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A new ranked set sampling design for estimating population mean and variance based on neoteric ranked set sampling

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Abstract

This paper proposes a new ranked set sampling (RSS) design, termed modified neoteric RSS (MNRSS), for efficient estimation of the population mean and variance. The proposed method extends the neoteric RSS (NRSS) framework by selecting sampled units in a more dispersed manner while avoiding extreme ranks, thereby improving population representation and estimation accuracy, particularly for the population variance. The theoretical properties of the proposed mean estimator are examined. Under perfect ranking, the estimator is shown to be unbiased when the underlying distribution is symmetric; for asymmetric distributions, it exhibits only a small bias. The performance of MNRSS was evaluated through an extensive simulation study involving several symmetric and asymmetric distributions. The proposed design was compared with simple random sampling (SRS), RSS, median RSS (MRSS), extreme RSS (ERSS), NRSS, systematic RSS (SRSS), and centralized RSS (CRSS). Performance was assessed using bias and mean square error (MSE). The results show that MNRSS generally outperforms the competing methods, particularly for estimating population variance and under asymmetric distributions. An application to Scots pine tree measurements further supports the practical usefulness of the proposed sampling design.

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1. Introduction

Sampling is essential for reliable statistical inference and is widely used when studying an entire population is impractical [1]. Ranked set sampling (RSS), originally proposed by McIntyre [2], is particularly useful when measuring the variable of interest is difficult or expensive but ranking units is efficient and inexpensive. Ranking may be based on expert judgment or a concomitant variable that is highly correlated with the variable of interest. High efficiency relative to simple random sampling (SRS) has been reported for the estimation of several population parameters [3,4]. Further details on RSS theory are provided by Takahasi and Wakimoto [5] and Stokes [6].

The RSS procedure can be described as follows:

- Randomly identify m^2 units from a continuous population with finite variance and divide them into m sets, each

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of size m.

- Rank the units in each set in increasing order using an inexpensive method or a concomitant variable.
- Select the ith-ranked unit from the ith set, where $i = 1, 2, \ldots, m$, for actual measurement.
- Repeat the preceding steps for r cycles to obtain a ranked set sample of size $n = mr$.

Several sampling designs based on the original RSS procedure have been proposed to improve statistical efficiency and reduce operational cost relative to SRS. These developments have produced numerous RSS-based estimators of population parameters, particularly the population mean and related finite-population characteristics.

Among the early developments, Muttlak [7,8] introduced pair RSS (PRSS) and median RSS (MRSS), respectively. Samawi *et al.* [9] proposed extreme RSS (ERSS), which uses extreme order statistics to improve estimation efficiency. Muttlak [10] subsequently introduced percentile RSS for estimating the population mean.

Further extensions and improvements to RSS have been proposed for different sampling conditions. Notable contributions include those of Al-Saleh and Al-Omari [11], Al-Nasser [12], Al-Omari and Al-Saleh [13], Bani-Mustafa *et al.* [14], Hanandeh *et al.* [15], and Taconeli [16].

More recent generalizations have improved the efficiency and flexibility of RSS. Zamanzade and Al-Omari [17] proposed neoteric RSS (NRSS). Inspired by this development, Khan *et al.* [18,19] introduced centralized RSS (CRSS) and two versions of systematic RSS (SRSS). Taconeli and Cabral [20] and Obeidat *et al.* [21] proposed two-stage sampling designs based on NRSS.

Despite these developments, more efficient designs are needed for accurate estimation of the population variance. Existing methods may select units that are too closely spaced or may rely heavily on extreme ranks, which can adversely affect variance estimation. The modified neoteric RSS (MNRSS) design addresses this limitation by selecting more dispersed units while strictly avoiding extreme ranks. This strategy is intended to improve population representation and the estimation of population variance. The proposed method is compared with SRS, RSS, MRSS, ERSS, NRSS, CRSS, and SRSS.

The remainder of this article is organized as follows. Section 2 reviews existing NRSS variations. Section 3 presents the proposed method. Section 4 describes the simulation study, and Section 5 presents the real-data application. Section 6 concludes the paper.

2. Existing neoteric ranked set sampling variations

This section reviews NRSS and related variations; further details are available in Chen *et al.* [22] and the references therein.

Table 1: Formulas for selecting the rank index k under the NRSS technique based on the parity of the set size (m) and set index (i).

Type of i	Odd m	Even m
Odd i	$(i-1)m + (m+1)/2$	$(i-1)m + (m+2)/2$
Even i	$(i-1)m + (m+1)/2$	$(i-1)m + m/2$

2.1. NRSS

NRSS was proposed by Zamanzade and Al-Omari [17]. Unlike classical RSS, NRSS does not divide the m^2 units into m sets of size m; instead, all m^2 units are ranked as a single set. The procedure is as follows:

- Randomly identify m^2 units from the population.
- Rank all m^2 units in increasing order using an inexpensive method or a concomitant variable.
- Select the units with ranks $(i-1)m+k$, where $i = 1, \ldots, m$. If m is odd, $k = (m+1)/2$. If m is even, $k = (m+2)/2$ for odd i and $k = m/2$ for even i, as summarized in Table 1.
- Repeat the preceding steps for r cycles to obtain an NRSS sample of size $n = mr$.

2.2. SRSS and CRSS

Two versions of SRSS were proposed by Khan *et al.* [18,19]. The first version, denoted by SRSS1, is described as follows [18]:

- Randomly identify m^2 units from the population.
- Rank the m^2 units in increasing order using an inexpensive method or a concomitant variable.
- Select the units with ranks $(m-1)i + m$, where $i = 0, \ldots, m-1$, for actual measurement.
- Repeat the preceding steps for r cycles to obtain an SRSS sample of size $n = mr$.

In the second version, denoted by SRSS2, the units selected in the third step have ranks $(m+1)i + 1$, where $i = 0, \ldots, m-1$.

CRSS was proposed in Ref. [18]. It differs from the preceding versions in the third step, where units with ranks

$$\frac{m^2 - m + 2i}{2}, \qquad i = 1, \ldots, m,$$

are selected for actual measurement. Tables 2 and 3 illustrate the selected units under these variations.

Table 2: The NRSS, SRSS1, SRSS2, and CRSS schemes for $m = 4$.

Scheme	$Y_{(1)}$	$Y_{(2)}$	$Y_{(3)}$	$Y_{(4)}$	$Y_{(5)}$	$Y_{(6)}$	$Y_{(7)}$	$Y_{(8)}$	$Y_{(9)}$	$Y_{(10)}$	$Y_{(11)}$	$Y_{(12)}$	$Y_{(13)}$	$Y_{(14)}$	$Y_{(15)}$	$Y_{(16)}$
NRSS			$\boxed{Y_{(3)}}$			$\boxed{Y_{(6)}}$					$\boxed{Y_{(11)}}$			$\boxed{Y_{(14)}}$		
SRSS1				$\boxed{Y_{(4)}}$			$\boxed{Y_{(7)}}$			$\boxed{Y_{(10)}}$			$\boxed{Y_{(13)}}$			
SRSS2	$\boxed{Y_{(1)}}$					$\boxed{Y_{(6)}}$					$\boxed{Y_{(11)}}$					$\boxed{Y_{(16)}}$
CRSS							$\boxed{Y_{(7)}}$	$\boxed{Y_{(8)}}$	$\boxed{Y_{(9)}}$	$\boxed{Y_{(10)}}$						

Table 3: The NRSS, SRSS1, SRSS2, and CRSS schemes for $m = 5$.

Scheme	$Y_{(1)}$	$Y_{(2)}$	$Y_{(3)}$	$Y_{(4)}$	$Y_{(5)}$	$Y_{(6)}$	$Y_{(7)}$	$Y_{(8)}$	$Y_{(9)}$	$Y_{(10)}$	$Y_{(11)}$	$Y_{(12)}$	$Y_{(13)}$	$Y_{(14)}$	$Y_{(15)}$	$Y_{(16)}$	$Y_{(17)}$	$Y_{(18)}$	$Y_{(19)}$	$Y_{(20)}$	$Y_{(21)}$	$Y_{(22)}$	$Y_{(23)}$	$Y_{(24)}$	$Y_{(25)}$
NRSS			$\boxed{Y_{(3)}}$					$\boxed{Y_{(8)}}$					$\boxed{Y_{(13)}}$					$\boxed{Y_{(18)}}$					$\boxed{Y_{(23)}}$		
SRSS1					$\boxed{Y_{(5)}}$				$\boxed{Y_{(9)}}$				$\boxed{Y_{(13)}}$				$\boxed{Y_{(17)}}$				$\boxed{Y_{(21)}}$				
SRSS2	$\boxed{Y_{(1)}}$						$\boxed{Y_{(7)}}$						$\boxed{Y_{(13)}}$						$\boxed{Y_{(19)}}$						$\boxed{Y_{(25)}}$
CRSS											$\boxed{Y_{(11)}}$	$\boxed{Y_{(12)}}$	$\boxed{Y_{(13)}}$	$\boxed{Y_{(14)}}$	$\boxed{Y_{(15)}}$										

Table 4: MSE of the mean estimators under symmetric and asymmetric distributions for $m = 5$.

Distribution	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
Uniform(0,1)	0.0027	0.0009	0.0012	0.0008	0.0006	0.0004	0.0008	0.0014	0.0006
Normal(0,1)	0.0331	0.0121	0.0095	0.014	0.007	0.0082	0.0072	0.0092	0.0072
Logistic	0.1079	0.0425	0.0265	0.0544	0.0223	0.0386	0.0209	0.024	0.025
Student's t	0.0677	0.03	0.0124	0.0455	0.0121	0.047	0.0101	0.0109	0.0156
Beta	0.0012	0.0004	0.0004	0.0004	0.0003	0.0002	0.0003	0.0004	0.0002
Skewed Beta	0.0042	0.0016	0.0079	0.003	0.0009	0.0015	0.0027	0.0188	0.0009
Half-Normal	0.0483	0.0183	0.0439	0.0354	0.0115	0.0477	0.0263	0.0647	0.0111
Gamma	0.1969	0.0854	0.2488	0.2129	0.0518	0.3549	0.148	0.3791	0.051
Lognormal	0.1553	0.0997	0.2602	0.3034	0.0493	0.7487	0.1689	0.3787	0.0423
Weibull	0.2952	0.1372	0.4857	0.396	0.0822	0.7031	0.2833	0.7638	0.0804

3. The proposed method

The proposed NRSS variation, termed modified neoteric RSS (MNRSS), is designed to produce a more dispersed sample than NRSS and CRSS while avoiding the extreme ranks used by the two SRSS versions. The objective is to obtain a representative sample and improve estimation of the population variance. To avoid extreme ranks strictly, the set size must satisfy $m \geq 4$.

An MNRSS sample of size $n = mr$ is obtained as follows:

1. Select m^2 units at random from the population of interest.
2. Rank the selected units in increasing order by visual judgment or another inexpensive method.
3. Select the unit with rank $l + m(i-1)$ for actual measurement, where $i = 1, \ldots, m$. The value of l depends on the parity of m and i.
   For even m,
   $$l = \begin{cases} m/2, & i \text{ odd}, \\ m/2+1, & i \text{ even}. \end{cases}$$
   For odd m, with $\lfloor x \rfloor$ denoting the integer part of x,
   $$l = \begin{cases} \lfloor m/2 \rfloor, & \begin{array}{l} i \text{ odd and } i < \lfloor m/2 \rfloor + 1, \\ \text{or } i \text{ even and } i > \lfloor m/2 \rfloor + 1, \end{array} \\ \lfloor m/2 \rfloor + 1, & i = \lfloor m/2 \rfloor + 1, \\ \lfloor m/2 \rfloor + 2, & \begin{array}{l} i \text{ even and } i < \lfloor m/2 \rfloor + 1, \\ \text{or } i \text{ odd and } i > \lfloor m/2 \rfloor + 1. \end{array} \end{cases}$$
4. Repeat Steps 1–3 for r cycles to obtain an MNRSS sample of size $n = mr$.

The procedure is illustrated by the following examples.

Example 1. Let $m = 4$ and $r = 1$, and let $\{Y_{1.1}, Y_{2.1}, \ldots, Y_{16.1}\}$ denote the units identified from the population. After ranking, the ordered units are $\{Y_{(1).1}, Y_{(2).1}, \ldots, Y_{(16).1}\}$. MNRSS selects

$$\{Y_{(2).1}, Y_{(7).1}, Y_{(10).1}, Y_{(15).1}\}$$

for actual measurement.

Example 2. Let $m = 5$ and $r = 1$, and let $\{Y_{1.1}, Y_{2.1}, \ldots, Y_{25.1}\}$ denote the units identified from the population. After ranking, MNRSS selects

$$\{Y_{(2).1}, Y_{(9).1}, Y_{(13).1}, Y_{(17).1}, Y_{(24).1}\}$$

for actual measurement.

3.1. Estimation of the population mean

Under MNRSS, the proposed estimator of the population mean μ is

$$\widehat{\mu}_{\mathrm{MNRSS}} = \frac{1}{mr} \sum_{j=1}^{r} \sum_{i=1}^{m} Y_{(l+m(i-1))j}. \tag{1}$$

Table 5: MSE of the mean estimators under symmetric and asymmetric distributions for $m = 6$.

Distribution	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
Uniform(0,1)	0.0023	0.0007	0.0009	0.0004	0.0004	0.0003	0.0005	0.001	0.0004
Normal(0,1)	0.0276	0.0088	0.0068	0.0117	0.0049	0.0058	0.005	0.0065	0.0049
Logistic	0.0904	0.0312	0.019	0.0514	0.0152	0.0277	0.0144	0.0168	0.016
Student's t	0.0556	0.0228	0.0088	0.0467	0.0079	0.0374	0.0069	0.0075	0.0089
Beta	0.001	0.0003	0.0003	0.0003	0.0002	0.0002	0.0002	0.0003	0.0002
Skewed Beta	0.0036	0.0012	0.0073	0.0082	0.0006	0.001	0.0019	0.0206	0.0006
Half-Normal	0.0405	0.0135	0.0395	0.0768	0.009	0.0421	0.0206	0.0635	0.0072
Gamma	0.1658	0.064	0.2352	0.5221	0.0446	0.3245	0.121	0.3856	0.0314
Lognormal	0.1289	0.0763	0.2561	0.7458	0.0524	0.7522	0.1484	0.3916	0.0271
Weibull	0.2495	0.1017	0.4677	1.046	0.075	0.6466	0.2345	0.7883	0.0479

Table 6: MSE of the mean estimators under symmetric and asymmetric distributions for $m = 7$.

Distribution	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
Uniform(0,1)	0.002	0.0005	0.0007	0.0003	0.0003	0.0002	0.0004	0.0007	0.0003
Normal(0,1)	0.0236	0.0066	0.005	0.0085	0.0035	0.0042	0.0036	0.0049	0.0036
Logistic	0.0774	0.0238	0.0134	0.0379	0.0113	0.0207	0.0105	0.0125	0.0119
Student's t	0.0477	0.0179	0.0062	0.0364	0.0061	0.031	0.0052	0.0056	0.0069
Beta	0.0009	0.0002	0.0002	0.0002	0.0001	0.0001	0.0001	0.0002	0.0001
Skewed Beta	0.003	0.0009	0.0097	0.0058	0.0004	0.0008	0.0013	0.0218	0.0004
Half-Normal	0.0346	0.0103	0.0444	0.0618	0.0059	0.0371	0.0167	0.0631	0.0052
Gamma	0.1417	0.0496	0.2723	0.4299	0.0274	0.2972	0.1009	0.3897	0.0223
Lognormal	0.1115	0.0606	0.2964	0.6324	0.0313	0.7431	0.1309	0.3986	0.0174
Weibull	0.2117	0.0797	0.5519	0.8605	0.0444	0.5949	0.1977	0.803	0.0336

Table 7: MSE of the mean estimators under symmetric and asymmetric distributions for $m = 8$.

Distribution	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
Uniform(0,1)	0.0017	0.0004	0.0005	0.0002	0.0002	0.0002	0.0003	0.0006	0.0002
Normal(0,1)	0.0206	0.0052	0.0039	0.0076	0.0027	0.0032	0.0027	0.0038	0.0027
Logistic	0.0685	0.0192	0.0105	0.0371	0.0086	0.0161	0.0081	0.0097	0.0088
Student's t	0.041	0.0144	0.0049	0.0384	0.0046	0.0263	0.004	0.0043	0.0049
Beta	0.0007	0.0002	0.0002	0.0002	0.0001	0.0001	0.0001	0.0002	0.0001
Skewed Beta	0.0027	0.0007	0.0094	0.0124	0.0003	0.0006	0.001	0.0226	0.0003
Half-Normal	0.0299	0.008	0.0425	0.1208	0.0049	0.033	0.0138	0.0624	0.0041
Gamma	0.1252	0.0387	0.2669	0.8481	0.0247	0.271	0.0859	0.3919	0.0182
Lognormal	0.0962	0.0528	0.2949	1.2484	0.0333	0.7319	0.1169	0.4038	0.019
Weibull	0.188	0.0634	0.5451	1.7423	0.042	0.5458	0.169	0.8117	0.0286

For even m, the rank offset is

$$l = \begin{cases} \lfloor m/2 \rfloor, & i \text{ odd}, \\ \lfloor m/2 \rfloor + 1, & i \text{ even}. \end{cases} \tag{2}$$

For odd m, it is

$$l = \begin{cases} \lfloor m/2 \rfloor, & i \text{ odd and } i < \lfloor m/2 \rfloor + 1, \\ & \text{or } i \text{ even and } i > \lfloor m/2 \rfloor + 1, \\ \lfloor m/2 \rfloor + 1, & i = \lfloor m/2 \rfloor + 1, \\ \lfloor m/2 \rfloor + 2, & i \text{ even and } i < \lfloor m/2 \rfloor + 1, \\ & \text{or } i \text{ odd and } i > \lfloor m/2 \rfloor + 1. \end{cases} \tag{3}$$

Because the r cycles are independent and identically distributed,

$$\mathrm{Var}(\widehat{\mu}_{\mathrm{MNRSS}}) = \frac{1}{m^2 r} \mathrm{Var}\left(\sum_{i=1}^{m} Y_{(l+m(i-1))} \right). \tag{4}$$

Within a cycle, however, the selected order statistics are dependent. Therefore,

$$\mathrm{Var}\left(\sum_{i=1}^{m} Y_{(l+m(i-1))} \right) = \sum_{i=1}^{m} \mathrm{Var}(Y_{(l+m(i-1))})$$

Table 8: Bias of the mean estimators under symmetric and asymmetric distributions for $m = 5$.

Distribution	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
Uniform(0,1)	-0.0004	-0.0001	-0.0001	$< 10^{-4}$	-0.0001	$< 10^{-4}$	-0.0001	-0.0001	$< 10^{-4}$
Normal(0,1)	-0.0005	-0.0011	-0.0006	-0.0009	-0.0004	-0.0005	-0.0002	-0.0002	-0.0003
Logistic	-0.0021	-0.0006	-0.0007	$< 10^{-4}$	-0.0004	0.0006	-0.0004	-0.0004	-0.0002
Student's t	0.0002	-0.0004	0.0002	0.0006	$< 10^{-4}$	0.0021	0.0003	0.0004	-0.0002
Beta	-0.0001	$< 10^{-4}$	-0.0002	0.0001	-0.0001	-0.0001	-0.0001	$< 10^{-4}$	$< 10^{-4}$
Skewed Beta	-0.0002	0.0005	0.0763	-0.041	0.0011	-0.0289	0.0379	0.1281	0.0028
Half-Normal	0.0001	-0.0001	-0.172	0.1178	-0.0354	0.1876	-0.1254	-0.225	0.0243
Gamma	$< 10^{-4}$	-0.0004	-0.4488	0.3115	-0.1026	0.5267	-0.333	-0.5803	0.0611
Lognormal	-0.0007	-0.0004	-0.4953	0.3628	-0.1634	0.7365	-0.3945	-0.607	0.0435
Weibull	0.0036	-0.0006	-0.6491	0.4495	-0.1477	0.7528	-0.4815	-0.8418	0.0863

Table 9: Bias of the mean estimators under symmetric and asymmetric distributions for $m = 6$.

Distribution	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
Uniform(0,1)	-0.0004	$< 10^{-4}$	0.0001	0.0002	$< 10^{-4}$	$< 10^{-4}$	$< 10^{-4}$	$< 10^{-4}$	$< 10^{-4}$
Normal(0,1)	-0.001	-0.0001	-0.0006	-0.0007	-0.0006	-0.0006	-0.0005	-0.0004	-0.0007
Logistic	-0.002	0.0009	0.0001	0.0019	$< 10^{-4}$	0.0003	-0.0001	-0.0001	-0.0002
Student's t	0.0006	0.0009	0.0003	0.0007	0.0002	0.0007	0.0001	$< 10^{-4}$	0.0001
Beta	-0.0002	-0.0001	-0.0002	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001
Skewed Beta	-0.0002	0.0002	0.0763	-0.0871	0.0012	-0.0243	0.0317	0.1374	0.0001
Half-Normal	-0.0007	0.0009	-0.1711	0.2422	-0.0447	0.1828	-0.1153	-0.2313	-0.0128
Gamma	0.0002	0.0023	-0.4482	0.6418	-0.1286	0.5183	-0.3086	-0.5965	-0.0419
Lognormal	-0.002	-0.0007	-0.4952	0.7434	-0.1951	0.7625	-0.3726	-0.6201	-0.0909
Weibull	0.0042	0.003	-0.6492	0.9267	-0.1849	0.7407	-0.4458	-0.866	-0.0607

Table 10: Bias of the mean estimators under symmetric and asymmetric distributions for $m = 7$.

Distribution	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
Uniform(0,1)	-0.0003	$< 10^{-4}$	-0.0001	0.0001	$< 10^{-4}$	$< 10^{-4}$	$< 10^{-4}$	$< 10^{-4}$	$< 10^{-4}$
Normal(0,1)	-0.001	-0.0009	-0.0002	-0.0003	-0.0001	0.0001	-0.0001	0.0001	$< 10^{-4}$
Logistic	-0.0021	0.0001	-0.0003	0.0009	-0.0001	$< 10^{-4}$	-0.0001	-0.0001	-0.0001
Student's t	0.0003	0.0002	0.0002	0.0014	0.0005	0.0015	0.0002	0.0003	0.0005
Beta	-0.0001	-0.0001	$< 10^{-4}$	$< 10^{-4}$	$< 10^{-4}$	$< 10^{-4}$	$< 10^{-4}$	$< 10^{-4}$	$< 10^{-4}$
Skewed Beta	-0.0001	0.0003	0.0922	-0.0724	0.0004	-0.0211	0.027	0.1431	-0.0011
Half-Normal	-0.0003	-0.0008	-0.1917	0.2196	-0.0272	0.1751	-0.1069	-0.2357	0.0024
Gamma	-0.0006	0.0014	-0.498	0.5876	-0.0813	0.5049	-0.2865	-0.6062	0.0008
Lognormal	-0.0014	-0.0037	-0.538	0.6917	-0.139	0.773	-0.3517	-0.6272	-0.034
Weibull	0.0036	-0.0003	-0.7211	0.8448	-0.1171	0.7217	-0.4141	-0.8802	0.0005

$$+\,2\sum_{1\le i<j\le m}\operatorname{Cov}\Big(Y_{(l+m(i-1))},Y_{(l+m(j-1))}\Big). \tag{5}$$

Combining Eqs. (4) and (5) gives

$$\operatorname{Var}(\widehat{\mu}_{\text{MNRSS}}) = \frac{1}{m^2 r}\Bigg[\sum_{i=1}^{m}\operatorname{Var}(Y_{(l+m(i-1))}) + 2\sum_{1\le i<j\le m}\operatorname{Cov}\Big(Y_{(l+m(i-1))},Y_{(l+m(j-1))}\Big)\Bigg]. \tag{6}$$

Theorem 1. Under perfect ranking, $\widehat{\mu}_{\text{MNRSS}}$ is an unbiased estimator of the population mean μ when the population distribution is symmetric.

Proof. For even m, set $r = 1$ without loss of generality. Then

$$\widehat{\mu}_{\text{MNRSS}} = \frac{1}{m}\sum_{i=1}^{m} Y_{(l_i)}, \qquad l_i = l + m(i-1). \tag{7}$$

Separate the sum into odd and even indices by setting $i = 2j-1$

Table 11: Bias of the mean estimators under symmetric and asymmetric distributions for $m = 8$.

Distribution	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
Uniform(0,1)	-0.0003	-0.0001	-0.0002	$< 10^{-4}$	-0.0001	-0.0001	-0.0001	-0.0002	-0.0001
Normal(0,1)	-0.0012	0.0003	-0.0002	0.0007	-0.0001	0.0002	-0.0001	$< 10^{-4}$	-0.0001
Logistic	-0.0021	-0.0008	-0.001	-0.0009	-0.0004	-0.0005	-0.0005	-0.0007	-0.0007
Student's t	0.0001	0.001	0.0002	0.0016	0.0005	0.0016	0.0004	0.0005	0.0006
Beta	-0.0002	$< 10^{-4}$	$< 10^{-4}$	0.0001	-0.0001	-0.0001	-0.0001	$< 10^{-4}$	-0.0001
Skewed Beta	-0.0001	0.0003	0.0921	-0.1103	0.0004	-0.0186	0.0234	0.147	0.0002
Half-Normal	-0.001	0.0008	-0.1914	0.3295	-0.0324	0.1676	-0.0991	-0.238	-0.0144
Gamma	-0.0003	0.0015	-0.4982	0.8759	-0.0958	0.4882	-0.2677	-0.6124	-0.0463
Lognormal	-0.0028	0.0022	-0.538	1.0388	-0.1573	0.7782	-0.3335	-0.6323	-0.0941
Weibull	0.0033	-0.0012	-0.7211	1.2646	-0.1372	0.6989	-0.3856	-0.8887	-0.0661

Table 12: MSE of the variance estimators under symmetric and asymmetric distributions for $m = 5$.

Distribution	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
Uniform(0,1)	0.0002	0.0001	0.0023	0.001	0.0001	0.0008	0.0009	0.0052	$< 10^{-4}$
Normal(0,1)	0.0689	0.0525	0.5147	0.3351	0.0426	0.7889	0.2953	0.8566	0.0292
Logistic	1.1861	0.9694	6.3042	5.1231	0.8167	15.1229	4.0491	9.5861	0.4141
Student's t	2.3856	1.8079	2.6685	5.9571	0.6187	21.7915	1.9419	3.6573	0.2006
Beta	0.0001	$< 10^{-4}$	0.0006	0.0003	$< 10^{-4}$	0.0005	0.0003	0.0011	$< 10^{-4}$
Skewed Beta	0.0005	0.0003	0.0044	0.0013	0.0001	0.0007	0.0014	0.0116	0.0001
Half-Normal	0.2069	0.1606	1.0503	0.7142	0.096	1.5369	0.5913	1.7896	0.0795
Gamma	7.1237	6.1701	21.2242	20.4543	3.2854	55.4032	13.8854	31.9785	2.3258
Lognormal	82.57	102.61	17.9333	160.264	8.1575	423.035	15.4352	21.0329	5.1186
Weibull	21.5306	18.5433	50.5813	56.1659	9.4815	157.899	34.8929	73.1517	6.5507

Table 13: MSE of the variance estimators under symmetric and asymmetric distributions for $m = 6$.

Distribution	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
Uniform(0,1)	0.0002	0.0001	0.0023	0.0037	0.0001	0.0006	0.0007	0.0057	$< 10^{-4}$
Normal(0,1)	0.0573	0.0403	0.5137	1.1408	0.0537	0.7582	0.2527	0.8991	0.016
Logistic	0.9887	0.7494	6.2942	16.066	1.0327	15.4386	3.5764	9.9624	0.3321
Student's t	1.8805	1.676	2.6641	11.9593	0.7534	22.0153	1.7827	3.762	0.3268
Beta	$< 10^{-4}$	$< 10^{-4}$	0.0006	0.0011	$< 10^{-4}$	0.0004	0.0002	0.0011	$< 10^{-4}$
Skewed Beta	0.0004	0.0002	0.0043	0.0041	0.0001	0.0005	0.001	0.0129	0.0001
Half-Normal	0.1704	0.1251	1.0464	2.1067	0.1122	1.4811	0.5038	1.8841	0.0425
Gamma	5.9562	4.9163	21.1471	50.5119	3.9356	56.7828	12.3356	33.1953	1.5613
Lognormal	67.274	55.5829	17.9129	195.694	9.2882	508.35	14.8118	21.2909	5.5546
Weibull	18.13	15.3348	50.519	124.968	11.324	163.816	31.4414	75.587	4.7279

and $i = 2j$, respectively, where $j = 1, 2, \ldots, m/2$:

$$\widehat{\mu}_{\text{MNRSS}} = \frac{1}{m}\Bigg[\sum_{j=1}^{m/2} Y_{(m/2+m(2j-2))} + \sum_{j=1}^{m/2} Y_{(m/2+1+m(2j-1))}\Bigg]. \tag{8}$$

Pair the jth term in the odd sequence with the $(m/2 - j + 1)$th term in the even sequence. The rank of the odd term is

$$k = \frac{m}{2} + 2mj - 2m,$$

and the rank of its matching even term is

$$k' = \frac{m}{2} + 1 + m\left[2\left(\frac{m}{2} - j + 1\right) - 1\right] = m^2 - 2mj + m + \frac{m}{2} + 1.$$

Hence,

$$k + k' = m^2 + 1.$$

For a distribution symmetric about μ,

$$E[Y_{(k)} + Y_{(N-k+1)}] = 2\mu, \qquad N = m^2.$$

Table 14: MSE of the variance estimators under symmetric and asymmetric distributions for $m = 7$.

Distribution	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
Uniform(0,1)	0.0001	0.0001	0.0031	0.0028	$< 10^{-4}$	0.0005	0.0005	0.006	$< 10^{-4}$
Normal(0,1)	0.0489	0.0321	0.6261	0.9482	0.0256	0.7121	0.218	0.9252	0.0086
Logistic	0.8368	0.6235	7.4304	13.6861	0.5489	15.4715	3.1795	10.1879	0.153
Student's t	2.2162	2.1174	3.0252	14.884	0.4896	29.0306	1.6418	3.8261	0.1673
Beta	$< 10^{-4}$	$< 10^{-4}$	0.0007	0.0009	$< 10^{-4}$	0.0004	0.0002	0.0012	$< 10^{-4}$
Skewed Beta	0.0004	0.0002	0.0062	0.0031	0.0001	0.0004	0.0007	0.0137	$< 10^{-4}$
Half-Normal	0.147	0.0991	1.285	1.7541	0.0569	1.3929	0.4335	1.9433	0.0261
Gamma	5.078	4.0293	24.9066	44.4951	2.2081	57.1404	11.0063	33.9367	0.8853
Lognormal	54.5168	43.0733	19.2039	208.6548	7.0756	512.4924	14.1967	21.4375	3.6918
Weibull	15.6513	12.469	58.5986	112.0505	6.6345	167.4335	28.4263	77.0265	2.6814

Table 15: MSE of the variance estimators under symmetric and asymmetric distributions for $m = 8$.

Distribution	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
Uniform(0,1)	0.0001	0.0001	0.0031	0.0061	$< 10^{-4}$	0.0004	0.0004	0.0062	$< 10^{-4}$
Normal(0,1)	0.0423	0.0265	0.6256	2.0366	0.0305	0.6671	0.1899	0.9425	0.012
Logistic	0.7229	0.5131	7.4279	28.9298	0.6575	15.2216	2.8443	10.3379	0.2814
Student's t	1.8235	1.4774	3.0257	25.5415	0.5644	29.0019	1.5191	3.8675	0.3126
Beta	$< 10^{-4}$	$< 10^{-4}$	0.0007	0.0019	$< 10^{-4}$	0.0003	0.0002	0.0012	$< 10^{-4}$
Skewed Beta	0.0003	0.0002	0.0061	0.0063	$< 10^{-4}$	0.0003	0.0005	0.0142	$< 10^{-4}$
Half-Normal	0.1274	0.0851	1.2824	3.5687	0.0646	1.3113	0.3763	1.9819	0.0297
Gamma	4.5435	3.396	24.9122	82.9272	2.5553	56.3096	9.9002	34.4198	1.2288
Lognormal	45.2734	62.8212	19.2056	288.7749	7.7885	512.3255	13.6285	21.5308	5.2536
Weibull	13.6601	10.2678	58.5757	198.7586	7.6399	167.7797	25.8203	77.973	3.8027

Table 16: Bias of the variance estimators under symmetric and asymmetric distributions for $m = 5$.

Distribution	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
Uniform(0,1)	0.0001	$< 10^{-4}$	-0.0476	0.0286	-0.0032	0.0283	-0.029	-0.0723	0.0037
Normal(0,1)	0.0004	0.0009	-0.7134	0.4966	-0.174	0.8494	-0.5387	-0.925	0.0892
Logistic	0.0038	0.0053	-2.5005	1.8355	-0.8136	3.6281	-2.0002	-3.095	0.241
Student's t	0.0047	-0.0006	-1.6296	1.2966	-0.7514	3.14	-1.3891	-1.912	-0.0538
Beta	$< 10^{-4}$	$< 10^{-4}$	-0.0237	0.0156	-0.0039	0.022	-0.0167	-0.0324	0.0032
Skewed Beta	0.0001	$< 10^{-4}$	-0.0643	0.0319	-0.0017	0.0257	-0.0349	-0.1073	0.0024
Half-Normal	0.0006	0.0003	-1.0173	0.6753	-0.2452	1.1462	-0.7599	-1.3368	0.1174
Gamma	0.0037	0.0097	-4.5812	3.1781	-1.5618	6.5352	-3.6945	-5.6526	0.4047
Lognormal	0.0197	0.0252	-4.2268	3.3431	-2.7461	10.05	-3.9205	-4.5859	-0.8736
Weibull	0.0119	-0.0007	-7.0735	4.9112	-2.6954	10.7311	-5.8604	-8.5501	0.5059

Thus, each pair has expectation 2μ. Because there are $m/2$ pairs,

$$E(\widehat{\mu}_{\text{MNRSS}}) = \frac{1}{m}\left(\frac{m}{2} \times 2\mu\right) = \mu.$$

The proof for odd m is given in Appendix A. □

The MNRSS estimator is next illustrated for uniform and exponential distributions, using the theory of order statistics in Balakrishnan and Cohen [23].

3.1.1. Uniform distribution

Let $Y_{(1)}, \ldots, Y_{(N)}$ be the order statistics of a sample of size N from the uniform distribution on $(0, 1)$. Their mean and variance are

$$E(Y_{(i)}) = \frac{i}{N+1}, \qquad \text{Var}(Y_{(i)}) = \frac{i(N-i+1)}{(N+1)^2(N+2)}. \tag{9}$$

For $i < j$, the covariance is

$$\text{Cov}\Big(Y_{(i)}, Y_{(j)}\Big) = \frac{i(N-j+1)}{(N+1)^2(N+2)}. \tag{10}$$

For $m = 4$ and $r = 1$, 16 units are ranked and the units with ranks 2, 7, 10, and 15 are measured. Therefore,

$$\widehat{\mu}_{\text{MNRSS}} = \frac{1}{4}\left(Y_{(2)} + Y_{(7)} + Y_{(10)} + Y_{(15)}\right), \tag{11}$$

Table 17: Bias of the variance estimators under symmetric and asymmetric distributions for $m = 6$.

Distribution	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
Uniform(0,1)	0.0001	0.0001	-0.0476	0.0596	-0.0043	0.0247	-0.0252	-0.0755	$< 10^{-4}$
Normal(0,1)	-0.0006	-0.0006	-0.7134	1.0216	-0.2144	0.8417	-0.4989	-0.948	-0.0753
Logistic	0.003	-0.0005	-2.5005	3.7461	-0.9705	3.7237	-1.8813	-3.1558	-0.4358
Student's t	0.002	0.0068	-1.6289	2.6187	-0.8515	3.4181	-1.3316	-1.9394	-0.5197
Beta	$< 10^{-4}$	$< 10^{-4}$	-0.0237	0.0322	-0.005	0.0207	-0.0151	-0.0334	-0.0011
Skewed Beta	0.0001	0.0001	-0.0642	0.0628	-0.002	0.0216	-0.0293	-0.1132	$< 10^{-4}$
Half-Normal	-0.0014	0.0003	-1.0167	1.3587	-0.3006	1.1443	-0.7023	-1.3722	-0.1093
Gamma	0.002	0.0152	-4.577	6.2863	-1.8595	6.7907	-3.4865	-5.7605	-0.8598
Lognormal	-0.0068	-0.0294	-4.226	6.4493	-3.0054	11.5639	-3.8421	-4.6141	-2.1829
Weibull	0.0161	0.0335	-7.076	9.5692	-3.1793	11.3161	-5.5691	-8.6929	-1.5707

Table 18: Bias of the variance estimators under symmetric and asymmetric distributions for $m = 7$.

Distribution	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
Uniform(0,1)	0.0001	$< 10^{-4}$	-0.0555	0.0516	-0.0017	0.0218	-0.0222	-0.0775	0.0017
Normal(0,1)	$< 10^{-4}$	-0.0011	-0.7899	0.9345	-0.1381	0.8211	-0.4636	-0.9617	-0.0056
Logistic	0.0028	-0.0001	-2.7226	3.4746	-0.6826	3.7628	-1.7748	-3.1916	-0.1473
Student's t	0.0018	0.0089	-1.7381	2.4793	-0.6786	3.6508	-1.2783	-1.9559	-0.3291
Beta	$< 10^{-4}$	$< 10^{-4}$	-0.0267	0.0289	-0.0028	0.0194	-0.0138	-0.034	0.0007
Skewed Beta	0.0001	-0.0002	-0.0776	0.0548	-0.0007	0.0187	-0.025	-0.1169	0.0005
Half-Normal	0.0001	-0.0034	-1.1311	1.2435	-0.1956	1.1211	-0.652	-1.3938	-0.0135
Gamma	$< 10^{-4}$	0.0143	-4.9829	5.9327	-1.3238	6.9274	-3.2956	-5.825	-0.3181
Lognormal	0.0076	-0.0687	-4.3806	6.3604	-2.5986	12.7326	-3.7623	-4.63	-1.6526
Weibull	0.0249	0.0003	-7.6443	9.1268	-2.3262	11.6857	-5.2989	-8.7759	-0.6825

Table 19: Bias of the variance estimators under symmetric and asymmetric distributions for $m = 8$.

Distribution	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
Uniform(0,1)	$< 10^{-4}$	$< 10^{-4}$	-0.0555	0.0778	-0.0022	0.0195	-0.0198	-0.0788	-0.0004
Normal(0,1)	-0.001	0.0001	-0.7898	1.3991	-0.1609	0.7982	-0.433	-0.9708	-0.0807
Logistic	0.0008	-0.0005	-2.7226	5.2103	-0.7751	3.7564	-1.6792	-3.2151	-0.4536
Student's t	0.0028	-0.0006	-1.7384	3.736	-0.7385	3.8344	-1.2299	-1.9665	-0.5323
Beta	$< 10^{-4}$	$< 10^{-4}$	-0.0267	0.0435	-0.0034	0.0182	-0.0127	-0.0344	-0.0013
Skewed Beta	0.0001	-0.0001	-0.0776	0.0786	-0.0007	0.0165	-0.0217	-0.1192	-0.0003
Half-Normal	-0.0015	0.0034	-1.1302	1.8329	-0.2265	1.0956	-0.6078	-1.4077	-0.1163
Gamma	0.0012	0.0084	-4.9847	8.5814	-1.4999	6.9599	-3.1272	-5.8665	-0.8926
Lognormal	-0.0082	0.0978	-4.3811	9.2645	-2.7578	13.8208	-3.6868	-4.6401	-2.205
Weibull	0.0197	-0.0309	-7.6441	13.0557	-2.6136	11.8682	-5.0525	-8.8299	-1.6181

Table 20: Descriptive statistics of tree heights (m).

N	Mean	Min	Q1	Median	Q3	Max	St. Dev
1678	9.566	1.9	6	8	11.7	28	5.289

and

$$E(\widehat{\mu}_{\text{MNRSS}}) = \frac{1}{4}\left(\frac{2}{17} + \frac{7}{17} + \frac{10}{17} + \frac{15}{17}\right) = \frac{1}{2}. \quad (12)$$

Thus, $\widehat{\mu}_{\text{MNRSS}}$ is unbiased for the population mean. Its variance is

$$\text{Var}(\widehat{\mu}_{\text{MNRSS}}) = \frac{1}{16}\left(\frac{30 + 70 + 70 + 30}{5202}\right)$$

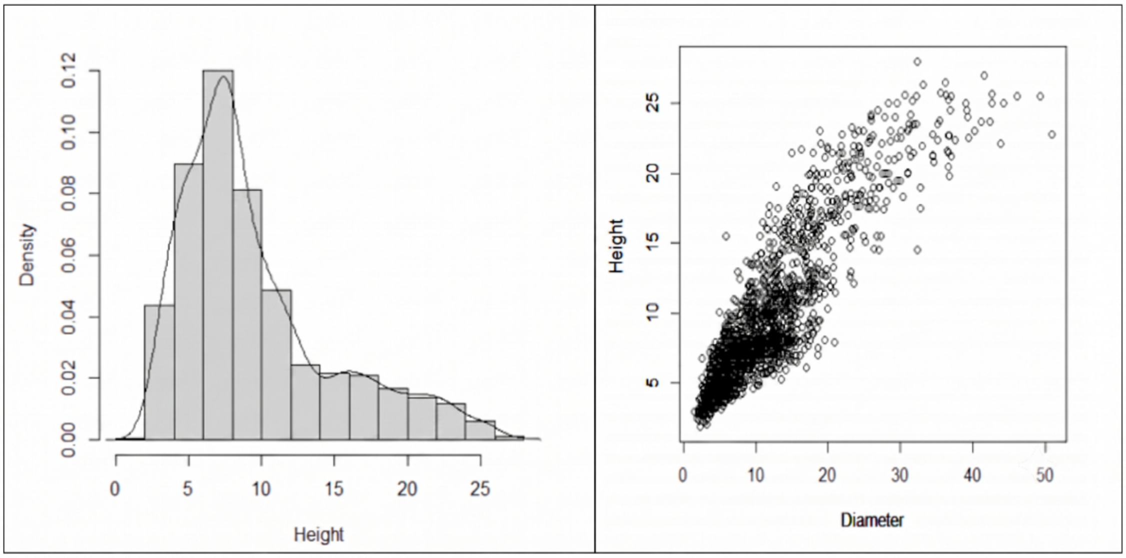

Figure 1: Histogram of tree heights (left) and scatter plot of diameter versus height (right).

Table 21: MSE of the mean estimators for different sampling methods and set sizes.

m	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
5	0.9307	0.3702	1.3055	0.8683	0.2256	0.7545	0.686	2.164	0.2021
6	0.7763	0.2742	1.2361	2.2061	0.1739	0.5851	0.5215	2.2508	0.1392
7	0.6732	0.2116	1.4895	1.7076	0.1114	0.4712	0.4085	2.3147	0.1002
8	0.5793	0.1656	1.4652	3.4927	0.0893	0.3845	0.3275	2.3597	0.0791

Table 22: Bias of the mean estimators for different sampling methods and set sizes.

m	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
5	-0.0083	-0.0021	-1.034	0.654	-0.0982	0.7608	-0.6871	-1.4205	0.1564
6	-0.0083	0.0034	-1.0307	1.3518	-0.1413	0.6817	-0.6103	-1.4685	0.00003
7	-0.0049	-0.0035	-1.1736	1.1971	-0.0623	0.6191	-0.5454	-1.4999	0.0403
8	-0.0074	-0.0061	-1.1733	1.8036	-0.0742	0.5645	-0.4933	-1.5205	-0.0227

Table 23: MSE of the variance estimators for different sampling methods and set sizes.

m	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
5	72.51	52.2607	452.2281	252.0523	32.25	430.5022	257.4613	716.1268	41.9434
6	60.5477	39.1606	448.6249	766.7556	36.9778	368.0664	214.8494	739.9931	13.4396
7	51.9205	31.1718	547.8939	645.6065	15.3516	315.0578	179.8022	752.8159	10.3186
8	44.9497	25.5085	547.7359	1298.4888	16.4732	268.9848	152.4039	760.9211	7.5396

Table 24: Bias of the variance estimators for different sampling methods and set sizes.

m	SRS	RSS	MRSS	ERSS	NRSS	SRSS1	SRSS2	CRSS	MNRSS
5	0.0825	0.0613	-21.06	13.4038	-3.6457	20.0264	-15.7561	-26.7503	4.4347
6	0.0632	0.0674	-21.0158	26.495	-4.932	18.6995	-14.4168	-27.1994	-0.5758
7	0.0741	-0.0547	-23.3435	24.3607	-2.4667	17.3997	-13.1944	-27.4358	1.0873
8	0.0555	-0.1014	-23.3512	35.3969	-3.068	16.1323	-12.159	-27.584	-0.8948

$$+ \frac{1}{8}\left(\frac{20+14+4+49+14+20}{5202}\right)$$
$$= 0.00531. \qquad (13)$$

The corresponding variances of the SRS and RSS estimators are 0.02083 and 0.00833, respectively. Consequently, MNRSS has efficiencies of 3.923 and 1.569 relative to SRS and RSS, respectively.

3.1.2. Exponential distribution

Let $Y_{(1)}, \ldots, Y_{(N)}$ be the order statistics of a sample of size N from the Exp(1) distribution. Their mean and variance are

$$E(Y_{(i)}) = \sum_{j=N-i+1}^{N} \frac{1}{j}, \qquad \mathrm{Var}(Y_{(i)}) = \sum_{j=N-i+1}^{N} \frac{1}{j^2}. \qquad (14)$$

For $i < j$,

$$\mathrm{Cov}\Big(Y_{(i)}, Y_{(j)}\Big) = \mathrm{Var}(Y_{(i)})\,. \qquad (15)$$

For $m = 4$ and $r = 1$, the measured units again have ranks 2, 7, 10, and 15. Hence,

$$\widehat{\mu}_{\mathrm{MNRSS}} = \frac{1}{4}\left(Y_{(2)} + Y_{(7)} + Y_{(10)} + Y_{(15)}\right), \qquad (16)$$

and

$$E(\widehat{\mu}_{\mathrm{MNRSS}}) = \frac{1}{4}\,(0.129166667 + 0.551760739$$
$$+0.930728993 + 2.380728993)$$
$$= 0.998096348. \qquad (17)$$

The bias is therefore -0.0019. The variance, to five decimal places, is

$$\mathrm{Var}(\widehat{\mu}_{\mathrm{MNRSS}}) = \frac{1}{16}(0.00835 + 0.04457 + 0.09295 + 0.58455)$$
$$+ \frac{1}{8}(0.00835 + 0.00835 + 0.00835$$
$$+ 0.04457 + 0.04457 + 0.09295)$$
$$= 0.07154. \qquad (18)$$

3.2. Estimation of the population variance

Variance estimation under RSS has been studied extensively. Stokes [6] proposed the estimator

$$\widehat{\sigma}^2 = \frac{1}{mr-1}\sum_{j=1}^{r}\sum_{i=1}^{m}\Big(Y_{(i)j} - \widehat{\mu}_{\mathrm{RSS}}\Big)^2, \qquad (19)$$

and showed that it is asymptotically unbiased and more efficient than the conventional SRS sample-variance estimator, irrespective of ranking quality.

MacEachern *et al.* [24] later introduced an unbiased variance estimator for standard RSS and showed that it is generally more efficient than Stokes' estimator, particularly under perfect ranking. Its implementation requires at least two cycles ($r \geq 2$).

The MacEachern estimator is adapted here for MNRSS. Although it is unbiased under standard RSS, unbiasedness does not automatically extend to MNRSS because the selected order statistics within a cycle arise from the same ranked set of m^2 units and are therefore dependent. The adapted estimator is

$$\widehat{\sigma}^2_{\mathrm{MNRSS}} = \frac{1}{2m^2r(r-1)}\sum_{k=1}^{m}\sum_{j=1}^{r}\sum_{i=1}^{r}\Big(Y_{l_k i} - Y_{l_k j}\Big)^2$$
$$+ \frac{1}{2m^2r^2}\sum_{\substack{k,t=1\\k\neq t}}^{m}\sum_{j=1}^{r}\sum_{i=1}^{r}\Big(Y_{l_k i} - Y_{l_t j}\Big)^2, \qquad (20)$$

where $l_s = l + m(s-1)$ for $s = 1, \ldots, m$, and l is defined in Eqs. (2) and (3). The first term measures variability among observations with the same selected order statistic across cycles, whereas the second measures variability between different selected order statistics across all cycles.

4. Simulation studies

The performance of the proposed sampling scheme for estimating the population mean and variance was examined using ten distributions representing a range of shapes and degrees of asymmetry. For reproducibility, the parameterizations were uniform(0, 1); normal(0, 1); logistic(0, 1) with location and scale parameters; Student's t distribution with 4 degrees of freedom; beta(3, 3) and beta(0.5, 0.25) with shape parameters α and β; half-normal(0, 2) with location and scale parameters; gamma(1.5, 2) with shape and scale parameters; lognormal(0, 1), where 0 and 1 are the mean and standard deviation of the natural logarithm; and Weibull(1, 3) with shape and scale parameters.

To examine the effect of set size m, values from 5 to 8 were considered, with $r = 3$ and $r = 6$. For each combination of parameter values and distribution, $M = 30{,}000$ datasets were simulated using MNRSS and the competing schemes: SRS, RSS, MRSS, ERSS, NRSS, SRSS1, SRSS2, and CRSS. Perfect ranking was assumed throughout.

For variance estimation, the standard unbiased sample variance was used for SRS, and the estimator of MacEachern *et al.* [24] was used for standard RSS. For MRSS, ERSS, NRSS, SRSS1, SRSS2, CRSS, and MNRSS, the MacEachern formula was adapted by substituting the order statistics selected under the respective scheme. Because the selected units within cycles are dependent under these modified designs, the adapted estimator is not necessarily unbiased.

Let $\widehat{\mu}_{B,i}$ denote the mean estimator under scheme B for simulation replication $i = 1, \ldots, M$. Bias and MSE were calculated as

$$\mathrm{Bias} = \frac{1}{M}\sum_{i=1}^{M}\left(\widehat{\mu}_{B,i} - \mu\right), \qquad (21)$$

and

$$\mathrm{MSE} = \frac{1}{M}\sum_{i=1}^{M}\left(\widehat{\mu}_{B,i} - \mu\right)^2. \qquad (22)$$

4.1. Results for estimation of the population mean

Tables 4–7 present the MSEs of the mean estimators for different values of m, whereas Tables 8–11 present the corresponding biases. Similar results were obtained for both values of r; therefore, only the results for $r = 3$ are reported. Based on bias and MSE, MNRSS yields the lowest error among the tested schemes for most asymmetric distributions, with NRSS generally performing second best. For symmetric distributions, MNRSS, NRSS, SRSS2, and CRSS perform similarly and yield lower MSEs than the older RSS variations.

4.2. Results for estimation of the population variance

Tables 12–15 present the MSEs of the variance estimators for different values of m, whereas Tables 16–19 present the corresponding biases. MNRSS provides the lowest variance-estimator MSE in most scenarios under both symmetric and asymmetric distributions. Exceptions occur; for example, NRSS produces a lower MSE than MNRSS when $m = 5$ for some distributions. Regarding bias, SRS and RSS perform better for some distributions, whereas MNRSS generally has the lowest bias among the neoteric variations.

5. Real-data application

The proposed sampling scheme was also evaluated using a real dataset. The `spati2` dataset from the R package `lmfor` [25], originally collected by Pukkala [26], contains height and diameter measurements for 1,678 Scots pine trees in Ilomantsi, Finland. Tree height was the primary variable of interest, and the objective was to estimate its population mean and variance. Figure 1 shows the distribution of tree heights and the relationship between diameter and height. The height distribution is unimodal and right-skewed, and diameter and height have a strong positive association, with a Pearson correlation coefficient of 0.866. Descriptive statistics are reported in Table 20.

Random samples were drawn with replacement from the dataset to emulate independent population sampling. For each cycle under the RSS-based schemes, independent random sets of size $N = m^2$ were drawn and ranked using exact height, corresponding to perfect ranking. This process was repeated for $r = 3$ independent cycles to complete one sample. A total of 10,000 samples were generated for each scheme and set size $m = 5, 6, 7, 8$.

Tables 21 and 22 present the MSE and bias of the mean estimators, respectively, whereas Tables 23 and 24 present the MSE and bias of the variance estimators. MNRSS has lower MSE and bias for mean estimation than the other tested techniques. For variance estimation, MNRSS has the lowest MSE for $m \geq 6$, whereas NRSS performs better for $m = 5$. These findings generally agree with the simulation results for asymmetric distributions.

6. Discussion and conclusion

This paper proposed the MNRSS design for estimating the population mean and variance. MNRSS estimators were compared with SRS and several RSS variations under symmetric and asymmetric distributions. The simulation and real-data results show that MNRSS estimators have lower MSEs than the competing methods in most evaluated scenarios, particularly for asymmetric distributions and larger set sizes ($m \geq 6$) under perfect ranking. Conclusions regarding superior performance are based primarily on MSE and relative efficiency; evaluation based solely on bias may lead to different interpretations, particularly for highly asymmetric distributions. The MNRSS mean estimator was proved to be unbiased for symmetric parent distributions. For asymmetric distributions, the simulation results indicate a small, practically negligible bias.

The practical application of MNRSS extends beyond the forestry example. MNRSS may be useful in environmental and agricultural studies in which exact measurement is expensive but visual or auxiliary-variable ranking is inexpensive. For example, soil-contamination studies could rank field plots using rapid assessments of vegetation health before chemical analysis of the selected plots. Agricultural researchers could similarly rank plants using canopy size before accurately measuring crop yield. By avoiding extreme ranks, MNRSS is intended to produce representative samples and accurate variance estimates.

The study has several limitations. The efficiency of MNRSS depends on ranking accuracy. Ranking errors resulting from poor visual judgment or a weakly correlated concomitant variable may reduce its efficiency gain over SRS. Furthermore, although the mean estimator performs well, it has a slight bias for highly skewed asymmetric distributions.

Future research could extend MNRSS to multivariate settings in which units are ranked using several auxiliary variables. Ratio and regression estimators could also be developed under MNRSS. Incorporating MNRSS into two-stage or stratified sampling designs may provide further precision for complex, large-scale population surveys.

Data availability

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Declaration of competing interest

The authors declare that they have no known competing financial or personal interests that could have appeared to influence the work reported in this article.

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Appendix A. Proof for odd m

Let

$$c = \left\lfloor \frac{m}{2} \right\rfloor = \frac{m-1}{2}, \qquad m = 2c+1.$$

The middle set index is $c+1$. The selected rank for this middle term is

$$\begin{aligned} l_{c+1} &= (c+1) + m(c+1-1) \\ &= c+1+mc \\ &= 2c^2+2c+1. \end{aligned} \tag{A.1}$$

The median rank in a sample of size m^2 is

$$\frac{m^2+1}{2} = \frac{(2c+1)^2+1}{2} = 2c^2+2c+1.$$

Thus, the middle term selects the exact sample median. For a symmetric distribution,

$$E(Y_{(\text{median})}) = \mu.$$

Pair each of the remaining $m-1$ terms indexed by $i < c+1$ with the term indexed by $i' = m-i+1$.

When i is odd, the rule gives $l = c$ and

$$k = c + m(i-1).$$

The matching index i' is also odd and satisfies $i' > c+1$; hence, the rule gives $l = c+2$ and

$$k' = c+2+m(m-i).$$

Therefore,

$$k + k' = 2c+2-m+m^2 = m^2+1,$$

because $m = 2c+1$.

When i is even, the rule gives $l = c+2$ and

$$k = c+2+m(i-1).$$

The matching index i' is even and satisfies $i' > c+1$; hence, the rule gives $l = c$ and

$$k' = c+m(m-i).$$

Again,

$$k+k' = 2c+2-m+m^2 = m^2+1.$$

Each pair therefore has expectation 2μ. Combining the median and the $(m-1)/2$ pairs gives

$$E(\widehat{\mu}_{\text{MNRSS}}) = \frac{1}{m}\left[\mu + \frac{m-1}{2}(2\mu)\right] = \mu.$$