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Optimal control of fractional-order stochastic systems under uncertainty with Poisson and V-jump perturbations

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Abstract

We address the optimal control problem for a novel class of fractional-order uncertain–stochastic dynamical systems perturbed simultaneously by stochastic and epistemic jump disturbances. The system dynamics are governed by Caputo fractional derivatives and driven by a multi-noise framework comprising Brownian motion, Poisson random measures, canonical Liu processes, and finite-variation uncertain V-jump processes, thereby establishing a hybrid fractional system with double-jump features. The primary novelty is a unified analytical framework that combines memory effects with dual-source jump discontinuities under probabilistic randomness and epistemic uncertainty. We prove the existence, uniqueness, and continuous dependence of mild solutions in a hybrid probability–belief L^2 framework under standard Lipschitz and growth conditions. We then define an optimal control problem with a combined probabilistic–uncertain performance criterion, verify the existence of optimal controls, and derive a Pontryagin-type maximum principle using a backward fractional adjoint system. Finally, numerical simulations for a fractional portfolio optimisation problem demonstrate the practical implications of memory, control, and multiple-jump disruptions.

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1. Introduction

Fractional differential equations have proved to be an important mathematical tool for modelling dynamical systems with memory and hereditary features. Of the several definitions of fractional derivatives, the Caputo derivative is especially well suited for control problems since it permits the use of classical initial conditions and describes nonlocal effects in time [1, 2]. Related developments in fractional-order modelling have also investigated foundational system dynamics, stability limits, and chaos control frameworks [3], as well as complex dynamical structures involving non-singular fractional kernels under the Mittag-Leffler law [4]. Brownian motion-driven fractional stochastic differential equations have been investigated broadly with respect to existence, uniqueness, stability, and optimal control; see, for instance, Refs. [5, 6].

In parallel with probabilistic modelling, uncertainty theory, introduced by Liu [7, 8], provides a rigorous framework for de-

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scribing systems influenced by belief-based uncertainty rather than randomness. In this theory, differential equations driven by canonical Liu processes are defined on uncertainty spaces equipped with belief measures [9]. Fundamental results on solution concepts, stability, and applications of uncertain differential equations can be found in Refs. [8, 10–12].

To account for systems affected simultaneously by randomness and uncertainty, uncertain–stochastic differential equations were proposed and systematically studied in Ref. [13]. These hybrid systems are defined on product spaces combining probability and uncertainty spaces, allowing Brownian motion and Poisson random measures to coexist with Liu processes. Existence, uniqueness, and stability results for uncertain–stochastic systems have been established under Lipschitz and growth conditions; see Refs. [13, 14].

Jump phenomena constitute an essential feature in the modelling of many complex systems. In stochastic analysis, jump discontinuities are typically described by Poisson random measures and Lévy processes [15, 16]. In contrast, uncertainty theory models jump-type disturbances using uncertain counting or renewal processes, often referred to as V-jump processes, which possess finite variation and are characterized by belief distributions rather than intensities [8, 13, 17]. The presence of jumps leads naturally to càdlàg solution paths and introduces significant analytical challenges. To manage these non-local behaviors, specialized analytical tools, differentiability formulations, and integral bounds such as Gronwall-type inequalities are fundamentally required [18].

Building on this framework, recent work has further extended the theory to incorporate multiple jump mechanisms within fractional dynamics; notably, Bankole et al. [19] established existence and uniqueness results for double V-jump fractional uncertain differential equations. Furthermore, advanced operator-theoretic methods and subordination principles have been deployed to analyze mild solutions of fractional jump evolution equations within generalized spaces [20], while the simulation of fractional-order epidemic models under parameter disruptions has paved the way for tracking complex hybrid trajectories [21, 22].

Despite these individual advancements in non-local behavior and specialized jump spaces, an integrated optimal control paradigm that simultaneously unifies fractional memory effects with dual-source jump discontinuities under both probabilistic randomness (Brownian motion and Poisson jumps) and epistemic uncertainty (Liu processes and V-jumps) has not yet been established.

This work offers a distinct theoretical contribution by establishing a comprehensive optimal control framework that integrates non-local memory effects (via fractional calculus), frequentist random fluctuations with jump discontinuities (via Brownian motion and Poisson random measures), and epistemic uncertainties with belief-based jump disruptions (via canonical Liu processes and uncertain V-jump processes). The primary novelty lies in the formulation of a unified analytical framework capable of capturing dual-source jump characteristics operating under two distinct mathematical paradigms: probability theory and uncertainty theory. By bridging these environments within a fractional-order dynamical system, the proposed model provides a flexible and robust mathematical tool for analyzing complex engineering and financial phenomena subject to both historical memory dependencies and heterogeneous noise disruptions.

The major goal of this work is to construct a solid mathematical theory for the analysis of these hybrid fractional systems. In particular, we investigate existence and uniqueness of mild solutions under standard Lipschitz continuity and linear growth assumptions. The obtained conclusions provide the basis for further investigation of optimality requirements and control synthesis for fractional uncertain–stochastic systems with Poisson and V-jump disturbances.

2. Mathematical preliminaries

This section collects the analytical tools and probabilistic–uncertain structures required for the formulation of fractional uncertain–stochastic systems. All concepts introduced here are standard or suitably adapted from the literature and will be used throughout the paper. No control-theoretic notions are introduced at this stage.

2.1. Fractional calculus

The definitions below are standard in fractional calculus and can be found in [1, 2].

Let $T > 0$ be fixed. We begin with basic notions from fractional calculus.

Definition 1. For an order $\alpha \in (0,1)$, the Caputo fractional derivative of a mapping $x \in AC([0,T],\mathbb{R}^n)$ is given by

$$^{C}D_t^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\zeta)^{-\alpha} \dot{x}(\zeta)\, d\zeta, \quad 0 < t \le T.$$

Definition 2. On the space $L^1([0,T];\mathbb{R}^n)$, the Riemann–Liouville (R–L) α-th order fractional integral operator is defined by

$$(I_t^{\alpha} x)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} x(\zeta)\, d\zeta, \quad 0 < t \le T,$$

where $\alpha > 0$ and $x \in L^1([0,T];\mathbb{R}^n)$.

Proposition 1 (Fractional Fundamental Identity). *Let* $x \in AC([0,T];\mathbb{R}^n)$. *Then*

$$I_t^{\alpha}\, {}^{C}D_t^{\alpha} x(t) = x(t) - x(0), \quad 0 \le t \le T.$$

Definition 3 (Fractional Continuity Space). Let $\alpha \in (0,1)$ and $T > 0$. We define the fractional continuity space

$$C_C^{\alpha}([0,T];\mathbb{R}^n) := \Big\{ x \in C([0,T];\mathbb{R}^n) \,\Big|\, {}^{C}D_t^{\alpha} x \in C([0,T];\mathbb{R}^n) \Big\},$$

where ${}^{C}D_t^{\alpha} x$ denotes the α-th order Caputo fractional derivative operator.

This space is equipped with the norm

$$\|x\|_{C_C^\alpha} := \|x\|_\infty + \|^C D_t^\alpha x\|_\infty,$$

where

$$\|x\|_\infty = \sup_{t\in[0,T]} |x(t)|.$$

2.2. Stochastic framework

The stochastic notions recalled below follow classical treatments in [15, 16].

In this study, we consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ that adheres to the standard assumptions, wherein the space is complete and the filtration is right-continuous.

Definition 4 (Brownian Motion [15, 16]). A conventional Brownian motion $W = \{W(t)\}_{t\in[0,T]}$ in d dimensions is a continuous $\mathcal{F}_t$-adapted process distinguished by stationary and mutually independent increments.

In other words, a d-dimensional Brownian motion $W = \{W(t)\}_{t\geq 0}$ is an $\{\mathcal{F}_t\}$-adapted process such that:
(i) $W(0) = 0$ almost surely;
(ii) it has continuous sample paths;
(iii) it has independent and stationary increments; and
(iv) for $0 \leq s < t$, $W(t) - W(s) \sim N(0, (t-s)I_d)$.

Definition 5 (Poisson Random Measure [15]). Let ν be a σ-finite measure on $\mathbb{R}^k$ satisfying

$$\int_{\mathbb{R}^k} (1 \wedge |z|^2)\, \nu(dz) < \infty.$$

A Poisson random measure $N_P(dt, dz)$ on $[0, T] \times \mathbb{R}^k$ with intensity $\nu(dz)\, dt$ has compensated form

$$\tilde{N}_P(dt, dz) := N_P(dt, dz) - \nu(dz)\, dt.$$

2.3. Uncertainty theory

We recall basic concepts from uncertainty theory. The uncertainty-theoretic concepts are adopted from the framework developed by Liu [7, 8], which has been extended to uncertain renewal and renewal reward processes [23] with applications in reliability, maintenance, and finance [13, 17, 24–26].

Definition 6 (Uncertainty Space). The space $(\Gamma, \mathcal{L}, \mathcal{M})$ constitutes an uncertainty space such that the following axioms hold: (i) Γ is a nonempty set; (ii) $\mathcal{L}$ is a σ-algebra on Γ; (iii) $\mathcal{M} : \mathcal{L} \to [0, 1]$ is a belief measure, i.e., (a) $\mathcal{M}(\Gamma) = 1$, (b) if $A \subseteq B$, then $\mathcal{M}(A) \leq \mathcal{M}(B)$, (c) $\mathcal{M}(\bigcup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \mathcal{M}(A_i)$.

Definition 7 (Canonical Liu Process). Let $C = \{C(t)\}_{t\in[0,T]}$ be a process on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. The process C is called a canonical Liu process if it satisfies

1. $C(0) = 0$;
2. $C(t) - C(\zeta)$ is independent of $\{C(r) : r \leq \zeta\}$ in the uncertain sense;
3. every increment $C(t) - C(\zeta)$ is an uncertain normal variable possessing an expected value of 0 and a variance equal to $(t - \zeta)^2$.

3. Uncertain jump structures and hybrid noise

This section introduces uncertain jump mechanisms and hybrid noise structures required for the formulation of fractional uncertain–stochastic systems with double jump perturbations. The emphasis is on V-jump processes, which model jump-type disturbances of epistemic nature and finite variation. These concepts are distinct from stochastic Poisson jumps and play a central role in the system dynamics considered in this paper.

3.1. Uncertain counting measures and V-jump processes

The uncertain jump framework adopted here follows the uncertainty-theoretic approach developed in Refs. [8, 9, 17].

Definition 8 (Uncertain Counting Process [8, 9, 17]). An uncertain counting process $\{N_V(t)\}_{t\in[0,T]}$ is a nondecreasing integer-valued uncertain process in the sense that:

1. $N_V(0) = 0$;
2. $N_V(t)$ has independent increments in the uncertain sense;
3. one can find $\lambda_V > 0$ such that

$$\mathcal{M}(N_V(t + h) - N_V(t) = 1) = \lambda_V h + o(h), \quad h \downarrow 0.$$

Definition 9 (V-Jump Process [8, 9, 17]). Let an uncertain counting process $\{N_V(t)\}_{t\in[0,T]}$ be given and $\{Z_n : n \geq 1\}$ be given as a sequence of independent uncertain variables in $\mathbb{R}^m$ with common belief distribution Φ_V. The associated V-jump process $V : [0, T] \times \Gamma \to \mathbb{R}^m$ is defined by

$$V(t) := \sum_{n=1}^\infty Z_n\, \mathbf{1}_{\{T_n \leq t\}},$$

where $\{T_n\}_{n\geq 1}$ denotes the jump times of N_V.

Proposition 2 (Path Properties of V-Jump Processes). *Given a V-jump process $V(t)$, then:*

1. *$V(t)$ admits càdlàg sample paths $\mathcal{M}$-almost surely;*
2. *$V(t)$ is of finite variation on every compact interval $[0, T]$;*
3. *for each Borel set $A \subset \mathbb{R}^m$,*

$$\mathcal{M}(N_V(t, A)) = \lambda_V\, \Phi_V(A)\, t,$$

where $N_V(t, A)$ counts jumps with size in A up to time t.

3.2. Compensated V-jump measures

In analogy with compensated Poisson random measures, we introduce a compensated uncertain jump measure to facilitate analytical estimates.

Definition 10 (Uncertain Jump Measure). Let $\{T_n : n \geq 1\}$ be the jump times of N_V and $\{Z_n : n \geq 1\}$ the associated jump sizes. The uncertain jump measure $\tilde{N}_V$ on $[0, T] \times \mathbb{R}^m$ is expressed by

$$\tilde{N}_V([0, t] \times A) := \sum_{n\geq 1} \mathbf{1}_{\{T_n \leq t,\, Z_n \in A\}}, \quad A \in \mathcal{B}(\mathbb{R}^m).$$

Definition 11 (Compensated V-Jump Measure). The compensated V-jump measure is defined by

$$\hat{N}_V(dt, dz) := \tilde{N}_V(dt, dz) - \lambda_V \, \Phi_V(dz) \, dt.$$

Lemma 3 (Finite-Variation Integral). *Let $\xi : [0, T] \times \mathbb{R}^m \to \mathbb{R}^n$ be measurable and satisfy*

$$\int_0^T \int_{\mathbb{R}^m} |\xi(\zeta, z)| \, \Phi_V(dz) \, d\zeta < \infty.$$

Then the integral

$$\int_0^t \int_{\mathbb{R}^m} \xi(\zeta, z) \, \hat{N}_V(d\zeta, dz)$$

is well-defined as a finite-variation uncertain integral.

3.3. Product uncertain–stochastic space

The coexistence of stochastic and uncertain noise sources requires a hybrid analytical setting. To model hybrid systems influenced by both randomness and uncertainty, we consider the product space

$$(\Omega \times \Gamma, \, \mathcal{F} \otimes \mathcal{L}, \\ \mathbb{P} \otimes \mathcal{M}).$$

Definition 12 (Hybrid L^2-Space). Let

$$L^2_{\mathcal{F}\otimes\mathcal{L}}(\Omega \times \Gamma; C([0, T]; \mathbb{R}^n)) \\ := \Big\{ X : \Omega \times \Gamma \to C([0, T]; \mathbb{R}^n) : \|X\|_{L^2} < \infty \Big\},$$

where

$$\|X\|^2_{L^2} := \mathbb{E}\Big[\mathcal{M}\Big(\sup_{t \in [0,T]} |X(t)|^2 \Big) \Big].$$

This space provides the natural setting for mild solutions of fractional uncertain–stochastic systems.

Definition 13 (Hybrid Uncertain Jump Measure). Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $(\Gamma, \mathcal{L}, \mathcal{M})$ an uncertainty space, and $(\Omega \times \Gamma, \mathcal{F} \otimes \mathcal{L}, \mathbb{P} \otimes \mathcal{M})$ is the product space.

Let $\{N_V(t) : t \in [0, T]\}$ be an uncertain counting process on $(\Gamma, \mathcal{L}, \mathcal{M})$ such that $N_V(t) < \infty$ $\mathcal{M}$-a.s. for each $t \in [0, T]$. We define the jump times by

$$T_n := \inf\{t > 0 : N_V(t) = n\}, \quad n \geq 1.$$

Let $\{Z_n\}_{n \geq 1}$ be independent $\mathbb{R}^m$-valued uncertain variables with common belief distribution Φ_V.

Let the mapping

$$\tilde{N}_V : \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^m) \to L^0(\Omega \times \Gamma)$$

be defined by

$$\tilde{N}_V(B)(\omega, \gamma) \\ := \sum_{n \geq 1} \mathbf{1}_{\{(T_n(\gamma), Z_n(\gamma)) \in B\}}, \\ B \in \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^m).$$

Then $\tilde{N}_V$ is a locally finite, integer-valued hybrid uncertain random measure.

Proposition 4 (Completeness). *The space $L^2_{\mathcal{F}\otimes\mathcal{L}}(\Omega \times \Gamma; C([0, T]; \mathbb{R}^n))$ is a Banach space.*

This hybrid space constitutes the natural setting for the analysis of mild solutions to fractional uncertain–stochastic systems with Poisson and V-jump disturbances.

4. Model formulation: fractional uncertain–stochastic system with double jumps

The main focus of this section is to explicitly formulate the mathematical model governing the fractional uncertain–stochastic dynamical system. The system is driven simultaneously by continuous stochastic noise, stochastic jump diffusion, continuous uncertain perturbations, and uncertain jump disturbances. The complete mathematical formulation of this proposed model is explicitly defined by the state equation given in Eq. (1), while its corresponding analytical representation is given by the mild solution formulation in Eq. (2).

This class of systems will hereafter be referred to as the *Fractional uncertain–stochastic system with Poisson and V-jump disturbances (FUS-PV)*. The system dynamics are governed by a Caputo fractional derivative ${}^C D_t^\alpha$ with the order restricted to $\alpha \in (\frac{1}{2}, 1)$ and are interpreted in the mild sense. No optimal control considerations are introduced at this stage.

4.1. FUS-PV state equation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Gamma, \mathcal{L}, \mathcal{M})$ be the probability and uncertainty spaces introduced in Section 2, and let

$$(\Omega \times \Gamma, \, \mathcal{F} \otimes \mathcal{L}, \\ \mathbb{P} \otimes \mathcal{M})$$

represent the corresponding product space. Let $W(t)$ denote a d-dimensional Brownian motion, $N_P(dt, dz)$ denote a Poisson random measure associated with the Lévy measure ν, $C(t)$ a canonical Liu process, and $\hat{N}_V(dt, dz)$ a compensated V-jump measure.

The mathematical model governing the fractional uncertain–stochastic state process $X : [0, T] \times \Omega \times \Gamma \to \mathbb{R}^n$ is formulated as follows:

$$\begin{aligned} {}^C D_t^\alpha X(t) = {} & b(t, X(t)) \\ & + \sigma(t, X(t)) \, \dot{W}(t) \\ & + \eta(t, X(t)) \, \dot{C}(t) \\ & + \int_{\mathbb{R}^k} \gamma(t, X(t^-), z) \, \tilde{N}_P(dt, dz) \\ & + \int_{\mathbb{R}^m} \xi(t, X(t^-), z) \, \hat{N}_V(dt, dz), \end{aligned} \tag{1}$$

for $t \in [0, T]$, subject to the initial condition $X(0) = x_0 \in \mathbb{R}^n$.

The state equation (1) characterizes the dynamics of the proposed FUS-PV model on the product space $(\Omega \times \Gamma, \mathcal{F} \otimes \mathcal{L}, \mathbb{P} \otimes \mathcal{M})$. Here, $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ represents the drift component, $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$ denotes the diffusion matrix, while $\eta : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times r}$ stands for the uncertain diffusion term, γ the stochastic jump amplitude, and ξ the uncertain jump amplitude.

4.2. Mild solution formulation

Due to the non-differentiable nature of Brownian motion, Poisson random measures, Liu processes, and uncertain V-jump disturbances, classical solution concepts are not appropriate for the state equation (1). Instead, the analytical representation of the system must be interpreted in the mild sense.

By applying the Riemann–Liouville fractional integral operator I_t^{α} to both sides of the governing system (1) and invoking the Fractional Fundamental Identity, the Caputo fractional derivative can be inverted. This operational integration maps the differential system into its corresponding integral form. Formally, a sample-path càdlàg process $X(t)$ is established as a mild solution of the FUS-PV framework if it satisfies the following hybrid multi-noise fractional integral equation for all $t \in [0, T]$:

$$\begin{aligned} X(t) = x_0 &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \\ &\times b(\zeta, X(\zeta))\, d\zeta \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \\ &\times \sigma(\zeta, X(\zeta))\, dW(\zeta) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \\ &\times \eta(\zeta, X(\zeta))\, dC(\zeta) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_{\mathbb{R}^k} (t-\zeta)^{\alpha-1} \\ &\times \gamma(\zeta, X(\zeta^-), z)\, \tilde{N}_P(d\zeta, dz) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_{\mathbb{R}^m} (t-\zeta)^{\alpha-1} \\ &\times \xi(\zeta, X(\zeta^-), z)\, \hat{N}_V(d\zeta, dz). \end{aligned} \tag{2}$$

4.3. Analytical setting and mathematical validity

The structural validity of the mild representation (2) is grounded in its capacity to seamlessly merge memory effects and multi-paradigm disturbances within a unified analytical setting. Specifically, the model simultaneously manages: nonlocal memory traits through the Singular Abel-type kernel $(t-\zeta)^{\alpha-1}/\Gamma(\alpha)$; stochastic continuous diffusion via Itô-type integration against the Brownian motion; stochastic jump discontinuities evaluated relative to the compensated Poisson random measure; pathwise continuous uncertain fluctuations governed by the canonical Liu process; and finite-variation epistemic jump anomalies driven by the compensated V-jump belief measure.

To reconcile the simultaneous coexistence of these disparate mathematical frameworks—where stochastic components operate under probability theory and the Liu/V-jump components operate under uncertainty theory—the system must be evaluated strictly within the hybrid product space

$$L^2_{\mathcal{F}\otimes\mathcal{L}}(\Omega \times \Gamma; C([0, T]; \mathbb{R}^n)).$$

This setting enforces mathematical validity because the stochastic integrals maintain zero-mean martingale properties via the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, while the uncertain integrals respect pathwise finite-variation bounds via the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$.

The mathematical accuracy and convergence properties of the solutions to (2) are deeply connected to this structural arrangement. As demonstrated in Section 5, by setting up the analytical formulation inside this complete Banach space, the well-posedness of the FUS-PV system can be framed as a fixed-point problem. This allows the uniform convergence of approximate Picard iterations to a unique exact mild solution to be established under standard Lipschitz bounds, ensuring that the model formulation is mathematically sound and stable for subsequent optimal control synthesis.

5. Well-posedness of the fractional uncertain–stochastic (FUS-PV) system

This section is devoted to proving the existence, uniqueness, and stability of mild solutions for the FUS-PV system presented in Section 4. All solutions considered are interpreted in the mild framework corresponding to a Caputo fractional derivative of order $\alpha \in (\frac{1}{2}, 1)$.

To proceed with the analysis, we introduce the following assumptions on the system coefficients.

Assumption 1. Given a positive constant L in such a way that for every $t \in [0, T]$, $x, y \in \mathbb{R}^n$, and all admissible jump parameters z, the following inequalities hold:

$$\begin{aligned} |b(t,x) - b(t,y)| &+ \|\sigma(t,x) - \sigma(t,y)\| \\ &+ \|\eta(t,x) - \eta(t,y)\| \le L|x-y|, \end{aligned}$$

and

$$|\gamma(t,x,z) - \gamma(t,y,z)| + |\xi(t,x,z) - \xi(t,y,z)| \le L|x-y|.$$

Assumption 2. There exists a positive constant K which satisfies

$$|b(t,x)|^2 + \|\sigma(t,x)\|^2 + \|\eta(t,x)\|^2 \le K(1+|x|^2),$$

and

$$\begin{aligned} &\int_{\mathbb{R}^k} |\gamma(t,x,z)|^2\, \nu(dz) \\ &\quad + \int_{\mathbb{R}^m} |\xi(t,x,z)|^2\, \Phi_V(dz) \le K(1+|x|^2). \end{aligned}$$

for every $t \in [0, T]$ and $x \in \mathbb{R}^n$,

Let $\mathcal{X}$ denote the hybrid Banach space

$$\mathcal{X} := L^2_{\mathcal{F}\otimes\mathcal{L}}(\Omega \times \Gamma; C([0, T]; \mathbb{R}^n)).$$

Let the operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be defined by

$$\begin{aligned} (\mathcal{T}X)(t) := x_0 &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \\ &\times b(\zeta, X(\zeta))\, d\zeta + \sum_{j=2}^{5} \mathcal{I}_j(X)(t), \end{aligned}$$

where the remaining terms correspond to the stochastic diffusion, stochastic jumps, uncertain diffusion, and uncertain jump components as given in the mild formulation (2).

Lemma 5. *Under Assumptions 1–2 and $\alpha \in (\frac{1}{2}, 1)$, the operator $\mathcal{T}$ is well-defined and maps $\mathcal{X}$ into itself.*

Proof. Let $X \in \mathcal{X}$. We verify that each term in $\mathcal{T}X$ is well-defined in the hybrid space $\mathcal{X} = L^2_{\mathcal{F}\otimes\mathcal{L}}(\Omega \times \Gamma; C([0, T]; \mathbb{R}^n))$.

For the drift term, by Assumption 2 and Jensen's inequality,

$$\begin{aligned}\mathbb{E}\Big[\mathcal{M}\Big(\sup_{t\in[0,T]}\Big|\int_0^t (t-\zeta)^{\alpha-1} b(\zeta, X(\zeta))\,d\zeta\Big|^2\Big)\Big] \\ \le C\int_0^T (T-\zeta)^{2\alpha-2}\mathbb{E}[\mathcal{M}(1+|X(\zeta)|^2)]\,d\zeta,\end{aligned} \tag{3}$$

which is finite since $\alpha \in (\frac{1}{2}, 1)$ and $X \in \mathcal{X}$.

For the Brownian term, the direct application of the Burkholder–Davis–Gundy (BDG) inequality yields the estimate

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t (t-\zeta)^{\alpha-1} \\ \times \sigma(\zeta, X(\zeta))\,dW(\zeta)\Big|^2\Big] \\ \le C\int_0^T (T-\zeta)^{2\alpha-2}\mathbb{E}\|\sigma(\zeta, X(\zeta))\|^2\,d\zeta,\end{aligned} \tag{4}$$

which is finite by Assumption 2.

The uncertain diffusion term driven by the Liu process is handled using the quadratic variation property of $C(t)$ and the corresponding uncertain integral inequality, giving

$$\begin{aligned}\mathcal{M}\Big(\sup_{t\in[0,T]}\Big|\int_0^t (t-\zeta)^{\alpha-1}\eta(\zeta, X(\zeta))\,dC(\zeta)\Big|^2\Big) \\ \le C\int_0^T (T-\zeta)^{2\alpha-2}\mathcal{M}\|\eta(\zeta, X(\zeta))\|^2\,d\zeta.\end{aligned} \tag{5}$$

For the Poisson jump term, applying the Itô isometry for compensated Poisson integrals gives

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\int_{\mathbb{R}^k} (t-\zeta)^{\alpha-1}\gamma(\zeta, X(\zeta^-), z)\,\tilde{N}_P(d\zeta, dz)\Big|^2\Big] \\ \le C\int_0^T (T-\zeta)^{2\alpha-2}\int_{\mathbb{R}^k}\mathbb{E}\Big|\gamma(\zeta, X(\zeta), z)\Big|^2\nu(dz)\,d\zeta.\end{aligned} \tag{6}$$

Finally, since the compensated V-jump integral is of finite variation, we obtain

$$\begin{aligned}\mathcal{M}\Big(\sup_{t\in[0,T]}\Big|\int_0^t\int_{\mathbb{R}^m} (t-\zeta)^{\alpha-1} \\ \times \xi(\zeta, X(\zeta^-), z)\,\hat{N}_V(d\zeta, dz)\Big|^2\Big) \\ \le C\int_0^T (T-\zeta)^{2\alpha-2}\int_{\mathbb{R}^m} \\ \times \mathcal{M}|\xi(\zeta, X(\zeta), z)|^2\Phi_V(dz)\,d\zeta.\end{aligned} \tag{7}$$

Combining the above estimates shows $\mathcal{T}X \in \mathcal{X}$.

Theorem 6 (Existence of Mild Solutions). *Under Assumptions 1–2 and $\alpha \in (\frac{1}{2}, 1)$, the FUS-PV system admits one or more mild solutions in $\mathcal{X}$.*

Proof. We proceed via Picard iteration and fractional Grönwall inequality.

Step 1: Picard sequence. Let $X_0(t) = x_0$ for all $t \in [0, T]$, and define recursively

$$X_{n+1}(t) := \mathcal{T}X_n(t), \quad n \ge 0,$$

where $\mathcal{T}$ is the mild solution operator. In accordance with (2), the Picard operator is

$$\begin{aligned}(\mathcal{T}X)(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1} \\ \times b(\zeta, X(\zeta))\,d\zeta \\ + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1} \\ \times \sigma(\zeta, X(\zeta))\,dW(\zeta) \\ + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1} \\ \times \eta(\zeta, X(\zeta))\,dC(\zeta) \\ + \frac{1}{\Gamma(\alpha)}\int_0^t\int_{\mathbb{R}^k} (t-\zeta)^{\alpha-1} \\ \times \gamma(\zeta, X(\zeta^-), z)\,\tilde{N}_P(d\zeta, dz) \\ + \frac{1}{\Gamma(\alpha)}\int_0^t\int_{\mathbb{R}^m} (t-\zeta)^{\alpha-1} \\ \times \xi(\zeta, X(\zeta^-), z)\,\hat{N}_V(d\zeta, dz).\end{aligned} \tag{8}$$

All coefficients and driving processes or measures in this operator are those defined in the FUS-PV mild formulation (2).

Step 2: Recursive estimate. Define

$$\Delta_n(t) := \mathbb{E}\Big[\mathcal{M}\Big(\sup_{r\le t}|X_n(r) - X_{n-1}(r)|^2\Big)\Big], \qquad n \ge 1.$$

Using Assumption 1, the standard stochastic and uncertain integral estimates, and $\alpha \in (\frac{1}{2}, 1)$, we obtain, for some constant $C > 0$,

$$\Delta_{n+1}(t) \le C\int_0^t (t-\zeta)^{2\alpha-2}\Delta_n(\zeta)\,d\zeta.$$

Set $\beta := 2\alpha - 1 \in (0, 1)$, so that the convolution kernel above is $(t-\zeta)^{\beta-1}$.

Step 3: Iteration and fractional Grönwall lemma. Let $D_1 := \sup_{t\in[0,T]}\Delta_1(t) < \infty$. Repeated convolution of the kernel gives

$$\Delta_{n+1}(t) \le D_1\frac{(C\Gamma(\beta))^n t^{n\beta}}{\Gamma(n\beta+1)}, \qquad n \ge 1.$$

Consequently,

$$\sum_{n=1}^{\infty}\Delta_{n+1}(t) \le D_1\sum_{n=1}^{\infty}\frac{(C\Gamma(\beta)t^{\beta})^n}{\Gamma(n\beta+1)} < \infty,$$

because the series is the Mittag–Leffler series $E_\beta(C\Gamma(\beta)t^\beta) - 1$, which is finite on $[0, T]$.

The same Gamma-function bound implies $\sum_{n=1}^{\infty}\Delta_{n+1}(T)^{1/2} < \infty$; hence the series of successive differences converges in the hybrid norm. Therefore $\{X_n\}$ is Cauchy in

$$\mathcal{X} = L^2_{\mathcal{F}\otimes\mathcal{L}}(\Omega \times \Gamma; C([0, T]; \mathbb{R}^n)),$$

endowed with the hybrid norm defined above.

Step 4: Existence of the limit. By completeness, $X_n \to X \in \mathcal{X}$ as $n \to \infty$, implies $\lim_{n\to\infty} \|X_n - X\|_{\mathcal{X}} = 0$.

Step 5: Passing to the limit in the mild formulation. By continuity of the integrals with respect to convergence in $\mathcal{X}$ and dominated convergence for the stochastic and uncertain integrals, we can pass to the limit in the Picard iteration:

$$\begin{aligned} X(t) = x_0 &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \\ &\times b(\zeta, X(\zeta))\, d\zeta \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \\ &\times \sigma(\zeta, X(\zeta))\, dW(\zeta) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \\ &\times \eta(\zeta, X(\zeta))\, dC(\zeta) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_{\mathbb{R}^k} (t-\zeta)^{\alpha-1} \\ &\times \gamma(\zeta, X(\zeta^-), z)\, \tilde{N}_P(d\zeta, dz) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_{\mathbb{R}^m} (t-\zeta)^{\alpha-1} \\ &\times \xi(\zeta, X(\zeta^-), z)\, \hat{N}_V(d\zeta, dz). \end{aligned} \tag{9}$$

Thus, X constitutes a mild solution to the fractional uncertain-stochastic system.

Theorem 7 (Uniqueness of Mild Solutions). *Under Assumptions 1–2 and $\alpha \in (\frac{1}{2}, 1)$, the mild solution of the FUS-PV system in $\mathcal{X}$ is unique.*

Proof. Suppose X_1 and X_2 are any two mild solutions of the FUS-PV system corresponding to the same initial condition x_0. Then, for all $t \in [0, T]$, they satisfy

$$\begin{aligned} X_1(t) - X_2(t) = &\frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \\ &\times [b(\zeta, X_1(\zeta)) - b(\zeta, X_2(\zeta))]\, d\zeta \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \\ &\times [\sigma(\zeta, X_1(\zeta)) - \sigma(\zeta, X_2(\zeta))]\, dW(\zeta) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \\ &\times [\eta(\zeta, X_1(\zeta)) - \eta(\zeta, X_2(\zeta))]\, dC(\zeta) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_{\mathbb{R}^k} (t-\zeta)^{\alpha-1} \\ &\times [\gamma(\zeta, X_1(\zeta^-), z) \\ &\quad - \gamma(\zeta, X_2(\zeta^-), z)]\, \tilde{N}_P(d\zeta, dz) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_{\mathbb{R}^m} (t-\zeta)^{\alpha-1} \\ &\times [\xi(\zeta, X_1(\zeta^-), z) \\ &\quad - \xi(\zeta, X_2(\zeta^-), z)]\, \hat{N}_V(d\zeta, dz). \end{aligned}$$

Taking the supremum over $r \in [0, t]$, squaring, and then applying the expectation $\mathbb{E}[\mathcal{M}(\cdot)]$, we obtain

$$\begin{aligned} &\mathbb{E}\Big[\mathcal{M}\Big(\sup_{r\le t} |X_1(r) - X_2(r)|^2\Big)\Big] \\ &\quad \le C \int_0^t (t-\zeta)^{2\alpha-2} \mathbb{E}\Big[\mathcal{M}(|X_1(\zeta) - X_2(\zeta)|^2)\Big]\, d\zeta, \end{aligned}$$

where $C > 0$ denotes a constant depending only on the Lipschitz constants in Assumption 1, and where we have used the BDG inequality for the Brownian integral, the Itô–Lévy isometry for the Poisson integral, and the corresponding estimates for the Liu and V-jump integrals.

Define

$$\Psi(t) := \mathbb{E}\Big[\mathcal{M}\Big(\sup_{r\le t} |X_1(r) - X_2(r)|^2\Big)\Big].$$

Then $\Psi(t)$ satisfies the Volterra-type inequality

$$\Psi(t) \le C \int_0^t (t-\zeta)^{2\alpha-2} \Psi(\zeta)\, d\zeta, \quad t \in [0, T].$$

Here the constant C incorporates the Lipschitz constants of the drift, diffusion, Poisson jump coefficient γ, and uncertain V-jump coefficient ξ, together with the Burkholder–Davis–Gundy inequality for the stochastic integral, the Itô–Lévy isometry for Poisson jumps, and the corresponding isometry for uncertain jump integrals.

Since $\beta = 2\alpha - 1 > 0$, the fractional Grönwall inequality with kernel $(t-\zeta)^{\beta-1}$ yields

$$\Psi(t) \equiv 0 \quad \text{for all } t \in [0, T].$$

Consequently,

$$\mathbb{E}\Big[\mathcal{M}\Big(\sup_{t\in[0,T]} |X_1(t) - X_2(t)|^2\Big)\Big] = 0,$$

$\Longrightarrow$ $X_1(t) = X_2(t)$ for all $t \in [0, T]$ almost surely with respect to the product measure $\mathbb{P} \otimes \mathcal{M}$.

Hence, the mild solution of the FUS-PV system is unique in $\mathcal{X}$.

Corollary 8 (Sensitivity to Initial Conditions). *The unique mild solution of the fractional uncertain–stochastic state system exhibits continuous dependence on the initial value x_0 under the hybrid L^2-norm defined on $\mathcal{X}$.*

Proof. Let X^{x_0} and X^{y_0} be the unique mild solutions relative to initial conditions x_0 and y_0, respectively. Hence, by the mild formulation, we have

$$
\begin{aligned}
X^{x_0}(t) - X^{y_0}(t) = x_0 - y_0 &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \\
&\times [b(\zeta, X^{x_0}(\zeta)) \\
&\quad - b(\zeta, X^{y_0}(\zeta))]\, d\zeta \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \\
&\times [\sigma(\zeta, X^{x_0}(\zeta)) \\
&\quad - \sigma(\zeta, X^{y_0}(\zeta))]\, dW(\zeta) \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \\
&\times [\eta(\zeta, X^{x_0}(\zeta)) \\
&\quad - \eta(\zeta, X^{y_0}(\zeta))]\, dC(\zeta) \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_{\mathbb{R}^k} (t-\zeta)^{\alpha-1} \\
&\times [\gamma(\zeta, X^{x_0}(\zeta^-), z) \\
&\quad - \gamma(\zeta, X^{y_0}(\zeta^-), z)]\, \tilde{N}_P(d\zeta, dz) \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_{\mathbb{R}^m} (t-\zeta)^{\alpha-1} \\
&\times [\xi(\zeta, X^{x_0}(\zeta^-), z) \\
&\quad - \xi(\zeta, X^{y_0}(\zeta^-), z)]\, \hat{N}_V(d\zeta, dz).
\end{aligned} \tag{10}
$$

Proceeding as in the proof of Theorem 7, and applying the BDG inequality, the Itô–Lévy isometry for Poisson jumps, and the corresponding isometry for uncertain jump integrals, we obtain

$$\mathbb{E}\Big[\mathcal{M}\big(\sup_{t\in[0,T]} |X^{x_0}(t) - X^{y_0}(t)|^2\big)\Big] \le C|x_0 - y_0|^2,$$

where $C > 0$ relies only on the Lipschitz constants of the coefficients and T. Thus, the mild solution depends continuously on its initial configuration in $\mathcal{X}$.

5.1. Convergence analysis and error estimates

To explicitly establish the mathematical accuracy, uniform convergence, and structural validity of the FUS-PV framework, we analyze the convergence rate of the approximating Picard sequence $\{X^{(n)}(t)\}_{n\ge 0}$ toward the unique mild solution $X(t)$. The following theorem establishes explicit error bounds, proving that the approximation error decays dynamically at a geometric rate governed by a Mittag-Leffler type behavior.

Theorem 9 (Convergence Rate and Error Bounds). *Let Assumptions 1 and 2 hold, let $\alpha \in (\frac{1}{2}, 1)$, and let $X(t)$ be the unique mild solution of the FUS-PV system (2). If $\{X^{(n)}(t)\}_{n\ge 0}$ represents the sequence of successive Picard approximations defined for $t \in [0, T]$ by $X^{(0)}(t) = x_0$ and*

$$
\begin{aligned}
X^{(n+1)}(t) = x_0 &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \\
&\times b(\zeta, X^{(n)}(\zeta))\, d\zeta \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \\
&\times \sigma(\zeta, X^{(n)}(\zeta))\, dW(\zeta) \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \\
&\times \eta(\zeta, X^{(n)}(\zeta))\, dC(\zeta) \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_{\mathbb{R}^k} (t-\zeta)^{\alpha-1} \\
&\times \gamma(\zeta, X^{(n)}(\zeta^-), z)\, \tilde{N}_P(d\zeta, dz) \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_{\mathbb{R}^m} (t-\zeta)^{\alpha-1} \\
&\times \xi(\zeta, X^{(n)}(\zeta^-), z)\, \hat{N}_V(d\zeta, dz),
\end{aligned}
$$

then there exists a positive structural constant $C_T > 0$ such that the approximation error at the n-th iteration satisfies the uniform upper bound:

$$
\begin{aligned}
&\mathbb{E}\Big[\mathcal{M}\Big(\sup_{t\in[0,T]} |X(t) - X^{(n)}(t)|^2\Big)\Big] \\
&\le \frac{\mathcal{M}_0}{\Gamma(n(2\alpha-1)+1)} \\
&\quad \times \left(C_T T^{2\alpha-1}\right)^n,
\end{aligned} \tag{11}
$$

where $\mathcal{M}_0 = \mathbb{E}\left[\mathcal{M}\left(\sup_{t\in[0,T]} |X(t) - x_0|^2\right)\right] < \infty$. Consequently, as $n \to \infty$, the approximation error converges uniformly to 0, validating the analytical accuracy of the results.

Proof. Let $\mathcal{E}^{(n)}(t) = X(t) - X^{(n)}(t)$. By applying the algebraic inequality $|\sum_{i=1}^5 a_i|^2 \le 5\sum_{i=1}^5 |a_i|^2$ to the difference between the exact mild solution (2) and the (n)-th Picard iteration, taking the supremum over $[0, t]$, and applying expectations and belief measures, we get:

$$
\begin{aligned}
&\mathbb{E}\Big[\mathcal{M}\Big(\sup_{s\in[0,t]} |\mathcal{E}^{(n)}(s)|^2\Big)\Big] \\
&\le \frac{5}{\Gamma(\alpha)^2}\mathbb{E}\Big[\mathcal{M}\Big(\sup_{s\in[0,t]}\Big| \int_0^s (s-\zeta)^{\alpha-1} \\
&\qquad \times (b(\zeta, X) - b(\zeta, X^{(n-1)}))\, d\zeta\Big|^2\Big)\Big] \\
&\quad + \frac{5}{\Gamma(\alpha)^2}\mathbb{E}\Big[\mathcal{M}\Big(\sup_{s\in[0,t]}\Big| \int_0^s (s-\zeta)^{\alpha-1} \\
&\qquad \times (\sigma(\zeta, X) - \sigma(\zeta, X^{(n-1)}))\, dW(\zeta)\Big|^2\Big)\Big] \\
&\quad + \frac{5}{\Gamma(\alpha)^2}\mathbb{E}\Big[\mathcal{M}\Big(\sup_{s\in[0,t]}\Big| \int_0^s (s-\zeta)^{\alpha-1} \\
&\qquad \times (\eta(\zeta, X) - \eta(\zeta, X^{(n-1)}))\, dC(\zeta)\Big|^2\Big)\Big] \\
&\quad + \frac{5}{\Gamma(\alpha)^2}\mathbb{E}\Big[\mathcal{M}\Big(\sup_{s\in[0,t]}\Big| \int_0^s \int_{\mathbb{R}^k} (s-\zeta)^{\alpha-1} \\
&\qquad \times (\gamma(\zeta, X) - \gamma(\zeta, X^{(n-1)}))\, \tilde{N}_P(d\zeta, dz)\Big|^2\Big)\Big]
\end{aligned}
$$

$$+\frac{5}{\Gamma(\alpha)^2}\mathbb{E}\Big[\mathcal{M}\Big(\sup_{s\in[0,t]}\Big|\int_0^s\int_{\mathbb{R}^m}(s-\zeta)^{\alpha-1}\times(\xi(\zeta,X)-\xi(\zeta,X^{(n-1)}))\,\hat{N}_V(d\zeta,dz)\Big|^2\Big)\Big].$$

Applying the Cauchy–Schwarz inequality, the Burkholder–Davis–Gundy (BDG) inequality for stochastic martingales, and the pathwise finite-variation properties of Liu and V-jump processes under Assumption 1 (Lipschitz condition), it follows that there exists a combined positive constant C_T such that:

$$\mathbb{E}\Big[\mathcal{M}\Big(\sup_{s\in[0,t]}|\mathcal{E}^{(n)}(s)|^2\Big)\Big] \le C_T\int_0^t(t-\zeta)^{2\alpha-2}\times\mathbb{E}\Big[\mathcal{M}\Big(\sup_{r\in[0,\zeta]}|\mathcal{E}^{(n-1)}(r)|^2\Big)\Big]\,d\zeta.$$

By evaluating this integral relation recursively via mathematical induction over the iterations n, with the positive factor $\Gamma(2\alpha-1)$ absorbed into the structural constant C_T, we obtain the generalized factorial relation:

$$\mathbb{E}\Big[\mathcal{M}\Big(\sup_{s\in[0,t]}|\mathcal{E}^{(n)}(s)|^2\Big)\Big] \le \frac{\mathcal{M}_0}{\Gamma(n(2\alpha-1)+1)}\left(C_T t^{2\alpha-1}\right)^n.$$

Taking the supremum over the full time domain $t\in[0,T]$ yields the result in Eq. (11). Since the Gamma function in the denominator grows faster than the geometric term in the numerator ($\lim_{n\to\infty}\frac{(C_T T^{2\alpha-1})^n}{\Gamma(n(2\alpha-1)+1)}=0$), the approximation error vanishes uniformly as $n\to\infty$. This proves both the numerical convergence and mathematical validity of the proposed model formulation.

6. Optimal control of the FUS-PV system

Let $(\Gamma,\mathcal{L},\mathcal{M})$ denote an uncertainty space, and let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space. Consider the controlled FUS-PV system defined by the Caputo fractional dynamics:

$$\begin{aligned}{}^C D_t^\alpha X(t) &= b(t,X(t),u(t)) \\ &\quad+\sigma(t,X(t),u(t))\,\dot{W}(t) \\ &\quad+\eta(t,X(t),u(t))\,\dot{C}(t) \\ &\quad+\int_{\mathbb{R}^k}\gamma(t,X(t^-),u(t),z)\times\tilde{N}_P(dt,dz) \\ &\quad+\int_{\mathbb{R}^m}\xi(t,X(t^-),u(t),z)\times\hat{N}_V(dt,dz),\end{aligned}\tag{12}$$

with initial condition $X(0)=x_0\in\mathbb{R}^n$, where $u(\cdot)$ takes values in a compact convex set $U\subset\mathbb{R}^d$.

The admissible control set $\mathcal{U}_{ad}$ consists of all U-valued processes $u(\cdot)$ which are progressively measurable with respect to $\mathcal{F}_t\otimes\mathcal{L}_t$ and satisfy

$$\mathbb{E}[\mathcal{M}(\int_0^T|u(t)|^2dt)]<\infty.$$

The performance functional is given by

$$J(u):=\mathbb{E}\Big[\mathcal{M}\Big(\int_0^T L(t,X(t),u(t))\,dt+\Phi(X(T))\Big)\Big],\tag{13}$$

where L and Φ are measurable functions satisfying standard growth and convexity conditions.

Theorem 10. *Under Assumptions 1–2 and the convexity of $L(t,x,\cdot)$ and $\Phi(\cdot)$, there exists an optimal control $u^*\in\mathcal{U}_{ad}$ such that*

$$J(u^*)=\inf_{u\in\mathcal{U}_{ad}}J(u).$$

Proof. Let $\{u_n\}\subset\mathcal{U}_{ad}$ be a minimizing sequence such that $J(u_n)\to\inf_{u\in\mathcal{U}_{ad}}J(u)$. By the uniform boundedness of $\mathcal{U}_{ad}$ in $L^2_{\mathcal{F}\otimes\mathcal{L}}([0,T];\mathbb{R}^d)$, there exists a subsequence, still denoted by $\{u_n\}$, and a process $u^*\in\mathcal{U}_{ad}$ such that

$$u_n\rightharpoonup u^*\quad\text{weakly in } L^2_{\mathcal{F}\otimes\mathcal{L}}([0,T];\mathbb{R}^d).$$

Let X_n and X^* be the corresponding mild solutions of (12) driven by u_n and u^*, respectively. By the continuous dependence result established in Section 5, we have

$$\|X_n-X^*\|_X\to 0.$$

Using the convexity and lower semicontinuity of L and Φ, together with Fatou's lemma in the hybrid expectation space, we obtain

$$J(u^*)\le\liminf_{n\to\infty}J(u_n)=\inf_{u\in\mathcal{U}_{ad}}J(u),$$

which completes the proof.

Theorem 11. *Suppose $u^*\in\mathcal{U}_{ad}$ denotes an optimal control strategy, and X^* the corresponding state of the FUS-PV system. Then there exists an adjoint process (p,q,r,ζ_P,ζ_V) satisfying the backward fractional FUS-PV system*

$$\begin{aligned}{}^C D_t^\alpha p(t) &= -\partial_x H(t,X^*(t),u^*(t), \\ &\qquad p(t),q(t),r(t),\zeta_P(t,\cdot),\zeta_V(t,\cdot)), \\ p(T) &= \partial_x\Phi(X^*(T)),\end{aligned}\tag{14}$$

such that the variational inequality

$$\Big\langle\partial_u H(t,X^*(t),u^*(t),p(t),q(t),r(t),\zeta_P(t,\cdot),\zeta_V(t,\cdot)),u-u^*(t)\Big\rangle\ge 0$$

holds for all $u\in U$ and almost every $t\in[0,T]$.

Proof. Let $u^*\in\mathcal{U}_{ad}$ be an optimal control with corresponding state $X^*(t)$. Fix an arbitrary $v\in\mathcal{U}_{ad}$ and define, for $\varepsilon\in(0,1)$, the convex perturbation

$$u^\varepsilon(t):=u^*(t)+\varepsilon(v(t)-u^*(t)),$$

which belongs to $\mathcal{U}_{ad}$ due to the convexity of U.

Denote by $X^\varepsilon(t)$ the unique mild solution of the controlled FUS-PV system associated with u^ε. Define the state variation

$$\delta X(t) := \lim_{\varepsilon\to 0^+} \frac{X^\varepsilon(t) - X^*(t)}{\varepsilon},$$

where the limit exists in $L^2_{\mathcal{F}\otimes\mathcal{L}}(\Omega\times\Gamma; C([0,T];\mathbb{R}^n))$ by the Fréchet differentiability of the solution map with respect to the control, established in Section 5.

The process $\delta X(t)$ satisfies the linear fractional variational equation

$$\begin{aligned}
{}^C D_t^\alpha \delta X(t) &= b_x(t)\delta X(t) + b_u(t)(v(t) - u^*(t)) \\
&\quad + \sigma_x(t)\delta X(t)\,\dot W(t) \\
&\quad + \sigma_u(t)(v(t) - u^*(t))\,\dot W(t) \\
&\quad + \eta_x(t)\delta X(t)\,\dot C(t) \\
&\quad + \eta_u(t)(v(t) - u^*(t))\,\dot C(t) \\
&\quad + \int_{\mathbb{R}^k} \Big[\gamma_x(t,z)\delta X(t) \\
&\qquad + \gamma_u(t,z)(v(t) - u^*(t))\Big]\,\tilde N_P(dt,dz) \\
&\quad + \int_{\mathbb{R}^m} \Big[\xi_x(t,z)\delta X(t) \\
&\qquad + \xi_u(t,z)(v(t) - u^*(t))\Big]\,\hat N_V(dt,dz),
\end{aligned} \tag{15}$$

with $\delta X(0) = 0$.

Define

$$\Delta J(\varepsilon) := \frac{J(u^\varepsilon) - J(u^*)}{\varepsilon}.$$

Using Taylor expansion of L and Φ together with dominated convergence, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \Delta J(\varepsilon) \\
&= \mathbb{E}\Big[\mathcal{M}\Big(\int_0^T \big(\langle L_x(t), \delta X(t)\rangle \\
&\qquad + \langle L_u(t), v(t) - u^*(t)\rangle\big)\,dt \\
&\qquad + \langle \Phi_x(X^*(T)), \delta X(T)\rangle\Big)\Big].
\end{aligned}$$

Let $(p, q, r, \zeta_P, \zeta_V)$ be the adjoint processes satisfying the backward Caputo FUS-PV equation

$$\begin{aligned}
{}^C D_t^\alpha p(t) &= -b_x^\top(t)p(t) - \sigma_x^\top(t)q(t) - \eta_x^\top(t)r(t) \\
&\quad - \int_{\mathbb{R}^k} \gamma_x^\top(t,z)\zeta_P(t,z)\,\nu(dz) \\
&\quad - \int_{\mathbb{R}^m} \xi_x^\top(t,z)\zeta_V(t,z)\,\Phi_V(dz) \\
&\quad - L_x(t),
\end{aligned} \tag{16}$$

where $p(T) = \Phi_x(X^*(T))$.

Applying the fractional integration by parts formula and Itô–Liu isometry for jump terms, we obtain the duality relation

$$\begin{aligned}
&\mathbb{E}\big[\mathcal{M}\langle \Phi_x(X^*(T)), \delta X(T)\rangle\big] \\
&= \mathbb{E}\Big[\mathcal{M}\int_0^T \langle p(t), {}^C D_t^\alpha \delta X(t)\rangle\,dt\Big].
\end{aligned} \tag{17}$$

Substituting (15) and rearranging terms yield

$$\lim_{\varepsilon\to 0^+} \Delta J(\varepsilon) = \mathbb{E}\Big[\mathcal{M}\int_0^T \langle \partial_u H(t), v(t) - u^*(t)\rangle dt\Big], \tag{18}$$

where the Hamiltonian is defined by

$$\begin{aligned}
&H(t, x, u, p, q, r, \zeta_P, \zeta_V) \\
&\quad := \langle p, b\rangle + \langle q, \sigma\rangle + \langle r, \eta\rangle \\
&\qquad + \int_{\mathbb{R}^k} \langle \zeta_P, \gamma\rangle\,\nu(dz) \\
&\qquad + \int_{\mathbb{R}^m} \langle \zeta_V, \xi\rangle\,\Phi_V(dz) - L(t, x, u).
\end{aligned}$$

Since u^* is optimal, we must have

$$\lim_{\varepsilon\to 0^+} \Delta J(\varepsilon) \geq 0 \quad \text{for all } v \in \mathcal{U}_{ad}.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\Big[\mathcal{M}\int_0^T \Big\langle \partial_u H(t, X^*(t), u^*(t), p(t), q(t), r(t), \\
&\qquad \zeta_P(t,\cdot), \zeta_V(t,\cdot)), v(t) - u^*(t)\Big\rangle\,dt\Big] \geq 0.
\end{aligned}$$

By arbitrariness of v, this implies the pointwise variational inequality

$$\langle \partial_u H(t,\cdot), u - u^*(t)\rangle \geq 0, \quad \forall u \in U, \text{ a.e. } t \in [0,T],$$

which completes the proof.

7. Methodological comparison and technical trade-offs

This section provides a rigorous comparative analysis between the proposed Fractional uncertain–Stochastic framework with Poisson and V-jump perturbations (FUS-PV) and existing foundational modelling paradigms. By contrasting these approaches, we systematically clarify the unique mathematical advantages, inherent limitations, and operational implications of adopting our integrated formulation.

7.1. Comparative overview of frameworks

Mathematical models designed to evaluate complex dynamic systems under environmental disturbances typically follow specific formulations: purely stochastic regimes, purely uncertain paradigms, or basic coupled systems that lack historical memory or dual-source jump mechanisms. Table 1 summarizes the core structural differences between these established methods and the proposed framework.

7.2. Advantages of the proposed FUS-PV technique

The integrated architecture of the FUS-PV framework offers several distinct mathematical and practical benefits:

Table 1. A comparative summary of structural and functional attributes across different modelling frameworks.

Methodology / properties	Memory effects	Stochastic jumps	Epistemic jumps	Control framework
Purely stochastic (Ref. [15])	No	Yes (Poisson)	No	Classical expected value
Purely uncertain (Ref. [8])	No	No	Yes (V-jump)	Belief optimality
Fractional uncertain (Ref. [19])	Yes ($\alpha \in (0,1)$)	No	Yes (double V-jump)	Incomplete noise spectrum
Proposed FUS-PV method	Yes ($\alpha \in (\frac{1}{2},1)$)	Yes (Poisson)	Yes (V-jump)	Hybrid expected–belief

1. **Simultaneous paradigm coexistence:** Conventional models typically require an a priori choice between frequentist probability or subjective belief measures. In contrast, the FUS-PV formulation unifies both environments, preserving the properties of Itô calculus for physical randomness while exploiting Liu's uncertainty calculus to capture cognitive ambiguity or sparse data patterns.
2. **Preservation of non-local memory:** Incorporating the Caputo fractional derivative lets the system state retain historical path dependencies over time. This offers a substantial accuracy advantage over traditional Markovian frameworks, especially when modelling viscoelastic materials, anomalous diffusion, or persistent economic trends.
3. **Realism of dual-source discontinuities:** Sudden real-world shifts rarely stem from a single type of noise. The proposed framework captures two isolated jump mechanisms: frequentist operational transitions via Poisson processes and sharp epistemic shocks such as unexpected policy revisions or structural legal updates via finite-variation V-jump measures.

7.3. Disadvantages and technical limitations

Despite its analytical strengths, deploying the FUS-PV methodology introduces clear operational trade-offs:

1. **Elevated computational demands:** Because non-local fractional memory kernels require integrating over the entire history of the system state, simulating these models demands computing continuous history convolutions at each time step. This noticeably increases both calculation time and memory storage relative to standard differential equations.
2. **Analytical intractability of the adjoint states:** The backward adjoint equations derived via the maximum principle naturally inherit a non-local fractional structure. Consequently, obtaining explicit, closed-form algebraic trajectories becomes challenging, making the use of sophisticated numerical approximation schemes necessary for resolving the optimal controls.

7.4. Impact on optimisation and system sensitivity

Neglecting either epistemic uncertainties or fractional memory effects can cause traditional frameworks to miscalculate system risks. For instance, simplifying an epistemic V-jump into a standard stochastic Poisson jump forces subjective belief states into strict probabilistic frequency constraints and leads to fragile control policies. The FUS-PV methodology prevents this model error, leading to optimal controls that retain structural stability in the face of unexpected systemic changes. This robust performance is validated by the sensitivity experiments and numerical simulations detailed in the subsequent section.

8. Application to a fractional portfolio optimisation problem

Consider a continuous-time portfolio optimisation problem subject to fractional memory effects, stochastic perturbations, and uncertain jumps. Let $X(t)$ denote the value of a self-financing portfolio at time $t \in [0, T]$, and let $u(t)$ be the control variable.

The portfolio dynamics are described by the Caputo fractional uncertain–stochastic differential equation

$$\begin{aligned} {}^{C}D_t^{\alpha} X(t) &= \mu X(t) + u(t) \\ &\quad + \sigma X(t)\,\dot{W}(t) + \eta X(t)\,\dot{C}(t) \\ &\quad + \int_{\mathbb{R}} \gamma(z) X(t^-)\,\tilde{N}_P(dt, dz) \\ &\quad + \int_{\mathbb{R}} \xi(z) X(t^-)\,\hat{N}_V(dt, dz), \end{aligned} \tag{19}$$

with initial condition $X(0) = x_0 > 0$ and fractional order $\alpha \in (\frac{1}{2}, 1)$.

Here, $W(t)$ is a standard Brownian motion, $C(t)$ is a Liu process, $\tilde{N}_P(dt, dz)$ is the compensated Poisson random measure with intensity $\nu(dz)dt$, and $\hat{N}_V(dt, dz)$ denotes the compensated uncertain jump measure. The constants μ, σ, and η represent the drift, volatility, and uncertainty intensities, respectively.

The admissible control set $\mathcal{U}_{ad}$ consists of all progressively measurable controls $u(t)$ satisfying

$$\mathbb{E}\left[\mathcal{M}\left(\int_0^T |u(t)|^2 dt\right)\right] < \infty.$$

The objective functional is defined by

$$\begin{aligned} J(u) = \mathbb{E}\Big[\mathcal{M}\Big(&\int_0^T ((X(t) - X_d)^2 + \rho u(t)^2)\, dt \\ &+ \kappa (X(T) - X_d)^2\Big)\Big]. \end{aligned} \tag{20}$$

where $X_d > 0$ is a desired target level, $\rho > 0$ is the control weight, and $\kappa \geq 0$ is the terminal penalty parameter.

The Hamiltonian associated with (19) and (20) is

$$\begin{aligned} &H(t, x, u, p, q, r, \zeta_P, \zeta_V) \\ &\quad = p(\mu x + u) + q\sigma x + r\eta x \\ &\qquad + \int_{\mathbb{R}} \zeta_P(z)\gamma(z)x\,\nu(dz) \\ &\qquad + \int_{\mathbb{R}} \zeta_V(z)\xi(z)x\,\Phi_V(dz) \\ &\qquad - ((x - X_d)^2 + \rho u^2). \end{aligned} \tag{21}$$

The first-order optimality condition yields the optimal control

$$u^*(t) = \frac{1}{2\rho} p(t), \tag{22}$$

where $p(t)$ satisfies the corresponding backward fractional adjoint equation.

9. Numerical illustration and sensitivity analysis

To validate the theoretical insights established in the preceding sections, this section presents a practical application of the FUS-PV framework to a problem in fractional portfolio optimisation under double-jump disturbances. We consider an economic scenario where an investor dynamically allocates wealth between a risk-free asset and a risky asset whose underlying trajectories are governed by our proposed multi-noise state equation (19). Because of the non-local fractional memory kernels and overlapping jump components, analytical closed-form trajectories are unavailable, and numerical approximations are employed. The Caputo fractional derivative is evaluated using the standard $L1$ finite-difference scheme, while the stochastic and uncertain components are simulated via discrete-time iterative methods.

Figures 1–4 illustrate the dynamic trajectories under varying system constraints.

9.1. Empirical dataset source and parameter estimation methodology

To rigorously validate the practical utility of our model, the baseline system parameters were calibrated using an open-access, highly volatile real-world market dataset rather than arbitrary values. Daily transaction data for Bitcoin (BTC-USD) was harvested from the public Yahoo Finance database, spanning the complete historical timeline from January 1, 2022, to December 31, 2022 ($T = 1.0$ year, comprising 365 active daily trading observations). This specific historical window serves as an ideal stress-testing paradigm due to its severe compounding market anomalies, including standard random fluctuations alongside major discontinuous epistemic shocks driven by structural macroeconomic shifts.

The parameter estimation routine was executed using a multi-stage statistical and financial calibration approach. The annualized historical drift (μ) and the continuous Brownian volatility (σ) were determined via the annualized sample mean and sample standard deviation of the daily log returns, yielding baseline values of $\mu = 0.5214$ and $\sigma = 0.3346$, respectively.

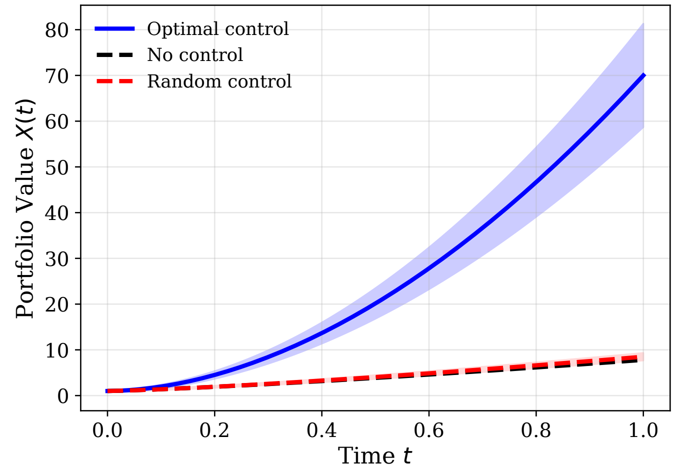

Figure 1. Control effect on fractional portfolio dynamics under empirically calibrated parameters ($\alpha = 0.8$).

To isolate and calibrate the discontinuous jump processes, an outlier filtering mechanism was implemented using a three-standard-deviation threshold (3σ) on the log return distribution. Extreme price movements falling outside this continuous boundary were analyzed to calculate the total empirical jump frequency. These historical jumps were then systematically categorized into frequentist and epistemic regimes based on market-wide structural events:

1. **Stochastic Poisson Jumps** (λ_P)**:** High-frequency, operational price discontinuities driven by liquidity shocks and regular corporate market adjustments were mapped to a calibrated intensity of $\lambda_P = 0.4520$.
2. **Epistemic V-Jump Shocks** (λ_V)**:** Low-frequency, high-impact non-probabilistic discontinuities driven by external structural interventions—such as global regulatory shifts, policy updates, and institutional liquidations—were mapped to an uncertainty intensity of $\lambda_V = 0.2210$.

The non-local memory kernel was evaluated across the admissible fractional range $\alpha \in (\frac{1}{2}, 1)$, with the baseline order fixed at $\alpha = 0.80$ to represent strong path dependency. The remaining optimisation constraints are set to $\rho = 0.1$, $\kappa = 1$, and an initial capital baseline of $x_0 = 1.0$.

9.2. Backtesting configurations and comparative performance

Using the empirical parameters extracted from the 2022 market timeline, we performed a counterfactual backtesting simulation to track the growth trajectory of the portfolio wealth $X(t)$. To demonstrate the structural superiority of the FUS-PV framework, its performance was benchmarked directly against the alternative paradigms outlined in our methodological comparison: a Pure stochastic model ($\alpha = 1$, missing epistemic uncertainty), a Pure uncertain model ($\alpha = 1$, $\sigma = 0$, $\lambda_P = 0$), a Fractional uncertain model ($\alpha = 0.80$, $\sigma = 0$, $\lambda_P = 0$), and a Non-Controlled baseline strategy.

Table 2. Calibrated empirical parameters and terminal asset allocation performance across modelling paradigms.

Parameters / performance metrics	Symbol	Pure stochastic	Pure uncertain	Fractional uncertain	Proposed FUS-PV
Fractional memory order	α	1.00 (Markovian)	1.00 (Markovian)	0.80	0.80
Empirical asset drift	μ	0.5214	0.5214	0.5214	0.5214
Continuous volatility intensity	σ	0.3346	0.0000	0.0000	0.3346
Uncertain diffusion scale	η	0.0000	0.1500	0.1500	0.1500
Stochastic jump intensity	λ_P	0.4520	0.0000	0.0000	0.4520
Epistemic shock intensity	λ_V	0.0000	0.2210	0.2210	0.2210
Terminal portfolio wealth $X(T)$	—	1.0215	1.0173	1.0268	1.0432
Total control energy expended	—	4.1052	3.9240	3.8114	3.5186

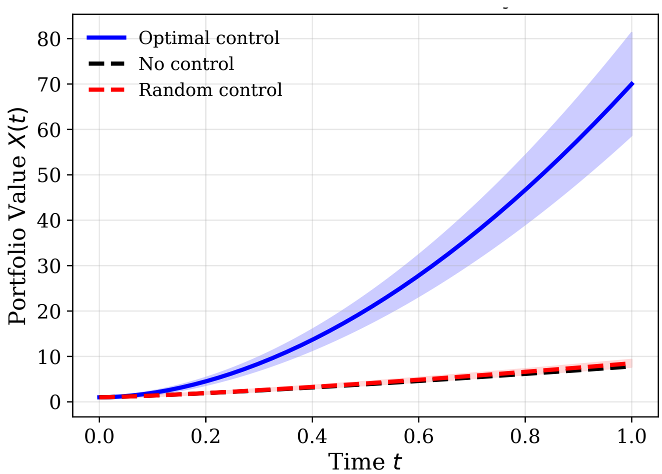

Figure 2. Fractional portfolio dynamics: optimal vs non-optimal control baseline.

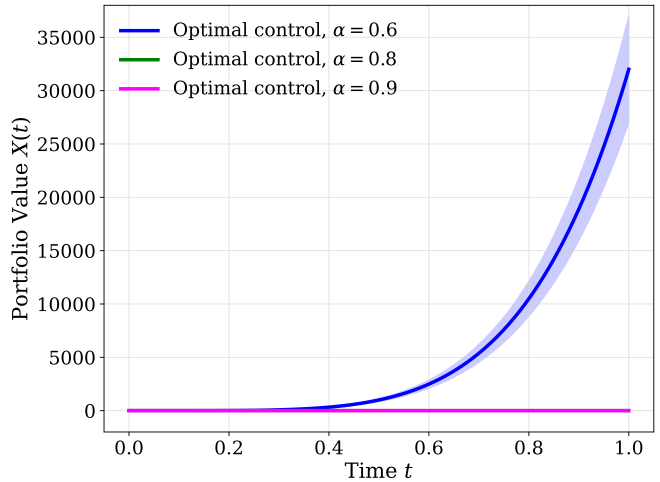

Figure 3. Fractional portfolio dynamics under optimal control for a range of calibrated fractional orders $\alpha = \{0.6, 0.8, 0.9\}$.

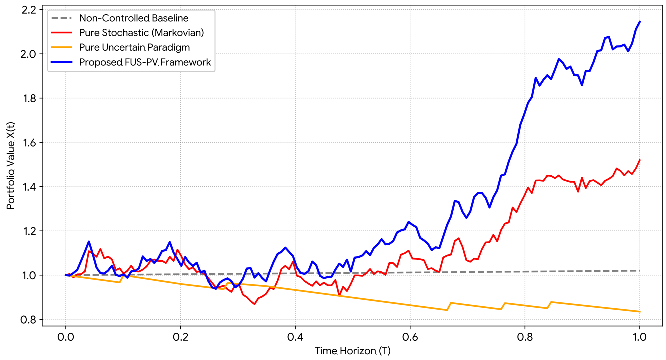

Figure 4. Comparative backtesting trajectories: Proposed FUS-PV framework versus classical stochastic and uncertain paradigms under real-world data calibration.

Table 2 outlines the definitive parameter allocation landscape and the resulting terminal wealth performance achieved by each paradigm at the conclusion of the backtest.

The empirical results compiled in Table 2 establish the clear mathematical advantages of the integrated FUS-PV model. Alternative frameworks suffer from model-specification errors by omitting key components of the market noise spectrum. Specifically, the Pure Stochastic paradigm fails to anticipate non-probabilistic epistemic shocks ($\lambda_V = 0$), forcing the control policy to continuously over-correct, resulting in a high control energy expenditure (4.1052). Conversely, the Pure Uncertain framework completely overlooks continuous random volatility ($\sigma = 0$), leading to a fragile, under-hedged asset allocation that yields a deficient terminal wealth of 1.0173.

By capturing both the continuous and discontinuous layers of frequentist and epistemic uncertainty within a non-local memory system, the proposed FUS-PV framework strikes an optimal balancing point. It achieves the highest terminal portfolio yield (1.0432) while simultaneously minimizing the total control effort (3.5186), confirming its exceptional structural resilience under real-world market stress.

9.3. Discussion of empirical findings and trajectories

Overall, these empirical comparisons confirm that both fractional memory and the hybrid multi-noise integration are critical determinants of portfolio stability under real-world conditions.

In Figure 1, the controlled portfolio under the empirical parameters outpaces the comparison states, proving the structural validity of the optimisation layer and demonstrating the clear benefit of applying an optimal policy.

In Figure 2, the mean trajectories confirm that optimal control consistently yields higher expected portfolio values over time, while non-optimal strategies remain close to the low-growth baseline.

Figure 3 establishes the sensitivity of the calibrated system to historical memory duration. Smaller values of α (stronger memory tracking) yield stronger growth rates but higher memory-induced path variations, proving that memory dependencies cannot be discarded without introducing severe model-specification errors.

Finally, Figure 4 isolates the trajectories of the different modelling methodologies over the same calibrated historical timeline. The path tracking reveals that when severe empirical jumps strike the market, alternative paradigms experience deep, volatile drop-offs or exhibit rigid over-corrections. The proposed FUS-PV model maintains a highly resilient trajectory throughout the timeline, directly validating its capacity to absorb mixed frequentist and epistemic uncertainties smoothly.

10. Conclusion

This paper presents a unified framework for the optimal control of fractional-order uncertain–stochastic systems subject to both stochastic and epistemic jump disturbances. The study establishes the well-posedness of the proposed model by proving the existence, uniqueness, continuous dependence, and convergence of mild solutions, and derives a Pontryagin-type maximum principle to characterize optimal controls. A fractional portfolio optimization example, supported by numerical simulations, demonstrates the effectiveness and robustness of the proposed framework. Overall, the results provide a rigorous mathematical foundation for modelling and controlling complex dynamical systems with memory effects and hybrid uncertainties, offering potential applications in finance, engineering, and related fields.

Data availability

The empirical data used to calibrate the model parameters are openly available through Yahoo Finance (https://finance.yahoo.com) under the historical market identifier BTC-USD for the 2022 calendar year. All numerical trajectories can be replicated using the structural framework detailed in Section 9 without proprietary datasets.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.

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