Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 8 (2026) 3590

Journal of the Nigerian Society of Physical Sciences

On the effect of fixed-bandwidth kernel density estimation on the exponential distribution

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Abstract

Kernel density estimation (KDE) is widely used as a nonparametric smoothing operator in statistics. In this work, we study fixed-bandwidth KDE as a convolution operator applied to an exponential baseline distribution with rate parameter $\beta > 0$. We show that, for any compactly supported kernel K on $[-1, 1]$ and any fixed bandwidth $h > 0$, the expected KDE admits an exact factorization in the interior region $y \geq h$: $\mathbb{E}[\widehat{f}(y)] = \beta e^{-\beta y} C_0$, where $C_0 = \int_{-1}^{1} K(u) e^{\beta h u}\, du$ depends only on the kernel and bandwidth. Thus, the exponential density is an eigenfunction of the kernel-smoothing operator in the interior domain: its shape is preserved up to multiplication by the eigenvalue C_0. After normalization on $[h, \infty)$, the resulting distribution reduces exactly to the shifted exponential density $\beta e^{-\beta(y-h)}$. The result clarifies the role of boundary effects in kernel smoothing and shows that fixed-bandwidth KDE does not generate new parametric families from exponential baselines under tail normalization.

DOI: 10.46481/jnsps.2026.3590

Keywords: Kernel density estimation, Shifted exponential distribution, Hazard rate, Exponential distribution, Compact support

Article History:
Received: 10 June 2026
Received in revised form: 14 July 2026
Accepted for publication: 17 July 2026
Available online: 04 August 2026

Communicated by: B. J. Falaye

1. Introduction and preliminaries

The exponential distribution is one of the simplest and most widely used distributions in the fields of reliability theory, survival analysis, and queuing processes due to its mathematical simplicity and memoryless property [1]. The probability density function (PDF) of the exponential distribution is defined as

$$g(y) = \beta e^{-\beta y}, \qquad y \geq 0, \quad \beta > 0.$$

A commonly used extension is the shifted exponential distribution, obtained by introducing a threshold parameter $h > 0$

$$f_{\mathrm{SE}}(y \mid \beta, h) = \beta e^{-\beta(y-h)}, \qquad y \geq h, \quad \beta > 0.$$

This model is useful in applications where lifetimes are assumed to begin after a specified threshold or delayed onset.

While the exponential distribution remains a cornerstone of reliability analysis, its constant hazard rate limits its ability to describe many observed lifetime patterns. Consequently, considerable effort has been devoted to developing more flexible extensions.

Among these are the exponentiated exponential distribution proposed by Gupta and Kundu [2], the beta exponential distribution [3], and the transmuted shifted exponential distribution developed by Ikechukwu and Eghwerido [4]. Detailed discussions of such models are available in Ref. [5].

The search for more flexible probabilistic models has also led to the development of several new distributions designed for bounded and heavy-tailed data. For example, the Epanechnikov Rayleigh distribution was proposed in Ref. [6], followed

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by the Epanechnikov Pareto distribution proposed by Odat [7] for heavy-tailed applications. Moreover, Fagbamigbe *et al.* [8] proposed the Exponentiated Log-Logistic Weibull distribution for censored lifetime data, while Osi *et al.* [9] introduced the Transmuted Cosine Topp-Leone G family for modeling lifetime data. Subsequently, Odat [10] introduced the Epanechnikov Kumaraswamy distribution to better accommodate bounded data exhibiting heavy-tailed behavior, while Odat [11] proposed a transmuted Ailamujia distribution for reliability studies. Collectively, these contributions reflect the ongoing interest in constructing new distributional families. They also motivate the need to examine carefully whether kernel-based procedures genuinely yield novel models.

Kernel density estimation (KDE) is a fundamental nonparametric technique for estimating unknown probability densities as described in Ref. [12]. Given a random sample

$$Y_1, Y_2, \ldots, Y_n,$$

from a distribution with density g, the KDE at a point y with bandwidth $h > 0$ is defined as

$$\widehat{f}(y) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{y - Y_i}{h}\right),$$

where K is a kernel function satisfying

$$K(u) \geq 0, \qquad \int_{-\infty}^{\infty} K(u)\,du = 1.$$

Throughout this paper, we assume that the kernel function satisfies the following conditions:

- $K(u) \geq 0$ for all u;
- $\int_{-\infty}^{\infty} K(u)\,du = 1$;
- $\mathrm{supp}(K) = [-1, 1]$;
- $K(u) = K(-u)$ (symmetry).

Typical examples include the uniform kernel,

$$K(u) = \frac{1}{2}, \qquad |u| \leq 1,$$

the triangular kernel,

$$K(u) = 1 - |u|, \qquad |u| \leq 1,$$

and the Epanechnikov kernel,

$$K(u) = \frac{3}{4}(1 - u^2), \qquad |u| \leq 1.$$

A natural question is whether applying KDE with a fixed bandwidth to an exponential baseline distribution can generate new parametric families. The underlying intuition is that different kernels may produce distinct smoothed density functions with different hazard rates.

The goal of this note is to show that this intuition does not hold in the classical KDE setting with fixed bandwidths and translation-invariant kernels. We therefore consider the expectation of the KDE estimator under the exponential model.

$$f_{\mathbb{E}}(y) = \mathbb{E}\left[\widehat{f}(y)\right] = \frac{1}{h}\int_{-\infty}^{\infty} K\left(\frac{y - t}{h}\right) g(t)\,dt. \tag{1}$$

We prove that, for any kernel satisfying the above assumptions and any fixed bandwidth $h > 0$, the expected KDE corresponding to an exponential baseline factors in the interior region as

$$f_{\mathbb{E}}(y) = C(\beta, h, K)e^{-\beta y},$$

where $C(\beta, h, K)$ is a constant depending only on the kernel, the bandwidth, and the exponential rate parameter. Consequently, after normalization on the interval $[h, \infty)$, the resulting density reduces exactly to the classical shifted exponential distribution.

Therefore, fixed-bandwidth KDE does not generate genuinely new distributions from an exponential baseline, nor does it alter the constant hazard-rate structure inherited from the exponential model.

Besides clarifying a misconception that may arise in distribution-generation studies, this negative result provides an instructive example of the interplay between kernel smoothing and the structural properties of the exponential distribution. It also highlights the limitations of fixed-bandwidth KDE as a mechanism for producing new parametric families and suggests that alternative constructions, such as variable bandwidth procedures or genuinely nonlinear transformations, are necessary to obtain novel hazard-rate behaviors.

The central aim is to address the possible misconception that fixed-bandwidth KDE applied to an exponential baseline creates new distribution families with different hazard rates. We prove that, for any fixed bandwidth and compactly supported kernel, the expected KDE retains the exponential form in the interior region and becomes a shifted exponential after normalization.

Table 1 summarizes the support properties of the kernels considered in this study, and Figure 1 provides a numerical comparison of their expected and normalized KDEs.

2. Main result

Theorem 1. *Let K be a kernel supported on $[-1, 1]$, and let $h > 0$. Then for all $y \geq h$,*

$$\mathbb{E}[\widehat{f}(y)] = \beta e^{-\beta y} C_0, \tag{2}$$

where

$$C_0 = \int_{-1}^{1} K(u)e^{\beta h u}\,du. \tag{3}$$

Proof. We start from Eq. (1). Since $y \geq h > 0$, we have

$$\mathbb{E}[\widehat{f}(y)] = \frac{1}{h}\int_{0}^{\infty} K\left(\frac{y - t}{h}\right)\beta e^{-\beta t}\,dt.$$

Since K is supported on $[-1, 1]$, the integrand is nonzero only when

$$-1 \leq \frac{y - t}{h} \leq 1 \quad \Rightarrow \quad y - h \leq t \leq y + h.$$

Table 1. Classification of common kernels by support. Only compactly supported kernels satisfy Theorem 1 and reduce exactly to shifted exponential densities after normalization.

Kernel	Compact support?	Support interval	Result
Uniform	Yes	$[-1, 1]$	Shifted exponential
Triangular	Yes	$[-1, 1]$	Shifted exponential
Epanechnikov	Yes	$[-1, 1]$	Shifted exponential
Gaussian	No	$(-\infty, \infty)$	Different (close, but not exact)

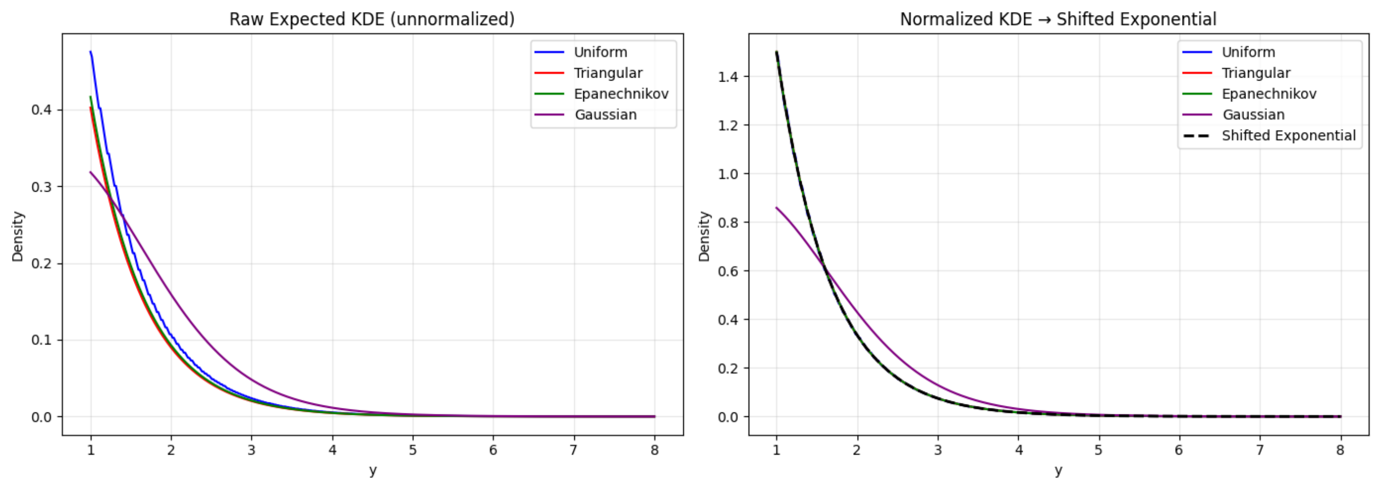

Figure 1. Numerical verification of Theorem 1. Left panel: Unnormalized expected KDEs for the uniform, triangular, Epanechnikov, and Gaussian kernels. Right panel: After normalization, the compactly supported kernels (uniform, triangular, and Epanechnikov) coincide exactly with the shifted exponential density (black curve). The Gaussian kernel (purple curve) differs because it has non-compact support.

For $y \geq h$, this interval lies inside $[0, \infty)$, so

$$\mathbb{E}[\widehat{f}(y)] = \frac{1}{h}\int_{y-h}^{y+h} K\Big(\frac{y-t}{h}\Big)\beta e^{-\beta t}\, dt.$$

Let $u = \frac{y-t}{h}$, so $t = y - hu$, $dt = -h du$. Then

$$\mathbb{E}[\widehat{f}(y)] = \beta e^{-\beta y}\int_{-1}^{1} K(u)e^{\beta h u}\, du.$$

Hence, by Eq. (3),

$$f_{\mathbb{E}}(y) = \beta e^{-\beta y} C_0.$$

□

Remark 1. *Theorem 1 characterizes the expected kernel density estimator. To connect this result with the random KDE used in practice, note that*

$$\widehat{f}(y) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h}K\Big(\frac{y - X_i}{h}\Big),$$

which is the average of i.i.d. random variables for each fixed y and fixed bandwidth h. By the strong law of large numbers,

$$\widehat{f}(y) \xrightarrow{\text{a.s.}} \mathbb{E}[\widehat{f}(y)], \qquad n \to \infty.$$

Therefore, the expression obtained in Theorem 1 may be interpreted as the deterministic large-sample target of the random KDE under fixed bandwidth smoothing. This differs from the classical KDE framework, where the bandwidth depends on the sample size and tends to zero, leading instead to convergence to the underlying density.

Remark 2. *Theorem 1 assumes the kernel is supported on $[-1, 1]$ for simplicity. If the kernel is supported on $[-A, A]$ for some $A > 0$, the result generalizes directly. Repeating the proof with $u = (y - t)/(h)$ yields, for $y \geq Ah$,*

$$f_{\mathbb{E}}(y) = \mathbb{E}[\widehat{f}(y)] = \beta e^{-\beta y}\int_{-A}^{A} K(u)e^{\beta h u}\, du,$$

where $K(u)$ is the kernel scaled to $[-A, A]$. After normalization on $[Ah, \infty)$, the resulting distribution is the shifted exponential $\beta e^{-\beta(y-Ah)}$. Thus, the shift equals Ah, i.e., the product of the bandwidth and the kernel's support radius. The constant C_0 in Theorem 1 generalizes to

$$C_0(A) = \int_{-A}^{A} K(u)e^{\beta h u}\, du.$$

The essential conclusion that only the shifted exponential emerges remains unchanged.

Theorem 2. *Let K be a kernel supported on $[-A, A]$ with $\int_{-A}^{A} K(u)\, du = 1$, and let $h > 0$. Then for all $y \geq Ah$,*

$$\mathbb{E}[\widehat{f}(y)] = \beta e^{-\beta y} C_0(A),$$

where

$$C_0(A) = \int_{-A}^{A} K(u)e^{\beta hu}\, du.$$

After normalization on $[Ah, \infty)$*, the resulting distribution is the shifted exponential*

$$f(y) = \beta e^{-\beta(y-Ah)}, \qquad y \geq Ah.$$

Proof. The proof follows the same steps as that of Theorem 1. Since K is supported on $[-A, A]$, the integrand is nonzero only when

$$-A \leq \frac{y-t}{h} \leq A,$$

which is equivalent to

$$y - Ah \leq t \leq y + Ah.$$

Since the density is supported on $[0, \infty)$, we further restrict to $t \geq 0$, hence

$$t \in [\max(0,\, y - Ah),\, y + Ah].$$

For $y \geq Ah$, we have $y - Ah \geq 0$, so the integration domain becomes

$$t \in [y - Ah,\, y + Ah].$$

Thus,

$$\mathbb{E}[\widehat{f}(y)] = \frac{1}{h}\int_{y-Ah}^{y+Ah} K\Big(\frac{y-t}{h}\Big)\beta e^{-\beta t}\, dt.$$

Let $u = \frac{y-t}{h}$, so that $t = y - hu$ and $dt = -h\, du$. The limits transform as $t = y - Ah \Rightarrow u = A$ and $t = y + Ah \Rightarrow u = -A$. Therefore,

$$\mathbb{E}[\widehat{f}(y)] = \beta e^{-\beta y}\int_{-A}^{A} K(u)e^{\beta hu}\, du.$$

Hence,

$$\mathbb{E}[\widehat{f}(y)] = C_0(A)\beta e^{-\beta y}. \tag{4}$$

Now, we have

$$f_{\text{norm}}(y) = \frac{\mathbb{E}[\widehat{f}(y)]}{\int_{Ah}^{\infty}\mathbb{E}[\widehat{f}(s)]\, ds}, \qquad y \geq Ah.$$

Using Eq. (4), we obtain

$$\int_{Ah}^{\infty}\mathbb{E}[\widehat{f}(s)]\, ds = \int_{Ah}^{\infty}\beta e^{-\beta s}C_0(A)\, ds = C_0(A)\, e^{-\beta Ah}.$$

Therefore,

$$f_{\text{norm}}(y) = \frac{\beta e^{-\beta y}C_0(A)}{C_0(A)e^{-\beta Ah}} = \beta e^{-\beta(y-Ah)}, \qquad y \geq Ah.$$

□

Remark 3. *Theorem 2 shows that, after normalization on* $[Ah, \infty)$*, the density becomes a shifted exponential with shift* Ah*, determined by the lower truncation point.*

The constant $C_0(A)$ *associated with the kernel cancels during normalization; therefore, the final shape is independent of* K*. Nevertheless, the kernel affects the unnormalized expected KDE through* $C_0(A)$*.*

In other words, the invariance applies to the normalized density, not to the original KDE.

2.1. *Relation to classical convolution eigenfunctions*

It is well-known that exponential functions are eigenfunctions of translation-invariant convolution operators on the whole real line. Indeed, for

$$(Tf)(y) = \int_{-\infty}^{\infty} K(y-t)f(t)\, dt,$$

and

$$f(t) = e^{-\beta t},$$

a change of variables yields

$$(Tf)(y) = e^{-\beta y}\int_{-\infty}^{\infty} K(u)e^{\beta u}\, du.$$

Hence

$$Tf = \lambda(\beta)f, \qquad \lambda(\beta) = \int_{-\infty}^{\infty} K(u)e^{\beta u}\, du.$$

The present work differs from this classical setting in two important respects. First, we consider the exponential probability density

$$g(y) = \beta e^{-\beta y},\ y \geq 0,$$

which is supported only on the positive half-line. Second, the expected kernel density estimator is given by

$$\mathbb{E}[\widehat{f}(y)] = \frac{1}{h}\int_{0}^{\infty} K\left(\frac{y-t}{h}\right)g(t)\, dt,$$

so the operator is not a full-line convolution and translation invariance is broken by the boundary at $t = 0$.

The main result of this paper shows that, despite the presence of this boundary, compactly supported kernels preserve an exact exponential shape for $y > h$ after normalization. In contrast, for non-compact kernels such as the Gaussian kernel, an additional factor involving the standard normal distribution function appears, and exact exponentiality is lost. Therefore, the novelty of the present analysis lies in understanding how the positive support constraint interacts with kernel smoothing and in identifying the special role played by compact kernel support.

This has practical consequences. For the traditional full-line case, convolution is translation-invariant and exponential functions are exact eigenfunctions of the corresponding smoothing operator. In contrast, on the positive half-line, the boundary truncation at zero leads to loss of translation invariance, and the corresponding KDE operator does not preserve exponential functions as global exact eigenfunctions.

In this work, we have established that compactly supported kernels provide exact control of boundary effects in the interior domain. Specifically, for $y \geq h$, Theorem 1 shows that the expected KDE admits an exact factorization of the form

$$\mathbb{E}[\widehat{f}(y)] = C_0(h)g(y),$$

where $C_0(h)$ depends only on the kernel and bandwidth, but it is independent of y. Therefore, the exponential density is preserved exactly by the smoothing operator in the interior region and acts as an exact eigenfunction with eigenvalue $C_0(h)$. The small-bandwidth approximation concerns only the eigenvalue, since $C_0(h) \approx 1$ as $h \to 0$, rather than the exponential shape itself.

For kernels with non-compact support, such as the Gaussian kernel, a similar expansion can be carried out, but due to the unbounded support, non-local contributions from the boundary region arise. Hence, the boundary effects are more diffuse compared with the sharply localized effects associated with compactly supported kernels.

Theorem 3. *For $0 \le y < h$, the expected KDE becomes*

$$\mathbb{E}[\widehat{f}(y)] = \beta e^{-\beta y} \int_{-1}^{y/h} K(u) e^{\beta h u}\, du,$$

where the upper limit depends on y. This no longer factors as $e^{-\beta y}$ times a constant, and boundary bias arises.

Proof. For $0 \le y < h$, we start from the expected KDE

$$\mathbb{E}[\widehat{f}(y)] = \frac{1}{h}\int_0^\infty K\Big(\frac{y-t}{h}\Big)\beta e^{-\beta t}\, dt.$$

Since K is supported on $[-1,1]$, the integrand is nonzero only when

$$-1 \le \frac{y-t}{h} \le 1 \quad \Rightarrow \quad y-h \le t \le y+h.$$

However, for $0 \le y < h$, the lower limit $y-h$ is negative. Since the exponential baseline $g(t) = \beta e^{-\beta t}$ is defined only for $t \ge 0$, the effective lower limit of integration is 0, not $y-h$. Thus,

$$\mathbb{E}[\widehat{f}(y)] = \frac{1}{h}\int_0^{y+h} K\Big(\frac{y-t}{h}\Big)\beta e^{-\beta t}\, dt.$$

Now make the change of variable $u = \frac{y-t}{h}$. Then $t = y - hu$, $dt = -h\,du$. When $t = 0$, we have $u = y/h$; when $t = y+h$, we have $u = -1$. Therefore,

$$\begin{aligned}\mathbb{E}[\widehat{f}(y)] &= \frac{1}{h}\int_{y/h}^{-1} K(u)\beta e^{-\beta(y-hu)}(-h)\, du \\ &= \beta e^{-\beta y}\int_{-1}^{y/h} K(u) e^{\beta h u}\, du.\end{aligned}$$

The upper limit of integration is y/h, which lies in $[0,1)$ for $0 \le y < h$. Since this limit depends on y, the integral is no longer a constant multiple of $e^{-\beta y}$. Consequently, the factorization $\mathbb{E}[\widehat{f}(y)] = C_0\beta e^{-\beta y}$ with a constant C_0 fails in the boundary region. The y-dependence of the upper limit is precisely the source of the boundary bias in kernel density estimation. □

To conclude the section, we note that while the density of interest is supported on $[0,\infty)$, the expected KDE is defined on all of $\mathbb{R}$ due to convolution. Hence, $\hat{f}_{\mathbb{E}}(y)$ for $y < 0$ can take positive values since the integral gets nonzero contribution from $t \ge 0$ via the kernel $K\big(\frac{y-t}{h}\big)$. The boundary leakage in the case of compactly supported kernels happens on the strip $-h \le y < 0$, while for non-compact kernels, e.g., the Gaussian kernel, it can happen on the whole negative half-line. Such a behavior illustrates boundary effects of kernel smoothing on a half-line rather than some deficiency of Gaussian kernels.

We use the Gaussian kernel as an example of a non-compact kernel to show that exact factorization may fail without compact support. Our results establish that compact support is a sufficient condition for exact factorization. Notably, these boundary effects do not affect the interior asymptotic results derived below for $y \ge h$.

2.2. Non-compact kernels: the Gaussian case

For a Gaussian kernel,

$$K(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}, \qquad u \in (-\infty,\infty).$$

The expected KDE for an exponential baseline $g(t) = \beta e^{-\beta t}$ is

$$\mathbb{E}[\widehat{f}(y)] = \frac{1}{h}\int_0^\infty K\Big(\frac{y-t}{h}\Big) g(t)\, dt = \frac{\beta}{h}\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-t)^2}{2h^2}} e^{-\beta t}\, dt.$$

First, let $s = t/h$, so $t = hs$, $dt = h\,ds$. Then

$$\mathbb{E}[\widehat{f}(y)] = \frac{\beta}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{(y/h-s)^2}{2}} e^{-\beta h s}\, ds.$$

Now, we complete the square in s. The exponent is

$$-\frac{1}{2}\left(\frac{y}{h} - s\right)^2 - \beta h s.$$

Expanding the square gives

$$-\frac{1}{2}\left(s^2 - 2s\frac{y}{h} + \frac{y^2}{h^2}\right) - \beta h s = -\frac{1}{2}s^2 + s\frac{y}{h} - \frac{y^2}{2h^2} - \beta h s.$$

Group the s terms

$$-\frac{1}{2}s^2 + s\left(\frac{y}{h} - \beta h\right) - \frac{y^2}{2h^2}.$$

So, by completing the square in s, we get

$$-\frac{1}{2}\left[s - \left(\frac{y}{h} - \beta h\right)\right]^2 + \frac{1}{2}\left(\frac{y}{h} - \beta h\right)^2 - \frac{y^2}{2h^2}.$$

Note that

$$\frac{1}{2}\left(\frac{y}{h} - \beta h\right)^2 - \frac{y^2}{2h^2} = \frac{1}{2}\left(\frac{y^2}{h^2} - 2\beta y + \beta^2 h^2\right) - \frac{y^2}{2h^2} = -\beta y + \frac{\beta^2 h^2}{2}.$$

Thus the exponent becomes

$$-\frac{1}{2}\left[s - \left(\frac{y}{h} - \beta h\right)\right]^2 - \beta y + \frac{\beta^2 h^2}{2}.$$

The expected KDE becomes

$$\mathbb{E}[\widehat{f}(y)] = \frac{\beta}{\sqrt{2\pi}} e^{-\beta y + \frac{\beta^2 h^2}{2}} \int_0^\infty e^{-\frac{1}{2}\left[s - \left(\frac{y}{h} - \beta h\right)\right]^2} ds.$$

Let $a = \frac{y}{h} - \beta h$. Then the integral is

$$\int_0^\infty e^{-\frac{1}{2}(s-a)^2} ds.$$

Using the standard normal cumulative distribution function (CDF) Φ,

$$\int_0^\infty e^{-\frac{1}{2}(s-a)^2}\,ds = \sqrt{2\pi}\,\Phi(a),$$

where $\Phi(a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-z^2/2}dz$.

To see this, note that $\int_{-\infty}^{\infty} e^{-\frac{1}{2}(s-a)^2}ds = \sqrt{2\pi}$, and the integral from $-\infty$ to 0 is $\sqrt{2\pi}\,\Phi(-a)$, so the integral from 0 to ∞ is $\sqrt{2\pi}(1-\Phi(-a)) = \sqrt{2\pi}\,\Phi(a)$.
Substitution gives the closed form

$$\mathbb{E}[\widehat{f}(y)] = \beta e^{-\beta y+\frac{\beta^2h^2}{2}}\;\Phi\Big(\frac{y}{h}-\beta h\Big), \qquad y \ge 0.$$

Interpretation. Unlike the compactly supported case, the Gaussian kernel does *not* yield a pure exponential form $\propto e^{-\beta y}$. The factor $\Phi(y/h-\beta h)$ depends on y and tends to 1 only as $y/h \to \infty$. Hence, for large y,

$$\mathbb{E}[\widehat{f}(y)] \approx \beta e^{-\beta y} e^{\frac{\beta^2 h^2}{2}},$$

which is consistent with the compact-support result after normalization (the constant $e^{\beta^2h^2/2}$ plays the role of C_0). For finite y, however, the y-dependence of Φ introduces a deviation from exact exponentiality, confirming that compact support is essential for Theorem 1.

Deviation bound. The discrepancy between the Gaussian-expected kernel density estimate and the exponential approximation $\beta e^{-\beta y}e^{\beta^2h^2/2}$ is given by

$$\Delta(y) = \beta\, e^{-\beta y+\frac{\beta^2h^2}{2}}\left(\Phi\Big(\frac{y}{h}-\beta h\Big)-1\right).$$

Applying the identity $\Phi(z)-1 = -\Phi(-z)$, which holds for every real z, we may rewrite this as

$$\Delta(y) = -\beta\, e^{-\beta y+\frac{\beta^2h^2}{2}}\;\Phi\Big(\beta h-\frac{y}{h}\Big).$$

Since Φ takes values in $[0,1]$, this immediately yields the bound

$$|\Delta(y)| = \beta\, e^{-\beta y+\frac{\beta^2h^2}{2}}\;\Phi\Big(\beta h-\frac{y}{h}\Big),$$

which is exact and holds for all $y \in \mathbb{R}$.

To understand how this bound behaves for large y, note that once $y \ge \beta h^2$ the argument $\beta h - y/h$ is non-positive, and the standard Gaussian tail estimate $\Phi(-t) \le \frac{1}{2}e^{-t^2/2}$ ($t \ge 0$) applies with $t = y/h-\beta h$. This gives

$$\Phi\Big(\beta h-\frac{y}{h}\Big) \le \frac{1}{2}\exp\Big(-\frac{1}{2}\Big(\frac{y}{h}-\beta h\Big)^2\Big),$$

and combining this with the expression for $|\Delta(y)|$ above, after simplifying the exponent, gives

$$|\Delta(y)| \le \frac{\beta}{2}\exp\Big(-\frac{y^2}{2h^2}\Big). \tag{5}$$

The derivative of the exponent in Eq. (5) is

$$\frac{d}{dy}\left(-\frac{y^2}{2h^2}\right) = -\frac{y}{h^2},$$

which is negative for $y > 0$. Therefore, the bound in Eq. (5) decreases monotonically and tends to zero as $y \to \infty$.

Remark 4. *The compact support condition is essential for the exact factorization established in Theorem 1. Kernels without compact support, such as the Gaussian kernel, do not generally satisfy this exact factorization. Figure 1 shows that while uniform, triangular, and Epanechnikov kernels reduce to the shifted exponential form, the Gaussian kernel deviates slightly. The Gaussian kernel provides a natural example for which Theorem 1 does not hold because exact factorization fails. This failure results from the unbounded support of the Gaussian kernel in the present half-line KDE problem; it does not imply that Gaussian kernels are generally inappropriate. Gaussian kernels remain valid in other settings or when boundary effects are handled appropriately. Under the present conditions, however, compact support is necessary.*

Theorem 4. *Let $g(y) = \beta e^{-\beta y}$, $y \ge 0$, and let K be a symmetric kernel supported on $[-1,1]$ satisfying*

$$\int_{-1}^{1} K(u)\,du = 1, \qquad \int_{-1}^{1} uK(u)\,du = 0.$$

Let $h > 0$. Then, for all $y \ge h$,

$$\mathbb{E}[\widehat{f}(y)] = \beta e^{-\beta y}\left(1+\frac{\beta^2h^2}{2}\mu_2 + O(h^4)\right),$$

where

$$\mu_2 = \int_{-1}^{1} u^2K(u)\,du.$$

Moreover, the $O(h^4)$ remainder is uniform in y for $y \ge h$.

Proof. From Theorem 1, for every $y \ge h$,

$$\mathbb{E}[\widehat{f}(y)] = \beta e^{-\beta y}\int_{-1}^{1} K(u)e^{\beta hu}\,du.$$

For each $u \in [-1,1]$, Taylor's theorem gives

$$e^{\beta hu} = 1+\beta hu+\frac{\beta^2h^2u^2}{2}+\frac{\beta^3h^3u^3}{6}+R_4(h,u),$$

where there exist constants $C > 0$ and $h_0 > 0$ such that

$$|R_4(h,u)| \le Ch^4, \qquad u \in [-1,1], \quad 0 < h < h_0.$$

Since $[-1,1]$ is compact, the above estimate is uniform in u.
Substituting this expansion into the integral yields

$$\begin{aligned}\int_{-1}^{1} K(u)e^{\beta hu}\,du = {}& \int_{-1}^{1} K(u)\,du + \beta h\int_{-1}^{1} uK(u)\,du \\ &+ \frac{\beta^2h^2}{2}\int_{-1}^{1} u^2K(u)\,du \\ &+ \frac{\beta^3h^3}{6}\int_{-1}^{1} u^3K(u)\,du \\ &+ \int_{-1}^{1} K(u)R_4(h,u)\,du.\end{aligned}$$

Since K is symmetric,

$$\int_{-1}^{1} uK(u)\,du = 0, \qquad \int_{-1}^{1} u^3K(u)\,du = 0,$$

and, by normalization,

$$\int_{-1}^{1} K(u)\,du = 1.$$

Hence,

$$\int_{-1}^{1} K(u)e^{\beta hu}\,du = 1 + \frac{\beta^2 h^2}{2}\mu_2 + \int_{-1}^{1} K(u)R_4(h,u)\,du,$$

where

$$\mu_2 = \int_{-1}^{1} u^2 K(u)\,du.$$

Furthermore,

$$\left|\int_{-1}^{1} K(u)R_4(h,u)\,du\right| \le \int_{-1}^{1} |K(u)|\,|R_4(h,u)|\,du \le Ch^4 \int_{-1}^{1} |K(u)|\,du,$$

which shows that

$$\int_{-1}^{1} K(u)e^{\beta hu}\,du = 1 + \frac{\beta^2 h^2}{2}\mu_2 + O(h^4).$$

Therefore,

$$\mathbb{E}[\widehat{f}(y)] = \beta e^{-\beta y}\left(1 + \frac{\beta^2 h^2}{2}\mu_2 + O(h^4)\right). \quad (6)$$

Subtracting $g(y) = \beta e^{-\beta y}$ from both sides of Eq. (6) gives the corresponding bias expansion:

$$\mathbb{E}[\widehat{f}(y)] - g(y) = \beta e^{-\beta y}\left(\frac{\beta^2 h^2}{2}\mu_2 + O(h^4)\right).$$

Finally, the estimate

$$\left|\int_{-1}^{1} K(u)R_4(h,u)\,du\right| \le Ch^4 \int_{-1}^{1} |K(u)|\,du$$

is independent of y. Since $0 < e^{-\beta y} \le 1$ for $y \ge h$, multiplication by $\beta e^{-\beta y}$ preserves this bound. Hence the $O(h^4)$ remainder is uniform in y for all $y \ge h$. □

Remark 5. *Theorem 1 shows that for any compactly supported kernel and any $y \ge h$,*

$$\mathbb{E}[\widehat{f}(y)] = C_0 \cdot \beta e^{-\beta y},$$

i.e., the exponential form is preserved exactly *(the eigenfunction property). The bandwidth h and kernel K affect only the multiplicative constant $C_0 = \int_{-1}^{1} K(u)e^{\beta hu}du$.*

What is approximate *is the closeness of $\mathbb{E}[\widehat{f}(y)]$ to the true density $g(y) = \beta e^{-\beta y}$. Theorem 4 shows that $C_0 = 1 + \frac{\beta^2 h^2}{2}\mu_2 + O(h^4)$, so the bias is of order $O(h^2)$. Thus the exponential is an exact eigenfunction, but the eigenvalue C_0 is close to 1 for small h.*

Corollary 1. *For $y \ge h$,*

$$(K_h * g)(y) = C_0 g(y),$$

so the exponential density is an eigenfunction of the kernel smoothing operator in the interior region.

Remark 6. *Theorem 1 shows that regardless of the kernel shape, the expected KDE is proportional to $e^{-\beta y}$. The kernel only affects the constant of proportionality C_0, not the functional form in y.*

Interpretation

The operator $T_h[f] = K_h * f$ defines a smoothing (convolution) operator. Theorem 1 shows that the exponential density is invariant under this operator up to a multiplicative constant in the interior region $y \ge h$. Thus:

- The kernel shape affects only the eigenvalue C_0;
- The exponential functional form is preserved exactly in the interior.

2.3. Normalization and the shifted exponential

The function $f(y) = \beta C_0 e^{-\beta y}$ is defined for $y \ge h$, but it does not necessarily integrate to 1 over $[h,\infty)$. Let C be the normalizing constant

$$\int_{h}^{\infty} \beta C_0 e^{-\beta y} dy = \beta C_0 \cdot \frac{e^{-\beta h}}{\beta} = C_0 e^{-\beta h}.$$

Therefore, the normalized PDF is

$$f_{\text{norm}}(y) = \frac{\beta C_0 e^{-\beta y}}{C_0 e^{-\beta h}} = \beta e^{-\beta(y-h)}, \quad y \ge h.$$

This is exactly the classical shifted exponential distribution.

Theorem 5. *After normalization to integrate to 1 over $[h,\infty)$, the expected KDE of an exponential baseline with any compactly supported kernel and any fixed bandwidth $h > 0$ yields the shifted exponential distribution*

$$f(y \mid \beta, h) = \beta e^{-\beta(y-h)}, \quad y \ge h.$$

The hazard rate is constant, specifically, $\lambda(y) = \beta$.

Remark 7. *Normalizing on $[h,\infty)$ can be interpreted as conditioning on the event $Y \ge h$, i.e., considering the tail conditional version of the expected KDE. Such a restriction is natural in settings where boundary effects are excluded from the analysis, as in threshold models and reliability contexts (e.g., warranty or survival regimes).*

More precisely, if $\mu_h(x) = \mathbb{E}[\hat{f}_h(x)]$, we consider the normalized tail form

$$\mu_h^{\text{tail}}(x) = \frac{\mu_h(x)}{\int_h^{\infty} \mu_h(u)\,du}, \quad x \ge h.$$

We emphasize that this normalization is not intended to alter the underlying probabilistic model or to define a new distribution family. Rather, it isolates the asymptotic shape of μ_h

away from boundary distortion, which affects the region $x < h$ (see Theorem 3). In this sense, the result characterizes a universal tail behavior shared by all compactly supported kernels, while no claim is made regarding global equivalence on $\mathbb{R}_+$.

2.4. Consequences for kernel selection

The results lead to the following observations:

1. No new distribution under fixed-bandwidth smoothing: For an exponential baseline density and compactly supported kernels satisfying the assumptions of the theorem, fixed-bandwidth kernel smoothing does not generate a new family of distributions. After normalization on the interior region, the result is again a shifted exponential form.
2. Preservation of the constant hazard structure: The normalized distribution obtained from this smoothing procedure retains the constant hazard rate β. Hence, within this framework, non-constant hazard behavior cannot be produced through fixed-bandwidth KDE of an exponential baseline.
3. Kernel-shape independence after normalization: Different compactly supported kernels, such as the uniform, triangular, and Epanechnikov kernels, influence the multiplicative constant C_0; however, this constant disappears after normalization, leaving the same shifted exponential form.
4. Role of the bandwidth: The bandwidth h determines the location of the boundary threshold and the resulting shift, but it does not modify the hazard rate shape in the normalized interior distribution.

2.5. Correct normalization

A common error in such derivations is miscalculating the normalizing constant. From Theorem 1, the unnormalized density is

$$f_{\text{unnorm}}(y) = \beta e^{-\beta y} \int_{-1}^{1} K(u) e^{\beta h u} du.$$

Let $I_K(\beta, h) = \int_{-1}^{1} K(u) e^{\beta h u} du$. Then

$$\int_h^\infty f_{\text{unnorm}}(y) dy = \beta I_K(\beta, h) \cdot \frac{e^{-\beta h}}{\beta} = I_K(\beta, h) e^{-\beta h}.$$

Thus, the correct normalizing constant is

$$C_{\text{norm}} = \frac{1}{I_K(\beta, h) e^{-\beta h}} = \frac{e^{\beta h}}{I_K(\beta, h)}.$$

Multiplying gives

$$f(y) = \frac{e^{\beta h}}{I_K(\beta, h)} \cdot \beta e^{-\beta y} I_K(\beta, h) = \beta e^{-\beta(y-h)}.$$

The constant cancels, confirming the result.

3. Examples for specific kernels

3.1. Uniform kernel

For $K(u) = 1/2$ on $[-1, 1]$

$$I_K(\beta, h) = \int_{-1}^{1} \frac{1}{2} e^{\beta h u} du = \frac{1}{2} \cdot \frac{e^{\beta h} - e^{-\beta h}}{\beta h} = \frac{\sinh(\beta h)}{\beta h}.$$

Then

$$f(y) = \beta e^{-\beta(y-h)}.$$

3.2. Triangular kernel

For $K(u) = 1 - |u|$ on $[-1, 1]$

$$I_K(\beta, h) = \int_{-1}^{1} (1 - |u|) e^{\beta h u} du = \frac{2}{\beta^2 h^2} (\cosh(\beta h) - 1).$$

Again, after normalization

$$f(y) = \beta e^{-\beta(y-h)}.$$

3.3. Epanechnikov kernel

For the Epanechnikov kernel

$$K(u) = \frac{3}{4}(1 - u^2), \qquad -1 \le u \le 1,$$

the normalization factor is given by

$$I_K(\beta, h) = \frac{3}{4} \int_{-1}^{1} (1 - u^2) e^{\beta h u} \, du.$$

Let

$$a = \beta h.$$

Then

$$\begin{aligned} I_K(\beta, h) &= \frac{3}{4} \int_{-1}^{1} (1 - u^2) e^{a u} \, du \\ &= \frac{3}{a^3} \left(a \cosh(a) - \sinh(a) \right). \end{aligned}$$

Hence,

$$I_K(\beta, h) = \frac{3 \left(a \cosh(a) - \sinh(a) \right)}{a^3}, \qquad a = \beta h.$$

Using the Taylor expansions of $\sinh(a)$ and $\cosh(a)$ around zero,

$$a \cosh(a) - \sinh(a) = \frac{a^3}{3} + \frac{a^5}{30} + O(a^7),$$

and therefore

$$I_K(\beta, h) = 1 + \frac{a^2}{10} + O(a^4).$$

Consequently,

$$I_K(\beta, h) \to 1 \qquad \text{as} \qquad \beta h \to 0,$$

which confirms the normalization of the Epanechnikov kernel.

Normalization again yields the shifted exponential.

4. How to obtain genuinely new distributions

The main result indicates that fixed-bandwidth, translation-invariant kernel smoothing of an exponential baseline cannot generate genuinely new distributional forms. In particular, the exponential structure is preserved up to a multiplicative factor, so no new hazard shapes arise within this framework. To move beyond this limitation, one must break the underlying factorization $f(y) \propto e^{-\beta y}$ that drives this rigidity.

One natural direction is to allow the bandwidth to vary with the evaluation point, that is, to take $h = h(y)$. In this case, the smoothed density takes the form

$$f(y) = \beta e^{-\beta y} \int_{-1}^{1} K(u)\, e^{\beta h(y) u}\, du,$$

and the exponential factor can no longer be separated from the kernel contribution, potentially leading to new structural behavior.

Another possibility is to change the baseline model itself. Replacing the exponential distribution with alternatives such as the Weibull, gamma, or lognormal distributions removes the memoryless structure and naturally allows for a richer variety of hazard-rate functions.

A further extension is to relax translation invariance by using kernels that depend separately on y and t, rather than only through the scaled difference $(y - t)/h$. Finally, one may consider mixture models or exponentially tilted constructions, where kernel smoothing is applied to more flexible underlying distributions.

Each of these approaches departs from the rigid structure of the classical fixed-bandwidth KDE and can therefore produce genuinely new distributional families.

5. Simulation verification

We conducted a small simulation to verify the theorem numerically. For $\beta = 1.5$ and $h = 1$, we generated 10,000 samples from an exponential distribution, applied uniform-kernel smoothing with bandwidth $h = 1$, and compared the empirical expected KDE with the theoretical shifted exponential density.

We present a numerical evaluation of the analytical expressions derived in the previous sections. Figure 1 illustrates the expected KDE for the uniform, triangular, Epanechnikov, and Gaussian kernels with $\beta = 1.5$ and $h = 1$, together with the corresponding normalized curves.

6. Conclusion

In this manuscript, we established a general negative result showing that fixed-bandwidth kernel density estimation does not generate new distributions when applied to an exponential baseline. More precisely, for any translation-invariant kernel with compact support on $[-1, 1]$ and any fixed bandwidth $h > 0$, the expected KDE retains the exponential form up to a multiplicative constant in the interior region $y \geq h$. After normalization, this leads back to the classical shifted exponential distribution. As a result, no genuinely new distribution is obtained, and the constant hazard rate property of the exponential model is preserved. This holds for all standard compactly supported kernels, including the uniform, triangular, and Epanechnikov kernels.

It is important to mention that the above statement does not hold for the whole positive half-line. In the range $0 \leq y < h$, one would not expect the expected KDE to decompose into a constant multiplying $e^{-\beta y}$. It is only by confining our attention to the tail part $[h, \infty)$ that we can get the shifted exponential form.

The central message of this article is cautionary. In the context of classical KDE with a finite bandwidth, where the kernel is compactly supported, trying to create new families of parametric distributions using the kernel smoother on the exponential distribution will not yield anything new; the same distributional structure will be obtained again. We believe that making such a statement may prevent unproductive exploration and demonstrates how tightly the memorylessness of the exponential distribution limits the impact of smoothing.

It is important to stress that this result is specific to the classical KDE framework with fixed bandwidth and translation-invariant kernels. Many well-known alternatives deliberately move beyond these restrictions. For instance, asymmetric kernel methods such as the gamma kernel estimator of Chen in Ref. [13], transformation-based approaches like log KDE as in Refs. [14–16], adaptive or variable bandwidth methods as described in Ref. [17], and asymmetric smoothing techniques on restricted domains as in Ref. [18] can produce much richer behavior. In these settings, the exponential structure is no longer preserved, and more flexible hazard shapes may appear. Our result should therefore be understood as describing the limitations of the classical KDE setting rather than restricting the broader field of density estimation. Finally, several directions for future work remain open. One possible extension is the development of neutrosophic kernel density estimation and its applications in reliability analysis.

Data availability

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Declaration of competing interest

The author declares that there are no known competing financial or personal interests that could have appeared to influence the work reported in this article.

Funding

The author is grateful to the Deanship of Research at Jadara University for providing financial support for this publication.

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