of Ω_1 and Ω_2, represented by $\Omega_1 * \Omega_2$ is a graph with vertex set $\Delta(\Omega_1 * \Omega_2) = \Delta_1 \times \Delta_2$ and edge set $\Xi(\Omega_1 * \Omega_2)$, where $(\vartheta_1, \vartheta_1')$ and $(\vartheta_2, \vartheta_2')$ are adjacent in $\Omega_1 * \Omega_2$ if and only if ϑ_1 and ϑ_2 are adjacent in Ω_1 or ϑ_1' and ϑ_2' are adjacent in Ω_2.

2.5. *Strong product*

Definition 2.5. Let $\Omega_1 = (\Delta_1, \Xi_1)$ and $\Omega_2 = (\Delta_2, \Xi_2)$ be two graphs. Then the *strong product* of Ω_1 and Ω_2, represented by $\Omega_1 \boxtimes \Omega_2$ is a graph with vertex set $\Delta(\Omega_1 \boxtimes \Omega_2) = \Delta_1 \times \Delta_2$ and edge set $\Xi(\Omega_1 \boxtimes \Omega_2)$, where $(\vartheta_1, \vartheta_1')$ and $(\vartheta_2, \vartheta_2')$ are adjacent in $\Omega_1 \boxtimes \Omega_2$ if and only if

1. $\vartheta_1 = \vartheta_2$ and ϑ_1' and ϑ_2' are adjacent in Ω_2 or
2. ϑ_1 and ϑ_2 are adjacent in Ω_1 and $\vartheta_1' = \vartheta_2'$ or
3. ϑ_1 and ϑ_2 are adjacent in Ω_1 and ϑ_1' and ϑ_2' are adjacent in Ω_2.

2.6. Tensor product

Definition 2.6. Let $\Omega_1 = (\Delta_1, \Xi_1)$ and $\Omega_2 = (\Delta_2, \Xi_2)$ be two graphs. Then the *tensor product* of Ω_1 and Ω_2, represented by $\Omega_1 \times \Omega_2$, is a graph with vertex set $\Delta(\Omega_1 \times \Omega_2) = \Delta_1 \times \Delta_2$ and edge set $\Xi(\Omega_1 \times \Omega_2)$, where $(\vartheta_1, \vartheta_1')$ and $(\vartheta_2, \vartheta_2')$ are adjacent in $\Omega_1 \times \Omega_2$ if and only if ϑ_1 and ϑ_2 are adjacent in Ω_1 and ϑ_1' and ϑ_2' are adjacent in Ω_2.

2.7. Homomorphic product

Definition 2.7. Let $\Omega_1 = (\Delta_1, \Xi_1)$ and $\Omega_2 = (\Delta_2, \Xi_2)$ be two graphs. Then the *homomorphic product* of Ω_1 and Ω_2, represented by $\Omega_1 \ltimes \Omega_2$ is a graph with vertex set $\Delta(\Omega_1 \ltimes \Omega_2) = \Delta_1 \times \Delta_2$ and edge set $\Xi(\Omega_1 \ltimes \Omega_2)$, where $(\vartheta_1, \vartheta_1')$ and $(\vartheta_2, \vartheta_2')$ are adjacent in $\Omega_1 \ltimes \Omega_2$ if and only if

1. $\vartheta_1 = \vartheta_2$ or
2. ϑ_1 and ϑ_2 are adjacent in Ω_1 and ϑ_1' and ϑ_2' are not adjacent in Ω_2.

2.8. Rooted product

Definition 2.8. Let $\Omega_1 = (\Delta_1, \Xi_1)$ be a graph and $\Omega_2 = (\Delta_2, \Xi_2)$ be a rooted graph with root vertex ϑ_r. Then, the *rooted product* of Ω_1 and Ω_2 denoted by $\Omega_1 \circ_{\vartheta_r} \Omega_2$ is a graph with vertex set $\Delta(\Omega_1 \circ_{\vartheta_r} \Omega_2) = \Delta_1 \times \Delta_2$ and edge set $\Xi(\Omega_1 \circ_{\vartheta_r} \Omega_2)$, where $e \in \Xi(\Omega_1 \circ_{\vartheta_r} \Omega_2)$ is an edge joining the vertices $(\vartheta_1, \vartheta_1')$ and $(\vartheta_2, \vartheta_2')$ if and only if

1. $\vartheta_1 = \vartheta_2$ and the vertices ϑ_1' and ϑ_2' are adjacent in Ω_2 or
2. the vertices ϑ_1 and ϑ_2 are adjacent in Ω_1 and $\vartheta_1' = \vartheta_2' = \vartheta_r$.

3. Subgraph relations among graph products

Theorem 3.1. Let $\Omega_1 = (\Delta_1, \Xi_1)$ and $\Omega_2 = (\Delta_2, \Xi_2)$ be two graphs. Then the Cartesian product $\Omega_1 \Box \Omega_2$ is a subgraph of

1. lexicographic product (composition) $\Omega_1 \circ \Omega_2$
2. co-normal product $\Omega_1 * \Omega_2$
3. strong product $\Omega_1 \boxtimes \Omega_2$

Proof.

1. The vertex set of the Cartesian product $\Omega_1 \Box \Omega_2$ is $\Delta_1 \times \Delta_2$ and there is an edge in the product joining the vertices $(\vartheta_1, \vartheta_1')$ and $(\vartheta_2, \vartheta_2')$ if and only if
   (a) $\vartheta_1 = \vartheta_2$ and the vertices ϑ_1' and ϑ_2' are adjacent in Ω_2 or
   (b) $\vartheta_1' = \vartheta_2'$ and the vertices ϑ_1 and ϑ_2 are adjacent in Ω_1.

   The vertex set of the lexicographic product $\Omega_1 \circ \Omega_2$ is $\Delta_1 \times \Delta_2$, and there is an edge in the product joining the vertices $(\vartheta_1, \vartheta_1')$ and $(\vartheta_2, \vartheta_2')$ if and only if
   (a) the vertices ϑ_1 and ϑ_2 are adjacent in Ω_1 or
   (b) $\vartheta_1 = \vartheta_2$ and the vertices ϑ_1' and ϑ_2' are adjacent in Ω_2.

   Comparing the definitions of Cartesian and lexicographic products, it follows that these two products have the same vertex set $\Delta_1 \times \Delta_2$. Also, the edge set $\Xi(\Omega_1 \Box \Omega_2)$ is a subset of the edge set $\Xi(\Omega_1 \circ \Omega_2)$. Therefore, $\Omega_1 \Box \Omega_2$ is a subgraph of $\Omega_1 \circ \Omega_2$.
2. The vertex set of the co-normal product $\Omega_1 * \Omega_2$ is $\Delta_1 \times \Delta_2$ and there is an edge in the product joining the vertices $(\vartheta_1, \vartheta_1')$ and $(\vartheta_2, \vartheta_2')$ if and only if ϑ_1 and ϑ_2 are adjacent in Ω_1 or ϑ_1' and ϑ_2' are adjacent in Ω_2. Comparing the definitions of Cartesian and co-normal products, it follows that these two products have the same vertex set $\Delta_1 \times \Delta_2$. Also, the edge set $\Xi(\Omega_1 \Box \Omega_2)$ is a subset of the edge set $\Xi(\Omega_1 * \Omega_2)$. Therefore, $\Omega_1 \Box \Omega_2$ is a subgraph of $\Omega_1 * \Omega_2$.
3. The vertex set of the strong product $\Omega_1 \boxtimes \Omega_2$ is $\Delta_1 \times \Delta_2$ and there is an edge in the product joining the vertices $(\vartheta_1, \vartheta_1')$ and $(\vartheta_2, \vartheta_2')$ if and only if
   (a) $\vartheta_1 = \vartheta_2$ and ϑ_1' and ϑ_2' are adjacent in Ω_2 or
   (b) ϑ_1 and ϑ_2 are adjacent in Ω_1 and $\vartheta_1' = \vartheta_2'$ or
   (c) ϑ_1 and ϑ_2 are adjacent in Ω_1 and ϑ_1' and ϑ_2' are adjacent in Ω_2.

   Comparing the definitions of Cartesian and strong products, it follows that these two products have the same vertex set $\Delta_1 \times \Delta_2$. Also, the edge set $\Xi(\Omega_1 \Box \Omega_2)$ is a subset of the edge set $\Xi(\Omega_1 \boxtimes \Omega_2)$. Therefore, $\Omega_1 \Box \Omega_2$ is a subgraph of $\Omega_1 \boxtimes \Omega_2$.

□

Theorem 3.2. Let $\Omega_1 = (\Delta_1, \Xi_1)$ and $\Omega_2 = (\Delta_2, \Xi_2)$ be two graphs. Then the lexicographic product $\Omega_1 \circ \Omega_2$ is a subgraph of the co-normal product $\Omega_1 * \Omega_2$.

Proof. The lexicographic and co-normal products have the same vertex set $\Delta_1 \times \Delta_2$. Also, the edge set $\Xi(\Omega_1 \circ \Omega_2)$ is a subset of the edge set $\Xi(\Omega_1 * \Omega_2)$. Therefore, $\Omega_1 \circ \Omega_2$ is a subgraph of $\Omega_1 * \Omega_2$. □

Theorem 3.3. Let $\Omega_1 = (\Delta_1, \Xi_1)$ and $\Omega_2 = (\Delta_2, \Xi_2)$ be two graphs. Then the tensor product $\Omega_1 \times \Omega_2$ is a subgraph of

1. lexicographic product (composition) $\Omega_1 \circ \Omega_2$
2. modular product $\Omega_1 \bigcirc \Omega_2$
3. strong product $\Omega_1 \boxtimes \Omega_2$
4. co-normal product $\Omega_1 * \Omega_2$

Proof.

1. The vertex set of the tensor product $\Omega_1 \times \Omega_2$ is $\Delta_1 \times \Delta_2$ and there is an edge in the product joining the vertices $(\vartheta_1, \vartheta_1')$ and $(\vartheta_2, \vartheta_2')$ if and only if ϑ_1 and ϑ_2 are adjacent in Ω_1 and ϑ_1' and ϑ_2' are adjacent in Ω_2. Comparing the definitions of tensor and lexicographic (given in part (1) of Theorem 3.1) products, it follows that these two products have the same vertex set $\Delta_1 \times \Delta_2$. Also, the edge set $\Xi(\Omega_1 \times \Omega_2)$ is a subset of the edge set $\Xi(\Omega_1 \circ \Omega_2)$. Therefore, $\Omega_1 \times \Omega_2$ is a subgraph of $\Omega_1 \circ \Omega_2$.
2. The vertex set of the modular product $\Omega_1 \bigcirc \Omega_2$ is $\Delta_1 \times \Delta_2$ and there is an edge in the product joining the vertices $(\vartheta_1, \vartheta_1')$ and $(\vartheta_2, \vartheta_2')$ if and only if

(a) ϑ_1 and ϑ_2 are adjacent in Ω_1 and ϑ_1' and ϑ_2' are adjacent in Ω_2 or
(b) ϑ_1 and ϑ_2 are not adjacent in Ω_1 and ϑ_1' and ϑ_2' are not adjacent in Ω_2.

Comparing the definitions of tensor and modular products, it follows that these two products have the same vertex set $\Delta_1 \times \Delta_2$. Also, the edge set $\Xi(\Omega_1 \times \Omega_2)$ is a subset of the edge set $\Xi(\Omega_1 \bigcirc \Omega_2)$. Therefore, $\Omega_1 \times \Omega_2$ is a subgraph of $\Omega_1 \bigcirc \Omega_2$.

3. The tensor and strong products have the same vertex set $\Delta_1 \times \Delta_2$. Also, the edge set $\Xi(\Omega_1 \times \Omega_2)$ is a subset of the edge set $\Xi(\Omega_1 \boxtimes \Omega_2)$. Therefore, $\Omega_1 \times \Omega_2$ is a subgraph of $\Omega_1 \boxtimes \Omega_2$.
4. The tensor and co-normal products have the same vertex set $\Delta_1 \times \Delta_2$. Also, the edge set $\Xi(\Omega_1 \times \Omega_2)$ is a subset of the edge set $\Xi(\Omega_1 * \Omega_2)$. Therefore, $\Omega_1 \times \Omega_2$ is a subgraph of $\Omega_1 * \Omega_2$.

□

Theorem 3.4. Let $\Omega_1 = (\Delta_1, \Xi_1)$ and $\Omega_2 = (\Delta_2, \Xi_2)$ be two graphs. Then the strong product $\Omega_1 \boxtimes \Omega_2$ is a subgraph of

1. the lexicographic product (composition) $\Omega_1 \circ \Omega_2$
2. the co-normal product $\Omega_1 * \Omega_2$

Proof. 1. Comparing the definitions of lexicographic and strong products, it follows that these two products have the same vertex set $\Delta_1 \times \Delta_2$. Also, the edge set $\Xi(\Omega_1 \boxtimes \Omega_2)$ is a subset of the edge set $\Xi(\Omega_1 \circ \Omega_2)$. Therefore, $\Omega_1 \boxtimes \Omega_2$ is a subgraph of $\Omega_1 \circ \Omega_2$.
2. The co-normal and strong products have the same vertex set $\Delta_1 \times \Delta_2$. Also, the edge set $\Xi(\Omega_1 \boxtimes \Omega_2)$ is a subset of the edge set $\Xi(\Omega_1 * \Omega_2)$. Therefore, $\Omega_1 \boxtimes \Omega_2$ is a subgraph of $\Omega_1 * \Omega_2$.

□

Theorem 3.5. Let $\Omega_1 = (\Delta_1, \Xi_1)$ and $\Omega_2 = (\Delta_2, \Xi_2)$ be two graphs. Then the rooted product $\Omega_1 \circ_{\vartheta_r} \Omega_2$ (where ϑ_r is the root vertex of Ω_2) is a subgraph of

1. the Cartesian product $\Omega_1 \square \Omega_2$
2. the lexicographic product (composition) $\Omega_1 \circ \Omega_2$
3. the co-normal product $\Omega_1 * \Omega_2$
4. the strong product $\Omega_1 \boxtimes \Omega_2$

Proof. 1. Comparing the definitions of rooted and Cartesian products, it follows that these two products have the same vertex set $\Delta_1 \times \Delta_2$. Also, the edge set $\Xi(\Omega_1 \circ_{\vartheta_r} \Omega_2)$ is a subset of the edge set $\Xi(\Omega_1 \square \Omega_2)$. Therefore, $\Omega_1 \circ_{\vartheta_r} \Omega_2$ is a subgraph of $\Omega_1 \square \Omega_2$.
2. The rooted and lexicographic products have the same vertex set $\Delta_1 \times \Delta_2$. Also, the edge set $\Xi(\Omega_1 \circ_{\vartheta_r} \Omega_2)$ is a subset of the edge set $\Xi(\Omega_1 \circ \Omega_2)$. Therefore, $\Omega_1 \circ_{\vartheta_r} \Omega_2$ is a subgraph of $\Omega_1 \circ \Omega_2$.
3. The rooted and co-normal products have the same vertex set $\Delta_1 \times \Delta_2$. Also, the edge set $\Xi(\Omega_1 \circ_{\vartheta_r} \Omega_2)$ is a subset of the edge set $\Xi(\Omega_1 * \Omega_2)$. Therefore, $\Omega_1 \circ_{\vartheta_r} \Omega_2$ is a subgraph of $\Omega_1 * \Omega_2$.
4. The rooted and strong products have the same vertex set $\Delta_1 \times \Delta_2$. Also, the edge set $\Xi(\Omega_1 \circ_{\vartheta_r} \Omega_2)$ is a subset of the edge set $\Xi(\Omega_1 \boxtimes \Omega_2)$. Therefore, $\Omega_1 \circ_{\vartheta_r} \Omega_2$ is a subgraph of $\Omega_1 \boxtimes \Omega_2$.

□

Example 1. Let $\Omega_1 = (\Delta_1, \Xi_1)$ be a graph with vertex set $\Delta_1 = \{v_1, v_2, v_3\}$ and edge set $\Xi_1 = \{v_1v_2, v_1v_3, v_2v_3\}$, as shown in Figure 1. Let $\Omega_2 = (\Delta_2, \Xi_2)$ be a graph with vertex set $\Delta_2 = \{u_1, u_2\}$ and edge set $\Xi_2 = \{u_1u_2\}$, as shown in Figure 2. Different products of the graphs Ω_1 and Ω_2 can then be formed to analyse the subgraph relationships among them. The Cartesian product $\Omega_1 \square \Omega_2$ is shown in Figure 3. The lexicographic product $\Omega_1 \circ \Omega_2$ is shown in Figure 4. The tensor product $\Omega_1 \times \Omega_2$ is shown in Figure 5. The strong product $\Omega_1 \boxtimes \Omega_2$ is shown in Figure 6. The co-normal product $\Omega_1 * \Omega_2$ is shown in Figure 7. The homomorphic product $\Omega_1 \ltimes \Omega_2$ is shown in Figure 8. The modular product $\Omega_1 \bigcirc \Omega_2$ is shown in Figure 9. The rooted product $\Omega_1 \circ_{u_1} \Omega_2$ of the graphs Ω_1 and Ω_2, where u_1 is the root vertex of Ω_2, is shown in Figure 10.

From these figures, we can identify that the Cartesian product of two graphs is a subgraph of the lexicographic, co-normal, and strong products. The lexicographic product is a subgraph of the co-normal product. The tensor product is a subgraph of the lexicographic, modular, strong, and co-normal products. The strong product is a subgraph of the lexicographic and co-normal products. Also, the rooted product is a subgraph of the Cartesian, lexicographic, co-normal, and strong products.

4. Conclusion

In this paper, we studied subgraph relations among different graph products. By comparing the adjacency conditions of various products, several inclusion relations were obtained. It was shown that the Cartesian product is a subgraph of the lexicographic, co-normal, and strong products. The lexicographic product was also proved to be a subgraph of the co-normal product. Furthermore, the tensor product was shown to be a subgraph of the lexicographic, modular, strong, and co-normal products. Similar relations involving strong and rooted products were also established. The results presented in this work provide a better understanding of the structural connections among graph products. The examples included in the paper help to visualise these relationships clearly. The study may be extended by considering subgraph relations for other graph products or by investigating similar properties in weighted graphs, fuzzy graphs, soft graphs, and other generalized graph structures.

Data availability

We do not have any research data outside the submitted manuscript file.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.

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