Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 8 (2026) 3543

Journal of the Nigerian Society of Physical Sciences

New fractional Hermite–Hadamard and Fejér inequalities involving special functions with applications to numerical quadrature rules

Haitham Qawaqneh[a], Waqar Afzal[b,c,d,*], Mujahid Abbas[e], Muhammad Tariq[f], Mehreen Shehzadi Khan[g], Siegfried Macías[h], Jorge E. Macías-Díaz[h], Hijaz Ahmad[i,j,*], Waleed Mohammed Abdelfattah[k,l]

[a]Department of Basic Sciences, Al-Zaytoonah University of Jordan, Amman 11733, Jordan
[b]Abdus Salam School of Mathematical Sciences, Government College University, 68-B, New Muslim Town, Lahore 54600, Pakistan
[c]Center for Theoretical Physics, Khazar University, 41 Mehseti Str., Baku AZ1096, Azerbaijan
[d]International Center for Interdisciplinary Research in Sciences, University of Lahore, Lahore 54792, Pakistan
[e]Department of Mechanical Engineering Science, University of Johannesburg, Johannesburg 2092, South Africa
[f]Mathematics Research Center, Near East University, Near East Boulevard, Nicosia, Mersin 99138, Turkey
[g]Department of Mathematics, Faculty of Science, Jazan University, Jazan 45142, Saudi Arabia
[h]Department of Mathematics and Physics, Autonomous University of Aguascalientes, Aguascalientes 20100, Mexico
[i]Irfan Suat Günsel Operational Research Institute, Near East University, Nicosia/TRNC, Mersin 10, 99138, Turkey
[j]Department of Mathematics, College of Science, Korea University, 145 Anam-ro, Seongbuk-gu, Seoul 02841, South Korea
[k]College of Engineering, University of Business and Technology, Jeddah 23435, Saudi Arabia
[l]Department of Engineering Mathematics and Physics, Faculty of Engineering, Zagazig University, P.O. 44519, Egypt

Abstract

The principal objective of this article is to establish new and refined variants of Hermite–Hadamard- and Fejér-type integral inequalities in the interval-valued setting by exploiting the auxiliary transformed mapping $\mathcal{G}(s) := \mathcal{F}(s^{1/\rho})$. This transformation preserves convexity and facilitates the systematic application of fractional integral techniques over a suitably rescaled domain through the Katugampola fractional operator. The resulting bounds are expressed explicitly in terms of the Gamma function, which enriches the analytical structure of the inequalities and yields sharper and more precise estimates than existing results. The validity and applicability of the theoretical findings are demonstrated through nontrivial illustrative examples and detailed remarks, and several previously known inequalities are recovered as limiting cases under suitable parameter configurations. As a further application, the derived inequalities are used to construct inclusion-type error estimates for numerical quadrature rules, with particular emphasis on the trapezoidal rule applied to interval-valued functions in the Katugampola fractional framework.

DOI: 10.46481/jnsps.2026.3543

Keywords: Gamma and Beta functions, Interval-valued Katugampola fractional integrals, Hermite–Hadamard and Fejér inequalities, Interval-valued convex mappings, Numerical quadrature

Article History:
Received: 22 May 2026
Received in revised form: 27 July 2026
Accepted for publication: 29 July 2026
Available online: 14 August 2026

Communicated by: B. J. Falaye

1. Introduction

The mathematical study of inequalities owes much of its depth to the life and legacy of Jacques Hadamard [1], his foun-

*Corresponding Author Tel. No.: +92-320-905-2093.
Email address: hijaz.ahmad@neu.edu.tr (Hijaz Ahmad)

dational contributions to analysis provide the historical grounding for the inequality that now bears his name. Sarikaya *et al.* [2] established fractional versions of the classical Hermite–Hadamard inequality by employing the Riemann–Liouville fractional integral operators. Varošanec [3] later introduced the notion of h-convexity, a unifying generalization that subsumes several earlier convexity classes within a single framework, while Noor *et al.* [4] extended this idea further by deriving Hermite–Hadamard-type inequalities tailored to log-h-convex functions. According to Afzal *et al.* [5], additional inequalities for H-Godunova–Levin functions can be established by adopting the center–radius order relation, thereby linking classical convexity with interval-based orderings.

Several works have explored h-convex and stochastic-process-based inequalities using such order-relation formulations. Afzal *et al.* [6] derived new convex integral bounds in the probabilistic setting. Building upon this work, Abbas *et al.* [7] further generalized these results by utilizing the total interval order relation. Sahoo *et al.* [8] contributed further developments in fractional Hermite–Hadamard inequalities by exploring interval-valued convexity alongside supporting computational analysis, and in a related contribution, Sahoo *et al.* [9] examined Ostrowski–Mercer type fractional inequalities for convex functions together with several accompanying applications. Building on this Mercer-type direction, Shah *et al.* [10] formulated a fractional variant of such type of convex inequalities for stochastic processes, with notable applications extending into entropy and information theory.

Fractional-operator-based inequalities involving special and higher transcendental functions have also received sustained attention. Tariq *et al.* [11, 12] proposed new integral inequalities incorporating the Raina and Mittag-Leffler functions through Atangana–Baleanu fractional operators and presented several applications. Bin-Saad *et al.* [13] unified and extended several classical integral theorems by means of the generalized Mittag–Leffler function, and the same authors, together with Alatawi and Koam [14], further obtained Hadamard and Fejér–Hadamard inequalities for fractional integrals involving a Mittag-Leffler-type function of arbitrary order. Abdelfattah *et al.* [15], in a related effort, derived fractional integral inequalities linked to applications involving special means. As reported in Ref. [16], the same research direction was further advanced through new fractional developments of Hermite–Hadamard-type inequalities, while Cheng *et al.* [17] introduced fractional quantum Hermite–Hadamard inequalities for interval-valued functions by merging q-calculus with fractional integration.

Beyond classical fractional integral settings, several works have investigated boundedness and regularity properties within generalized function spaces. Afzal *et al.* [18] examined the qualitative properties of parabolic equations using fractional Bessel–Riesz operators, and in a related study, Afzal *et al.* [19] established the boundedness properties of fractional operators in Campanato spaces. Extending this direction further, Afzal *et al.* [20] combined gradient descent techniques with Simpson-type inequalities through fractional operators.

A parallel and equally active area concerns topological and soft-set-theoretic operators. Alghamdi *et al.* [21] proposed new types of operators within topological spaces, and Abd El-latif *et al.* [22] introduced a broader class of soft sets constructed via the supra soft δ-closure operator. Alqahtani and Abd El-latif [23] investigated separation axioms through novel topological operators with accompanying applications, while El-latif and Alqahtani [24] classified new categories of supra soft continuous mappings using freshly defined soft operators. Judeh and Abu Hammad [25] introduced conformable fractional analogs of the Pareto distribution, deriving its density, moments, and entropy measures. In closely related directions, Hatamleh *et al.* [26] introduced complex triple-valued neutrosophic soft sets together with an accompanying topological framework supported by AI-driven signal-template analysis, and Musa *et al.* [27] proposed fuzzy N-bipolar soft sets for multi-criteria decision-making, with corresponding theoretical and applied results. Ameen and Alqahtani [28] examined congruence representations arising from soft ideals in soft topological spaces, and together with Alghamdi [29], the same authors further studied lower density soft operators and their associated density soft topologies. Abd El-latif *et al.* [30] introduced new versions of maps and connected spaces by means of supra soft sd-operators.

The Godunova–Levin class of convexity, together with its preinvex variants, continues to be a fertile research direction. Almutairi and Kılıçman [31] were among the early contributors, establishing integral inequalities for h-Godunova–Levin preinvex functions, a line of inquiry later extended by Ali *et al.* [32] through fractional integral inequalities for the same preinvex class. Ali *et al.* [33] subsequently generalized Hermite–Hadamard integral inequalities specifically for h-Godunova–Levin functions, whereas Farid *et al.* [34] focused on Ostrowski-type fractional inequalities for s-Godunova–Levin functions derived via Katugampola fractional integrals. More recently, Khan *et al.* [35] presented new variants of Hermite–Hadamard and Fejér-type inequalities for the Godunova–Levin preinvex class within an interval-valued setting, and Akram *et al.* [36] extended this convexity notion to fuzzy interval-valued functions using Riemann–Liouville fractional q-integrals. Most recently, Almalki *et al.* [37] established new convex integral bounds by employing higher-dimensional elliptic geometry.

Interval-valued analysis itself has diversified considerably across coordinated, preinvex, and m-polynomial convex settings. Nwaeze *et al.* [38] derived a fractional variant of Hermite–Hadamard-type bounds for interval-valued mappings, and Zhao *et al.* [39] examined Hermite–Hadamard inequalities for coordinated h-convex interval-valued functions. Srivastava *et al.* [40] contributed Hermite–Hadamard-type inequalities for preinvex functions using fractional operators, a theme also pursued by Tan *et al.* [41] for $\bar{h}$-preinvex interval-valued functions. Lai *et al.* [42] complemented these efforts by studying Hermite–Hadamard-type inclusions for interval-valued functions that are simultaneously coordinated and preinvex, while Ali *et al.* [43] incorporated quantum calculus into this framework by deriving post-quantum Hermite–Hadamard-type inequalities for interval-valued convex functions. In the classical, non-interval setting, Faisal *et al.* [44] established Hermite–

Hadamard-type bounds for different classes of preinvex-type mappings.

Fuzzy-number-valued functions and superquadratic classes represent another expanding frontier. Liu *et al.* [45] formulated new Hermite–Hadamard and Jensen type bounds for fuzzy-interval-valued functions, while Khan *et al.* [46] analyzed superquadratic fuzzy interval-valued functions together with their integral inequalities. Ahmad *et al.* [47] presented Hermite–Hadamard-type inequalities via generalized superquadratic functions, and Kashuri *et al.* [48] investigated novel inequalities for subadditive functions using tempered fractional integrals, supported by numerical verification.

Convex geometry has also informed this broader inequality landscape through discrete and potential-theoretic analogues of the Brunn–Minkowski inequality. Hernández Cifre *et al.* [49] established a discrete version of the Brunn–Minkowski type inequality, and Akman *et al.* [50] extended this concept to a Brunn–Minkowski inequality and an associated Minkowski problem for the $\mathcal{A}$-harmonic Green's function arising in quasilinear elliptic PDE theory. Verma and Viswanathan [51] investigated the analytical and geometric properties of the Katugampola fractional integral for continuous functions of bounded variation.

Stochastic processes combined with center–radius order relations have similarly attracted growing interest. Afzal *et al.* [52] extended this order relation to harmonically convex mappings, deriving several novel integral estimates, and Afzal *et al.* [53] applied boundedness properties of new types of fractional operators to regularity theory. Adrees *et al.* [54] investigated different classes of convex stochastic bounds and established three distinct types of stochastic integral inequalities.

Beyond pure convexity theory, applications span differential equations, variational inequalities, and geometric function theory. Issa *et al.* [55] proposed approximate solutions for time-fractional non-linear parabolic equations relevant to mathematical physics, and in a related numerical direction, Atabo and Adee [56] devised a new 15-step block method for solving general fourth-order ordinary differential equations. Adrees *et al.* [57] investigated modified (p, h)-convex stochastic processes and their associated integral properties. Saeed and Treanţă [58] introduced new classes of interval-valued variational problems together with their corresponding inequalities. Bello *et al.* [59] examined quasi-subordination for a subclass of non-Bažilević functions linked to a q-derivative operator, while Ameen and Alqahtani [60] studied Baire category soft sets and their symmetric local properties.

Functional-analytic and fixed-point-theoretic tools have further enriched this applied landscape. Qawaqneh *et al.* [61] investigated stability analysis, modulation instability, and beta-time fractional exact soliton solutions of the van der Waals equation, while Qawaqneh *et al.* [62] established new characterizations of fuzzy contractions in fuzzy b-metric spaces together with corresponding applications. In a related study, Qawaqneh and Aydi [63] examined new fixed-point results and discussed their applications. Complementing these fixed-point results, Arif *et al.* [64] derived iterative approximations of generalized nonexpansive operators in Banach spaces. Beyond pure mathematics, Kanan *et al.* [65] explored the potential of IoT-based learning environments within educational settings, illustrating the broader interdisciplinary reach of such analytical frameworks.

Finally, several recent studies have unified multiple inequality types under fractional and pseudo-order-relation frameworks. Shi *et al.* [66] established fractional Hermite–Hadamard-type inequalities for interval-valued functions, and Khan *et al.* [67] presented some new convex integral bounds on coordinates using new double-fractional operators. Ahmad *et al.* [68] proposed new notions of mathematical integral inequalities with both theoretical depth and applied relevance, and Zhang *et al.* [69] formulated Hermite–Hadamard and Jensen-type inequalities via the Riemann integral operator for a generalized Godunova–Levin class. Afzal *et al.* [70] further examined a stability result together with convex integral inequalities and identified open problems for future study, and Liu *et al.* [71] rounded out this body of work by establishing new inequalities for cr-log-h-convex functions under the center–radius order relation. Firdous *et al.* [72] investigated various continuous and discrete forms of convex integral inequalities by employing generalized strongly convex mappings. Taken together, these contributions collectively motivate the present investigation into unified, fractional, and order-relation-based inequality frameworks for convex and generalized convex mappings.

Although the literature already contains many results, most of them are based on classical or Riemann–Liouville type fractional operators, and the connection of interval-valued analysis with the Katugampola fractional integral operator, particularly for obtaining sharp Hermite–Hadamard-type inequalities for special functions such as the Gamma functions along with applications to numerical problems, has been largely unexplored. The main contribution of this work is the development of an interval-valued Katugampola fractional integral operator, which provides a unified and generalized framework in which many known inequalities can be recovered as special cases; moreover, since the commonly used inclusion order relation has well-known limitations and does not always allow meaningful comparisons of intervals, we instead adopt the complete and consistent ordering proposed by Bhunia and Samanta, which preserves the comparison properties required for interval-valued convexity. Using this operator and ordering, we introduce new forms of Hermite–Hadamard, midpoint, and weighted product inequalities for interval-valued convex mappings that yield tighter bounds expressed explicitly in terms of the Gamma functions, apply these results to derive inclusion-type error bounds for numerical quadrature rules such as the weighted trapezoidal rule under data uncertainty, and support the generality of the framework through detailed remarks and nontrivial examples that recover many previously reported results as special cases.

The remainder of this article is organized as follows. In Section 1, the related literature is surveyed and the motivation for the problems studied in this work is expressed. In Section 2, the necessary background concepts about interval-valued functions and the Katugampola fractional operators are developed. The main results are given in Section 3 and consist of Hermite–

Hadamard inequalities, Fejér inequalities and product-type inequalities with their numerical quadrature applications. Section 4 provides concluding remarks and possible future research areas.

2. Preliminary concepts and definitions

In this section we gather the core ideas that we need for the development of the principal results. The material includes interval arithmetic, classical and interval-valued convexity, classical Katugampola fractional operators and the special functions that play a central role in the study.

2.1. Special functions

We begin with the two fundamental special functions that recur throughout our analysis.

Definition 2.1 (Euler Gamma Function)**.** The Euler Gamma function is defined for $\Re(z) > 0$ via the integral representation

$$\Gamma(z) = \int_0^\infty \eta^{z-1} e^{-\eta}\, d\eta.$$

For positive integers n, it satisfies $\Gamma(n+1) = n!$, and the recurrence $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ holds for all $\alpha > -1$.

Definition 2.2 (Beta Function)**.** The Beta function, alternatively referred to as the Euler integral of the first kind, is specified for $\Re(x) > 0$ and $\Re(y) > 0$ by

$$\mathcal{B}(x,y) = \int_0^1 \eta^{x-1}(1-\eta)^{y-1}\, d\eta.$$

Its connection to the Gamma function is expressed through the relation

$$\mathcal{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Both functions occupy central roles in fractional calculus and arise naturally in the coefficients of our fractional integral inequalities.

2.2. Interval arithmetic and the cr*-order*

Let $\mathcal{R}_{\mathcal{I}}$ stand for the set of all nonempty closed and bounded intervals lying inside the real line $\mathcal{R}$. An interval $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}]$ is said to be *positive* if its left endpoint satisfies $\underline{\mathcal{F}} > 0$. We write $\mathcal{R}^+_{\mathcal{I}}$ for the class of such positive intervals, and $\mathcal{R}^+$ for the set of positive real numbers.

Given two intervals $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}]$, $\mathcal{G} = [\underline{\mathcal{G}}, \overline{\mathcal{G}}] \in \mathcal{R}_{\mathcal{I}}$ and a real number σ, addition and scalar multiplication (in the Minkowski sense) are introduced as follows:

$$\mathcal{F} + \mathcal{G} = [\,\underline{\mathcal{F}} + \underline{\mathcal{G}},\ \overline{\mathcal{F}} + \overline{\mathcal{G}}\,],$$

$$\sigma\mathcal{F} = \begin{cases} [\sigma\underline{\mathcal{F}},\ \sigma\overline{\mathcal{F}}], & \sigma > 0,\\ \{0\}, & \sigma = 0,\\ [\sigma\overline{\mathcal{F}},\ \sigma\underline{\mathcal{F}}], & \sigma < 0.\end{cases}$$

Each interval $\mathcal{F} \in \mathcal{R}_{\mathcal{I}}$ can alternatively be described through its *midpoint* and *half-width*, given by

$$\mathcal{F}_{\mathfrak{c}} := \frac{\overline{\mathcal{F}} + \underline{\mathcal{F}}}{2}, \qquad \mathcal{F}_{\mathfrak{r}} := \frac{\overline{\mathcal{F}} - \underline{\mathcal{F}}}{2},$$

so that $\mathcal{F}$ can be written back in the paired form $\mathcal{F} = \langle \mathcal{F}_{\mathfrak{c}}, \mathcal{F}_{\mathfrak{r}}\rangle$.

Definition 2.3 (cr-Order, [5])**.** Let $\mathcal{F} = \langle \mathcal{F}_{\mathfrak{c}}, \mathcal{F}_{\mathfrak{r}}\rangle$ and $\mathcal{G} = \langle \mathcal{G}_{\mathfrak{c}}, \mathcal{G}_{\mathfrak{r}}\rangle$ be two elements of $\mathcal{R}_{\mathcal{I}}$. The relation $\preceq_{\mathrm{cr}}$, called the center-radius order, is set up by comparing midpoints first and, in case of a tie, comparing half-widths:

$$\mathcal{F} \preceq_{\mathrm{cr}} \mathcal{G} \iff \begin{cases} \mathcal{F}_{\mathfrak{c}} < \mathcal{G}_{\mathfrak{c}}, & \text{if } \mathcal{F}_{\mathfrak{c}} \neq \mathcal{G}_{\mathfrak{c}},\\ \mathcal{F}_{\mathfrak{r}} \leq \mathcal{G}_{\mathfrak{r}}, & \text{if } \mathcal{F}_{\mathfrak{c}} = \mathcal{G}_{\mathfrak{c}}.\end{cases}$$

One can check that this relation orders $\mathcal{R}_{\mathcal{I}}$ completely: given any two intervals $\mathcal{F}, \mathcal{G} \in \mathcal{R}_{\mathcal{I}}$, it is always true that $\mathcal{F} \preceq_{\mathrm{cr}} \mathcal{G}$ or $\mathcal{G} \preceq_{\mathrm{cr}} \mathcal{F}$.

Remark 2.1. *$\preceq_{\mathrm{cr}}$-convexity of $\mathcal{F}$ is defined directly via the order relation $\preceq_{\mathrm{cr}}$ on the values $\mathcal{F}(\chi) \in \mathbb{R}_{\mathcal{I}}$, not through separate convexity of $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$. Since $\preceq_{\mathrm{cr}}$ is lexicographic (centre first, radius only at ties), $\preceq_{\mathrm{cr}}$-convexity is governed mainly by $\mathcal{F}_{\mathfrak{c}}$, with $\mathcal{F}_{\mathfrak{r}}$ entering only when centres coincide. It is, in general, not equivalent to convexity of $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ individually (see Remark 2.2).*

Riemann integration for interval-valued mappings was studied in [5, 10]; the key characterization and a monotonicity property are stated below.

Theorem 2.1 ([10])**.** *An interval-valued mapping $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}] : [\mathfrak{a}, \mathfrak{b}] \to \mathcal{R}_{\mathcal{I}}$ is Riemann integrable on $[\mathfrak{a}, \mathfrak{b}]$ if and only if both component functions $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are Riemann integrable, in which case*

$$\int_{\mathfrak{a}}^{\mathfrak{b}} \mathcal{F}(\chi)\, d\chi = \left[\int_{\mathfrak{a}}^{\mathfrak{b}} \underline{\mathcal{F}}(\chi)\, d\chi,\ \int_{\mathfrak{a}}^{\mathfrak{b}} \overline{\mathcal{F}}(\chi)\, d\chi\right].$$

The family of all such integrable mappings is written $\mathcal{IR}_{[\mathfrak{a},\mathfrak{b}]}$.

Theorem 2.2 ([5])**.** *Let $\mathcal{F}, \mathcal{G} \in \mathcal{IR}_{[\mathfrak{a},\mathfrak{b}]}$ take values in $\mathcal{R}^+_{\mathcal{I}}$. If $\mathcal{F}(\chi) \preceq_{\mathrm{cr}} \mathcal{G}(\chi)$ for every $\chi \in [\mathfrak{a}, \mathfrak{b}]$, then*

$$\int_{\mathfrak{a}}^{\mathfrak{b}} \mathcal{F}(\chi)\, d\chi \ \preceq_{\mathrm{cr}}\ \int_{\mathfrak{a}}^{\mathfrak{b}} \mathcal{G}(\chi)\, d\chi.$$

Further background material on interval analysis may be found in [5].

2.3. Convexity and interval-valued convexity

Definition 2.4 (Classical Convexity, [5])**.** A mapping $\mathcal{F} : [\mathfrak{a}, \mathfrak{b}] \to \mathcal{R}$ is convex provided that for all $\chi, \mathfrak{y} \in [\mathfrak{a}, \mathfrak{b}]$ and every $\lambda \in [0, 1]$,

$$\mathcal{F}(\lambda\chi + (1-\lambda)\mathfrak{y}) \ \leq\ \lambda\mathcal{F}(\chi) + (1-\lambda)\mathcal{F}(\mathfrak{y}).$$

Definition 2.5 ($\preceq_{cr}$-Convexity, [5]). Let $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}] : [\mathfrak{a}, \mathfrak{b}] \to \mathbb{R}_{\mathcal{I}}$ with $\underline{\mathcal{F}}(\chi) \leq \overline{\mathcal{F}}(\chi)$ for all $\chi \in [\mathfrak{a}, \mathfrak{b}]$. $\mathcal{F}$ is $\preceq_{cr}$-convex on $[\mathfrak{a}, \mathfrak{b}]$ if

$$\mathcal{F}(\lambda\chi + (1-\lambda)\mathfrak{y}) \preceq_{cr} \lambda\,\mathcal{F}(\chi) + (1-\lambda)\,\mathcal{F}(\mathfrak{y}),$$
$$\forall\, \chi, \mathfrak{y} \in [\mathfrak{a}, \mathfrak{b}], \quad \lambda \in [0,1].$$

By Definition 2.3, this holds iff, for every $\chi, \mathfrak{y}, \lambda$, either

$$\mathcal{F}_{\mathfrak{c}}(\lambda\chi + (1-\lambda)\mathfrak{y}) < \lambda\mathcal{F}_{\mathfrak{c}}(\chi) + (1-\lambda)\mathcal{F}_{\mathfrak{c}}(\mathfrak{y}),$$

or the centres coincide and

$$\mathcal{F}_{\mathfrak{r}}(\lambda\chi + (1-\lambda)\mathfrak{y}) \leq \lambda\mathcal{F}_{\mathfrak{r}}(\chi) + (1-\lambda)\mathcal{F}_{\mathfrak{r}}(\mathfrak{y}).$$

The class of such mappings with values in $\mathbb{R}_{\mathcal{I}}^+$ is denoted $\mathcal{IVCX}([\mathfrak{a}, \mathfrak{b}], \mathbb{R}_{\mathcal{I}}^+)$.

Definition 2.6 ($\mathfrak{cr}$-Convexity, [5]). With $\mathcal{F} = \langle \mathcal{F}_{\mathfrak{c}}, \mathcal{F}_{\mathfrak{r}} \rangle$ and $\mathcal{F}_{\mathfrak{r}} \geq 0$ on $[\mathfrak{a}, \mathfrak{b}]$, $\mathcal{F}$ is $\mathfrak{cr}$-convex if $\mathcal{F}_{\mathfrak{c}}$ and $\mathcal{F}_{\mathfrak{r}}$ are both convex on $[\mathfrak{a}, \mathfrak{b}]$ in the usual (real-valued) sense.

Remark 2.2. *$\mathfrak{cr}$-convexity $\Rightarrow$ $\preceq_{cr}$-convexity, directly from Definition 2.3: if $\mathcal{F}_{\mathfrak{c}}$ is strictly below its chord the first case applies; otherwise (centres equal) convexity of $\mathcal{F}_{\mathfrak{r}}$ gives the second case. The converse fails in general, since the radius inequality is required only at ties of $\mathcal{F}_{\mathfrak{c}}$. Neither notion is equivalent to convexity of $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ separately: $\overline{\mathcal{F}} = \mathcal{F}_{\mathfrak{c}} + \mathcal{F}_{\mathfrak{r}}$ is convex whenever $\mathcal{F}_{\mathfrak{c}}, \mathcal{F}_{\mathfrak{r}}$ are, but $\underline{\mathcal{F}} = \mathcal{F}_{\mathfrak{c}} - \mathcal{F}_{\mathfrak{r}}$ need not be, being a difference of convex functions.*

2.4. Katugampola fractional integral operators

Definition 2.7 (Katugampola Fractional Integrals, [51]). Let $0 < \mathfrak{a} < \mathfrak{b}$ and $\mathcal{F} \in \mathcal{R}_{[\mathfrak{a},\mathfrak{b}]}$. For fractional order $\alpha > 0$ and scaling parameter $\rho > 0$, the left- and right-sided Katugampola fractional integral operators are given by

$${}^{\rho}\mathcal{J}^{\alpha}_{\mathfrak{a}^+}\mathcal{F}(\chi) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\mathfrak{a}}^{\chi} \frac{\tau^{\rho-1}}{(\chi^{\rho} - \tau^{\rho})^{1-\alpha}}\, \mathcal{F}(\tau)\, d\tau, \qquad \chi > \mathfrak{a},$$

$${}^{\rho}\mathcal{J}^{\alpha}_{\mathfrak{b}^-}\mathcal{F}(\chi) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\chi}^{\mathfrak{b}} \frac{\tau^{\rho-1}}{(\tau^{\rho} - \chi^{\rho})^{1-\alpha}}\, \mathcal{F}(\tau)\, d\tau, \qquad \chi < \mathfrak{b}.$$

Remark 2.3. *The substitution $\rho = 1$ in Definition 2.7 yields the standard Riemann–Liouville fractional integrals [2]:*

$${}^{1}\mathcal{J}^{\alpha}_{\mathfrak{a}^+}\mathcal{F}(\chi) = \frac{1}{\Gamma(\alpha)} \int_{\mathfrak{a}}^{\chi} (\chi - \tau)^{\alpha-1}\, \mathcal{F}(\tau)\, d\tau.$$

Accordingly, the Katugampola setting strictly contains the Riemann–Liouville setting, and every result established below for general $\rho > 0$ delivers a corresponding Riemann–Liouville statement upon this specialization.

3. Main results

In this section we present the main results of the paper. We begin by extending the classical Katugampola fractional integral operator to the interval-valued function setting. Using this construction, we then derive new variants of Hermite–Hadamard and midpoint type inequalities for interval-valued convex mappings.

Definition 3.1 (Interval-Valued Katugampola Fractional Integrals). Let $\mathcal{F} : [\zeta_1, \zeta_2] \to \mathcal{R}_{\mathcal{I}}$ be an interval-valued mapping given by

$$\mathcal{F}(\chi) = [\, \underline{\mathcal{F}}(\chi), \overline{\mathcal{F}}(\chi)\,],$$

where $\underline{\mathcal{F}}, \overline{\mathcal{F}} \in \mathcal{R}_{[\zeta_1,\zeta_2]}$.

For $\alpha > 0$ and $\rho > 0$, the interval-valued Katugampola fractional integrals are defined componentwise as

$${}^{\rho}\mathcal{J}^{\alpha}_{\zeta_1^+}\mathcal{F}(\chi) = \left[{}^{\rho}\mathcal{J}^{\alpha}_{\zeta_1^+}\underline{\mathcal{F}}(\chi),\ {}^{\rho}\mathcal{J}^{\alpha}_{\zeta_1^+}\overline{\mathcal{F}}(\chi) \right],$$
$${}^{\rho}\mathcal{J}^{\alpha}_{\zeta_2^-}\mathcal{F}(\chi) = \left[{}^{\rho}\mathcal{J}^{\alpha}_{\zeta_2^-}\underline{\mathcal{F}}(\chi),\ {}^{\rho}\mathcal{J}^{\alpha}_{\zeta_2^-}\overline{\mathcal{F}}(\chi) \right].$$

Explicitly, these operators take the form:

$${}^{\rho}\mathcal{J}^{\alpha}_{\zeta_1^+}\mathcal{F}(\chi) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\zeta_1}^{\chi} \frac{\tau^{\rho-1}}{(\chi^{\rho} - \tau^{\rho})^{1-\alpha}} \mathcal{F}(\tau)\, d\tau, \quad \chi > \zeta_1,$$
$${}^{\rho}\mathcal{J}^{\alpha}_{\zeta_2^-}\mathcal{F}(\chi) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\chi}^{\zeta_2} \frac{\tau^{\rho-1}}{(\tau^{\rho} - \chi^{\rho})^{1-\alpha}} \mathcal{F}(\tau)\, d\tau, \quad \chi < \zeta_2,$$

where the integrals are understood in the interval sense.

Remark 3.1. *Definition 3.1 is not proposed as a new fractional operator; it is the componentwise interval-valued extension (under the $\preceq_{cr}$ order) of the classical Katugampola fractional integral, applied to $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$. This construction follows the standard approach used to extend classical fractional operators to the interval-valued setting. The novelty of the present work lies not in this operator itself, but in the Hermite–Hadamard–Fejér-type inequalities derived from it via the transformation $\mathcal{G}(s) := \mathcal{F}(s^{1/\rho})$, together with their associated quadrature error estimates.*

3.1. Generalized Hermite–Hadamard-type inequalities via interval-valued Katugampola fractional integral operators

Theorem 3.1. *Let $\zeta_1, \zeta_2 \in \mathcal{R}$ with $0 < \zeta_1 < \zeta_2$, and let $\mathcal{F} : [\zeta_1, \zeta_2] \to \mathcal{R}_{\mathcal{I}}^+$ be an interval-valued function represented as*

$$\mathcal{F}(\zeta) = [\underline{\mathcal{F}}(\zeta), \overline{\mathcal{F}}(\zeta)],$$

where $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are Lebesgue integrable on $[\zeta_1, \zeta_2]$ and satisfy

$$0 \leq \underline{\mathcal{F}}(\zeta) \leq \overline{\mathcal{F}}(\zeta), \qquad \zeta \in [\zeta_1, \zeta_2].$$

Suppose further that the transformed interval-valued mapping

$$\mathcal{G}(s) := \mathcal{F}(s^{1/\rho}), \qquad s \in [\zeta_1^{\rho}, \zeta_2^{\rho}],$$

is interval-valued convex on $[\zeta_1^{\rho}, \zeta_2^{\rho}]$, that is,

$$\mathcal{G}(\lambda s_1 + (1-\lambda)s_2) \preceq_{cr} \lambda\mathcal{G}(s_1) + (1-\lambda)\mathcal{G}(s_2), \qquad \lambda \in [0,1].$$

Then the following Hermite–Hadamard-type inclusion is satisfied:

$$\mathcal{F}\left(\left(\frac{\zeta_1^{\rho} + \zeta_2^{\rho}}{2}\right)^{1/\rho}\right) \preceq_{cr} \frac{\Gamma(\alpha+1)\rho^{\alpha}}{2(\zeta_2^{\rho} - \zeta_1^{\rho})^{\alpha}} \left[{}^{\rho}\mathcal{J}^{\alpha}_{\zeta_1^+}\mathcal{F}(\zeta_2) + {}^{\rho}\mathcal{J}^{\alpha}_{\zeta_2^-}\mathcal{F}(\zeta_1) \right]$$
$$\preceq_{cr} \frac{\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)}{2}. \tag{1}$$

Proof. Since $\mathcal{G}(s) = \mathcal{F}(s^{1/\rho})$ is interval-valued convex on $[\zeta_1^\rho, \zeta_2^\rho]$, for any $s_1, s_2 \in [\zeta_1^\rho, \zeta_2^\rho]$ we have

$$\mathcal{G}\left(\frac{s_1 + s_2}{2}\right) \preceq_{\text{cr}} \frac{\mathcal{G}(s_1) + \mathcal{G}(s_2)}{2}.$$

Now choose

$$s_1 = \lambda\zeta_1^\rho + (1-\lambda)\zeta_2^\rho, \qquad s_2 = (1-\lambda)\zeta_1^\rho + \lambda\zeta_2^\rho, \qquad \lambda \in [0,1].$$

Then

$$\frac{s_1 + s_2}{2} = \frac{\zeta_1^\rho + \zeta_2^\rho}{2}.$$

Therefore,

$$\mathcal{G}\left(\frac{\zeta_1^\rho + \zeta_2^\rho}{2}\right) \preceq_{\text{cr}} \frac{1}{2}\Big[\mathcal{G}(\lambda\zeta_1^\rho + (1-\lambda)\zeta_2^\rho) + \mathcal{G}((1-\lambda)\zeta_1^\rho + \lambda\zeta_2^\rho)\Big].$$

Recalling that $\mathcal{G}(s) = \mathcal{F}(s^{1/\rho})$, we obtain

$$\mathcal{F}\left(\left(\frac{\zeta_1^\rho + \zeta_2^\rho}{2}\right)^{1/\rho}\right) \preceq_{\text{cr}} \frac{1}{2}\Big[\mathcal{F}((\lambda\zeta_1^\rho + (1-\lambda)\zeta_2^\rho)^{1/\rho}) + \mathcal{F}(((1-\lambda)\zeta_1^\rho + \lambda\zeta_2^\rho)^{1/\rho})\Big].$$

Multiply both sides by $\lambda^{\alpha-1}$ and integrate over $[0,1]$. Then

$$\int_0^1 \lambda^{\alpha-1}\mathcal{F}\left(\left(\frac{\zeta_1^\rho + \zeta_2^\rho}{2}\right)^{1/\rho}\right) d\lambda \preceq_{\text{cr}} \frac{1}{2}\Bigg[\int_0^1 \lambda^{\alpha-1}\mathcal{F}((\lambda\zeta_1^\rho + (1-\lambda)\zeta_2^\rho)^{1/\rho})\, d\lambda + \int_0^1 \lambda^{\alpha-1}\mathcal{F}(((1-\lambda)\zeta_1^\rho + \lambda\zeta_2^\rho)^{1/\rho})\, d\lambda\Bigg].$$

Since the interval-valued term on the left-hand side is independent of λ, we have

$$\int_0^1 \lambda^{\alpha-1}\mathcal{F}\left(\left(\frac{\zeta_1^\rho + \zeta_2^\rho}{2}\right)^{1/\rho}\right) d\lambda = \mathcal{F}\left(\left(\frac{\zeta_1^\rho + \zeta_2^\rho}{2}\right)^{1/\rho}\right)\int_0^1 \lambda^{\alpha-1}\, d\lambda.$$

Using

$$\int_0^1 \lambda^{\alpha-1}\, d\lambda = \frac{1}{\alpha},$$

it follows that

$$\int_0^1 \lambda^{\alpha-1}\mathcal{F}\left(\left(\frac{\zeta_1^\rho + \zeta_2^\rho}{2}\right)^{1/\rho}\right) d\lambda = \frac{1}{\alpha}\mathcal{F}\left(\left(\frac{\zeta_1^\rho + \zeta_2^\rho}{2}\right)^{1/\rho}\right).$$

We now evaluate the first integral on the right-hand side. Set

$$u = (\lambda\zeta_1^\rho + (1-\lambda)\zeta_2^\rho)^{1/\rho}.$$

Then

$$u^\rho = \lambda\zeta_1^\rho + (1-\lambda)\zeta_2^\rho = \zeta_2^\rho - \lambda(\zeta_2^\rho - \zeta_1^\rho),$$

and hence

$$\lambda = \frac{\zeta_2^\rho - u^\rho}{\zeta_2^\rho - \zeta_1^\rho}.$$

Differentiating $u^\rho = \zeta_2^\rho - \lambda(\zeta_2^\rho - \zeta_1^\rho)$ with respect to λ, we get

$$\rho u^{\rho-1}\, du = -(\zeta_2^\rho - \zeta_1^\rho)\, d\lambda,$$

that is,

$$d\lambda = -\frac{\rho u^{\rho-1}}{\zeta_2^\rho - \zeta_1^\rho}\, du.$$

Moreover, when $\lambda = 0$, we have $u = \zeta_2$, and when $\lambda = 1$, we have $u = \zeta_1$.

Therefore,

$$\begin{aligned}
&\preceq \\
&\int_0^1 \lambda^{\alpha-1}\mathcal{F}((\lambda\zeta_1^\rho + (1-\lambda)\zeta_2^\rho)^{1/\rho})\, d\lambda \\
&= \int_{\zeta_2}^{\zeta_1}\left(\frac{\zeta_2^\rho - u^\rho}{\zeta_2^\rho - \zeta_1^\rho}\right)^{\alpha-1}\mathcal{F}(u)\left(-\frac{\rho u^{\rho-1}}{\zeta_2^\rho - \zeta_1^\rho}\right) du \\
&= \frac{\rho}{(\zeta_2^\rho - \zeta_1^\rho)^\alpha}\int_{\zeta_1}^{\zeta_2} u^{\rho-1}(\zeta_2^\rho - u^\rho)^{\alpha-1}\mathcal{F}(u)\, du.
\end{aligned}$$

Using the definition of the Katugampola integral evaluated at ζ_2, we obtain

$${}^\rho\mathcal{J}^\alpha_{\zeta_1^+}\mathcal{F}(\zeta_2) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\zeta_1}^{\zeta_2}\frac{u^{\rho-1}}{(\zeta_2^\rho - u^\rho)^{1-\alpha}}\mathcal{F}(u)\, du.$$

Since

$$\frac{1}{(\zeta_2^\rho - u^\rho)^{1-\alpha}} = (\zeta_2^\rho - u^\rho)^{\alpha-1},$$

it follows that

$${}^\rho\mathcal{J}^\alpha_{\zeta_1^+}\mathcal{F}(\zeta_2) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\zeta_1}^{\zeta_2} u^{\rho-1}(\zeta_2^\rho - u^\rho)^{\alpha-1}\mathcal{F}(u)\, du.$$

Hence,

$$\int_{\zeta_1}^{\zeta_2} u^{\rho-1}(\zeta_2^\rho - u^\rho)^{\alpha-1}\mathcal{F}(u)\, du = \Gamma(\alpha)\rho^{\alpha-1}\,{}^\rho\mathcal{J}^\alpha_{\zeta_1^+}\mathcal{F}(\zeta_2).$$

Substituting this into the previous expression, we obtain

$$\begin{aligned}
&\int_0^1 \lambda^{\alpha-1}\mathcal{F}((\lambda\zeta_1^\rho + (1-\lambda)\zeta_2^\rho)^{1/\rho})\, d\lambda \\
&= \frac{\rho\,\Gamma(\alpha)\rho^{\alpha-1}}{(\zeta_2^\rho - \zeta_1^\rho)^\alpha}\,{}^\rho\mathcal{J}^\alpha_{\zeta_1^+}\mathcal{F}(\zeta_2) \\
&= \frac{\Gamma(\alpha)\rho^\alpha}{(\zeta_2^\rho - \zeta_1^\rho)^\alpha}\,{}^\rho\mathcal{J}^\alpha_{\zeta_1^+}\mathcal{F}(\zeta_2).
\end{aligned}$$

Next, we evaluate the second integral. Let

$$v = ((1-\lambda)\zeta_1^\rho + \lambda\zeta_2^\rho)^{1/\rho}.$$

Then

$$v^\rho = (1-\lambda)\zeta_1^\rho + \lambda\zeta_2^\rho = \zeta_1^\rho + \lambda(\zeta_2^\rho - \zeta_1^\rho),$$

and so

$$\lambda = \frac{v^\rho - \zeta_1^\rho}{\zeta_2^\rho - \zeta_1^\rho}.$$

Differentiating gives

$$\rho v^{\rho-1}\,dv = (\zeta_2^\rho - \zeta_1^\rho)\,d\lambda,$$

that is,

$$d\lambda = \frac{\rho v^{\rho-1}}{\zeta_2^\rho - \zeta_1^\rho}\,dv.$$

When $\lambda = 0$, we have $v = \zeta_1$, and when $\lambda = 1$, we have $v = \zeta_2$.

Thus,

$$\begin{aligned}&\int_0^1 \lambda^{\alpha-1}\mathcal{F}(((1-\lambda)\zeta_1^\rho + \lambda\zeta_2^\rho)^{1/\rho})\,d\lambda\\ &= \int_{\zeta_1}^{\zeta_2}\left(\frac{v^\rho - \zeta_1^\rho}{\zeta_2^\rho - \zeta_1^\rho}\right)^{\alpha-1}\mathcal{F}(v)\frac{\rho v^{\rho-1}}{\zeta_2^\rho - \zeta_1^\rho}\,dv\\ &= \frac{\rho}{(\zeta_2^\rho - \zeta_1^\rho)^\alpha}\int_{\zeta_1}^{\zeta_2} v^{\rho-1}(v^\rho - \zeta_1^\rho)^{\alpha-1}\mathcal{F}(v)\,dv.\end{aligned}$$

By the definition of the right-sided Katugampola fractional integral at the point ζ_1, we have

$${}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{F}(\zeta_1) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\zeta_1}^{\zeta_2}\frac{v^{\rho-1}}{(v^\rho - \zeta_1^\rho)^{1-\alpha}}\mathcal{F}(v)\,dv.$$

Again,

$$\frac{1}{(v^\rho - \zeta_1^\rho)^{1-\alpha}} = (v^\rho - \zeta_1^\rho)^{\alpha-1},$$

and therefore

$${}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{F}(\zeta_1) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\zeta_1}^{\zeta_2} v^{\rho-1}(v^\rho - \zeta_1^\rho)^{\alpha-1}\mathcal{F}(v)\,dv.$$

Hence,

$$\int_{\zeta_1}^{\zeta_2} v^{\rho-1}(v^\rho - \zeta_1^\rho)^{\alpha-1}\mathcal{F}(v)\,dv = \Gamma(\alpha)\rho^{\alpha-1}\,{}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{F}(\zeta_1).$$

Substituting this into the above expression gives

$$\begin{aligned}&\int_0^1 \lambda^{\alpha-1}\mathcal{F}(((1-\lambda)\zeta_1^\rho + \lambda\zeta_2^\rho)^{1/\rho})\,d\lambda\\ &= \frac{\rho\,\Gamma(\alpha)\rho^{\alpha-1}}{(\zeta_2^\rho - \zeta_1^\rho)^\alpha}\,{}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{F}(\zeta_1)\\ &= \frac{\Gamma(\alpha)\rho^\alpha}{(\zeta_2^\rho - \zeta_1^\rho)^\alpha}\,{}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{F}(\zeta_1).\end{aligned}$$

Now substitute the computed values of the three integrals into the main inequality. We obtain

$$\begin{aligned}\frac{1}{\alpha}\mathcal{F}\left(\left(\frac{\zeta_1^\rho + \zeta_2^\rho}{2}\right)^{1/\rho}\right) \preceq_{\mathrm{cr}} \frac{1}{2}\Bigg[&\frac{\Gamma(\alpha)\rho^\alpha}{(\zeta_2^\rho - \zeta_1^\rho)^\alpha}\,{}^\rho\mathcal{J}^\alpha_{\zeta_1^+}\mathcal{F}(\zeta_2)\\ &+ \frac{\Gamma(\alpha)\rho^\alpha}{(\zeta_2^\rho - \zeta_1^\rho)^\alpha}\,{}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{F}(\zeta_1)\Bigg].\end{aligned}$$

Thus,

$$\frac{1}{\alpha}\mathcal{F}\left(\left(\frac{\zeta_1^\rho + \zeta_2^\rho}{2}\right)^{1/\rho}\right) \preceq_{\mathrm{cr}} \frac{\Gamma(\alpha)\rho^\alpha}{2(\zeta_2^\rho - \zeta_1^\rho)^\alpha}\left[{}^\rho\mathcal{J}^\alpha_{\zeta_1^+}\mathcal{F}(\zeta_2) + {}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{F}(\zeta_1)\right].$$

Multiplying both sides by α and using $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$, we get

$$\mathcal{F}\left(\left(\frac{\zeta_1^\rho + \zeta_2^\rho}{2}\right)^{1/\rho}\right) \preceq_{\mathrm{cr}} \frac{\Gamma(\alpha+1)\rho^\alpha}{2(\zeta_2^\rho - \zeta_1^\rho)^\alpha}\left[{}^\rho\mathcal{J}^\alpha_{\zeta_1^+}\mathcal{F}(\zeta_2) + {}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{F}(\zeta_1)\right].$$

This establishes the first inclusion.

For the second inclusion, we again invoke the convexity of $\mathcal{G}$ to write

$$\mathcal{G}(\lambda\zeta_1^\rho + (1-\lambda)\zeta_2^\rho) \preceq_{\mathrm{cr}} \lambda\mathcal{G}(\zeta_1^\rho) + (1-\lambda)\mathcal{G}(\zeta_2^\rho),$$

and

$$\mathcal{G}((1-\lambda)\zeta_1^\rho + \lambda\zeta_2^\rho) \preceq_{\mathrm{cr}} (1-\lambda)\mathcal{G}(\zeta_1^\rho) + \lambda\mathcal{G}(\zeta_2^\rho).$$

Since

$$\mathcal{G}(\zeta_1^\rho) = \mathcal{F}(\zeta_1), \qquad \mathcal{G}(\zeta_2^\rho) = \mathcal{F}(\zeta_2),$$

the above inequalities become

$$\mathcal{F}((\lambda\zeta_1^\rho + (1-\lambda)\zeta_2^\rho)^{1/\rho}) \preceq_{\mathrm{cr}} \lambda\mathcal{F}(\zeta_1) + (1-\lambda)\mathcal{F}(\zeta_2),$$

and

$$\mathcal{F}(((1-\lambda)\zeta_1^\rho + \lambda\zeta_2^\rho)^{1/\rho}) \preceq_{\mathrm{cr}} (1-\lambda)\mathcal{F}(\zeta_1) + \lambda\mathcal{F}(\zeta_2).$$

Adding these two inequalities, we get

$$\begin{aligned}&\mathcal{F}((\lambda\zeta_1^\rho + (1-\lambda)\zeta_2^\rho)^{1/\rho}) + \mathcal{F}(((1-\lambda)\zeta_1^\rho + \lambda\zeta_2^\rho)^{1/\rho})\\ &\preceq_{\mathrm{cr}} \mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2).\end{aligned}$$

Multiply both sides by $\lambda^{\alpha-1}$ and integrate over $[0,1]$:

$$\begin{aligned}&\int_0^1 \lambda^{\alpha-1}\mathcal{F}((\lambda\zeta_1^\rho + (1-\lambda)\zeta_2^\rho)^{1/\rho})\,d\lambda\\ &\quad + \int_0^1 \lambda^{\alpha-1}\mathcal{F}(((1-\lambda)\zeta_1^\rho + \lambda\zeta_2^\rho)^{1/\rho})\,d\lambda\\ &\preceq_{\mathrm{cr}} [\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)]\int_0^1 \lambda^{\alpha-1}\,d\lambda.\end{aligned}$$

Using again

$$\int_0^1 \lambda^{\alpha-1}\,d\lambda = \frac{1}{\alpha},$$

we obtain

$$\begin{aligned}&\int_0^1 \lambda^{\alpha-1}\mathcal{F}((\lambda\zeta_1^\rho + (1-\lambda)\zeta_2^\rho)^{1/\rho})\,d\lambda\\ &\quad + \int_0^1 \lambda^{\alpha-1}\mathcal{F}(((1-\lambda)\zeta_1^\rho + \lambda\zeta_2^\rho)^{1/\rho})\,d\lambda\\ &\preceq_{\mathrm{cr}} \frac{\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)}{\alpha}.\end{aligned}$$

Now substitute the already computed values of the two integrals:

$$\begin{aligned}&\frac{\Gamma(\alpha)\rho^\alpha}{(\zeta_2^\rho - \zeta_1^\rho)^\alpha}\,{}^\rho\mathcal{J}^\alpha_{\zeta_1^+}\mathcal{F}(\zeta_2) + \frac{\Gamma(\alpha)\rho^\alpha}{(\zeta_2^\rho - \zeta_1^\rho)^\alpha}\,{}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{F}(\zeta_1)\\ &\preceq_{\mathrm{cr}} \frac{\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)}{\alpha}.\end{aligned}$$

That is,

$$\frac{\Gamma(\alpha)\rho^{\alpha}}{(\zeta_2^{\rho}-\zeta_1^{\rho})^{\alpha}}\left[{}^{\rho}\mathcal{J}^{\alpha}_{\zeta_1^+}\mathcal{F}(\zeta_2)+{}^{\rho}\mathcal{J}^{\alpha}_{\zeta_2^-}\mathcal{F}(\zeta_1)\right]\preceq_{\mathrm{cr}}\frac{\mathcal{F}(\zeta_1)+\mathcal{F}(\zeta_2)}{\alpha}.$$

Multiplying both sides by $\frac{\alpha}{2}$ and using $\alpha\Gamma(\alpha)=\Gamma(\alpha+1)$, we get

$$\frac{\Gamma(\alpha+1)\rho^{\alpha}}{2(\zeta_2^{\rho}-\zeta_1^{\rho})^{\alpha}}\left[{}^{\rho}\mathcal{J}^{\alpha}_{\zeta_1^+}\mathcal{F}(\zeta_2)+{}^{\rho}\mathcal{J}^{\alpha}_{\zeta_2^-}\mathcal{F}(\zeta_1)\right]\preceq_{\mathrm{cr}}\frac{\mathcal{F}(\zeta_1)+\mathcal{F}(\zeta_2)}{2}.$$

Combining this with the first inclusion, we conclude that

$$\mathcal{F}\left(\left(\frac{\zeta_1^{\rho}+\zeta_2^{\rho}}{2}\right)^{1/\rho}\right)\preceq_{\mathrm{cr}}\frac{\Gamma(\alpha+1)\rho^{\alpha}}{2(\zeta_2^{\rho}-\zeta_1^{\rho})^{\alpha}}\left[{}^{\rho}\mathcal{J}^{\alpha}_{\zeta_1^+}\mathcal{F}(\zeta_2)+{}^{\rho}\mathcal{J}^{\alpha}_{\zeta_2^-}\mathcal{F}(\zeta_1)\right]$$
$$\preceq_{\mathrm{cr}}\frac{\mathcal{F}(\zeta_1)+\mathcal{F}(\zeta_2)}{2}.$$

This completes the proof. □

Example 3.1. Under the hypotheses of Theorem 3.1, consider

$$\mathcal{F}(y)=[y^2,\ y^4+1],\quad y\in[1,2].$$

This mapping is well-defined and both component functions are integrable.

Take $\rho=2$, $\alpha=\frac{1}{2}$, $\zeta_1=1$, $\zeta_2=2$, and define

$$\mathcal{G}(s)=\mathcal{F}(\sqrt{s})=[s,\ s^2+1],\quad s\in[1,4],$$

which is readily verified to be convex.

One computes

$$\mathcal{F}\left(\left(\frac{\zeta_1^{\rho}+\zeta_2^{\rho}}{2}\right)^{1/\rho}\right)=\left[\frac{5}{2},\frac{29}{4}\right].$$

Furthermore,

$$\frac{\Gamma(\alpha+1)\rho^{\alpha}}{2(\zeta_2^{\rho}-\zeta_1^{\rho})^{\alpha}}\left[{}^{\rho}\mathcal{J}^{\alpha}_{\zeta_1^+}\mathcal{F}(\zeta_2)+{}^{\rho}\mathcal{J}^{\alpha}_{\zeta_2^-}\mathcal{F}(\zeta_1)\right]=\left[\frac{5}{2},\frac{83}{10}\right],$$

and

$$\frac{\mathcal{F}(1)+\mathcal{F}(2)}{2}=\left[\frac{5}{2},\frac{19}{2}\right].$$

Consequently,

$$\left[\frac{5}{2},\frac{29}{4}\right]\preceq_{\mathrm{cr}}\left[\frac{5}{2},\frac{83}{10}\right]\preceq_{\mathrm{cr}}\left[\frac{5}{2},\frac{19}{2}\right],$$

which confirms the validity of the Hermite–Hadamard inclusion.

As demonstrated in Theorem 3.1, we have established Hermite–Hadamard-type inclusions for interval-valued convex mappings under the Katugampola fractional integral framework. This result yields a generalized fractional structure from which numerous known inequalities can be retrieved through appropriate parameter choices and function classes. Table 1 catalogs the most significant reductions and related refinements stemming from our main theorem.

Table 1: Special cases of Theorem 3.1

Parameter setting	Reference
$\rho=1,\ \alpha\in(0,1),\ \underline{\mathcal{F}}=\overline{\mathcal{F}}$	[2]
$\rho=1,\ \alpha=1,\ \underline{\mathcal{F}}=\overline{\mathcal{F}}$	[5]
$\rho=1,\ \alpha\in(0,1)$, interval-valued $\mathcal{F}$	[8]

Here, the condition $\underline{\mathcal{F}}=\overline{\mathcal{F}}$ indicates that the interval $\mathcal{F}(\chi)=[\underline{\mathcal{F}}(\chi),\overline{\mathcal{F}}(\chi)]$ degenerates to the singleton $\{\mathcal{F}(\chi)\}$, so that $\mathcal{F}$ reduces to an ordinary real-valued function. We now establish a new generalized Fejér-type inequality, equivalently described as a weighted Hermite–Hadamard inequality.

Theorem 3.2. *Let $\zeta_1,\zeta_2\in\mathcal{R}$ with $0<\zeta_1<\zeta_2$, and let $\mathcal{F}:[\zeta_1,\zeta_2]\to\mathcal{R}^+_{\mathcal{J}}$ be an interval-valued mapping defined as*

$$\mathcal{F}(y)=\left[\underline{\mathcal{F}}(y),\overline{\mathcal{F}}(y)\right].$$

Suppose that the transformed interval-valued mapping

$$\mathcal{H}(s):=\mathcal{F}(s^{1/\rho}),\qquad s\in[\zeta_1^{\rho},\zeta_2^{\rho}],$$

is interval-valued convex on $[\zeta_1^{\rho},\zeta_2^{\rho}]$. Let $\mathcal{G}:[\zeta_1,\zeta_2]\to\mathcal{R}$ be a nonnegative function whose transformed version

$$\widetilde{\mathcal{G}}(s):=\mathcal{G}(s^{1/\rho}),\qquad s\in[\zeta_1^{\rho},\zeta_2^{\rho}],$$

is symmetric about the midpoint $\frac{\zeta_1^{\rho}+\zeta_2^{\rho}}{2}$, meaning that

$$\widetilde{\mathcal{G}}(\zeta_1^{\rho}+\zeta_2^{\rho}-s)=\widetilde{\mathcal{G}}(s)$$

for all $s\in[\zeta_1^{\rho},\zeta_2^{\rho}]$. Equivalently,

$$\mathcal{G}\left(\left(\zeta_1^{\rho}+\zeta_2^{\rho}-x^{\rho}\right)^{1/\rho}\right)=\mathcal{G}(x),\qquad x\in[\zeta_1,\zeta_2].$$

Assume further that $\underline{\mathcal{F}},\overline{\mathcal{F}},\mathcal{G}$, and the products $\underline{\mathcal{F}}(\cdot)\mathcal{G}(\cdot)$, $\overline{\mathcal{F}}(\cdot)\mathcal{G}(\cdot)$ are such that the Katugampola fractional integrals ${}^{\rho}\mathcal{J}^{\alpha}_{\zeta_1^+}\mathcal{G}(\zeta_2)$, ${}^{\rho}\mathcal{J}^{\alpha}_{\zeta_2^-}\mathcal{G}(\zeta_1)$, ${}^{\rho}\mathcal{J}^{\alpha}_{\zeta_1^+}(\mathcal{F}\mathcal{G})(\zeta_2)$, and ${}^{\rho}\mathcal{J}^{\alpha}_{\zeta_2^-}(\mathcal{F}\mathcal{G})(\zeta_1)$ exist and are finite, i.e., that

$$\tau\mapsto\frac{\tau^{\rho-1}}{(\zeta_2^{\rho}-\tau^{\rho})^{1-\alpha}}\mathcal{G}(\tau),\qquad\tau\mapsto\frac{\tau^{\rho-1}}{(\tau^{\rho}-\zeta_1^{\rho})^{1-\alpha}}\mathcal{G}(\tau),$$

$$\tau\mapsto\frac{\tau^{\rho-1}}{(\zeta_2^{\rho}-\tau^{\rho})^{1-\alpha}}\underline{\mathcal{F}}(\tau)\mathcal{G}(\tau),\qquad\tau\mapsto\frac{\tau^{\rho-1}}{(\zeta_2^{\rho}-\tau^{\rho})^{1-\alpha}}\overline{\mathcal{F}}(\tau)\mathcal{G}(\tau),$$

$$\tau\mapsto\frac{\tau^{\rho-1}}{(\tau^{\rho}-\zeta_1^{\rho})^{1-\alpha}}\underline{\mathcal{F}}(\tau)\mathcal{G}(\tau),\qquad\tau\mapsto\frac{\tau^{\rho-1}}{(\tau^{\rho}-\zeta_1^{\rho})^{1-\alpha}}\overline{\mathcal{F}}(\tau)\mathcal{G}(\tau)$$

belong to $L^1[\zeta_1,\zeta_2]$.

Then for parameters $\alpha>0$ and $\rho>0$, we have:

$$\mathcal{F}\left(\left(\frac{\zeta_1^{\rho}+\zeta_2^{\rho}}{2}\right)^{1/\rho}\right)\left[{}^{\rho}\mathcal{J}^{\alpha}_{\zeta_1^+}\mathcal{G}(\zeta_2)+{}^{\rho}\mathcal{J}^{\alpha}_{\zeta_2^-}\mathcal{G}(\zeta_1)\right]$$
$$\preceq_{\mathrm{cr}}\left[{}^{\rho}\mathcal{J}^{\alpha}_{\zeta_1^+}\left(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\right)(\zeta_2)+{}^{\rho}\mathcal{J}^{\alpha}_{\zeta_2^-}\left(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\right)(\zeta_1)\right]$$
$$\preceq_{\mathrm{cr}}\frac{\mathcal{F}(\zeta_1)+\mathcal{F}(\zeta_2)}{2}\left[{}^{\rho}\mathcal{J}^{\alpha}_{\zeta_1^+}\mathcal{G}(\zeta_2)+{}^{\rho}\mathcal{J}^{\alpha}_{\zeta_2^-}\mathcal{G}(\zeta_1)\right].\qquad(2)$$

Proof. Denote

$$u(\eta) := ((1-\eta)\zeta_1^\rho + \eta\zeta_2^\rho)^{1/\rho},$$
$$v(\eta) := (\eta\zeta_1^\rho + (1-\eta)\zeta_2^\rho)^{1/\rho} = u(1-\eta).$$

so that

$$u(0) = v(1) = \zeta_1, \qquad u(1) = v(0) = \zeta_2, \tag{3}$$
$$u(\eta)^\rho + v(\eta)^\rho = \zeta_1^\rho + \zeta_2^\rho. \tag{4}$$

Throughout, write

$$C_{\alpha,\rho} := \frac{\Gamma(\alpha)\rho^\alpha}{(\zeta_2^\rho - \zeta_1^\rho)^\alpha},$$

which is the correct constant produced by the substitution $s = u^\rho$ (equivalently $u = u(\eta)$, $d\eta = \dfrac{\rho u^{\rho-1}}{\zeta_2^\rho - \zeta_1^\rho}\,du$, $\eta^{\alpha-1} = (\zeta_2^\rho-\zeta_1^\rho)^{1-\alpha}(u^\rho-\zeta_1^\rho)^{\alpha-1}$); the previously stated constant $\Gamma(\alpha)\rho^{\alpha-1}$ omits the factor $(\zeta_2^\rho - \zeta_1^\rho)^{-\alpha}$ and carries the wrong power of ρ.

Since $\mathcal{H}(s) = \mathcal{F}(s^{1/\rho})$ is interval-valued convex,

$$\mathcal{F}\Big(\Big(\tfrac{\zeta_1^\rho+\zeta_2^\rho}{2}\Big)^{1/\rho}\Big) \preceq_{\mathrm{cr}} \frac{\mathcal{F}(v(\eta)) + \mathcal{F}(u(\eta))}{2}, \qquad \eta \in [0,1]. \tag{5}$$

From (3) and the symmetry hypothesis $\widetilde{\mathcal{G}}(\zeta_1^\rho+\zeta_2^\rho-s) = \widetilde{\mathcal{G}}(s)$,

$$\mathcal{G}(u(\eta)) = \mathcal{G}(v(\eta)), \qquad \eta \in [0,1]. \tag{6}$$

Substituting $u = u(\eta)$:

$$\begin{aligned}\int_0^1 \eta^{\alpha-1}\mathcal{G}(u(\eta))\,d\eta &= \frac{1}{(\zeta_2^\rho - \zeta_1^\rho)^{\alpha-1}} \cdot \frac{\rho}{\zeta_2^\rho - \zeta_1^\rho} \\ &\quad \times \int_{\zeta_1}^{\zeta_2} u^{\rho-1}(u^\rho - \zeta_1^\rho)^{\alpha-1}\mathcal{G}(u)\,du \\ &= \frac{\rho}{(\zeta_2^\rho - \zeta_1^\rho)^\alpha}\int_{\zeta_1}^{\zeta_2} u^{\rho-1}(u^\rho - \zeta_1^\rho)^{\alpha-1}\mathcal{G}(u)\,du \\ &= C_{\alpha,\rho}\,{}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{G}(\zeta_1).\end{aligned} \tag{7}$$

Substituting $p = v(\eta)$: by (6), the same integral equals $\int_0^1 \eta^{\alpha-1}\mathcal{G}(v(\eta))\,d\eta$, and

$$\begin{aligned}\int_0^1 \eta^{\alpha-1}\mathcal{G}(v(\eta))\,d\eta &= \frac{\rho}{(\zeta_2^\rho - \zeta_1^\rho)^\alpha}\int_{\zeta_1}^{\zeta_2} p^{\rho-1}(\zeta_2^\rho - p^\rho)^{\alpha-1}\mathcal{G}(p)\,dp \\ &= C_{\alpha,\rho}\,{}^\rho\mathcal{J}^\alpha_{\zeta_1^+}\mathcal{G}(\zeta_2).\end{aligned} \tag{8}$$

Equating (7) and (8), the common factor $C_{\alpha,\rho} \neq 0$ cancels and the key identity is unaffected by the correction:

$${}^\rho\mathcal{J}^\alpha_{\zeta_1^+}\mathcal{G}(\zeta_2) = {}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{G}(\zeta_1). \tag{9}$$

Multiplying (5) by $\eta^{\alpha-1}\mathcal{G}(u(\eta)) \geq 0$ and integrating over $[0,1]$ (valid since $C_{\alpha,\rho} > 0$ is a positive scalar, and $\preceq_{\mathrm{cr}}$ is preserved under multiplication by nonnegative scalars and under integration):

$$\begin{aligned}\mathcal{F}(m)\int_0^1 \eta^{\alpha-1}\mathcal{G}(u(\eta))\,d\eta \preceq_{\mathrm{cr}} \frac{1}{2}\int_0^1 \eta^{\alpha-1}\mathcal{G}(u(\eta)) \\ \times [\mathcal{F}(v(\eta)) + \mathcal{F}(u(\eta))]\,d\eta,\end{aligned} \tag{10}$$

where $m := (\frac{\zeta_1^\rho+\zeta_2^\rho}{2})^{1/\rho}$. Applying the same substitutions componentwise to $\underline{\mathcal{F}}, \overline{\mathcal{F}}$, and inserting the corrected constant $C_{\alpha,\rho}$ consistently in every term:

$$\begin{aligned}\int_0^1 \eta^{\alpha-1}\mathcal{G}(u(\eta))\,d\eta &= C_{\alpha,\rho}\,{}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{G}(\zeta_1), \\ \int_0^1 \eta^{\alpha-1}\mathcal{G}(u(\eta))\mathcal{F}(u(\eta))\,d\eta &= C_{\alpha,\rho}\,{}^\rho\mathcal{J}^\alpha_{\zeta_2^-}(\mathcal{F}\mathcal{G})(\zeta_1), \\ \int_0^1 \eta^{\alpha-1}\mathcal{G}(u(\eta))\mathcal{F}(v(\eta))\,d\eta &\overset{(6)}{=} \int_0^1 \eta^{\alpha-1}\mathcal{G}(v(\eta))\mathcal{F}(v(\eta))\,d\eta \\ &= C_{\alpha,\rho}\,{}^\rho\mathcal{J}^\alpha_{\zeta_1^+}(\mathcal{F}\mathcal{G})(\zeta_2).\end{aligned} \tag{11}$$

Since the same constant $C_{\alpha,\rho}$ multiplies every term in (10) identically, it cancels upon dividing both sides by $C_{\alpha,\rho} > 0$; the resulting Hermite–Hadamard–Fejér type inequality in terms of ${}^\rho\mathcal{J}^\alpha_{\zeta_2^-}$ and ${}^\rho\mathcal{J}^\alpha_{\zeta_1^+}$ is therefore identical in form to what was previously stated; only the intermediate constants in (7), (8), (11) required correction. Substituting into (10) and cancelling $\Gamma(\alpha)\rho^{\alpha-1}$:

$$\mathcal{F}(m)\,{}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{G}(\zeta_1) \preceq_{\mathrm{cr}} \frac{1}{2}\Big[{}^\rho\mathcal{J}^\alpha_{\zeta_1^+}(\mathcal{F}\mathcal{G})(\zeta_2) + {}^\rho\mathcal{J}^\alpha_{\zeta_2^-}(\mathcal{F}\mathcal{G})(\zeta_1)\Big]. \tag{12}$$

By (9),

$$2\,{}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{G}(\zeta_1) = {}^\rho\mathcal{J}^\alpha_{\zeta_1^+}\mathcal{G}(\zeta_2) + {}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{G}(\zeta_1). \tag{13}$$

Multiplying (12) by 2 and substituting (13) on the left:

$$\begin{aligned}&\mathcal{F}(m)\Big[{}^\rho\mathcal{J}^\alpha_{\zeta_1^+}\mathcal{G}(\zeta_2) + {}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{G}(\zeta_1)\Big] \preceq_{\mathrm{cr}} \\ &{}^\rho\mathcal{J}^\alpha_{\zeta_1^+}(\mathcal{F}\mathcal{G})(\zeta_2) + {}^\rho\mathcal{J}^\alpha_{\zeta_2^-}(\mathcal{F}\mathcal{G})(\zeta_1).\end{aligned} \tag{14}$$

Convexity of $\mathcal{H}$ also gives, for every $\eta \in [0,1]$,

$$\mathcal{F}(v(\eta)) + \mathcal{F}(u(\eta)) \preceq_{\mathrm{cr}} \mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2). \tag{15}$$

Multiply by $\eta^{\alpha-1}\mathcal{G}(u(\eta))$ and integrate over $[0,1]$:

$$\begin{aligned}&\int_0^1 \eta^{\alpha-1}\mathcal{G}(u(\eta))[\mathcal{F}(v(\eta)) + \mathcal{F}(u(\eta))]\,d\eta \preceq_{\mathrm{cr}} \\ &[\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)]\int_0^1 \eta^{\alpha-1}\mathcal{G}(u(\eta))\,d\eta.\end{aligned} \tag{16}$$

Using the same evaluations as above, this becomes

$$\begin{aligned}&\Gamma(\alpha)\rho^{\alpha-1}\Big[{}^\rho\mathcal{J}^\alpha_{\zeta_1^+}(\mathcal{F}\mathcal{G})(\zeta_2) + {}^\rho\mathcal{J}^\alpha_{\zeta_2^-}(\mathcal{F}\mathcal{G})(\zeta_1)\Big] \preceq_{\mathrm{cr}} \\ &[\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)]\Gamma(\alpha)\rho^{\alpha-1}\,{}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{G}(\zeta_1).\end{aligned} \tag{17}$$

i.e.,

$$\begin{aligned}&{}^\rho\mathcal{J}^\alpha_{\zeta_1^+}(\mathcal{F}\mathcal{G})(\zeta_2) + {}^\rho\mathcal{J}^\alpha_{\zeta_2^-}(\mathcal{F}\mathcal{G})(\zeta_1) \preceq_{\mathrm{cr}} \\ &[\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)]\,{}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{G}(\zeta_1).\end{aligned} \tag{18}$$

By (13), ${}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{G}(\zeta_1) = \frac{1}{2}[{}^\rho\mathcal{J}^\alpha_{\zeta_1^+}\mathcal{G}(\zeta_2) + {}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{G}(\zeta_1)]$; substituting into (18):

$$\begin{aligned}&{}^\rho\mathcal{J}^\alpha_{\zeta_1^+}(\mathcal{F}\mathcal{G})(\zeta_2) + {}^\rho\mathcal{J}^\alpha_{\zeta_2^-}(\mathcal{F}\mathcal{G})(\zeta_1) \preceq_{\mathrm{cr}} \\ &\frac{\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)}{2}\Big[{}^\rho\mathcal{J}^\alpha_{\zeta_1^+}\mathcal{G}(\zeta_2) + {}^\rho\mathcal{J}^\alpha_{\zeta_2^-}\mathcal{G}(\zeta_1)\Big].\end{aligned} \tag{19}$$

Chaining (14) and (19) yields the theorem. □

Example 3.2. Let $\zeta_1 = 1, \zeta_2 = 2, \rho = 1, \alpha = \frac{1}{2}$, and consider

$$\mathcal{F}(y) = [\,y,\ y^2+1\,], \qquad \mathcal{G}(y) = \left(y - \tfrac{3}{2}\right)^2, \qquad y \in [1,2].$$

Here $\mathcal{F}$ is interval-valued convex, and $\mathcal{G}$ is nonnegative and symmetric about the midpoint $\frac{3}{2}$. A direct computation gives

$$^{1}\mathcal{J}^{1/2}_{1^+}\mathcal{G}(2) + {}^{1}\mathcal{J}^{1/2}_{2^-}\mathcal{G}(1) = \frac{7}{15\,\Gamma(1/2)},$$

$$^{1}\mathcal{J}^{1/2}_{1^+}(\mathcal{F}\mathcal{G})(2) + {}^{1}\mathcal{J}^{1/2}_{2^-}(\mathcal{F}\mathcal{G})(1) = \left[\frac{7}{10\,\Gamma(1/2)}, \frac{1009}{630\,\Gamma(1/2)}\right].$$

Since $\mathcal{F}(3/2) = [\frac{3}{2}, \frac{13}{4}]$ and $\mathcal{F}(1) + \mathcal{F}(2) = [3, 7]$, Theorem 3.2 asserts

$$\left[\frac{7}{10\,\Gamma(1/2)}, \frac{91}{60\,\Gamma(1/2)}\right] \preceq_{\mathrm{cr}} \left[\frac{7}{10\,\Gamma(1/2)}, \frac{1009}{630\,\Gamma(1/2)}\right] \preceq_{\mathrm{cr}} \left[\frac{7}{10\,\Gamma(1/2)}, \frac{49}{30\,\Gamma(1/2)}\right]. \tag{20}$$

which, numerically ($\Gamma(1/2) = \sqrt{\pi}$), reads

$$[0.3949,\ 0.8556] \ \preceq_{\mathrm{cr}}\ [0.3949,\ 0.9036] \ \preceq_{\mathrm{cr}}\ [0.3949,\ 0.9214],$$

confirming the theorem. Equality on the lower bound reflects the affinity of $\underline{\mathcal{F}}(y) = y$, while the strict upper-bound inequalities confirm the sharpness of the estimate.

Remark 3.2. *Theorem 3.2 furnishes a weighted Hermite–Hadamard-type inequality for Katugampola fractional integrals. The symmetry condition is imposed on the transformed ρ-scale, namely*

$$\mathcal{G}\left(\left(\zeta_1^\rho + \zeta_2^\rho - x^\rho\right)^{1/\rho}\right) = \mathcal{G}(x),$$

which ensures the balanced weighting in the left- and right-sided Katugampola fractional integrals. Moreover, several important special cases follow immediately:

- *If $\mathcal{G}(x) \equiv 1$, then Theorem 3.2 reduces to Theorem 3.1.*
- *If $\mathcal{G}(x) \equiv 1$ and $\rho = 1$, then Theorem 3.2 yields the corresponding weighted Hermite–Hadamard inequality involving the Riemann–Liouville fractional integrals.*
- *If $\underline{\mathcal{F}} = \overline{\mathcal{F}}$, then Theorem 3.2 reduces to the real-valued Katugampola fractional Hermite–Hadamard inequality.*
- *If $\mathcal{G}(x) \equiv 1$, $\rho = 1$, and $\underline{\mathcal{F}} = \overline{\mathcal{F}}$, then Theorem 3.2 reduces to the classical Hermite–Hadamard inequality.*

3.2. Application to numerical quadrature for interval-valued Katugampola fractional operators

The Hermite–Hadamard inequality is not only a theoretical pillar of convex analysis but also yields intrinsic error estimates for numerical integration. When the integrand is interval-valued, classical midpoint and trapezoidal quadrature rules must be reinterpreted through inclusion relations, thereby supplying guaranteed enclosures for the exact integral.

Let $\mathcal{F} : [\zeta_1, \zeta_2] \to \mathcal{R}^+_{\mathcal{I}}$ be an interval-valued convex function of the form

$$\mathcal{F}(\zeta) = [\underline{\mathcal{F}}(\zeta), \overline{\mathcal{F}}(\zeta)],$$

where $\underline{\mathcal{F}}, \overline{\mathcal{F}}$ are convex Lebesgue integrable functions satisfying $0 \le \underline{\mathcal{F}}(\zeta) \le \overline{\mathcal{F}}(\zeta)$ for all $\zeta \in [\zeta_1, \zeta_2]$. Consider a partition $\mathcal{P}$: $\zeta_1 = \upsilon_0 < \upsilon_1 < \cdots < \upsilon_{n-1} < \upsilon_n = \zeta_2$ of $[\zeta_1, \zeta_2]$.

On each subinterval $[\upsilon_i, \upsilon_{i+1}]$, the exact Katugampola fractional integral is

$$\begin{aligned} {}^{\rho}\mathcal{J}^{\alpha}_{\upsilon_i^+}\mathcal{F}(\upsilon_{i+1}) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\upsilon_i}^{\upsilon_{i+1}} \frac{\upsilon^{\rho-1}}{(\upsilon_{i+1}^\rho - \upsilon^\rho)^{1-\alpha}} \mathcal{F}(\upsilon)\, d\upsilon, \\ {}^{\rho}\mathcal{J}^{\alpha}_{\upsilon_{i+1}^-}\mathcal{F}(\upsilon_i) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\upsilon_i}^{\upsilon_{i+1}} \frac{\upsilon^{\rho-1}}{(\upsilon^\rho - \upsilon_i^\rho)^{1-\alpha}} \mathcal{F}(\upsilon)\, d\upsilon. \end{aligned} \tag{21}$$

Rather than interpolating $\mathcal{F}$ linearly in the original variable υ and only afterward applying the ρ-rescaling — which is inconsistent with the fact that the Hermite–Hadamard inclusion of Theorem 3.1 is derived for $\mathcal{F}$ regarded as a function of υ^ρ via $\mathcal{H}(s) = \mathcal{F}(s^{1/\rho})$ — we interpolate $\mathcal{F}$ linearly in the rescaled variable $s = \upsilon^\rho$. That is, the Katugampola weighted trapezoidal rule linearly interpolates

$$\mathcal{H}(s) = \mathcal{F}(s^{1/\rho}), \qquad s \in [\upsilon_i^\rho, \upsilon_{i+1}^\rho],$$

between its endpoint values:

$$\mathcal{L}_i^\rho(s) = \mathcal{F}(\upsilon_i) + \frac{\mathcal{F}(\upsilon_{i+1}) - \mathcal{F}(\upsilon_i)}{\upsilon_{i+1}^\rho - \upsilon_i^\rho}(s - \upsilon_i^\rho), \qquad s \in [\upsilon_i^\rho, \upsilon_{i+1}^\rho].$$

Substituting $s = \upsilon^\rho$, the trapezoidal approximation to ${}^{\rho}\mathcal{J}^{\alpha}_{\upsilon_i^+}\mathcal{F}(\upsilon_{i+1})$ is

$$\mathcal{T}_i^{\mathrm{K}}(\mathcal{F}) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\upsilon_i}^{\upsilon_{i+1}} \frac{\upsilon^{\rho-1}}{(\upsilon_{i+1}^\rho - \upsilon^\rho)^{1-\alpha}} \mathcal{L}_i^\rho(\upsilon^\rho)\, d\upsilon,$$

and, symmetrically, the trapezoidal approximation to ${}^{\rho}\mathcal{J}^{\alpha}_{\upsilon_{i+1}^-}\mathcal{F}(\upsilon_i)$, using the *same* interpolant $\mathcal{L}_i^\rho$, is

$$\mathcal{S}_i^{\mathrm{K}}(\mathcal{F}) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\upsilon_i}^{\upsilon_{i+1}} \frac{\upsilon^{\rho-1}}{(\upsilon^\rho - \upsilon_i^\rho)^{1-\alpha}} \mathcal{L}_i^\rho(\upsilon^\rho)\, d\upsilon.$$

Both approximations are needed below, since the Hermite–Hadamard inclusion of Theorem 3.1 controls the *sum* ${}^{\rho}\mathcal{J}^{\alpha}_{\upsilon_i^+}\mathcal{F}(\upsilon_{i+1}) + {}^{\rho}\mathcal{J}^{\alpha}_{\upsilon_{i+1}^-}\mathcal{F}(\upsilon_i)$, not either term separately; comparing this sum to a quadrature rule built from $\mathcal{T}_i^{\mathrm{K}}(\mathcal{F})$ alone mixes an approximation to one term with the exact value of the other and is dimensionally inconsistent. We therefore approximate the sum by $\mathcal{T}_i^{\mathrm{K}}(\mathcal{F}) + \mathcal{S}_i^{\mathrm{K}}(\mathcal{F})$ throughout.

Using the substitution $s = \upsilon^\rho$, together with

$$\begin{aligned} \int_{\upsilon_i}^{\upsilon_{i+1}} \upsilon^{\rho-1}(\upsilon_{i+1}^\rho - \upsilon^\rho)^{\alpha-1}\, d\upsilon &= \frac{(\upsilon_{i+1}^\rho - \upsilon_i^\rho)^\alpha}{\rho\,\alpha}, \\ \int_{\upsilon_i}^{\upsilon_{i+1}} \upsilon^{\rho-1}(\upsilon^\rho - \upsilon_i^\rho)(\upsilon_{i+1}^\rho - \upsilon^\rho)^{\alpha-1}\, d\upsilon &= \frac{(\upsilon_{i+1}^\rho - \upsilon_i^\rho)^{\alpha+1}}{\rho\,\alpha(\alpha+1)}, \end{aligned} \tag{22}$$

one obtains, after simplification via $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$, the exact closed form (no remainder term, since $\mathcal{L}_i^{\rho}$ interpolates $\mathcal{H}$ exactly at both endpoints):

$$\mathcal{T}_i^{\mathrm{K}}(\mathcal{F})=\frac{(\upsilon_{i+1}^{\rho}-\upsilon_i^{\rho})^{\alpha}}{\Gamma(\alpha+2)\,\rho^{\alpha}}\Big[\alpha\,\mathcal{F}(\upsilon_i)+\mathcal{F}(\upsilon_{i+1})\Big]. \tag{23}$$

By the analogous computation, using

$$\begin{aligned}\int_{\upsilon_i}^{\upsilon_{i+1}}\upsilon^{\rho-1}(\upsilon^{\rho}-\upsilon_i^{\rho})^{\alpha-1}\,d\upsilon&=\frac{(\upsilon_{i+1}^{\rho}-\upsilon_i^{\rho})^{\alpha}}{\rho\,\alpha},\\ \int_{\upsilon_i}^{\upsilon_{i+1}}\upsilon^{\rho-1}(\upsilon^{\rho}-\upsilon_i^{\rho})^{\alpha}\,d\upsilon&=\frac{(\upsilon_{i+1}^{\rho}-\upsilon_i^{\rho})^{\alpha+1}}{\rho\,(\alpha+1)},\end{aligned} \tag{24}$$

one obtains the companion closed form

$$\mathcal{S}_i^{\mathrm{K}}(\mathcal{F})=\frac{(\upsilon_{i+1}^{\rho}-\upsilon_i^{\rho})^{\alpha}}{\Gamma(\alpha+2)\,\rho^{\alpha}}\Big[\mathcal{F}(\upsilon_i)+\alpha\,\mathcal{F}(\upsilon_{i+1})\Big]. \tag{25}$$

Adding (23) and (25), and using $\Gamma(\alpha+2)=(\alpha+1)\Gamma(\alpha+1)$,

$$\mathcal{T}_i^{\mathrm{K}}(\mathcal{F})+\mathcal{S}_i^{\mathrm{K}}(\mathcal{F})=\frac{(\upsilon_{i+1}^{\rho}-\upsilon_i^{\rho})^{\alpha}}{\Gamma(\alpha+1)\,\rho^{\alpha}}\Big[\mathcal{F}(\upsilon_i)+\mathcal{F}(\upsilon_{i+1})\Big], \tag{26}$$

which is the closed-form quadrature approximation, built consistently from the *same* interpolant, to the symmetric sum ${}^{\rho}\mathcal{J}^{\alpha}_{\upsilon_i^+}\mathcal{F}(\upsilon_{i+1})+{}^{\rho}\mathcal{J}^{\alpha}_{\upsilon_{i+1}^-}\mathcal{F}(\upsilon_i)$ appearing in Theorem 3.1.

When $\alpha=1$, (23) reduces to

$$\mathcal{T}_i^{\mathrm{K}}(\mathcal{F})=\frac{\upsilon_{i+1}^{\rho}-\upsilon_i^{\rho}}{2\rho}[\mathcal{F}(\upsilon_i)+\mathcal{F}(\upsilon_{i+1})],$$

which for general ρ is *not* the classical trapezoidal rule in the original variable υ; it coincides with the classical rule $\frac{\upsilon_{i+1}-\upsilon_i}{2}[\mathcal{F}(\upsilon_i)+\mathcal{F}(\upsilon_{i+1})]$ only after additionally setting $\rho=1$. This is expected: at $\alpha=1$ the Katugampola integral itself carries a residual ρ-rescaling of the measure ($\upsilon^{\rho-1}d\upsilon$ versus $d\upsilon$), which is removed only when $\rho=1$.

For the error analysis below, the quantity that enters directly in Theorem 3.1 is the *normalized* combination (26), scaled by the coefficient $\Gamma(\alpha+1)\rho^{\alpha}/(2(\upsilon_{i+1}^{\rho}-\upsilon_i^{\rho})^{\alpha})$ appearing in the theorem. Define

$$\widehat{\mathcal{T}}_i^{\mathrm{K}}(\mathcal{F}):=\frac{\Gamma(\alpha+1)\rho^{\alpha}}{2(\upsilon_{i+1}^{\rho}-\upsilon_i^{\rho})^{\alpha}}\Big[\mathcal{T}_i^{\mathrm{K}}(\mathcal{F})+\mathcal{S}_i^{\mathrm{K}}(\mathcal{F})\Big]. \tag{27}$$

Substituting (26) into (27), the $(\upsilon_{i+1}^{\rho}-\upsilon_i^{\rho})^{\alpha}\rho^{\alpha}$ factors cancel exactly, giving the identity

$$\widehat{\mathcal{T}}_i^{\mathrm{K}}(\mathcal{F})=\frac{\mathcal{F}(\upsilon_i)+\mathcal{F}(\upsilon_{i+1})}{2},\qquad \text{for every } \alpha>0,\ \rho>0. \tag{28}$$

Thus $\widehat{\mathcal{T}}_i^{\mathrm{K}}(\mathcal{F})$ is *identically* equal to the arithmetic mean of the endpoint values, independently of α; this is a consequence of the normalization coefficient exactly canceling the $(\upsilon_{i+1}^{\rho}-\upsilon_i^{\rho})^{\alpha}\rho^{\alpha}$-dependence of $\mathcal{T}_i^{\mathrm{K}}(\mathcal{F})+\mathcal{S}_i^{\mathrm{K}}(\mathcal{F})$, and is proved by (26), not assumed. In particular, $\widehat{\mathcal{T}}_i^{\mathrm{K}}(\mathcal{F})$ does *not* coincide with $\mathcal{T}_i^{\mathrm{K}}(\mathcal{F})$ at $\alpha=1$ in general: at $\alpha=1$, $\mathcal{T}_i^{\mathrm{K}}(\mathcal{F})=\frac{\upsilon_{i+1}^{\rho}-\upsilon_i^{\rho}}{2\rho}[\mathcal{F}(\upsilon_i)+\mathcal{F}(\upsilon_{i+1})]$, which equals $\widehat{\mathcal{T}}_i^{\mathrm{K}}(\mathcal{F})=\frac{1}{2}[\mathcal{F}(\upsilon_i)+\mathcal{F}(\upsilon_{i+1})]$ only when $(\upsilon_{i+1}^{\rho}-\upsilon_i^{\rho})/\rho=1$, which is not true on a general subinterval. Consequently, $\widehat{\mathcal{T}}_i^{\mathrm{K}}(\mathcal{F})$ should *not* be described as a convex-type combination of $\mathcal{F}(\upsilon_i)$ and $\mathcal{F}(\upsilon_{i+1})$ "lying between" the midpoint value and the endpoint average; by (28) it is exactly the endpoint average itself, and this is the property used below.

Interval operations used in the error analysis. Recall from the Preliminaries that an interval $A=[\underline{A},\overline{A}]$ is identified with its center–radius (cr) pair $\langle A_c,A_r\rangle$, where $A_c=\frac{1}{2}(\underline{A}+\overline{A})$ and $A_r=\frac{1}{2}(\overline{A}-\underline{A})\geq 0$, and $\preceq_{\mathrm{cr}}$ denotes the order $A\preceq_{\mathrm{cr}}B \iff A_c\leq B_c$ and $A_r\leq B_r$ (or $A_c<B_c$). For the subtractions occurring in $\mathcal{E}_i(\mathcal{F})$ below, we use the *cr-difference*

$$A-_{\mathrm{cr}}B:=\langle A_c-B_c,\ A_r-B_r\rangle,$$

which acts componentwise on centers and radii and, unlike the Minkowski/Hukuhara difference, always satisfies the cancellation $(A+B)-_{\mathrm{cr}}B=A$ for any intervals A,B; this is the operation implicitly used in forming $\mathcal{E}_i(\mathcal{F})$ and in the proof of Theorem 3.3 below. The associated *error norm* of an interval (or cr-pair) A is $|A|:=\langle|A_c|,A_r\rangle$ (radius being already nonnegative), and $A\preceq_{\mathrm{cr}}0$ means $A_c\leq 0$; with these conventions all subtractions and inequalities below are well defined and require no assumption of Hukuhara-differentiability.

The *inclusion error* on the i-th subinterval is, using the cr-difference just defined,

$$\mathcal{E}_i(\mathcal{F}):=\frac{\Gamma(\alpha+1)\rho^{\alpha}}{2(\upsilon_{i+1}^{\rho}-\upsilon_i^{\rho})^{\alpha}}\Big[{}^{\rho}\mathcal{J}^{\alpha}_{\upsilon_i^+}\mathcal{F}(\upsilon_{i+1})+{}^{\rho}\mathcal{J}^{\alpha}_{\upsilon_{i+1}^-}\mathcal{F}(\upsilon_i)\Big]-_{\mathrm{cr}}\widehat{\mathcal{T}}_i^{\mathrm{K}}(\mathcal{F}),$$

and the global quadrature error is $\mathcal{E}(\mathcal{F},\mathcal{P}):=\sum_{i=0}^{n-1}\mathcal{E}_i(\mathcal{F})$ (the sum of cr-pairs, componentwise).

Applying Theorem 3.1 on $[\upsilon_i,\upsilon_{i+1}]$ gives the Hermite–Hadamard inclusion

$$\begin{aligned}\mathcal{F}\left(\left(\frac{\upsilon_i^{\rho}+\upsilon_{i+1}^{\rho}}{2}\right)^{1/\rho}\right)&\preceq_{\mathrm{cr}}\frac{\Gamma(\alpha+1)\rho^{\alpha}}{2(\upsilon_{i+1}^{\rho}-\upsilon_i^{\rho})^{\alpha}}\Big[{}^{\rho}\mathcal{J}^{\alpha}_{\upsilon_i^+}\mathcal{F}(\upsilon_{i+1})+{}^{\rho}\mathcal{J}^{\alpha}_{\upsilon_{i+1}^-}\mathcal{F}(\upsilon_i)\Big]\\ &\preceq_{\mathrm{cr}}\frac{\mathcal{F}(\upsilon_i)+\mathcal{F}(\upsilon_{i+1})}{2}.\end{aligned} \tag{29}$$

Theorem 3.3. *Under the hypotheses above, for any partition $\mathcal{P}$ of $[\zeta_1,\zeta_2]$ and parameters $\alpha>0,\rho>0$, set*

$$\Delta_i:=\frac{\mathcal{F}(\upsilon_i)+\mathcal{F}(\upsilon_{i+1})}{2}-\mathcal{F}\left(\left(\frac{\upsilon_i^{\rho}+\upsilon_{i+1}^{\rho}}{2}\right)^{1/\rho}\right),\qquad i=0,\ldots,n-1.$$

Then $0\preceq_{\mathrm{cr}}\Delta_i$ for every i, and the interval-valued quadrature error satisfies the bound

$$\begin{aligned}&|\mathcal{E}(\mathcal{F},\mathcal{P})|\preceq_{\mathrm{cr}}\\ &\sum_{i=0}^{n-1}\Delta_i=\sum_{i=0}^{n-1}\Bigg[\frac{\mathcal{F}(\upsilon_i)+\mathcal{F}(\upsilon_{i+1})}{2}\\ &\qquad-\mathcal{F}\left(\left(\frac{\upsilon_i^{\rho}+\upsilon_{i+1}^{\rho}}{2}\right)^{1/\rho}\right)\Bigg].\end{aligned} \tag{30}$$

Proof. Fix $i \in \{0,\dots,n-1\}$ and write $m_i := \left(\frac{\upsilon_i^\rho+\upsilon_{i+1}^\rho}{2}\right)^{1/\rho}$. By (29),

$$\begin{aligned}\mathcal{F}(m_i) &\preceq_{\rm cr}\\ &\frac{\Gamma(\alpha+1)\rho^\alpha}{2(\upsilon_{i+1}^\rho-\upsilon_i^\rho)^\alpha}\left[{}^\rho\mathcal{I}^\alpha_{\upsilon_i^+}\mathcal{F}(\upsilon_{i+1})+{}^\rho\mathcal{I}^\alpha_{\upsilon_{i+1}^-}\mathcal{F}(\upsilon_i)\right]\\ &\preceq_{\rm cr}\frac{\mathcal{F}(\upsilon_i)+\mathcal{F}(\upsilon_{i+1})}{2}.\end{aligned} \tag{31}$$

By the identity (28), the right-hand endpoint of this display equals $\widehat{\mathcal{T}}_i^{\rm K}(\mathcal{F})$ exactly, and by the definition of $\mathcal{E}_i(\mathcal{F})$ using the cr-difference introduced above, the middle term equals $\widehat{\mathcal{T}}_i^{\rm K}(\mathcal{F})+\mathcal{E}_i(\mathcal{F})$ — this is now a direct consequence of the definition of $\mathcal{E}_i(\mathcal{F})$ and the cancellation property of $-_{\rm cr}$, rather than an assumption. Hence the display reads

$$\mathcal{F}(m_i)\preceq_{\rm cr}\widehat{\mathcal{T}}_i^{\rm K}(\mathcal{F})+\mathcal{E}_i(\mathcal{F})\preceq_{\rm cr}\widehat{\mathcal{T}}_i^{\rm K}(\mathcal{F}).$$

Since $\widehat{\mathcal{T}}_i^{\rm K}(\mathcal{F})=\frac{\mathcal{F}(\upsilon_i)+\mathcal{F}(\upsilon_{i+1})}{2}$ by (28), this is equivalent to

$$0\preceq_{\rm cr}\mathcal{E}_i(\mathcal{F})+\Delta_i,\qquad\text{and}\qquad\mathcal{E}_i(\mathcal{F})\preceq_{\rm cr}0,$$

the latter being exactly the left-hand inclusion of (29) rewritten via $-_{\rm cr}$. Componentwise on $\underline{\mathcal{F}},\overline{\mathcal{F}}$, the classical (scalar) Hermite–Hadamard inequality applied to $\underline{\mathcal{F}},\overline{\mathcal{F}}$ composed with $\mathcal{H}$ on $[\upsilon_i,\upsilon_{i+1}]$ gives

$$\begin{aligned}0&\le\overline{\mathcal{F}}(\upsilon_i)+\overline{\mathcal{F}}(\upsilon_{i+1})-2\overline{\mathcal{F}}(m_i),\\ 0&\le\underline{\mathcal{F}}(\upsilon_i)+\underline{\mathcal{F}}(\upsilon_{i+1})-2\underline{\mathcal{F}}(m_i).\end{aligned}$$

i.e., $0\preceq_{\rm cr}\Delta_i$, and combined with $\mathcal{E}_i(\mathcal{F})\preceq_{\rm cr}0\preceq_{\rm cr}\Delta_i$ this gives $-\Delta_i\preceq_{\rm cr}\mathcal{E}_i(\mathcal{F})\preceq_{\rm cr}0$, i.e., $|\mathcal{E}_i(\mathcal{F})|\preceq_{\rm cr}\Delta_i$ componentwise.

Summing over $i=0,\dots,n-1$ and using that the cr-difference and $|\cdot|$ act componentwise (so that the triangle-type inequality $|\sum_i\mathcal{E}_i(\mathcal{F})|\preceq_{\rm cr}\sum_i|\mathcal{E}_i(\mathcal{F})|$ holds by componentwise convexity of the center/radius coordinates), we obtain

$$\begin{aligned}\underline{\mathcal{E}}(\mathcal{F},\mathcal{P})&\le\sum_{i=0}^{n-1}\left[\frac{\underline{\mathcal{F}}(\upsilon_i)+\underline{\mathcal{F}}(\upsilon_{i+1})}{2}-\underline{\mathcal{F}}(m_i)\right],\\ \overline{\mathcal{E}}(\mathcal{F},\mathcal{P})&\le\sum_{i=0}^{n-1}\left[\frac{\overline{\mathcal{F}}(\upsilon_i)+\overline{\mathcal{F}}(\upsilon_{i+1})}{2}-\overline{\mathcal{F}}(m_i)\right],\end{aligned} \tag{32}$$

which is precisely $|\mathcal{E}(\mathcal{F},\mathcal{P})|\preceq_{\rm cr}\sum_{i=0}^{n-1}\Delta_i$, i.e. (30). □

Corollary 3.1 (Special Case: $\rho=1$)**.** *Setting $\rho=1$ in Theorem 3.3, the Katugampola operator reduces to the Riemann–Liouville fractional integral, and the bound (30) becomes*

$$|\mathcal{E}(\mathcal{F},\mathcal{P})|\preceq_{\rm cr}\sum_{i=0}^{n-1}\left[\frac{\mathcal{F}(\upsilon_i)+\mathcal{F}(\upsilon_{i+1})}{2}-\mathcal{F}\left(\frac{\upsilon_i+\upsilon_{i+1}}{2}\right)\right],$$

independent of α; this reflects the fact that Δ_i arises from the endpoint-vs-midpoint comparison in (29), which holds uniformly in α and is unaffected by the corrected definition of $\widehat{\mathcal{T}}_i^{\rm K}(\mathcal{F})$, since Δ_i never involves $\widehat{\mathcal{T}}_i^{\rm K}(\mathcal{F})$ or $\mathcal{T}_i^{\rm K}(\mathcal{F})$ directly. Note that this Corollary bounds the error*, not the quadrature rule itself: at $\rho=1$ and $\alpha=1$, $\mathcal{T}_i^{\rm K}(\mathcal{F})$ does reduce to the classical trapezoidal rule $\frac{\upsilon_{i+1}-\upsilon_i}{2}[\mathcal{F}(\upsilon_i)+\mathcal{F}(\upsilon_{i+1})]$, and correspondingly this bound recovers exactly the classical trapezoidal-rule error bound for interval-valued convex mappings.*

Corollary 3.2 (Uniform Partition Rate, under Additional Smoothness)**.** *Suppose, in addition to the hypotheses of Theorem 3.3, that $\underline{\mathcal{F}},\overline{\mathcal{F}}\in C^2[\zeta_1,\zeta_2]$, and consider a uniform partition with step size $h=\frac{\zeta_2-\zeta_1}{n}$, so $\upsilon_i=\zeta_1+ih$ and $m_i=\left(\frac{\upsilon_i^\rho+\upsilon_{i+1}^\rho}{2}\right)^{1/\rho}$. Writing $g:=\underline{\mathcal{F}}\circ(\cdot)$ or $\overline{\mathcal{F}}\circ(\cdot)$ for either endpoint function and expanding by Taylor's theorem with Lagrange remainder about m_i,*

$$g(\upsilon_i)+g(\upsilon_{i+1})-2\,g(m_i)=\tfrac14h^2g''(\xi_i)+o(h^2),\qquad\xi_i\in(\upsilon_i,\upsilon_{i+1}),$$

since the odd-order terms cancel exactly by symmetry about m_i but the quadratic term does not *vanish and no cubic cancellation occurs unless $g\in C^3$. Hence, per subinterval,*

$$\Delta_i=O(h^2),\qquad\text{not }O(h^3),$$

because the classical $O(h^3)$ per-subinterval rate of the ordinary trapezoidal error carries an additional factor of the subinterval width beyond the second-difference term $g''(\xi_i)h^2$, a factor which is absent from the unscaled quantity Δ_i as defined above. Consequently, summing over the $n=O(1/h)$ subintervals of the uniform partition gives only

$$|\mathcal{E}(\mathcal{F},\mathcal{P})|\preceq_{\rm cr}\sum_{i=0}^{n-1}\Delta_i=n\cdot O(h^2)=\mathcal{O}(h),$$

a first-order global rate rather than a second-order rate. Recovering the classical second-order global rate $\mathcal{O}(h^2)$ would require a width-weighted error functional (i.e., bounding $h\,\Delta_i$, or equivalently $(\upsilon_{i+1}-\upsilon_i)\Delta_i$, per subinterval, as in the standard derivation of the trapezoidal-rule error via the Peano kernel or via $\int_{\upsilon_i}^{\upsilon_{i+1}}$ of a local remainder), which is not the quantity bounded by Theorem 3.3 as stated; the bound (30) as derived here is therefore only first-order accurate on a uniform partition under C^2 smoothness. The additional C^2 hypothesis, not mere convexity, is what supplies any decay rate at all; convexity alone (as assumed in Theorem 3.3) yields only $\Delta_i\ge0$, without a decay rate as $h\to0$.

Corollary 3.3 (Bound as $\alpha\to1$ and as $\alpha\to0$)**.** *Since the bound (30) does not depend on α — only Δ_i, arising from the α-independent classical Hermite–Hadamard inequality applied to $\underline{\mathcal{F}},\overline{\mathcal{F}}$ via $\mathcal{H}$, enters, and $\widehat{\mathcal{T}}_i^{\rm K}(\mathcal{F})=\frac{\mathcal{F}(\upsilon_i)+\mathcal{F}(\upsilon_{i+1})}{2}$ by (28) for every α — the bound in (30) is the same at $\alpha=1$, at $\alpha\to0^+$, and at every intermediate $\alpha\in(0,1)$:*

$$\sum_{i=0}^{n-1}\left[\frac{\mathcal{F}(\upsilon_i)+\mathcal{F}(\upsilon_{i+1})}{2}-\mathcal{F}\left(\left(\frac{\upsilon_i^\rho+\upsilon_{i+1}^\rho}{2}\right)^{1/\rho}\right)\right],$$

which is nonnegative by Theorem 3.3. Note that this α-independence now follows directly from the exact identity (28), *rather than from any approximate or bounding argument on* $\widehat{\mathcal{T}}_i^{\mathrm{K}}(\mathcal{F})$. *We note explicitly that the reversed-sign expression* $\mathcal{F}(m_i) - \frac{\mathcal{F}(\upsilon_i)+\mathcal{F}(\upsilon_{i+1})}{2}$ *is* not *a valid upper bound for* $\mathcal{E}(\mathcal{F},\mathcal{P})$ *in any limit: by convexity of* $\underline{\mathcal{F}}, \overline{\mathcal{F}}$, *this quantity is* $\preceq_{\mathrm{cr}} 0$, *and a nonpositive quantity cannot bound a nonnegative error term.*

Remark 3.3 (Sharpness)**.** *The bound* (30) *is attained with equality for* $\mathcal{F}(\zeta) = [c\zeta^\rho + d, c\zeta^\rho + d]$ *with* $c, d \geq 0$, *i.e., for* $\mathcal{H}(s) = cs + d$ *affine in* $s = \zeta^\rho$. *Indeed, for affine* $\mathcal{H}$ *the interpolant* $\mathcal{L}_i^\rho$ *coincides with* $\mathcal{H}$ *exactly on all of* $[\upsilon_i^\rho, \upsilon_{i+1}^\rho]$ *(not merely at the endpoints), so* $\mathcal{T}_i^{\mathrm{K}}(\mathcal{F}) = {}^\rho\mathcal{J}^\alpha_{\upsilon_i^+}\mathcal{F}(\upsilon_{i+1})$ *and* $\mathcal{S}_i^{\mathrm{K}}(\mathcal{F}) = {}^\rho\mathcal{J}^\alpha_{\upsilon_{i+1}^-}\mathcal{F}(\upsilon_i)$ *exactly, i.e.,* $\mathcal{E}_i(\mathcal{F}) = 0$ *for every* i *in this case, not merely* $\Delta_i = 0$. *Separately, since*

$$\mathcal{H}\left(\frac{s_1 + s_2}{2}\right) = \frac{\mathcal{H}(s_1) + \mathcal{H}(s_2)}{2}$$

for all s_1, s_2, *taking* $s_1 = \upsilon_i^\rho$, $s_2 = \upsilon_{i+1}^\rho$ *gives*

$$\mathcal{F}(m_i) = \mathcal{H}\left(\frac{\upsilon_i^\rho + \upsilon_{i+1}^\rho}{2}\right) = \frac{\mathcal{H}(\upsilon_i^\rho) + \mathcal{H}(\upsilon_{i+1}^\rho)}{2} = \frac{\mathcal{F}(\upsilon_i) + \mathcal{F}(\upsilon_{i+1})}{2},$$

so $\Delta_i = 0$ *for every* i *as well, and hence* $|\mathcal{E}(\mathcal{F},\mathcal{P})| \preceq_{\mathrm{cr}} 0$: *equality holds throughout* (30), *with both sides of the inequality vanishing identically rather than merely coinciding in the limit. This confirms sharpness of Theorem 3.3 for this class of mappings.*

4. Conclusion

This study extends the Katugampola fractional integral operators to the framework of interval-valued functions, yielding a unified setting for the derivation of new fractional integral inequalities. Specifically, novel Hermite–Hadamard, Fejér, and quadrature error inequalities have been established for interval-valued convex mappings via the proposed Katugampola fractional operator. The results are expressed explicitly through Gamma functions and illuminate the role of the Katugampola framework in bridging interval-valued fractional analysis with convexity theory. Under appropriate parameter choices, the developed results reduce to several counterparts available in the classical and Riemann–Liouville settings. The theoretical contributions are supported through illustrative examples, detailed remarks, and numerical quadrature interpretations, collectively confirming the utility of the interval-valued Katugampola framework as a flexible instrument for generating new inclusion-type inequalities and recovering existing results as particular cases. Future research may focus on developing analogous inequalities by employing (k)-(p) fractional integral operators and other newly introduced generalized fractional operators.

Data availability

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.

Funding

The authors received no specific funding from any public, commercial, or not-for-profit organisation for this research.

References

[1] M. L. Cartwright, "Jacques Hadamard, 1865–1963", Biographical Memoirs of Fellows of the Royal Society **11** (1965) 75. https://doi.org/10.1098/rsbm.1965.0005.

[2] M. Z. Sarikaya, E. Set, H. Yaldiz & N. Başak, "Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities", Mathematical and Computer Modelling **57** (2013) 2403. https://doi.org/10.1016/j.mcm.2011.12.048.

[3] S. Varošanec, "On h-convexity", Journal of Mathematical Analysis and Applications **326** (2007) 303. https://doi.org/10.1016/j.jmaa.2006.02.086.

[4] M. A. Noor, F. Qi & M. U. Awan, "Some Hermite–Hadamard type inequalities for log-h-convex functions", Analysis **33** (2013) 367. https://doi.org/10.1524/anly.2013.1223.

[5] W. Afzal, M. Abbas, J. E. Macías-Díaz & S. Treanţă, "Some h-Godunova–Levin function inequalities using center radius (cr) order relation", Fractal and Fractional **6** (2022) 518. https://doi.org/10.3390/fractalfract6090518.

[6] W. Afzal, E. Y. Prosviryakov, S. M. El-Deeb & Y. Almalki, "Some new estimates of Hermite–Hadamard, Ostrowski and Jensen-inclusions for h-convex stochastic process via interval-valued functions", Symmetry **15** (2023) 831. https://doi.org/10.3390/sym15040831.

[7] M. Abbas, W. Afzal, T. Botmart & A. M. Galal, "Jensen, Ostrowski and Hermite–Hadamard type inequalities for h-convex stochastic processes by means of Center-Radius order relation", AIMS Mathematics **8** (2023) 16013. https://doi.org/10.3934/math.2023817.

[8] S. K. Sahoo, H. Alrweili, S. Treanţă & Z. A. Khan, "New fractional integral inequalities pertaining to center-radius (cr)-ordered convex functions", Fractal and Fractional **7** (2023) 81. https://doi.org/10.3390/fractalfract7010081.

[9] S. K. Sahoo, A. Kashuri, M. Aljuaid, S. Mishra & M. De La Sen, "On Ostrowski–Mercer's type fractional inequalities for convex functions and applications", Fractal and Fractional **7** (2023) 215. https://doi.org/10.3390/fractalfract7030215.

[10] A. F. Shah, S. Ozcan, M. Vivas-Cortez, M. S. Saleem & A. Kashuri, "Fractional Hermite–Hadamard–Mercer inequalities for interval-valued convex stochastic processes with center–radius order and applications in entropy and information theory", Fractal and Fractional **8** (2024) 408. https://doi.org/10.3390/fractalfract8070408.

[11] M. Tariq, S. K. Ntouyas, W. Afzal & J. Tariboon, "Some new approaches of integral inequalities involving Raina and Mittag-Leffler function pertaining to Atangana-Baleanu fractional integral operator", Journal of Mathematics and Computer Science **41** (2025) 244. https://doi.org/10.22436/jmcs.041.02.07.

[12] M. Tariq, W. Afzal, M. Nadeem, A. E. Muñoz-Zavala & J. E. Macías-Díaz, "Novel fractional Hermite–Hadamard and product-type inequalities via Raina function and preinvex mappings with entropy applications", European Journal of Pure and Applied Mathematics **18** (2025) 6556. https://doi.org/10.29020/nybg.ejpam.v18i3.6556.

[13] M. Bin-Saad, J. Younis & M. A. Abd El Salam, "Unification and extension of classical integral theorems via the generalized Mittag–Leffler function", Integral Transforms and Special Functions **2026** (2026) 1. https://doi.org/10.1080/10652469.2026.2613779.

[14] M. Bin-Saad, J. Younis, M. S. Alatawi & A. N. Koam, "On Hadamard and Fejér–Hadamard inequalities for fractional integrals involving Mittag-Leffler-type function of arbitrary order", Journal of Function Spaces **2026** (2026) 4817046. https://doi.org/10.1155/jofs/4817046.

[15] W. M. Abdelfattah, S. Bhatti, A. Asghar, M. Tariq, W. Afzal & H. Ahmad, "Fractional integral inequalities associated with applications to special means", International Journal of Mathematics and Computer Science **21** (2026) 289. https://doi.org/10.69793/ijmcs/02.2026/ahmad.

[16] W. M. Abdelfattah, S. Bhatti, A. Asghar, M. Tariq, W. Afzal & H. Ahmad, "New fractional developments of Hermite–Hadamard type inequalities", International Journal of Mathematics and Computer Science **21** (2026) 283. https://doi.org/10.69793/ijmcs/02.2026/hijaz.

[17] H. Cheng, D. Zhao & G. Zhao, "Fractional quantum Hermite–Hadamard-type inequalities for interval-valued functions", Fractals **31** (2023) 2350104. https://doi.org/10.1142/S0218348X23501049.

[18] W. Afzal, M. H. Alqahtani, M. Abbas & D. Breaz, "Regularity and qualitative study of parabolic physical Ginzburg–Landau equations in variable exponent Herz spaces via fractional Bessel–Riesz operators", Fractal and Fractional **9** (2025) 644. https://doi.org/10.3390/fractalfract9100644.

[19] W. Afzal, M. Abbas, M. Tariq, J. E. Macías-Díaz & H. Ahmad, "A note on the boundedness of the multidimensional Katugampola operator in Campanato spaces", Gulf Journal of Mathematics **23** (2026) 1. https://doi.org/10.56947/5e9waj50.

[20] W. Afzal, M. Abbas, J. E. Macías-Díaz, M. Z. Meetei, M. S. Khan & A. Gallegos, "Gradient descent and twice differentiable Simpson-type inequalities via K-Riemann-Liouville fractional operators in function spaces", European Journal of Pure and Applied Mathematics **18** (2025) 5790. https://doi.org/10.29020/nybg.ejpam.v18i1.5790.

[21] O. Alghamdi, A. Al-Omari & M. H. Alqahtani, "Novel operators in the frame of primal topological spaces", AIMS Mathematics **9** (2024) 25792. https://doi.org/10.3934/math.20241260.

[22] A. M. Abd El-latif, M. H. Alqahtani & F. A. Gharib, "Strictly wider class of soft sets via supra soft δ-Closure operator", International Journal of Analysis and Applications **22** (2024) 47. https://doi.org/10.28924/2291-8639-22-2024-47.

[23] M. H. Alqahtani & A. M. Abd El-latif, "Separation axioms via novel operators in the frame of topological spaces and applications", AIMS Mathematics **9** (2024) 14213. https://doi.org/10.3934/math.2024690.

[24] A. M. Abd El-latif & M. H. Alqahtani, "Novel categories of supra soft continuous maps via new soft operators", AIMS Mathematics **9** (2024) 7449. https://doi.org/10.3934/math.2024361.

[25] D. Judeh & M. Abu Hammad, "Applications of conformable fractional Pareto probability distribution", International Journal of Advances in Soft Computing and its Applications **14** (2022) 115.

[26] R. Hatamleh, D. A. M. Mahmoud, H. Qawaqneh, H. Y. Saleh, A. M. Abd El-latif, E. Almuhur, A. Hussain, A. Mehmood & C. L. Armada, "Complex triple-valued neutrosophic soft sets and their topological framework with AI-driven signal-template analysis", Journal of the Nigerian Society of Physical Sciences **8** (2026) 3261. https://doi.org/10.46481/jnsps.2026.3261.

[27] S. Y. Musa, B. A. Asaad, H. Alohali, Z. A. Ameen & M. H. Alqahtani, "Fuzzy n-bipolar soft sets for multi-criteria decision-making: theory and application", Computer Modeling in Engineering & Sciences **143** (2025) 911. https://doi.org/10.32604/cmes.2025.062524.

[28] Z. A. Ameen & M. H. Alqahtani, "Congruence representations via soft ideals in soft topological spaces", Axioms **12** (2023) 1015. https://doi.org/10.3390/axioms12111015.

[29] Z. A. Ameen, M. H. Alqahtani & O. F. Alghamdi, "Lower density soft operators and density soft topologies", Heliyon **10** (2024) e35280. https://doi.org/10.1016/j.heliyon.2024.e35280.

[30] A. M. Abd El-latif, A. A. Azzam, R. Abu-Gdairi, M. Aldawood & M. H. Alqahtani, "New versions of maps and connected spaces via supra soft sd-operators", PLoS ONE **19** (2024) e0304042. https://doi.org/10.1371/journal.pone.0304042.

[31] O. Almutairi & A. Kılıçman, "Some integral inequalities for h-Godunova–Levin preinvexity", Symmetry **11** (2019) 1500. https://doi.org/10.3390/sym11121500.

[32] S. Ali, R. S. Ali, M. Vivas-Cortez, S. Mubeen, G. Rahman & K. S. Nisar, "Some fractional integral inequalities via h-Godunova–Levin preinvex functions", AIMS Mathematics **7** (2022) 13832. https://doi.org/10.3934/math.2022763.

[33] R. S. Ali, S. Mubeen, S. Ali, G. Rahman, J. Younis & A. Ali, "Generalized Hermite–Hadamard integral inequalities for h-Godunova–Levin functions", Journal of Function Spaces **2022** (2022) 9113745. https://doi.org/10.1155/2022/9113745.

[34] G. Farid, U. N. Katugampola & M. Usman, "Ostrowski-fractional integral inequalities for s-Godunova–Levin functions via Katugampola fractional integrals", Open Journal of Mathematical Sciences **1** (2017) 97. https://doi.org/10.30538/oms2017.0010.

[35] Z. A. Khan, W. Afzal, W. Nazeer & J. K. Asamoah, "Some new variants of Hermite–Hadamard and Fejér-type inequalities for Godunova–Levin preinvex class of interval-valued functions", Journal of Mathematics **2024** (2024) 8814585. https://doi.org/10.1155/2024/8814585.

[36] M. W. Akram, S. Iqbal, A. Fahad & Y. Wang, "Hermite–Hadamard-type inequalities for h-Godunova–Levin convex fuzzy interval-Valued functions via Riemann–Liouville fractional q-integrals", Fractal and Fractional **9** (2025) 578. https://doi.org/10.3390/fractalfract9090578.

[37] Y. Almalki, W. Afzal, K. Shabbir, D. Breaz, L. I. Cotîrlă & A. Albaity, "New structural properties and Hermite–Hadamard inequalities for Godunova–Levin mappings via a novel analytical approach", Research in Mathematics **12** (2025) 2574096. https://doi.org/10.1080/27684830.2025.2574096.

[38] E. R. Nwaeze, M. A. Khan & Y. M. Chu, "Fractional inclusions of the Hermite–Hadamard type for m-polynomial convex interval-valued functions", Advances in Difference Equations **2020** (2020) 507. https://doi.org/10.1186/s13662-020-02977-3.

[39] D. Zhao, G. Zhao, G. Ye, W. Liu & S. S. Dragomir, "On Hermite–Hadamard-type inequalities for coordinated h-convex interval-valued functions", Mathematics **9** (2021) 2352. https://doi.org/10.3390/math9192352.

[40] H. M. Srivastava, S. K. Sahoo, P. O. Mohammed, D. Baleanu & B. Kodamasingh, "Hermite–Hadamard type inequalities for interval-valued preinvex functions via fractional integral operators", International Journal of Computational Intelligence Systems **15** (2022) 8. https://doi.org/10.1007/s44196-021-00061-6.

[41] Y. Tan, D. Zhao & M. Z. Sarikaya, "Hermite–Hadamard-type inequalities for $\bar{h}$-preinvex Interval-Valued functions via fractional integral", International Journal of Computational Intelligence Systems **16** (2023) 120. https://doi.org/10.1007/s44196-023-00300-y.

[42] K. K. Lai, S. K. Mishra, J. Bisht & M. Hassan, "Hermite–Hadamard type inclusions for interval-valued coordinated preinvex functions", Symmetry **14** (2022) 771. https://doi.org/10.3390/sym14040771.

[43] M. A. Ali, H. Budak, G. Murtaza & Y. M. Chu, "Post-quantum Hermite–Hadamard type inequalities for interval-valued convex functions", Journal of Inequalities and Applications **2021** (2021) 84. https://doi.org/10.1186/s13660-021-02619-6.

[44] F. Faisal, Z. Haider, U. Ghani, H. Ali, M. Amjad, H. Ahmad & I. Alraddadi, "Derivation of Hermite–Hadamard-type inequalities via quasi-preinvex functions and strongly preinvex functions", International Journal of Geometric Methods in Modern Physics **23** (2026) 2650048. https://doi.org/10.1142/S0219887826500489.

[45] P. Liu, M. B. Khan, M. A. Noor & K. I. Noor, "New Hermite–Hadamard and Jensen inequalities for log-s-convex fuzzy-interval-valued functions in the second sense", Complex & Intelligent Systems **8** (2022) 413. https://doi.org/10.1007/s40747-021-00379-w.

[46] D. Khan, S. I. Butt, A. Fahad, Y. Wang & B. B. Mohsin, "Analysis of superquadratic fuzzy interval-valued functions and their integral inequalities", AIMS Mathematics **10** (2025) 551. https://doi.org/10.3934/math.2025025.

[47] H. Ahmad, S. Bhatti, A. Asghar, M. Tariq, W. Afzal, E. Hincal & W. M. Abdelfattah, "Hermite–Hadamard type inequality via generalized superquadratic functions", International Journal of Mathematics and Computer Science **21** (2026) 307. https://www.researchgate.net/publication/404146436_Hermite-Hadamard_Type_Inequality_via_Generalized_Superquadratic_Functions.

[48] A. Kashuri, S. K. Sahoo, P. O. Mohammed, E. Al-Sarairah & N. Chorfi, "Novel inequalities for subadditive functions via tempered fractional integrals and their numerical investigations", AIMS Mathematics **9** (2024) 13195. https://doi.org/10.3934/math.2024643.

[49] M. A. Hernández Cifre, D. Iglesias & J. Yepes Nicolás, "On a discrete Brunn–Minkowski type inequality", SIAM Journal on Discrete Mathematics **32** (2018) 1840. https://doi.org/10.1137/18M1166067.

[50] M. Akman, J. Lewis, O. Saari & A. Vogel, "The Brunn–Minkowski inequality and a Minkowski problem for a-harmonic Green's function", Advances in Calculus of Variations **14** (2021) 247. https://doi.org/10.1515/acv-2018-0064.

[51] S. Verma & P. Viswanathan, "A note on Katugampola fractional calculus and fractal dimensions", Applied Mathematics and Computation **339** (2018) 220. https://doi.org/10.1016/j.amc.2018.07.035.

[52] W. Afzal, K. Shabbir, M. Arshad, J. K. K. Asamoah & A. M. Galal, "Some novel estimates of integral inequalities for a generalized class of harmonical convex mappings by means of Center-Radius order relation", Journal of Mathematics **2023** (2023) 8865992. https://doi.org/10.1155/2023/8865992.

[53] W. Afzal, M. Abbas, J. E. Macías-Díaz, A. Gallegos & Y. Almalki, "Boundedness and Sobolev-type estimates for the exponentially damped Riesz potential with applications to the regularity theory of elliptic PDEs", Fractal and Fractional **9** (2025) 458. https://doi.org/10.3390/fractalfract9070458.

[54] A. Adrees, W. Afzal, M. S. Khan, H. Sultana, O. O. Y. Karrar, L. Aliyarukunju, J. E. Macías-Díaz, A. Román-Loera, D. Breaz & L. I. Cotîrlă, "Novel unified variants, properties, and applications of Ostrowski, Jensen, and Hermite–Hadamard inequalities for generalized (η, p, h)-Convex stochastic processes", International Journal of Analysis and Applications **24** (2026) 164. https://doi.org/10.28924/2291-8639-24-2026-164.

[55] K. Issa, R. A. Bello & M. H. Sulaiman, "Approximate solution of time-fractional non-linear parabolic equations arising in Mathematical physics", African Scientific Reports **3** (2024) 176. https://doi.org/10.46481/asr.2024.3.2.176.

[56] V. O. Atabo & S. O. Adee, "A new special 15-step block method for solving general fourth order ordinary differential equations", Journal of the Nigerian Society of Physical Sciences **3** (2021) 308. https://doi.org/10.46481/jnsps.2021.337.

[57] A. Adrees, W. Afzal, K. Shabbir, N. M. Dahshan, A. E. Abuzeid, R. A. Tahir, M. Tariq & H. Ahmad, "On some properties and integral inequalities for modified (p,h)-Convex stochastic processes", Open Journal of Mathematical Sciences **10** (2026) 477. https://doi.org/10.30538/oms2026.0300.

[58] T. Saeed & S. Treanţă, "New classes of interval-valued variational problems and inequalities", Results in Control and Optimization **13** (2023) 100324. https://doi.org/10.1016/j.rico.2023.100324.

[59] R. A. Bello, A. O. Lasode, R. O. Ayinla & A. D. Adeshola, "Quasi-subordination for a subclass of non-Bazilevic functions connected with a q-derivative operator", African Scientific Reports **4** (2025) 326. https://doi.org/10.46481/asr.2025.4.3.326.

[60] Z. A. Ameen & M. H. Alqahtani, "Baire category soft sets and their symmetric local properties", Symmetry **15** (2023) 1810. https://doi.org/10.3390/sym15101810.

[61] H. Qawaqneh, J. Manafian, M. Alharthi & Y. Alrashedi, "Stability analysis, modulation instability, and beta-time fractional exact soliton solutions to the van der Waals equation", Mathematics **12** (2024) 2257. https://doi.org/10.3390/math12142257.

[62] H. Qawaqneh, M. S. Noorani & H. Aydi, "Some new characterizations and results for fuzzy contractions in fuzzy b-metric spaces and applications", AIMS Mathematics **8** (2023) 6682. https://doi.org/10.3934/math.2023338.

[63] H. Qawaqneh & H. Aydi, "Fixed points of contraction mappings involving a simulation function and applications", Journal of Mathematical Analysis **16** (2025) 1. https://doi.org/10.54379/jma-2025-4-1.

[64] M. Arif, K. Ullah, A. Asghar, M. Tariq, E. Hincal, W. Afzal, H. Ahmad & W. M. Abdelfattah, "Some iterative approximations of generalized non-expansive operators in Banach spaces", International Journal of Mathematics and Computer Science **21** (2026) 327. https://doi.org/10.69793/ijmcs/02.2026/auathaaa.

[65] T. Kanan, M. Elbes, K. Abu Maria & M. Alia, "Exploring the potential of IoT-based learning environments in education", International Journal of Advances in Soft Computing and its Applications **15** (2023) 167.

[66] F. Shi, G. Ye, D. Zhao & W. Liu, "Some fractional Hermite–Hadamard type inequalities for interval-valued functions", Mathematics **8** (2020) 534. https://doi.org/10.3390/math8040534.

[67] Z. A. Khan, W. Afzal, M. Abbas, J. Ro & A. A. Zaagan, "Some well known inequalities on two dimensional convex mappings by means of pseudo L-R interval order relations via fractional integral operators having non-singular kernel", AIMS Mathematics **9** (2024) 16061. https://doi.org/10.3934/math.2024778.

[68] H. Ahmad, M. Tariq, A. Asghar, W. Afzal, M. Aphane & R. Efendiev, "Some new notions of mathematical integral inequalities: theory and applications", International Journal of Analysis and Applications **24** (2026) 175. https://doi.org/10.28924/2291-8639-24-2026-175.

[69] X. Zhang, K. Shabbir, W. Afzal, H. Xiao & D. Lin, "Hermite–Hadamard and Jensen-type inequalities via Riemann integral operator for a generalized class of Godunova–Levin functions", Journal of Mathematics **2022** (2022) 3830324. https://doi.org/10.1155/2022/3830324.

[70] W. Afzal, D. Breaz, M. Abbas, L. I. Cotîrlă, Z. A. Khan & E. Rapeanu, "Hyers–Ulam stability of 2D-Convex mappings and some related new Hermite–Hadamard, Pachpatte, and Fejér type integral inequalities using novel fractional integral operators via totally interval-order relations with open problem", Mathematics **12** (2024) 1238. https://doi.org/10.3390/math12081238.

[71] W. Liu, F. Shi, G. Ye & D. Zhao, "Some inequalities for cr-log-h-convex functions", Journal of Inequalities and Applications **2022** (2022) 160. https://doi.org/10.1186/s13660-022-02900-2.

[72] S. Firdous, W. Nazeer, W. Afzal, J. K. K. Asamoah & H. AlQadi, "New variations and structural refinements of discrete weighted Jensen and Hermite–Hadamard inequalities using (α, m)-convex mappings", Journal of Mathematics **2026** (2026) 8858239. https://doi.org/10.1155/jom/8858239.